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ON INVERSE γ-SYSTEMS AND THE NUMBER OF $L_{\infty\lambda}$-EQUIVALENT, NON-ISOMORPHIC MODELS FOR λ SINGULAR

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Abstract. Suppose λ is a singular cardinal of uncountable cofinality κ. For a model $\mathscr{M}$ of cardinality λ, let $\mathrm{No}(\mathscr{M})$ denote the number of isomorphism types of models $\mathscr{N}$ of cardinality λ which are $L_{\infty\lambda}$-equivalent to $\mathscr{M}$. In [7] Shelah considered inverse κ-systems $\mathscr{A}$ of abelian groups and their certain kind of quotient limits $\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A})$. In particular Shelah proved in [7, Fact 3.10] that for every cardinal μ there exists an inverse κ-system $\mathscr{A}$ such that $\mathscr{A}$ consists of abelian groups having cardinality at most μ^κ and $\mathrm{card}(\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A})) = \mu$. Later in [8, Theorem 3.3] Shelah showed a strict connection between inverse κ-systems and possible values of No (under the assumption that $\theta^\kappa < \lambda$ for every $\theta < \lambda$): if $\mathscr{A}$ is an inverse κ-system of abelian groups having cardinality $< \lambda$, then there is a model $\mathscr{M}$ such that $\mathrm{card}(\mathscr{M}) = \lambda$ and $\mathrm{No}(\mathscr{M}) = \mathrm{card}(\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A}))$. The following was an immediate consequence (when $\theta^\kappa < \lambda$ for every $\theta < \lambda$): for every nonzero $\mu < \lambda$ or $\mu = \lambda^\kappa$ there is a model $\mathscr{M}_\mu$ of cardinality λ with $\mathrm{No}(\mathscr{M}_\mu) = \mu$. In this paper we show: for every nonzero $\mu \leq \lambda^\kappa$ there is an inverse κ-system $\mathscr{A}$ of abelian groups having cardinality $< \lambda$ such that $\mathrm{card}(\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A})) = \mu$ (under the assumptions $2^\kappa < \lambda$ and $\theta^{<\kappa} < \lambda$ for all $\theta < \lambda$ when $\mu > \lambda$), with the obvious new consequence concerning the possible value of No. Specifically, the case $\mathrm{No}(\mathscr{M}) = \lambda$ is possible when $\theta^\kappa < \lambda$ for every $\theta < \lambda$.

§1. Introduction. Suppose λ is a cardinal. For a model $\mathscr{M}$ we let $\mathrm{card}(\mathscr{M})$ denote the cardinality of the universe of $\mathscr{M}$. When $\mathscr{M}$ and $\mathscr{N}$ are models of the same vocabulary and they satisfy the same sentences of the infinitary language $L_{\infty\lambda}$, we write $\mathscr{M} \equiv_{\infty\lambda} \mathscr{N}$. For any model $\mathscr{M}$ of cardinality λ we define $\mathrm{No}(\mathscr{M})$ to be the cardinality of the set

$$\{\, \mathscr{N}/\cong \mid \mathrm{card}(\mathscr{N}) = \lambda \text{ and } \mathscr{N} \equiv_{\infty\lambda} \mathscr{M} \,\},$$

where $\mathscr{N}/\cong$ is the equivalence class of $\mathscr{N}$ under the isomorphism relation. Our principal purpose is to study the possible values of $\mathrm{No}(\mathscr{M})$ for models $\mathscr{M}$ of singular cardinality with uncountable cofinality.

When $\mathscr{M}$ is countable, $\mathrm{No}(\mathscr{M}) = 1$ by [4]. This result extends to structures of cardinality λ when λ is a singular cardinal of countable cofinality [1].

If $V = L$, λ is an uncountable regular cardinal which is not weakly compact, and $\mathscr{M}$ is a model of cardinality λ, then $\mathrm{No}(\mathscr{M})$ has either the value 1 or 2^λ. For $\lambda = \aleph_1$

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this result was first proved in [2]. Later in [5] Shelah extended this result to all other regular non-weakly compact cardinals. The possibility $\mathrm{No}(\mathscr{M}) = \aleph_0$ is consistent with ZFC+GCH in case $\lambda = \aleph_1$, as remarked in [5]. The values $\mathrm{No}(\mathscr{M}) \in \omega \setminus \{0, 1\}$ are proved to be consistent with ZFC+GCH in the forthcoming paper of the authors [11] (number 646 in Shelah's publications).

The case $\mathscr{M}$ has cardinality of a weakly compact cardinal is dealt with in [6] by Shelah. The result is that for κ weakly compact there is for every $1 \leq \mu \leq \kappa$ a model $\mathscr{M}_\mu$ such that $\mathrm{No}(\mathscr{M}_\mu) = \mu$. There is in preparation by the authors a paper where the question for κ weakly compact is revisited.

The case $\mathscr{M}$ is of singular cardinality λ with uncountable cofinality κ was first treated in [7], where the relations of $\mathscr{M}$ have infinitely many places. Later in [8] Shelah improved the result by showing that if $\theta^\kappa < \lambda$ for every $\theta < \lambda$ and $0 < \mu < \lambda$ then $\mathrm{No}(\mathscr{M}) = \mu$ is possible for a model $\mathscr{M}$ having cardinality λ and relations of finitely many places only. The main idea in those papers was to transform the problem of possible values of $\mathrm{No}(\mathscr{M})$ into a question concerning possible cardinalities of "quotient limit" $\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A})$ of an inverse system $\mathscr{A}$ of groups [8, Theorem 3.3]:

THEOREM 1 (λ cardinal with $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$). *If $\theta^\kappa < \lambda$ for every $\theta < \lambda$ and $\mathscr{A}$ is an inverse κ-system of abelian groups having cardinality $< \lambda$, then there is a model $\mathscr{M}$ of cardinality λ (with relations having finitely many places only) such that*

$$\mathrm{No}(\mathscr{M}) = \mathrm{card}(\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A})).$$

Actually the groups in [8, Theorem 3.3] are not limited to be abelian. However, abelian groups suffice for the present purposes.

The recent paper fills a gap left open since the paper [8]. We present a uniform way to construct inverse κ-system of abelian groups having a quotient limit of desired cardinality. The most important new case is that the cardinality of a quotient limit can be λ for some inverse system (in other cases, where the result below can be applied, the Singular Cardinal Hypothesis fails). The result of this paper is:

THEOREM 2 (λ cardinal with $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$). *For every nonzero $\mu \leq \lambda$ there is an inverse κ-system $\mathscr{A} = \langle G_i, h_{i,j} \mid i < j < \kappa \rangle$ of abelian groups satisfying that* $\mathrm{card}(G_i) < \lambda$ *for every $i < \kappa$ and*

$$\mathrm{card}(\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A})) = \mu.$$

The same conclusion holds also for the values $\lambda < \mu \leq \lambda^\kappa$ under the assumption that $2^\kappa < \lambda$ and $\theta^{<\kappa} < \lambda$ for every $\theta < \lambda$.

So the general method used here to find new possibilities for the values of $\mathrm{No}(\mathscr{M})$ is the same as in [8]. As an immediate consequence of the last theorem we get:

THEOREM 3. *Suppose λ is a singular cardinal of uncountable cofinality κ. For each nonzero $\mu \leq \lambda^\kappa$ there is a model $\mathscr{M}$ (with relations having finitely many places only) satisfying* $\mathrm{card}(\mathscr{M}) = \lambda$ *and* $\mathrm{No}(\mathscr{M}) = \mu$, *provided that $\theta^\kappa < \lambda$ for every $\theta < \lambda$.*

We give all necessary definitions concerning inverse κ-systems $\mathscr{A}$ of abelian groups and their special kind of quotient limits $\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A})$ in the next section.

§2. Preliminaries.

DEFINITION 2.1. Suppose γ is a limit ordinal and for every $i < j < \gamma$, G_i is a group and $h_{i,j}$ is a homomorphism from G_j into G_i. The family $\mathscr{A} = \langle G_i, h_{i,j} \mid i < j < \gamma \rangle$ is called an inverse γ-system when the equation $h_{i,j} \circ h_{j,k} = h_{i,k}$ holds for every $i < j < k < \gamma$. As in [7] we assume that all the groups G_i, $i < \gamma$, are additive abelian groups.

To simplify our notation we make an agreement that the letters i, j, k, and l always denote ordinals smaller than γ. Hence "for all $i < j$" means "for all ordinals i and j with $i < j < \gamma$" and so on.

The main objects of our study are the following two sets:

$$\mathrm{Gr}(\mathscr{A}) = \{ \langle \boldsymbol{a}^{i,j} \mid i < j < \gamma \rangle \mid \boldsymbol{a}^{i,j} \in G_i \text{ and for all } k > j,\ \boldsymbol{a}^{i,k} = \boldsymbol{a}^{i,j} + h_{i,j}(\boldsymbol{a}^{j,k}) \};$$

$$\mathrm{Fact}(\mathscr{A}) = \left\{ \langle \boldsymbol{a}^{i,j} \mid i < j < \gamma \rangle \,\middle|\, \text{for some } \bar{y} \in \prod_{k<\gamma} G_k,\ \boldsymbol{a}^{i,j} = \bar{y}^i - h_{i,j}(\bar{y}^j) \right\}.$$

We consider $\mathrm{Gr}(\mathscr{A})$ and $\mathrm{Fact}(\mathscr{A})$ as additive abelian groups where the group operation $+$ and the unit element 0 are pointwise defined. The factor group $\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A})$ is well-defined since $\mathrm{Fact}(\mathscr{A}) \subseteq \mathrm{Gr}(\mathscr{A})$ by the requirements $h_{i,j} \circ h_{j,k} = h_{i,k}$ for all $i < j < k$. For any inverse γ-system $\mathscr{A}$, the group $\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A})$ is called the quotient limit of $\mathscr{A}$.

DEFINITION 2.2. We let $\gamma \star \gamma$ be the set $\{ (i, j) \in \gamma \times \gamma \mid i < j \}$. For every subset I of $\gamma * \gamma$ we define

$$I^{1\text{st}} = \{ i < \gamma \mid (i, j) \in I \text{ for some } j < \gamma \}$$

and for each $i \in I^{1\text{st}}$,

$$I[i] = \{ j < \gamma \mid (i, j) \in I \}.$$

We also say that

- I is cobounded if $\gamma \setminus I^{1\text{st}}$ and $\gamma \setminus I[i]$, for all $i \in I^{1\text{st}}$, are bounded subsets of γ;
- I is coherent if $I^{1\text{st}}$ is unbounded in γ and for every $i \in I^{1\text{st}}$, $I[i] = I^{1\text{st}} \setminus (i+1)$;
- I is eventually coherent if it is unbounded and for every $i \in I^{1\text{st}}$, $I^{1\text{st}} \setminus I[i]$ is a bounded subset of γ.

REMARK. Suppose I is an eventually coherent subset of $\gamma \star \gamma$ and S is a subset of $I^{1\text{st}}$. If $\mathrm{card}(S) < \mathrm{cf}(\gamma)$, then $I^{1\text{st}} \setminus (\bigcap_{i \in S} I[i])$ is a bounded subset of γ. If S is unbounded in γ, then $I \cap (S \times S)$ is an eventually coherent subset of I.

In [8, Claim 1.12] Shelah proved (note the remark given after the following lemma) that if two sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ from $\mathrm{Gr}(\mathscr{A})$ agree on a coherent set of indices, then $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathscr{A})$. The following slight improvement of this condition has an essential role in the proof of Theorem 2.

LEMMA 2.3. *Suppose $\mathscr{A}$ is an inverse γ-system, and $\boldsymbol{a}$, $\boldsymbol{b} \in \mathrm{Gr}(\mathscr{A})$. Then $\boldsymbol{a} \equiv \boldsymbol{b}$ mod $\mathrm{Fact}(\mathscr{A})$ holds if there is an eventually coherent subset I of $\gamma \star \gamma$ such that $\boldsymbol{a}^{i,j} = \boldsymbol{b}^{i,j}$ for all $(i, j) \in I$.*

PROOF. We shall need an eventually coherent subset J of I having the property that $\langle J[i] \mid i \in J^{1st} \rangle$ is a decreasing chain of end segments of J^{1st}. Let S be an unbounded subset of I having the order type $\mathrm{cf}(\gamma)$. Define a subset J of I by $J^{1st} = S$ and for all $j \in S$,

$$J[j] = S \cap \bigcap_{i \in S \cap (j+1)} (I[i] \setminus (i^* + 1)),$$

where i^* is the supremum of the bounded subset $I^{1st} \setminus I[i]$ of γ. The set J is well-defined since I is eventually coherent and $\mathrm{card}(S \cap (j+1)) < \mathrm{cf}(\gamma)$ for all $j < \gamma$. Now J is also eventually coherent, and furthermore, for all $i \in J^{1st}$, $J[i] = S \setminus \min(J[i])$ and for all $j \in J^{1st} \setminus i$, $\min(J[i]) \leq \min(J[j])$.

Define for every $i < \gamma$, i' to be $\min(J^{1st} \setminus (i+1))$ and $i'' = \min(J[i'])$. Then the following are satisfied for all $i < j$:

- $i < i' < i''$, $j < j' < j''$, $i' \leq j'$, $i'' \leq j''$, and also $i' < j''$;
- $j'' \in I^{1st}$ and (i', i''), (j', j''), $(i', j'') \in I$.

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are in $\mathrm{Gr}(\mathscr{A}_R^T)$ we have

$$\boldsymbol{a}^{i,j''} = \boldsymbol{a}^{i,j} + h_{i,j}(\boldsymbol{a}^{j,j''}),$$
$$\boldsymbol{b}^{i,j''} = \boldsymbol{b}^{i,j} + h_{i,j}(\boldsymbol{b}^{j,j''}).$$

Therefore the following equations hold:

$$\text{(A)} \qquad \begin{aligned} \boldsymbol{a}^{i,j} - \boldsymbol{b}^{i,j} &= (\boldsymbol{a}^{i,j''} - \boldsymbol{b}^{i,j''}) - (h_{i,j}(\boldsymbol{a}^{j,j''}) - h_{i,j}(\boldsymbol{b}^{j,j''})) \\ &= (\boldsymbol{a}^{i,j''} - \boldsymbol{b}^{i,j''}) - h_{i,j}(\boldsymbol{a}^{j,j''} - \boldsymbol{b}^{j,j''}). \end{aligned}$$

Because of $i < i' < j''$ we also have that

$$\boldsymbol{a}^{i,j''} = \boldsymbol{a}^{i,i'} + h_{i,i'}(\boldsymbol{a}^{i',j''}),$$
$$\boldsymbol{b}^{i,j''} = \boldsymbol{b}^{i,i'} + h_{i,i'}(\boldsymbol{b}^{i',j''}).$$

Since $(i', j'') \in I$, $\boldsymbol{a}^{i',j''} = \boldsymbol{b}^{i',j''}$ holds. Hence we get

$$\text{(B)} \qquad \boldsymbol{a}^{i,j''} - \boldsymbol{b}^{i,j''} = \boldsymbol{a}^{i,i'} - \boldsymbol{b}^{i,i'}.$$

Moreover, $i < i' < i''$ yields

$$\boldsymbol{a}^{i,i''} = \boldsymbol{a}^{i,i'} + h_{i,i'}(\boldsymbol{a}^{i',i''}),$$
$$\boldsymbol{b}^{i,i''} = \boldsymbol{b}^{i,i'} + h_{i,i'}(\boldsymbol{b}^{i',i''}).$$

Now $(i', i'') \in I$ implies that $\boldsymbol{a}^{i',i''} = \boldsymbol{b}^{i',i''}$, and consequently

$$\boldsymbol{a}^{i,i'} - \boldsymbol{b}^{i,i'} = \boldsymbol{a}^{i,i''} - \boldsymbol{b}^{i,i''}.$$

This equation together with (A) and (B) implies that for all $i < j$

$$\boldsymbol{a}^{i,j} - \boldsymbol{b}^{i,j} = (\boldsymbol{a}^{i,i''} - \boldsymbol{b}^{i,i''}) - h_{i,j}(\boldsymbol{a}^{j,j''} - \boldsymbol{b}^{j,j''}).$$

So the sequence $\bar{y} = \langle \boldsymbol{a}^{i,i''} - \boldsymbol{b}^{i,i''} \mid i < \gamma \rangle \in \prod_{i<\gamma} G_i$ exemplifies that $\boldsymbol{a} - \boldsymbol{b} \in \mathrm{Fact}(\mathscr{A})$, and we have $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathscr{A})$. ⊣

REMARK. In [8, Claim 1.12] the groups of an inverse system $\mathscr{A}$ need not to be abelian groups. Hence instead of the factor group $\mathrm{Gr}(\mathscr{A})/\mathrm{Fact}(\mathscr{A})$ a partition $\mathrm{Gr}(\mathscr{A})/\approx_{\mathscr{A}}$ with a special kind of equivalence relation $\approx_{\mathscr{A}}$ were considered there. However, it is straightforward to prove, by means of the preceding proof, also the more general case of Lemma 2.3 where "equivalent modulo $\mathrm{Fact}(\mathscr{A})$" is replaced by $\approx_{\mathscr{A}}$.

In the next section we shall need a notion of a tree, so we shortly describe our notation.

DEFINITION 2.4. Suppose $T = \langle T, \lhd \rangle$ is a tree of height γ. For every $i < \gamma$, T_i is the i^{th} level of the tree. When $i < j < \gamma$ and $\eta \in T_j$, then $\eta \restriction i$ denotes the unique element $\nu \in T_i$ for which $\nu \lhd \eta$ holds. For each $i < \gamma$ and $\nu \in T_i$, $T_j[\nu]$ is the set $\{ \eta \in T_j \mid \nu \lhd \eta \}$. The set of all γ-branches of T, i.e., the set

$$\left\{ t \in \prod_{i<\gamma} T_i \;\middle|\; \text{for all } i < j,\ t(i) \lhd t(j) \right\},$$

is denoted by $\mathrm{Br}_\gamma(T)$.

§3. **The inverse γ-system of free R-modules.** In this section we define special kind of inverse γ-systems $\mathscr{A}_R^T$ and prove a result concerning cardinalities of their quotient limit $\mathrm{Gr}(\mathscr{A}_R^T)/\mathrm{Fact}(\mathscr{A}_R^T)$ (Conclusion 3.12). A direct consequence of the result will be Theorem 2.

DEFINITION 3.1. Suppose γ is a limit ordinal, R is a ring, and T is a tree of height γ. We define an inverse γ-system $\mathscr{A}_R^T = \langle G_i, h_{i,j} \mid i < j < \gamma \rangle$ by the following stipulations:

(a) for each $i < \gamma$, G_i is the R-module freely generated by

$$\{ \boldsymbol{x}_{\nu,l} \mid \nu \in T_i \text{ and } i < l < \gamma \};$$

(b) for every $i < j < \gamma$, $h_{i,j}$ is the homomorphism from G_j into G_i determined by the values

$$h_{i,j}(\boldsymbol{x}_{\eta,l}) = \boldsymbol{x}_{\eta\restriction i,l} - \boldsymbol{x}_{\eta\restriction i,j},$$

for all $\eta \in T_j$ and $l > j$. (It is easy to check that the equations $h_{i,k} = h_{i,j} \circ h_{j,k}$ are satisfied for all $i < j < k$.)

We consider $\mathrm{Gr}(\mathscr{A}_R^T)$, $\mathrm{Fact}(\mathscr{A}_R^T)$, and $\mathrm{Gr}(\mathscr{A}_R^T)/\mathrm{Fact}(\mathscr{A}_R^T)$ as R-modules where the operations $+$, $\cdot$, and the unit element 0 for addition are pointwise defined.

For each $t \in \mathrm{Br}_\gamma(T)$, we define $\boldsymbol{t}$ to be the sequence $\langle \boldsymbol{x}_{t(i),j} \mid i < j < \gamma \rangle$. Directly by the definitions of G_i and $h_{i,j}$, $\boldsymbol{t}$ belongs to $\mathrm{Gr}(\mathscr{A}_R^T)$ for every $t \in \mathrm{Br}_\gamma(T)$. We let $\langle \boldsymbol{t} \rangle_R^{t \in \mathrm{Br}_\gamma(T)}$ be the submodule of $\mathrm{Gr}(\mathscr{A}_R^T)$ generated by the elements $\boldsymbol{t}$, $t \in \mathrm{Br}_\gamma(T)$. When $\mathrm{Br}_\gamma(T)$ is empty $\langle \boldsymbol{t} \rangle_R^{t \in \mathrm{Br}_\gamma(T)}$ is the trivial submodule $\{0\}$.

REMARK. Each G_i is nonempty when T has height γ. Hence $\prod_{i<\gamma} G_i$ is nonempty, and also

$$\mathrm{Fact}(\mathscr{A}_R^T) = \left\{ \langle \bar{y}^i - h_{i,j}(\bar{y}^j) \mid i < j < \gamma \rangle \;\middle|\; \bar{y} \in \prod_{i<\gamma} G_i \right\}$$

is nonempty. So $\mathrm{Gr}(\mathscr{A}_R^T) \supseteq \mathrm{Fact}(\mathscr{A}_R^T$ is nonempty for every ring R and tree T of height γ.

Observe also that the inverse γ-system $\mathscr{A}_R^T$ is the same as used in [7, Claim 3.8] when R is the trivial ring $\{0,1\}$ and T consists of μ many disjoint γ-branches. So the proof given in this section offers an alternative proof for [7, Claim 3.8], and even more information, namely that $\mathrm{card}(\mathrm{Gr}(\mathscr{A}_R^T)/\,\mathrm{Fact}(\mathscr{A}_R^T))$ must be exactly μ not only $\geq \mu$.

DEFINITION 3.2. Suppose $\boldsymbol{a} \in \mathrm{Gr}(\mathscr{A}_R^T)$ and $i < j < \gamma$. By the definition of G_i and the requirement $\boldsymbol{a}^{i,j} \in G_i$, we define $a_{\nu,l}^{i,j}$ for $\nu \in T_i$ and $l > i$, to be the coefficients from R (with only finitely many of them nonzero) which satisfy the equation

$$\boldsymbol{a}^{i,j} = \sum_{\substack{l>i\\ \nu\in T_i}} \boldsymbol{a}_{i,j}^{\nu,l} \cdot \boldsymbol{x}_{\nu,l}.$$

The finite set

$$\{\,(\nu,l) \in T_i \times (\gamma \smallsetminus (i+1)) \mid a_{\nu,l}^{i,j} \neq 0\,\}$$

is called the support of $\boldsymbol{a}^{i,j}$, and it is denoted by $\mathrm{supp}(\boldsymbol{a}^{i,j})$.

Suppose S is a subset of γ, $e \in G_i$, and $e_{\nu,l} \in R$ for every $\nu \in T_i$ and $l > i$ are elements such that

$$e = \sum_{\substack{\nu\in T_i\\ l>i}} e_{\nu,l} \cdot \boldsymbol{x}_{\nu,l}.$$

Then we write $e \restriction S$ for the following element of G_i:

$$\sum_{\substack{\nu\in T_i\\ l\in S\smallsetminus(i+1)}} e_{\nu,l} \cdot \boldsymbol{x}_{\nu,l}.$$

The following simple lemma has an important corollary.

LEMMA 3.3.

(a) *The restriction $h_{i,j}(e)\restriction j$ equals 0 for every $i < j$ and $e \in G_j$.*

(b) *For every $\boldsymbol{a} \in \mathrm{Fact}(\mathscr{A}_R^T) \smallsetminus \{0\}$, there are $i < j < \gamma$ such that $\boldsymbol{a}^{i,j}\restriction j \neq 0$.*

PROOF.

(a) Straightforwardly by the definitions of G_j and $h_{i,j}$.

(b) By the definition of $\mathrm{Fact}(\mathscr{A}_R^T)$, let $\bar{y} \in \prod_{i<\gamma} G_i$ be such that for all $i < j$, $\boldsymbol{a}^{i,j} = \bar{y}^i - h_{i,j}(\bar{y}^j)$. In addition to that let $y_{\nu,l}^i \in R$, for $i < \gamma$, $\nu \in T_i$ and $l > i$, be such that

$$\bar{y}^i = \sum_{\substack{\nu\in T_i\\ l>i}} y_{\nu,l}^i \cdot \boldsymbol{x}_{\nu,l}.$$

Since $\boldsymbol{a} \neq 0$ there must be $i < \gamma$ with $\bar{y}^i \neq 0$. Define j to be

$$\min\{\, l > i \mid y_{\nu,l}^i \neq 0 \text{ for some } \nu \in T_i\,\} + 1.$$

Then $\bar{y}^i\restriction j$ is nonzero and because $h_{i,j}(\bar{y}^j)\restriction j = 0$, we have

$$\boldsymbol{a}^{i,j}\restriction j = \bar{y}^i\restriction j - h_{i,j}(\bar{y}^j)\restriction j = \bar{y}^i\restriction j \neq 0.$$

COROLLARY 3.4. *The elements* $\boldsymbol{t}$, $t \in \mathrm{Br}_\gamma(T)$, *are independent over* $\mathrm{Fact}(\mathscr{A}_R^T)$, *i.e.*,

$$\langle \boldsymbol{t} \rangle_R^{t \in \mathrm{Br}_\gamma(T)} \cap \mathrm{Fact}(\mathscr{A}_R^T) = \{0\}.$$

Hence $\mathscr{A}_R^T$ *satisfies*

$$\mathrm{card}(\mathrm{Gr}(\mathscr{A}_R^T)/\,\mathrm{Fact}(\mathscr{A}_R^T)) \geq \mathrm{card}(\langle \boldsymbol{t} \rangle_R^{t \in \mathrm{Br}_\gamma(T)}).$$

PROOF. Directly by the definition of $\boldsymbol{t}$, $\boldsymbol{t}^{i,j} = \boldsymbol{x}_{t(i),j}$ and hence $\boldsymbol{t}^{i,j} \restriction j = 0$, for all $t \in \mathrm{Br}_\gamma(T)$ and $i < j$. So for any nonzero $\boldsymbol{a} = \sum_{1 \leq m \leq n} d_m \cdot \boldsymbol{t_m}$, where $n < \omega$, $d_m \in R \setminus \{0\}$, and $t_m \in \mathrm{Br}_\gamma(T)$, the restrictions $\boldsymbol{a}^{i,j} \restriction j$ are equal to 0 for all $i < j$. So by the preceding lemma $\boldsymbol{a}$ can not be in $\mathrm{Fact}(\mathscr{A}_R^T)$. ⊣

Next we derive equations of weighty significance.

LEMMA 3.5. *Suppose* $\boldsymbol{b} \in \mathrm{Gr}(\mathscr{A}_R^T)$ *and* $i < j < k < \gamma$. *Then the following equations are satisfied for all* $\nu \in T_i$:

(A) $$b_{\nu,l}^{i,k} = b_{\nu,l}^{i,j} \qquad \textit{when } i < l < j;$$

(B) $$b_{\nu,j}^{i,k} = b_{\nu,j}^{i,j} - \sum_{\substack{\eta \in T_j[\nu] \\ l > j}} b_{\eta,l}^{j,k};$$

(C) $$b_{\nu,l}^{i,k} = b_{\nu,l}^{i,j} + \sum_{\eta \in T_j[\nu]} b_{\eta,l}^{j,k} \qquad \textit{when } l > j.$$

PROOF. By dividing the sum into groups we get that

$$\begin{aligned}\boldsymbol{b}^{i,j} &= \sum_{\substack{l > i \\ \nu \in T_i}} \boldsymbol{b}_{i,j}^{\nu,l} \cdot \boldsymbol{x}_{\nu,l} \\ &= \sum_{\nu \in T_i} \Big(\sum_{i<l<j} b_{\nu,l}^{i,j} \cdot \boldsymbol{x}_{\nu,l} + b_{\nu,j}^{i,j} \cdot \boldsymbol{x}_{\nu,j} + \sum_{l>j} b_{\nu,l}^{i,j} \cdot \boldsymbol{x}_{\nu,l} \Big).\end{aligned}$$

Similarly the following equation is satisfied,

$$\boldsymbol{b}^{i,k} = \sum_{\nu \in T_i} \Big(\sum_{i<l<j} b_{\nu,l}^{i,k} \cdot \boldsymbol{x}_{\nu,l} + b_{\nu,j}^{i,k} \cdot \boldsymbol{x}_{\nu,j} + \sum_{l>j} b_{\nu,l}^{i,k} \cdot \boldsymbol{x}_{\nu,l} \Big).$$

From the definition of $h_{i,j}$ we may infer that

$$\begin{aligned} h_{i,j}(\boldsymbol{b}^{j,k}) &= \sum_{\substack{\eta \in T_j \\ l > j}} b_{\eta,l}^{j,k} \cdot h_{i,j}(\boldsymbol{x}_{\eta,l}) \\ &= \sum_{\substack{\eta \in T_j \\ l > j}} b_{\eta,l}^{j,k} \cdot (\boldsymbol{x}_{\eta \restriction i,l} - \boldsymbol{x}_{\eta \restriction i,j}) \\ &= \sum_{\substack{\eta \in T_j \\ l > j}} b_{\eta,l}^{j,k} \cdot \boldsymbol{x}_{\eta \restriction i,l} - \sum_{\substack{\eta \in T_j \\ l > j}} b_{\eta,l}^{j,k} \cdot \boldsymbol{x}_{\eta \restriction i,j} \\ &= \sum_{\nu \in T_i} \Bigg(\sum_{l > j} \Big(\sum_{\eta \in T_j[\nu]} b_{\eta,l}^{j,k} \Big) \cdot \boldsymbol{x}_{\nu,l} - \Big(\sum_{\substack{\eta \in T_j[\nu] \\ l > j}} b_{\eta,l}^{j,k} \Big) \cdot \boldsymbol{x}_{\nu,j} \Bigg). \end{aligned}$$

So the equations (A), (B), and (C) for all $i < j < k$ follow by comparing the coefficients of each generator $\boldsymbol{x}_{\nu,l}$ in the equation $\boldsymbol{b}^{i,k} = \boldsymbol{b}^{i,j} + h_{i,j}(\boldsymbol{b}^{j,k})$. ⊣

LEMMA 3.6. *Suppose* $\boldsymbol{a} \in \mathrm{Gr}(\mathscr{A}_R^T)$.

(a) *For all* $i < j < k$, $\boldsymbol{a}^{i,j}\restriction j = \boldsymbol{a}^{i,k}\restriction j$.

(b) $(\mathrm{cf}(\gamma) > \aleph_0)$. *For every* $i < \gamma$, *the union* $\bigcup_{i<j<\gamma} \mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j)$ *is of finite cardinality* (*where* $\mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j) = \mathrm{supp}(\boldsymbol{a}^{i,j}) \cap (T_i \times j)$ *of course*).

(c) $(\mathrm{cf}(\gamma) > \aleph_0)$. *There is* $\boldsymbol{b} \in \mathrm{Gr}(\mathscr{A}_R^T)$ *satisfying the following conditions*:

- $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathscr{A}_R^T)$,
- $I = \{ (i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i,j}\restriction j = 0 \}$ *is cobounded* (*in fact* $I^{1\text{st}} = \gamma$), *and*
- *for every* $(i,j) \in I$, $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j}\restriction\{j\}$.

PROOF.

(a) The claim holds directly by Lemma 3.5 (A).

(b) Suppose the union is infinite. Since $\mathrm{cf}(\gamma) > \aleph_0$ there is some $k < \gamma$ for which already $\bigcup_{j<k} \mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j)$ is infinite. By (a), $\mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j) \subseteq \mathrm{supp}(\boldsymbol{a}^{i,k})$ for each $j < k$. Consequently $\bigcup_{j<k} \mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j) \subseteq \mathrm{supp}(\boldsymbol{a}^{i,k})$ contrary to the finiteness of $\mathrm{supp}(\boldsymbol{a}^{i,k})$.

(c) By (a) and (b) there must be for every $i < \gamma$ a bound $i^* \in \gamma \setminus (i+1)$ such that for every $j \geq i^*$, $\boldsymbol{a}^{i,i^*}\restriction i^* = \boldsymbol{a}^{i,j}\restriction j$. Define an element $\boldsymbol{c} \in \mathrm{Fact}(\mathscr{A}_R^T)$ by

$$\boldsymbol{c}^{i,j} = \boldsymbol{a}^{i,i^*}\restriction i^* - h_{i,j}(\boldsymbol{a}^{j,j^*}\restriction j^*),$$

for all $i < j$. Let $\boldsymbol{b}$ be $\boldsymbol{a} - \boldsymbol{c}$. Then $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathscr{A}_R^T)$ and for every $i < \gamma$ and $j \geq i^*$

$$\begin{aligned}\boldsymbol{b}^{i,j} &= \boldsymbol{a}^{i,j} - \boldsymbol{c}^{i,j}\\ &= \boldsymbol{a}^{i,j}\restriction(\gamma \setminus j) + \boldsymbol{a}^{i,j}\restriction j - \boldsymbol{a}^{i,i^*}\restriction i^* + h_{i,j}(\boldsymbol{a}^{j,j^*}\restriction j^*)\\ &= \boldsymbol{a}^{i,j}\restriction(\gamma \setminus j) + h_{i,j}(\boldsymbol{a}^{j,j^*}\restriction j^*).\end{aligned}$$

It follows from Lemma 3.3 (a) that $\boldsymbol{b}^{i,j}\restriction j = 0$ for all $i < \gamma$ and $j \geq i^*$, and thus I is cobounded.

Now suppose, contrary to the last claim in (c), that $b^{i,j}_{\nu,l} \neq 0$ for some $i < \gamma$, $j \geq i^*$, $\nu \in T_i$, and $l > j$. Let k be $\max\{i^*, j^*, l+1\}$. Then both $\boldsymbol{b}^{i,k}\restriction k$ and $\boldsymbol{b}^{j,k}\restriction k$ are 0. By Lemma 3.5 (C) the following equation holds:

$$\sum_{\eta \in T_j[\nu]} b^{j,k}_{\eta,l} = b^{i,k}_{\nu,l} - b^{i,j}_{\nu,l}.$$

Since $b^{i,j}_{\nu,l} \neq 0$ and $l < k$ implies $b^{i,k}_{\nu,l} = 0$ the sum $\sum_{\eta \in T_j[\nu]} b^{j,k}_{\eta,l}$ must be nonzero. So there is $\eta \in T_j[\nu]$ with $b^{j,k}_{\eta,l} \neq 0$. This contradicts the facts $l < k$ and $\boldsymbol{b}^{j,k}\restriction k$ equals 0. ⊣

LEMMA 3.7. *Suppose* $\boldsymbol{b} \in \mathrm{Gr}(\mathscr{A}_R^T)$ *and* I *is a subset of*

$$\{ (i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i,j}\restriction\{j\} = \boldsymbol{b}^{i,j} \}.$$

Then for all $(i,j) \in I$, $\nu \in T_i$, *and* $k \in I[i] \cap I[j]$,

$$b^{i,j}_{\nu,j} = \sum_{\eta \in T_j[\nu]} b^{j,k}_{\eta,k} = b^{i,k}_{\nu,k}.$$

PROOF. Since (i,k) and (j,k) are in I, both $b^{i,k}_{\nu,j}$ and $b^{j,k}_{\eta,l}$ are equal to 0 for all $\eta \in T_j$ when $l \neq k$. Hence Lemma 3.5 (B) can be reduced to the form $b^{i,j}_{\nu,j} = \sum_{\eta \in T_j[\nu]} b^{j,k}_{\eta,k}$. Now $(i,j) \in I$ guarantees that $b^{i,j}_{\nu,k} = 0$. Thus the reduced form together with Lemma 3.5 (C) (applied for $l = k$) yield $b^{i,j}_{\nu,j} = b^{i,k}_{\nu,k}$. ⊣

LEMMA 3.8. *Suppose $\boldsymbol{b}$ is an element of* $\mathrm{Gr}(\mathscr{A}^T_R)$.

(a) *If $\boldsymbol{b}$ is not in* $\mathrm{Fact}(\mathscr{A}^T_R)$ *and I is an eventually coherent subset of $\gamma \star \gamma$ such that $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j} \restriction \{j\}$ for all $(i,j) \in I$, then there is an eventually coherent subset J of I with $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j} \restriction \{j\} \neq 0$ whenever $(i,j) \in J$.*

(b) $(\mathrm{cf}(\gamma) > \aleph_0)$. *If J is an eventually coherent subset of $\gamma \star \gamma$ such that $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j} \restriction \{j\} \neq 0$ for all $(i,j) \in J$, then there are a bound $n^* < \omega$ and an eventually coherent subset K of J such that* $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$ *for all $(i,j) \in K$.*

PROOF.

(a) Since $\boldsymbol{b} \not\equiv 0 \mod \mathrm{Fact}(\mathscr{A}^T_R)$ it follows by Lemma 2.3 that there is no subset of $\{ (i,j) \in I \mid \boldsymbol{b}^{i,j} = 0 \}$ which would be eventually coherent. Hence there is an unbounded subset S of $I^{1\mathrm{st}}$ such that for each $i \in S$ there is $j_i \in I[i]$ with $\boldsymbol{b}^{i,j_i}$ nonzero. Fix any $i \in S$. Since $\boldsymbol{b}^{i,j_i} = \boldsymbol{b}^{i,j_i} \restriction \{j_i\} \neq 0$, let ν_i be an element of T_i with $b^{i,j_i}_{\nu_i,j_i} \neq 0$. By Lemma 3.7, $b^{i,k}_{\nu_i,k} = b^{i,j_i}_{\nu_i,j_i} \neq 0$ for all $k \in I[i] \cap I[j_i]$. Because I was eventually coherent, we have shown that $J = I \cap (S \times S)$ is an eventually coherent set as wanted in the claim.

(b) First of all we claim that for each $i \in J^{1\mathrm{st}}$ the union $\bigcup_{j \in J[i]} \mathrm{supp}(\boldsymbol{b}^{i,j})$ is of finite cardinality. Observe that for every $(i,j) \in J$,

$$\mathrm{supp}(\boldsymbol{b}^{i,j}) = \mathrm{supp}(\boldsymbol{b}^{i,j}) \cap (T_i \times \{j\}).$$

Assume, contrary to this subclaim, that $i \in J^{1\mathrm{st}}$, $\langle j_m \mid m < \omega \rangle$ is an increasing sequence of ordinals in $J[i]$, and $\{ \nu_m \mid m < \omega \}$ is a set of distinct elements from T_i such that $b^{i,j_m}_{\nu_m,j_m}$ nonzero for every $m < \omega$. Since J is eventually coherent and γ is of uncountable cofinality let $k < \gamma$ be the minimal element in $J[i] \cap \bigcap_{m<\omega} J[j_m]$. Now for each $m < \omega$, the pairs (i,j_m), (i,k), and (j_m,k) are in J, and by Lemma 3.7, the equation $b^{i,j_m}_{\nu_m,j_m} = b^{i,k}_{\nu_m,k} \neq 0$ holds. So the infinite set $\{ (\nu_m,k) \mid m < \omega \}$ is a subset of $\mathrm{supp}(\boldsymbol{b}^{i,k})$, a contradiction.

It follows from the subclaim that for each $i \in J^{1\mathrm{st}}$, the finite ordinal

$$n_i = \mathrm{card}\left(\bigcup_{j \in J[i]} \mathrm{supp}(\boldsymbol{b}^{i,j}) \right) + 1$$

satisfies $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n_i$ for all $j \in J[i]$. Since $J^{1\mathrm{st}}$ is uncountable, there are $n^* < \omega$ and an unbounded subset S of $J^{1\mathrm{st}}$ such that $n_i = n^*$ for all $i \in S$. So n^* and the set $K = J \cap (S \times S)$ meet the requirements of the claim. ⊣

LEMMA 3.9 $(\mathrm{cf}(\gamma) > \mathrm{card}(R))$. *Suppose $\boldsymbol{b}$ is in* $\mathrm{Gr}(\mathscr{A}^T_R)$ *and I is an eventually coherent subset of*

$$\{ (i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i,j} \restriction \{j\} = \boldsymbol{b}^{i,j} \neq 0 \}.$$

Then there are $d \in R$, $t \in \mathrm{Br}_\gamma(T)$, and an eventually coherent subset J of I for which $b^{i,j}_{t(i),j} = d \neq 0$ whenever $(i,j) \in J$.

PROOF. We define by induction on $\alpha < \mathrm{cf}(\gamma)$ the following objects:

- an increasing sequence $\langle i_\alpha \mid \alpha < \mathrm{cf}(\gamma) \rangle$ of ordinals in $I^{1\mathrm{st}}$ with limit γ;
- an increasing sequence $\langle \nu_\alpha \mid \alpha < \mathrm{cf}(\gamma) \rangle \in \prod_{\alpha<\mathrm{cf}(\gamma)} T_{i_\alpha}$;
- subsets K_α of $I[i_\alpha]$ such that $I^{1\mathrm{st}} \setminus K_\alpha$ are bounded in γ;
- elements $d_\alpha \in R \setminus \{0\}$ such that for every $k \in K_\alpha$, $b^{i_\alpha.k}_{\nu_\alpha.k} = d_\alpha$.

This suffices since $\mathrm{card}(R) < \mathrm{cf}(\gamma)$ implies that there are $d \in R$ and $H \subseteq \mathrm{cf}(\gamma)$ unbounded in $\mathrm{cf}(\gamma)$ such that $d_\alpha = d$ for every $\alpha \in H$. Moreover, the claim is satisfied by $t \in \mathrm{Br}_\gamma(T)$ and $J \subseteq I$ defined as follows. For every $i < \gamma$, $t(i) = \nu_{\beta_i}\restriction i$, where $\beta_i = \min\{\,\alpha < \mathrm{cf}(\gamma) \mid i_\alpha \geq i\,\}$, and $J = \bigcup_{\alpha\in H}(\{i_\alpha\}\times(S\cap K_\alpha))$, where S is $\{\, i_\alpha \mid \alpha \in H\,\}$.

Let $\langle \gamma_\alpha \mid \alpha < \mathrm{cf}(\gamma)\rangle$ be an increasing sequence with limit γ. Define

$$i_\alpha = \min\left(\left(I^{1\mathrm{st}} \cap \bigcap_{\beta<\alpha} K_\beta\right) \setminus \gamma_\alpha\right)$$

and

$$j = \min\left(I^{1\mathrm{st}} \cap I[i_\alpha] \cap \bigcap_{\beta<\alpha} I[i_\beta]\right),$$

where both $\bigcap_{\beta<\alpha} K_\beta$ and $\bigcap_{\beta<\alpha} I[i_\beta]$ are equal to γ when $\alpha = 0$. This pair (i_α, j) is well-defined since I is eventually coherent, $\alpha < \mathrm{cf}(\gamma)$, and when $\alpha > 0$, $I^{1\mathrm{st}} \setminus K_\beta$ is bounded for each $\beta < \alpha$ by the induction hypothesis.

If $\alpha = 0$, then $(i_0, j) \in I$ guarantees that $\boldsymbol{b}^{i_0.j}\restriction\{j\} = \boldsymbol{b}^{i_0.j} \neq 0$. Hence we can find $\nu_0 \in T_{i_0}$ with $b^{i_0.j}_{\nu_0.j} \neq 0$.

When $\alpha > 0$ we define elements $\eta_\beta \in T_{i_\alpha}[\nu_\beta]$ for each $\beta < \alpha$ as follows. Fix $\beta < \alpha$. Since $i_\alpha \in K_\beta$ we get by the induction hypothesis that $b^{i_\beta.i_\alpha}_{\nu_\beta.i_\alpha} = d_\beta \neq 0$. Furthermore $(i_\beta, i_\alpha) \in I$ (because $K_\beta \subseteq I[i_\beta]$), $(i_\beta, j) \in I$, and $(i_\alpha, j) \in I$ together with Lemma 3.7 yield

$$\sum_{\eta\in T_{i_\alpha}[\nu_\beta]} b^{i_\alpha.j}_{\eta.j} = b^{i_\beta.i_\alpha}_{\nu_\beta.i_\alpha} \neq 0.$$

Therefore we can find $\eta_\beta \in T_{i_\alpha}[\nu_\beta]$ for which $b^{i_\alpha.j}_{\eta_\beta.j} \neq 0$.

If $\alpha > 0$ is a successor ordinal define ν_α to be $\eta_{\alpha-1}$. When α is a limit ordinal, the finiteness of the support $\mathrm{supp}(\boldsymbol{b}^{i_\alpha.j})$ ensures that there are $\nu_\alpha \in T_{i_\alpha}$ and an unbounded subset H of α such that $\eta_{\beta'} = \nu_\alpha$ for all $\beta' \in H$. By the induction hypothesis $\nu_\beta \lhd \nu_{\beta'}$ for all $\beta < \beta' < \alpha$. Hence $\nu_\beta \lhd \nu_{\beta'} \lhd \eta_{\beta'} = \nu_\alpha$ holds for every $\beta < \alpha$ and $\beta' = \min(H \setminus \beta)$.

Let d_α be $b^{i_\alpha.j}_{\nu_\alpha.j}$. By Lemma 3.7, every $k \in I[i_\alpha] \cap I[j]$ satisfies that $b^{i_\alpha.k}_{\nu_\alpha.k} = b^{i_\alpha.j}_{\nu_\alpha.j} = d_\alpha$. Hence i_α, ν_α, and d_α together with the set $K_\alpha = I[i_\alpha] \cap I[j]$ meet the requirements given at the beginning of the proof. ⊣

COROLLARY 3.10 ($\mathrm{cf}(\gamma) > \aleph_0$). *If* $\mathrm{Br}_\gamma(T)$ *is empty, then* $\mathrm{Gr}(\mathscr{A}^T_R) = \mathrm{Fact}(\mathscr{A}^T_R)$.

PROOF. Suppose $\boldsymbol{a} \in \mathrm{Gr}(\mathscr{A}^T_R) \setminus \mathrm{Fact}(\mathscr{A}^T_R)$. By Lemma 3.6 (c) together with Lemma 3.8 (a) there is $\boldsymbol{b} \in \mathrm{Gr}(\mathscr{A}^T_R)$ such that $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathscr{A}^T_R)$ and the set

$$\{\,(i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i.j}\restriction\{j\} = \boldsymbol{b}^{i.j} \neq 0\,\}$$

is eventually coherent. By Lemma 3.9 there is a γ-branch through the tree T, i.e., $\mathrm{Br}_\gamma(T) \neq \emptyset$. Observe that the assumption $\mathrm{card}(R) < \mathrm{cf}(\gamma)$ is not needed, as can be seen from the proof of Lemma 3.9. ⊣

LEMMA 3.11 ($\mathrm{cf}(\gamma) > \max\{\aleph_0, \mathrm{card}(R)\}$). *The elements* $\boldsymbol{t}$, $t \in \mathrm{Br}_\gamma(T)$, *generate* $\mathrm{Gr}(\mathscr{A}_R^T)$ *modulo* $\mathrm{Fact}(\mathscr{A}_R^T)$.

PROOF. We show that for every $\boldsymbol{a} \in \mathrm{Gr}(\mathscr{A}_R^T)$ with $\boldsymbol{a} \notin \mathrm{Fact}(\mathscr{A}_R^T)$ we can find $n < \omega$, $d_1, \dots, d_n \in R \setminus \{0\}$ and $t_1, \dots, t_n \in \mathrm{Br}_\gamma(T)$ satisfying

(A) $$\boldsymbol{a} \equiv \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \mod \mathrm{Fact}(\mathscr{A}_R^T).$$

Suppose $\boldsymbol{a} \in \mathrm{Gr}(\mathscr{A}_R^T) \setminus \mathrm{Fact}(\mathscr{A}_R^T)$. By Lemma 3.6 (c) and Lemma 3.8 (a) let $\boldsymbol{b}$ be an element of $\mathrm{Gr}(\mathscr{A}_R^T)$ and I_1 an eventually coherent subset of $\gamma \star \gamma$ such that $\boldsymbol{a} \equiv \boldsymbol{b}$ mod $\mathrm{Fact}(\mathscr{A}_R^T)$ and for each $(i,j) \in I_1$, $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j} \restriction \{j\} \neq 0$. Furthermore, we may assume by Lemma 3.8 (b) that $n^* < \omega$ is a bound for which $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$ hold for all $(i,j) \in I_1$.

By Lemma 3.9 there are $d_1 \in R$, $t_1 \in \mathrm{Br}_\gamma(T)$, and an eventually coherent set $J_1 \subseteq I_1$ having the property that $b^{i,j}_{t_1(i),j} = d_1 \neq 0$ whenever $(i,j) \in J_1$. Since $d_1 \cdot \boldsymbol{t_1} \in \mathrm{Gr}(\mathscr{A}_R^T)$, the sequence $\boldsymbol{c} = \boldsymbol{b} - d_1 \cdot \boldsymbol{t_1}$ is in $\mathrm{Gr}(\mathscr{A}_R^T)$. If $\boldsymbol{c}$ is in $\mathrm{Fact}(\mathscr{A}_R^T)$, then $\boldsymbol{b} \equiv d_1 \cdot \boldsymbol{x}St_1 \mod \mathrm{Fact}(\mathscr{A}_R^T)$, and because of $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathscr{A}_R^T)$, also (A) holds for $n = 1$.

Suppose $1 \le n < \omega$ and objects $d_m \in R \setminus \{0\}$, $t_m \in \mathrm{Br}_\gamma(T)$, and $J_m \subseteq J_1$ for $m \le n$ are already defined. Assume also that these objects satisfy the following conditions:

(1) $J_{m'} \supseteq J_m$ for all $1 \le m' \le m \le n$;
(2) for all $1 \le m' < m \le n$ and $i \in (J_m)^{1\mathrm{st}}$, $t_{m'}(i) \neq t_m(i)$;
(3) for every $1 \le m \le n$ and $(i,j) \in J_m$, $b^{i,j}_{t_m(i),j} = d_m \neq 0$;
(4) $\boldsymbol{c} = \boldsymbol{b} - \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \notin \mathrm{Fact}(\mathscr{A}_R^T)$.

Clearly $\boldsymbol{c}^{i,j} = \boldsymbol{c}^{i,j} \restriction \{j\}$ and $\mathrm{card}(\mathrm{supp}(\boldsymbol{c}^{i,j})) \le \mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$ for all $(i,j) \in J_n$. Again by Lemma 3.8 (a), there is an eventually coherent set $I_{n+1} \subseteq J_n$ such that for each $(i,j) \in I_{n+1}$, $\boldsymbol{c}^{i,j} \neq 0$. Moreover, by Lemma 3.9, there are $d_{n+1} \in R$, $t_{n+1} \in \mathrm{Br}_\gamma(T)$, and an eventually coherent set $J_{n+1} \subseteq \{\, (i,j) \in I_{n+1} \mid c^{i,j}_{t_{n+1}(i),j} = d_{n+1} \neq 0 \,\}$.

The properties (2), (3) and (4) above imply that $c^{i,j}_{t_m(i),j} = b^{i,j}_{t_m(i),j} - d_m = 0$ for every $m \le n$ and $(i,j) \in J_m$. On the other hand, $c^{i,j}_{t_{n+1}(i),j}$ is nonzero for each $(i,j) \in J_{n+1}$. Thus $t_{n+1}(i)$ can not be in $\{\, t_m(i) \mid 1 \le m \le n \,\}$ if $i \in (J_{n+1})^{1\mathrm{st}}$. So for all $(i,j) \in J_{n+1}$, $\boldsymbol{x}_{t_{n+1}(i),j} \notin \{\, \boldsymbol{x}_{t_m(i),j} \mid 1 \le m \le n \,\}$, and consequently $b^{i,j}_{t_{n+1}(i),j} = c^{i,j}_{t_{n+1}(i),j}$. Thus also J_{n+1}, t_{n+1}, and d_{n+1} satisfy the properties (1), (2), and (3) (but not necessarily (4)).

We claim that there must be $n < n^*$ such that

(B) $$\boldsymbol{b} - \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \in \mathrm{Fact}(\mathscr{A}_R^T).$$

Assume, contrary to this subclaim, that the process introduced above has been carried out n^* many times and objects J_m, t_m, d_m for $i \leq m \leq n^*$ are defined. In addition to that suppose they satisfy the conditions (1), (2), and (3). Define $i = \min((J_{n^*})^{1\text{st}})$ and $j = \min(J_{n^*}[i])$. Then for every $m \leq n^*$, $(i, j) \in J_m$ yields $b^{i,j}_{t_m(i).j} = d_m \neq 0$. This contradicts the condition $\operatorname{card}(\operatorname{supp}(\boldsymbol{b}^{i,j})) < n^*$, since the set $\{ (t_m(i), j) \mid m \leq n^* \} \subseteq \operatorname{supp}(\boldsymbol{b}^{i,j})$ is of cardinality n^*.

Now suppose $n < \omega$ is a finite ordinal satisfying (B). Then $\boldsymbol{b} \equiv \sum_{1 \leq m \leq n} d_m \cdot \boldsymbol{t_m}$ mod $\operatorname{Fact}(\mathscr{A}^T_R)$, and because $\boldsymbol{a} \equiv \boldsymbol{b}$ mod $\operatorname{Fact}(\mathscr{A}^T_R)$ also (A) is satisfied. ⊣

CONCLUSION 3.12. *For any ordinal γ of uncountable cofinality, ring R with*

$$\operatorname{card}(R) < \operatorname{cf}(\gamma),$$

and tree T of height γ, the inverse γ-system $\mathscr{A}^T_R = \langle G_i, h_{i,j} \mid i < j < \gamma \rangle$ has the properties that

$$\operatorname{card}(G_i) = \max\{\operatorname{card}(\gamma), \operatorname{card}(T_i), \operatorname{card}(R)\}$$

for all $i < \gamma$, and

$$\operatorname{card}(\operatorname{Gr}(\mathscr{A}^T_R)/\operatorname{Fact}(\mathscr{A}^T_R)) = \operatorname{card}(\langle \boldsymbol{t} \rangle_R^{t \in \operatorname{Br}_\gamma(T)}).$$

PROOF OF THEOREM 2. Remember that λ and κ were cardinals with $\aleph_0 < \kappa = \operatorname{cf}(\lambda) < \lambda$. We wanted to study possible cardinalities μ of the quotient limit $\operatorname{Gr}(\mathscr{A})/\operatorname{Fact}(\mathscr{A})$, where $\mathscr{A}$ is an inverse κ-system consisting of abelian groups having cardinality $< \lambda$. Now Conclusion 3.12 gives a complete solution to this problem because of $\lambda > \operatorname{cf}(\lambda) = \kappa = \operatorname{cf}(\kappa) > \aleph_0$. Namely, in order to meet the requirements $\operatorname{card}(G_i) < \lambda$ for all $i < \kappa$, it is needed only to ensure that R and the i^{th} level of T are small enough. On the other hand, a suitable choice of R and T yields any desired value for $\mu = \operatorname{card}(\operatorname{Gr}(\mathscr{A}^T_R)/\operatorname{Fact}(\mathscr{A}^T_R))$. We briefly describe methods to choose suitable R and T for every nonzero $\mu \leq \lambda^\kappa$.

For any R, $\operatorname{card}(\operatorname{Gr}(\mathscr{A}^T_R)/\operatorname{Fact}(\mathscr{A}^T_R))$ equals 1 when $\operatorname{Br}_\kappa(T)$ is empty. So $\mu = 1$ is possible since obviously there exists a tree of height κ without κ-branches and having levels of cardinality $< \lambda$ when λ singular of cofinality κ. Also all the finite values $\mu > 1$ are possible by taking T with only one κ-branch and R with $\operatorname{card}(R) = \mu$.

Furthermore the case of infinite $\mu < \lambda$ is satisfied by any R with $\operatorname{card}(R) < \min\{\kappa, \mu\}$ and T with exactly μ many κ-branches. The value $\mu = \lambda$ is possible for any R with $\operatorname{card}(R) < \kappa$ because a suitable tree can be constructed, for example, as follows. Let $\langle \lambda_i \mid i < \kappa \rangle$ be an increasing sequence of ordinals $< \lambda$ with limit λ. Then the tree

$$T = \left\{ t \restriction \alpha \;\middle|\; \alpha < \kappa,\ t \in \prod_{i<\kappa} \lambda_i, \text{ and } t(i) \text{ is nonzero only for finitely many } i < \kappa \right\},$$

ordered by inclusion, satisfies $\operatorname{card}(\operatorname{Br}_\kappa(T)) = \lambda$ and $\operatorname{card}(T_i) = \lambda_i < \lambda$ for each $i < \kappa$.

Also the cardinalities μ of the quotient limit, when $\lambda < \mu \leq \lambda^\kappa$, are possible for any ring of cardinality $< \kappa$. Existence of a suitable tree is proved for example in [9, Fact 10] under the assumption that $2^\kappa < \lambda$ and $\theta^{<\kappa} < \lambda$ for every $\theta < \lambda$ (other sources for a proof are given in [10, Analytical Guide §10]). ⊣

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