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Cardinalities of topologies with small base

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Abstract

Let T be the family of open subsets of a topological space (not necessarily Hausdorff or even T_0). We prove that if T has a base of cardinality $\leqslant \mu$, $\lambda \leqslant \mu < 2^\lambda$, λ strong limit of cofinality $\aleph_0$, then T has cardinality $\leqslant \mu$ or $\geqslant 2^\lambda$. This is our main conclusion (21). In Theorem 2 we prove it under some set-theoretic assumption, which is clear when $\lambda = \mu$; then we eliminate the assumption by a theorem on pcf from [Sh 460] motivated originally by this. Next we prove that the simplest examples are the basic ones; they occur in every example (for $\lambda = \aleph_0$ this fulfills a promise from [Sh 454]). The main result for the case $\lambda = \aleph_0$ was proved in [Sh 454].

Why do we deal with λ strong limit of cofinality $\aleph_0$? Essentially as other cases are closed.

Example 1. *If I is a linear order of cardinality μ with λ Dedekind cuts then there is a topology T of cardinality $\lambda > \mu$ with a base B of cardinality μ.*

Construction. Let B be $\{[-\infty, x)_I : x \in I\}$ where $[-\infty, x)_I = \{y \in I : I \vDash y < x\}$. $\square_1$

Remarks. As is well known, if $\mu = \mu^{<\mu}$, $\mu < \lambda \leqslant \chi = \chi^\mu$ then there is a μ^+-c.c. μ-complete forcing notion Q, of cardinality χ such that in V^Q we have $2^\mu = \chi$, there is a λ-tree with exactly μ λ-branches (and $\leqslant \mu$ other branches) hence a linear order of cardinality μ with exactly λ Dedekind cuts. As possibly $\lambda^{\aleph_0} > \lambda$, this limits possible

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generalizations of our Main Theorem. Also there are results guaranteeing the existence of such trees and linear or$^{\omega}$ders, e.g. if μ is strong limit singular of uncountable cofinality, $\mu < \lambda \leqslant 2^{\mu}$ (see [14], [15, 3.5 + §5]) and more (see [17]).

So we naturally concentrate on strong limit cardinals of countable cofinality. We do not try to 'save' the natural numbers like $n(*) + 6$ used during the proof.

Theorem 2 (Main). *Assume*

(a) *λ_n for $n < \omega$ are regular or finite cardinals, $2^{\lambda_n} < \lambda_{n+1}$ and $\lambda = \sum_{n<\omega} \lambda_n$ $(\geqslant \aleph_0)$.*

(b) *$\lambda = \sum_{n<\omega} \mu_n$ (even $\mu_{n+1} \geqslant \lambda_n$) and $\beth_3(\mu_n) < \lambda_n$, $\lambda \leqslant \mu < \lambda^{\aleph_0}$ $(= 2^{\lambda})$ and $\mathrm{cov}(\mu, \lambda_n^+, \lambda_n^+, \mu_n^+) \leqslant \mu$ (see Definition 3 below, trivial when $\lambda = \aleph_0$ and easy when $\mu = \lambda$).*

(c) *Let T be the family of open subsets of a topological space (not necessarily Hausdorff or even T_0), and suppose that T has a base B of cardinality $\leqslant \mu$ (i.e., B is a subset of T which is closed under finite intersections, and the sets in T are the unions of subfamilies of B).*

Then

(1) *The cardinality of T is either at least $\lambda^{\aleph_0}$ $(=2^{\lambda})$ of at most μ.*

(2) *In fact, if $|T| > \mu$ then for some set X_0 of λ points, $\{U \cap X_0 : U \in T\}$ has cardinality 2^{λ}. Moreover, for some $B' \subseteq T$ of cardinality λ, $\{X_0 \cap U : U$ is the union of a subfamily of $B'\}$ has cardinality 2^{λ}.*

Definition 3 ([15, 5.1]). $\mathrm{cov}(\mu, \lambda^+, \lambda^+, \kappa) = \min\{|P| : P$ a family of subsets of μ each of cardinality $\leqslant \lambda$, such that if $a \subseteq \mu$, $|a| \leqslant \lambda$ then for some $\alpha < \kappa$ and $a_i \in P$ (for $i < \alpha$) we have $a \subseteq \bigcup_{i<\alpha} a_i\}$.

Proof of Theorem 2. Suppose we have a counterexample T to 2(2) (as 2(1) follows from 2(2)) with a base B and let Ω be the set of points of the space, so w.l.o.g. $\lambda \leqslant \mu = |B| < 2^{\lambda}$. Our result, as explained in the abstract, for the case $\lambda = \aleph_0$ was proved in [18], and see background there; the proof as written here applies to this case too but we usually do not mention when things trivialize for the case $\lambda = \aleph_0$; w.l.o.g. $\Omega = \bigcup B$, $\emptyset \in B$ and B is closed under finite intersections and unions. So T is the set of all unions of subfamilies of B.

We prove first that:

Observation 4. *For each n there is a family R of cardinality $\leqslant \mu$ of partial functions from λ_n to μ such that:* ***for every*** *function f from λ_n to μ* ***there is*** *a partition $\langle r_\zeta \mid \zeta < \mu_n \rangle$ of λ_n (i.e., pairwise disjoint subsets of λ_n with union λ_n) for which $\bigwedge_{\zeta<\mu_n} f \restriction r_\zeta \in R$.*

Proof. By assumption (b) and $2^{\lambda_n} < \lambda \leqslant \mu$ and λ is strong limit of cofinality $\aleph_0 \leqslant \mu_n$. $\square_4$

Claim 5. *Assume Z^* is a subset of Ω of cardinality at most μ and T' is a subfamily of T satisfying*

$$(*) \qquad (\forall U_1, U_2 \in T')[U_1 = U_2 \Leftrightarrow U_1 \cap Z^* = U_2 \cap Z^*],$$

$|T'| > \mu$ and $n < \omega$.

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Then we can find a subset Z of Z^* of cardinality μ_n, subsets Z_α of Z and members U_α of T and subfamilies T_α of T' each of cardinality $> \mu$ for $\alpha < \mu_n$ such that:

(a) *the sets Z_α for $\alpha < \mu_n$ are pairwise distinct;*

(b) *for $\alpha < \mu_n$ and $V \in T'$ we have: $V \in T_\alpha$ iff $V \cap Z = Z_\alpha \subseteq U_\alpha \subseteq V$.*

Proof. We shall use (*) freely. Define an equivalence relation E on Z^*:

$$xEy \text{ iff } |\{U \in T' : x \in U \Leftrightarrow y \notin U\}| \leqslant \mu$$

(check that E is indeed an equivalence relation).

Let $Z^\otimes \subseteq Z^*$ be a set of representatives. Now for $V \in T'$ we have:

$$(*) \qquad \{U \in T' : U \cap Z^\otimes = V \cap Z^\otimes\} \subseteq \bigcup_{xEz,\, \{x,z\} \subseteq Z^*} \{U \in T' : z \in U \equiv x \notin U \text{ but } U \cap Z^\otimes = V \cap Z^\otimes\} \cup \{V^*\}$$

where

$$V^* = \{y \in Z^* : \text{for the } x \in Z^\otimes \text{ such that } yEx \text{ we have } x \in V\}.$$

[Why? Assume U is in the left side, i.e., $U \in T'$ and $U \cap Z^\otimes = V \cap Z^\otimes$; now we shall prove that U is in the right side; if $U = V^*$ this is straightforward, otherwise for some $x \in Z^*$, $x \in U \equiv x \notin V^*$; as $Z^\otimes$ is a set of representitives for E for some $z \in Z^\otimes$, we have zEx so by the definition of V^*, $x \in V^* \Leftrightarrow z \in V$. But as $U \cap Z^\otimes = V \cap Z^\otimes$ we have $z \in V \Leftrightarrow z \in U$. Together $x \in U \Leftrightarrow z \notin U$ and we are done.]

Now the right side of $(*)$ is the union of $\leqslant |Z^*|^2$ sets, each of cardinality $\leqslant \mu$ (by the definition of xEz). Hence the left side in $(*)$ has cardinality $\leqslant |Z^*|^2 \times \mu \leqslant \mu$. Let $\{V_i : i < i^*\} \subseteq T'$ be maximal such that: $V_i \cap Z^\otimes$ are pairwise distinct and $V_i \in T'$. So clearly

$$|T'| = \left| \bigcup_{i<i^*} \{U \in T' : U \cap Z^\otimes = V_i \cap Z^\otimes\} \right| \leqslant \sum_{i<i^*} \mu = \mu |i^*|,$$

but $|T'| > \mu$ hence $|i^*| = |\{U \cap Z^\otimes : U \in T'\}| > \mu$. Hence (as λ is strong limit) necessarily $|Z^\otimes| \geqslant \lambda$, so we can let $z_\beta \in Z^\otimes$ for $\beta < \lambda_n$ be distinct. For $\alpha < \beta < \lambda_n$ we know that $\neg z_\alpha E z_\beta$ hence for some truth value $\mathbf{t}_{\alpha,\beta}$ we have $|\{U \in T' : z_\alpha \in U \equiv z_\beta \notin U \equiv \mathbf{t}_{\alpha,\beta}\}| > \mu$. But B is a base of T of cardinality $\leqslant \mu$, hence for some $V_{\alpha,\beta} \in B$ the set

$$S_{\alpha,\beta} = \{U \in T' : z_\alpha \in U \equiv z_\beta \notin U \equiv \mathbf{t}_{\alpha,\beta}, \text{ and } \{z_\alpha, z_\beta\} \cap U \subseteq V_{\alpha,\beta} \subseteq U\}$$

has cardinality $> \mu$.

Choose $U^1_{\alpha,\beta} \in S_{\alpha,\beta}$ such that $\mu < |S^1_{\alpha,\beta}|$ where

$$S^1_{\alpha,\beta} \stackrel{\text{def}}{=} \{U \in S_{\alpha,\beta} : U \cap \{z_\zeta : \zeta < \lambda_n\} = U^1_{\alpha,\beta} \cap \{z_\zeta : \zeta < \lambda_n\}\},$$

note that $U^1_{\alpha,\beta}$ exists as $2^{\lambda_n} < \lambda \leqslant \mu < |S_{\alpha,\beta}|$.

By Observation 4 we can find a family R of cardinality $\leqslant \mu$, members of R have the form $\bar{u} = \langle u_\alpha : \alpha \in r \rangle$, where $r \subseteq \lambda_n$, $u_\alpha \in B$ such that for every sequence $\bar{u} = \langle u_\alpha : \alpha < \lambda_n \rangle$ of members of B, there is a partition $\langle r_\zeta : \zeta < \mu_n \rangle$ of λ_n (so $r_\zeta = r_\zeta[\bar{u}] \subseteq \lambda_n$ for $\zeta < \mu_n$) such that $\bar{u} \restriction r_\zeta \in R$ (remember $\emptyset \in B$). W.l.o.g. if $\bar{u}^l = \langle u^l_\alpha : \alpha \in r^l \rangle \in R$ for $l = 1, 2$ then $\bar{u} = \langle u_\alpha : \alpha \in r \rangle \in R$ where $r = r^1 \cup r^2$ and

$$u_\alpha = \begin{cases} u^1_\alpha & \text{if } \alpha \in r^1, \\ u^2_\alpha & \text{if } \alpha \in r^2 \setminus r^1. \end{cases}$$

For each $V \in T'$ we can find $\bar{u}[V] = \langle u_\gamma[V] : \gamma < \lambda_n \rangle$, such that (remember $\emptyset \in B$):

$$u_\gamma[V] \in B,$$

$$z_\gamma \in V \;\Rightarrow\; z_\gamma \in u_\gamma[V] \subseteq V,$$

$$z_\gamma \notin V \;\Rightarrow\; u_\gamma[V] = \emptyset.$$

Clearly there is $U^2_{\alpha,\beta} \in S^1_{\alpha,\beta}$ such that:

(**) for any finite subset $w \subseteq \mu_n$ and $\alpha < \beta < \lambda_n$, the following family has cardinality $> \mu$:

$$S^2_{\alpha,\beta,w} \stackrel{\text{def}}{=} \{U \in S^1_{\alpha,\beta} : (\forall \zeta < \mu_n)(r_\zeta[\bar{u}[U]] = r_\zeta[\bar{u}[U^2_{\alpha,\beta}]]) \text{ and}$$
$$(\forall \zeta \in w)(\bar{u}[U] \restriction r_\zeta = \bar{u}[U^2_{\alpha,\beta}] \restriction r_\zeta)\}.$$

By the Erdös–Rado theorem for some set $M \in [\lambda_n]^{\mu_n^+}$:

(a) for every $\alpha < \beta$ from M, $\mathbf{t}_{\alpha,\beta}$ are the same;
(b) for every $\alpha < \beta \in M$, γ, $\varepsilon \in M$ the truth values of "$z_\gamma \in V_{\alpha,\beta}$", "$z_\gamma \in U^2_{\alpha,\beta}$", "$z_\varepsilon \in u_\gamma[U^2_{\alpha,\beta}]$" and the value of "$\min\{\zeta < \mu_n : \gamma \in r_\zeta[\bar{u}[U^2_{\alpha,\beta}]]\}$" depend just on the order and equalities between α, β, γ and ε.

Let $M = \{\alpha(i) : i < \mu_n^+\}$ where $[i < j \;\Rightarrow\; \alpha(i) < \alpha(j)]$, let t be 0 if $i < j \Rightarrow \mathbf{t}_{\alpha(i),\alpha(j)} = \text{truth}$ and 1 if $i < j \Rightarrow \mathbf{t}_{\alpha(i),\alpha(j)} = \text{false}$.

Case 1. If $i < j < \mu_n^+$ and $\varepsilon < i \vee \varepsilon > j$ then $z_{\alpha(\varepsilon)} \notin U^2_{\alpha(i),\alpha(j)}$. So for some $\zeta_1 < \mu_n$

($\otimes$) for every $i < \mu_n^+$, $\zeta_1 = \min\{\zeta : \alpha(i+t) \in r_\zeta[\bar{u}[U^2_{\alpha(i),\alpha(i+1)}]]\}$.

Let $Z = \{z_{\alpha(i)} : i < \mu_n\}$, $Z_i = \{z_{\alpha(2i+t)}\}$, $U_i = u_{\alpha(2i+t)}[U^2_{\alpha(2i),\alpha(2i+1)}]$. Clearly $U_i \cap Z U_i \subseteq U^2_{\alpha(2i),(2i+1)}$ and $U_i \cap Z = Z_i$, lastly let $T_i = S^2_{\alpha(2i),\alpha(2i+1),\{\zeta_1\}}$; now Z, Z_i, U_i, T_i are as required.

Case 2. If $i < j < \mu_n^+$ then

$$\varepsilon < i \;\Rightarrow\; z_{\alpha(\varepsilon)} \in U^2_{\alpha(i),\alpha(j)},$$

$$\varepsilon > j \;\Rightarrow\; z_{\alpha(\varepsilon)} \notin U^2_{\alpha(i),\alpha(j)}.$$

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So for some $\zeta_1 < \mu_n$, $\zeta_2 < \mu_n$

$(\otimes)$ (a) for $\varepsilon < i < j < \mu_n^+$, $\zeta_1 = \min\{\zeta : \alpha(\varepsilon) \in r_\zeta[\bar{u}[U^2_{\alpha(i),\alpha(j)}]]\}$,
(b) for $i < \mu_n^+$, $\zeta_2 = \min\{\zeta : \alpha(i+t) \in r_\zeta[\bar{u}[U^2_{\alpha(i),\alpha(i+1)}]]\}$.

Let $Z = \{z_{\alpha(i)} : i < \mu_n\}$, $Z_i = \{z_{\alpha(\varepsilon)} : \varepsilon < 2i\} \cup \{z_{\alpha(2i+t)}\}$, $U_i = \bigcup\{u_{\alpha(\varepsilon)}[U^2_{\alpha(2i),\alpha(2i+1)}] : \varepsilon \leqslant 2i+1\}$ and $T_i = S^2_{\alpha(2i),\alpha(2i+1),\{\zeta_1,\zeta_2\}}$.

Case 3. If $i < j < \mu_n^+$ then

$$\varepsilon < i \Rightarrow z_{\alpha(\varepsilon)} \notin U^2_{\alpha(i),\alpha(j)},$$

$$\mu_n^+ > \varepsilon > j \Rightarrow z_{\alpha(\varepsilon)} \in U^2_{\alpha(i),\alpha(j)}.$$

So for some $\zeta_1 < \mu_n$, $\zeta_2 < \mu_n$

$(\otimes)$ (a) for $i < \mu_n^+$, $\zeta_1 = \min\{\zeta : \alpha(i+t) \in r_\zeta[\bar{u}[U^2_{\alpha(i),\alpha(i+1)}]]\}$,
(b) for $i < j < \varepsilon < \mu_n^+$, $\zeta_2 = \min\{\zeta : \alpha(\varepsilon) \in r_\zeta[u[U^2_{\alpha(i),\alpha(j)}]]\}$.

Let $Z = \{z_{\alpha(i)} : i < \mu_n\}$, $Z_i = \{z_{\alpha(\varepsilon)} : \varepsilon = 2i+t$ or $2i+1 < \varepsilon < \mu_n\}$, $U_i = \bigcup\{u_{\alpha(\varepsilon)}[U^2_{\alpha(2i),\alpha(2i+1)}] : \varepsilon = 2i+t$ or $2i+1 < \varepsilon < \mu_n\}$ and $T_i = S^2_{\alpha(2i),\alpha(2i+1),\{\zeta_1,\zeta_2\}}$.

Case 4. If $i < j < \mu_n^+$ then

$$\varepsilon < i \Rightarrow z_{\alpha(\varepsilon)} \in U^2_{\alpha(i),\alpha(j)},$$

$$\varepsilon > j \Rightarrow z_{\alpha(\varepsilon)} \in U^2_{\alpha(i),\alpha(j)}.$$

So for some $\zeta_1, \zeta_2, \zeta_3 < \mu_n$

$(\otimes)$ (a) for $\varepsilon < i < j < \mu_n^+$, $\zeta_1 = \min\{\zeta : \alpha(\varepsilon) \in r_\zeta[\bar{u}[U^2_{\alpha(i),\alpha(j)}]]\}$,
(b) for $i < \mu_n^+$, $\zeta_2 = \min\{\zeta : \alpha(i+t) \in r_\zeta[\bar{u}[U^2_{\alpha(i),\alpha(i+1)}]]\}$,
(c) for $i < j < \varepsilon < \mu_n^+$, $\zeta_3 = \min\{\zeta : \alpha(\varepsilon) \in r_\zeta[\bar{u}[U^2_{\alpha(i),\alpha(j)}]]\}$.

Let $Z = \{z_{\alpha(i)} : i < \mu_n\}$. $Z_i = \{z_{\alpha(\varepsilon)} : \varepsilon < \mu_n$ and $\varepsilon \neq 2i+1-t\}$, $U_i = \bigcup\{u_{\alpha(\varepsilon)}[U^2_{\alpha(2i),\alpha(2i+1)}] : \varepsilon > \mu_n, \varepsilon \neq 2i+1-t\}$ and $T_i = S^2_{\alpha(2i),\alpha(2i+1),\{\zeta_1,\zeta_2,\zeta_3\}}$. Now in all cases we have chosen Z, T_α, U_α, Z_α ($\alpha < \mu_n$) as required thus finishing the proof of the claim. $\square_5$

Claim 6. *If $Z^* \subseteq \Omega$, $|Z^*| \leqslant \mu$, then $\{U \cap Z^* : U \in T\}$ has cardinality $\leqslant \mu$.*

Proof. Assume not. We can find $T' \subseteq T$ such that:

(α) for $U_1, U_2 \in T'$ we have $U_1 = U_2 \Leftrightarrow U_1 \cap Z^* = U_2 \cap Z^*$.
(β) $|T'| > \mu$.

By induction on n we define $\langle T_\eta, Z^1_\eta, Z^2_\eta, U_\eta : \eta \in \prod_{l<n} \mu_l \rangle$ such that:

(a) T_η is a subset of T' of cardinality $> \mu$;
(b) if $\nu \triangleleft \eta$ then $T_\eta \subseteq T_\nu$;
(c) if $\eta = \langle\ \rangle$ then $T_\eta = T'$, $Z^1_\eta = Z^2_\eta = \emptyset$, $U_\eta = \emptyset$;
(d) $Z^1_\eta \subseteq Z^2_\eta \subseteq Z^*$ and $|Z^2_\eta| \leqslant \mu_{\lg\eta}$, Z^2_η disjoint to $\bigcup\{U_{\eta\restriction l} : l < \lg\eta\}$;
(e) $U_\eta \in T$;
(f) if $V \in T_\eta$ then $U_\eta \subseteq V$ and $V \cap Z^2_\eta = U_\eta \cap Z^2_\eta = Z^1_\eta$;
(g) if $\lg(\eta) = \lg(\nu) = n+1$ and $\eta\restriction n = \nu\restriction n$ then $Z^2_\eta = Z^2_\nu$ but
(h) if $\lg(\eta) = \lg(\nu) = n+1$, $\eta\restriction n = \nu\restriction n$ but $\eta \neq \nu$ then $Z^1_\eta \neq Z^1_\nu$.

Why this is sufficient? Let $Z \stackrel{\text{def}}{=} \bigcup\{Z^2_\eta : \eta \in \bigcup_n \prod_{l<n}\mu_l\}$. It is a subset of Z^* of cardinality $\leqslant \lambda$. The set $B' \stackrel{\text{def}}{=} \{U_\eta : \eta \in \bigcup_{n<\omega}\prod_{l<n}\mu_l\}$ is included in T and has cardinality $\leqslant \lambda$. For $\eta \in \prod_n \mu_n$ we let $U_\eta = \bigcup_{n<\omega} U_{\eta\restriction n}$. Now as $U_{\eta\restriction n} \in T$ (by clause (e)), clearly $U_\eta \in T$. Now suppose $\eta \neq \nu$ are in $\prod_{n<\omega}\mu_n$ and we shall prove that $U_\eta \cap Z \neq U_\nu \cap Z$, as $|\prod_n \mu_n| = 2^\lambda$ this suffices (giving (1) + (2) from Theorem 2). Let n be minimal such that $\eta(n) \neq \nu(n)$, so $\eta\restriction n = \nu\restriction n$. By clause (g), $Z^2_{\eta\restriction(n+1)} = Z^2_{\nu\restriction(n+1)}$. So (by clause (h)) $Z^1_{\eta\restriction(n+1)}, Z^1_{\nu\restriction(n+1)}$ are distinct subsets of $Z^2_{\eta\restriction(n+1)} = Z^2_{\nu\restriction(n+1)} \subseteq Z$. So it suffices to show $U_\eta \cap Z^2_{\eta\restriction(n+1)} = Z^1_{\eta\restriction(n+1)}$ and $U_\nu \cap Z^2_{\nu\restriction(n+1)} = Z^1_{\nu\restriction(n+1)}$ and by symmetry it suffices to prove the first. Now $Z^1_{\eta\restriction(n+1)} \subseteq U_{\eta\restriction(n+1)}$ by clause (f), hence $Z^1_{\eta\restriction(n+1)} \subseteq U_\eta$ so it suffices to prove that $U_\eta \cap Z^2_{\eta\restriction(n+1)} \subseteq Z^1_{\eta\restriction(n+1)}$; for this it suffices to prove that for $l < \omega$

$$(*) \qquad U_{\eta\restriction l} \cap Z^2_{\eta\restriction(n+1)} \subseteq Z^1_{\eta\restriction(n+1)}.$$

Case 1. $l = n+1$. This holds by clause (f).

Case 2. $l > n+1$. Then choose any $V \in T_{\eta\restriction l}$, so we know $U_{\eta\restriction l} \subseteq V$ (by clause (f)) and $V \in T_{\eta\restriction(n+1)}$ (by clause (b)), and $V \cap Z^2_{\eta\restriction(n+1)} = Z^1_{\eta\restriction(n+1)}$ (by clause (f)), together finishing.

Case 3. $l \leqslant n$. By clause (d), $Z^2_{\eta\restriction(n+1)}$ is disjoint from $U_{\eta\restriction l}$.

So we have finished to prove sufficiency, but we still have to carryout the induction. For $n = 0$ try to apply (c), the main point being $|T_{\langle\ \rangle}| > \mu$ which holds by the choice of T' (which was possible by the assumption that the claim fails). Suppose we have defined for n and let $\eta \in \prod_{l<n}\mu_l$. We apply Claim 5 with T_η, $Z^* \setminus \bigcup_{l<n} U_{\eta\restriction l}$ and n here standing for T', Z^*, n there.

We get there $Z, Z_\alpha, T_\alpha, U_\alpha$ $(\alpha < \mu_n)$ satisfying (a) + (b) there. We choose $T_{\eta^\frown\langle\alpha\rangle}$ to be T_α, $U_{\eta^\frown\langle\alpha\rangle}$ to be U_α, $Z^2_{\eta^\frown\langle\alpha\rangle}$ to be Z and $Z^1_{\eta^\frown\langle\alpha\rangle}$ to be Z_α. You can check the induction hypotheses, so we have finished. $\square_6$

Definition 7. $X \subseteq \Omega$ is *small* if $\{X \cap U : U \in T\}$ has cardinality $\leqslant \mu$. The family of small $X \subseteq \Omega$ will be denoted by $\mathscr{I} = \mathscr{I}_T$ (or more exactly, $\mathscr{I}_{T,\Omega}$).

Claim 8. *The family of small sets, $\mathscr{I}$, is a μ^+-complete ideal (on Ω, including all singletons of course).*

Proof. Clearly $\mathscr{I}$ is a family of subsets of Ω, and it is trivial to check that ($X \in \mathscr{I}$ and $Y \subseteq X$) $\Rightarrow$ $Y \in \mathscr{I}$. So assume $X_\alpha \in \mathscr{I}$ for $\alpha < \alpha(*)$, $\alpha(*) \leqslant \mu$ and we shall prove that $X = \bigcup_\alpha X_\alpha \in \mathscr{I}$. Each X_α has a subset Y_α such that

(a) $|Y_\alpha| \leqslant \mu$ and
(b) if V, W are elements of T with $V \cap X_\alpha \neq W \cap X_\alpha$ then there is some element $y \in Y_\alpha$ which is in exactly one of V, W (possible as $X_\alpha \in \mathscr{I}$).

Now if V, W are elements of T which differ on $X = \bigcup_{\alpha<\alpha(*)} X_\alpha$, then they already differ on some X_α and hence they differ on some Y_α hence on $Y \stackrel{\text{def}}{=} \bigcup_{\alpha<\alpha(*)} Y_\alpha$. So $|\{U \cap X : U \in T\}| = |\{U \cap Y : U \in T\}|$, so it suffices to prove that Y is small. But Y has cardinality $\leqslant |\bigcup_\alpha Y_\alpha| \leqslant \sum_\alpha |Y_\alpha| \leqslant \mu \times \mu = \mu$; so Claim 6 implies that Y is small and hence X is small. $\square_8$

Conclusion 9. *W.l.o.g.* $\operatorname{card}(\Omega) = \mu^+$.

Proof. As obviously $\{x\} \in \mathscr{I}$ for $x \in \Omega$, by Claim 8 we know $|\Omega| > \mu$. Let $T' \subseteq T$ be of cardinality μ^+ and let $\Omega' \subseteq \Omega$ be of cardinality μ^+ such that: if $U \neq V$ are from T' then $U \cap \Omega' \neq V \cap \Omega'$. Let T'' be $\{U \cap \Omega' : U \in T\}$ and $B' = \{U \cap \Omega' : U \in B\}$. Now T'', B', Ω' are also a counterexample to the Main Theorem and satisfy the additional demand. $\square_9$

Claim 10. *W.l.o.g. for some $n(*)$, for no $Z \subseteq \Omega$ of cardinality $\mu_{n(*)}$ and U_α, T_α, Z_α ($\alpha < \mu_{n(*)}$) does the conclusion of Claim 5 (with Ω, T here standing for Z^*, T' there) hold.*

Proof. Repeat the proof of Claim 6. I.e. we let $Z^* \stackrel{\text{def}}{=} \Omega$, and add the demand

(i) $\qquad T_\eta = \{U \in T : U_{\eta \restriction l} \subseteq U \text{ and } U \cap Z^2_{\eta \restriction l} \subseteq U_{\eta \restriction l} \text{ for } l < \lg \eta\}.$

The only change is in the end of the paragraph before the last one where we have used Claim 5, now instead we say that if we fail then for our n, replacing T, Ω by T_η, Z^* resp. gives the desired conclusion (note T_η is closed under union and finite intersection, and has a basis of cardinality $\leqslant \mu$:

$$B_\eta \stackrel{\text{def}}{=} \left\{U \cup \bigcup_{l<\lg \eta} U_{\eta \restriction l} : U \in B \text{ and } U \cap Z^2_{\eta \restriction l} \subseteq U_{\eta \restriction l} \text{ for } l < \lg \eta\right\}$$

which is included in T_η). (Pedantically, we have to replace T_η, Z^* by $\{U \setminus \bigcup_{l<n} U_{n \restriction l} : U \in T_n\}$, $\bigcap_{U \in T_\eta} U \setminus \bigcup_{l<n} U_{n \restriction l}$.) $\square$

Observation 11. *Suppose λ is strong limit of cofinality $\aleph_0$, I is a linear order of cardinality $\leqslant \mu$, $\lambda \leqslant \mu < \lambda^{\aleph_0}$, and I has $> \mu$ Dedekind cuts. Then I has $\geqslant \mu^{\aleph_0}$ $(= \lambda^{\aleph_0})$ Dedekind cuts.*

Remark. This observation does not rely on the assumptions of Theorem 2.

Proof. We define by induction on α when $\mathrm{rk}_I(x, y) = \alpha$ holds for $x < y$ in I.

For $\alpha = 0$: $\mathrm{rk}_I(x, y) = \alpha$ iff $(x, y)_I = \{z \in I : x < z < y\}$ has cardinality $< \lambda$.

For $\alpha > 0$: $\mathrm{rk}_I(x, y) = \alpha$ iff for $\beta < \alpha$, $\neg[\mathrm{rk}_I(x, y) = \beta]$ but *for any* (x_i, y_i) $(i < \lambda)$, pairwise disjoint subintervals of (x, y), *there is* i such that $\bigvee_{\beta<\alpha} \mathrm{rk}_I(x_i, y_i) = \beta$.

$(*)_1$ Note that by thinning the family, w.l.o.g., $[x_i, y_i]$ are pairwise disjoint.

[Why? E.g. as for every j the set $\{i : [x_i, y_i] \cap [x_j, y_j] \neq \emptyset\}$ has at most three members.]

$(*)_2$ For $\alpha > 0$ and $x < y$ from I, $\mathrm{rk}_I(x, y) = \alpha$ iff for $\beta < \alpha, \neg[\mathrm{rk}_I(x, y) = \beta]$ and for some $\lambda' < \lambda$ for any (x_i, y_i) $(i < \lambda')$, pairwise disjoint subintervals of (x, y), there are $i < \lambda'$ and $\beta < \alpha$ such that $\mathrm{rk}_I(x_i, y_i) = \beta$.

[Why? The demand in $(*)_2$ certainly implies the demand in the definition; for the other direction assume that the definition holds but the demand in $(*)_2$ fails, and we shall derive a contradiction. So for each $n < \omega$ there are pairwise disjoint subintervals (x_i^n, y_i^n) of (x, y), for $i < \lambda_n$ such that $\neg[\mathrm{rk}_I(x_i^n, y_i^n) = \beta]$ (when $\beta < \alpha$ and $i < \lambda_n$). As we can successively replace $\{(x_i^n, y_i^n) : i < \lambda_n\}$ by any subfamily of the same cardinality (when the λ_n's are finite – by a subfamily of cardinality λ_{n-1}), w.l.o.g., for each n, all members of $\{x_i^n : i < \lambda_n\}$ realize the same Dedekind cut of $\{x_j^m, y_j^m : m < n, j < \lambda_m\}$ and similarly for all members of $\{y_i^n : i < \lambda_n\}$. So for $m < n$, $i < \lambda_n$, the interval (x_i^n, y_i^n) cannot contain a point from $\{x_j^m, y_j^m : j < \lambda_m\}$ (as then the same occurs for all such i's, for the same point contradicting the "pairwise disjoint") so either our interval (x_i^n, y_i^n) is disjoint to all the intervals (x_j^m, y_j^m) for $j < \lambda_m$ or it is contained in one of the intervals (x_j^m, y_j^m); as j does not depend on i we denote it by $j(m, n)$; if $\lambda = \aleph_0$, by the Ramsey theorem w.l.o.g. for $m < n$, $j(m, n)$ does not depend on n; now the family $\{(x_i^m, y_i^m) : m < \omega, i < \lambda_m$ and for every $n < \omega$ which is $> m$ we have $i \neq j(m, n)\}$ contradicts the definition.]

If $\mathrm{rk}_I(x, y)$ is not equal to any ordinal let it be ∞. Let $\alpha^* = \sup\{\mathrm{rk}_I(x, y) + 1 : x < y$ in I and $\mathrm{rk}_I(x, y) < \infty\}$. Clearly $\mathrm{rk}_I(x, y) \in \alpha^* \cup \{\infty\}$ for every $x < y$ in I (and in fact $\alpha^* < \mu^+$). As we can add to I the first and the last elements it suffices to prove:

(A) if $\mathrm{rk}_I(x, y) = \alpha < \infty$ then $(x, y)_I$ has $\leqslant \mu$ Dedekind cuts, and

(B) if $\mathrm{rk}_I(x, y) = \infty$ then it has $\geqslant \lambda^{\aleph_0}$ Dedekind cuts.

Since (B) is straightforward, we only prove (A).

Proof of (A). We prove this by induction on α. If α is zero, this is trivial. So assume that $\alpha > 0$, hence by $(*)_2$ for some $\lambda' < \lambda$ there are no pairwise disjoint subintervals (x_i, y_i) for $i < \lambda'$ such that $\beta < \alpha$ implies $\neg[\mathrm{rk}_I(x_i, y_i) = \beta]$. Let J be the completion of I, so each member of $J \setminus I$ realizes on I a Dedekind cut with no last element in the lower half and no first element in the upper half, and $|J| > \mu \geqslant |I|$. Let $J^+ \overset{\text{def}}{=} \{z \in J : z \notin I$ and if $x \in I$, $y \in I$ and $x <_J z <_J y$ and $\beta < \alpha$ then $\neg[\mathrm{rk}_I(x, y) = \beta]\}$. By the induction hypothesis, as $|I| \leqslant \mu$, easily $|J \setminus J^+| \leqslant \mu$ hence the cardinality of J^+ is $> \mu$. By the Erdös–Rado theorem (remembering λ is strong limit and $\lambda' < \lambda$), there is a monotonic (by $<_J$) sequence $\langle z_i : i < \lambda' \rangle$ of members of J^+; by symmetry w.l.o.g. $\langle z_i : i < \lambda' \rangle$ is $<_J$-increasing. Now for each $i < \lambda'$ as $z_i <_J z_{i+1}$ are both in J^+, necessarily there is

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a member x_i of I such that $z_i <_J x_i <_J z_{i+1}$. So $x_i <_J z_{i+1} <_J x_{i+1}$ and $x_i \in I$, $x_{i+1} \in I$ and $z_{i+1} \in J^+$ hence by the definition of J^+ we know that for no $\beta < \alpha$ holds $\mathrm{rk}_I(x_i, x_{i+1}) = \beta$. So finally the family $\{(x_i, x_{i+1}) : i < \lambda'\}$ of subintervals of (x, y) gives the desired contradiction to $(*)_2$. $\square_{11}$

Definition 12. We define an equivalence relation E on Ω: xEy iff $\{U \in T : x \in U \equiv y \notin U\}$ has cardinality $\leqslant \mu$.

Conclusion 13. (0) *The equivalence relation E has $< \lambda_{n(*)} < \lambda$ equivalence classes (for some $n(*) < \omega$, which w.l.o.g. is as required in Claim 10 too).*

(1) *W.l.o.g. for each $x \in \Omega$ one of the following sets has cardinality $\leqslant \mu$:*

(a) $\{U \in T : x \in U\}$, (b) $\{U \in T : x \notin U\}$.

(2) *W.l.o.g. for all $x \in \Omega$ we get the same case above, in fact it is case* (b).

(3) *W.l.o.g. for any two distinct members x, y of Ω for some $U \in B$ we have $x \in U$ iff $y \notin U$.*

Proof. (0) By Claim 10 and the proof of Claim 5 (if E has $\geqslant \lambda$ equivalence classes we can repeat the proof of Claim 5 and get a contradiction to Claim 10).

(1), (2), (3) Let $\langle X_\zeta : \zeta < \zeta^* \rangle$ list the E-equivalence classes, so $\zeta^* < \lambda_{n(*)}$. As $\Omega \notin \mathscr{I}$, and $\mathscr{I}$ is μ^+-complete (Claim 8) for some ζ, $X_\zeta \notin \mathscr{I}$. Let $\Omega' = X_\zeta$, $T' = \{U \cap \Omega' : U \in T\}$, $B' = \{U \cap \Omega' : U \in B\}$; so Ω', B', T' have all the properties we attribute to Ω, B, T and in addition now E has one equivalence class. So we assume this.

Fix any $x_0 \in \Omega$, let $B^0 = \{U \in B : x_0 \notin U\}$, $T^0 = \{U \in T : x_0 \notin U\} \cup \{\Omega\}$, $B^1 = \{U \in B : x_0 \in U\}$, $T^1 = \{U \in T : x_0 \in U\} \cup \{\emptyset\}$. For some $l \in \{0, 1\}$, $|T^l| > \mu$, and then Ω, B^l, T^l satisfy the earlier requirements and the demands in (1) and (2). For (3) define an equivalence relation E' on Ω: $xE'y$ iff $(\forall U \in B)\,[x \in U \equiv y \in U]$, let $\Omega' \subseteq \Omega$ be a set of representatives, $B' = \{U \cap \Omega' : U \in B\}$ and finish as in the beginning of the proof of Claim 5. The only thing that is left is the second phrase in (2). But if it fails then for every $U \in T \backslash \{\emptyset\}$ choose a nonempty subset $V[U]$ from B. As the number of possible $V[U]$ is $\leqslant |B| \leqslant \mu$, for some $V \in B \backslash \{\emptyset\}$, for $> \mu$ members U of T, $V = V[U]$ and hence $V \subseteq U$. Choose $x \in V$; so for x clause (a) of (2) fails and hence for all $y \in \Omega$ clause (b) of (2) holds, as required. $\square_{13}$

Proof 14 (of Theorem 2). Consider for $n = n(*)$ (from Claim 13(0) and as in Claim 10) the following:

$(*)$ there are an open set V and a subset Z of V and for each $\alpha < \lambda_n$ there are $Z_\alpha \subseteq Z$ and open subsets V_α, U_α of V such that:

(a) for $\alpha < \beta < \lambda_n$ the sets $V_\alpha \cap Z$, $V_\beta \cap Z$ are distinct;

(b) $U_\alpha \cap Z = Z_\alpha$;

(c) the number of sets $U \in T$ satisfying $U \cap Z = V_\alpha \cap Z$ and $U_\alpha \subseteq U$ is $> \mu$.

So by Claim 10 we know that this fails for n.

Let χ be large enough and let $\bar{N} = \langle N_i : i < \mu^+ \rangle$ be an elementary chain of submodels of $(H(\chi), \in)$ of cardinality μ (and B, Ω, T belong to N_0 of course) increasing fast enough hence e.g.: if $X \in N_i$ is a small set, $U \in T$ then there is $U' \in N_i \cap T$ with $U \cap X = U' \cap X$ (you can avoid the name "elementary submodel" if you agree to list the closure properties actually used; as done in [18]. For $x \in \Omega$ let $i(x)$ be the unique i such that x belongs to $N_{i+1} \backslash N_i$ or $i = -1$ if $x \in N_0$ (remember $|\Omega| = \mu^+$).

Definition 15. We define: $x \in \Omega$ is *$\bar{N}$-pertinent* if it belongs to some small subset of Ω which belongs to $N_{i(x)}$ (e.g. $i(x) = -1$) and *$\bar{N}$-impertinent* otherwise.

Observation 16. *$\Omega_{\mathrm{ip}} = \{x \in \Omega : x$ is $\bar{N}$-impertinent$\}$ is not small (see Definition 7).*

Proof. As $N_0 \cap \Omega$ is small by Claim 8, for some U^*, $T' \overset{\mathrm{def}}{=} \{U \in T: U \cap N_0 \cap \Omega = U^* \cap N_0 \cap \Omega\}$ has cardinality $> \mu$. So it suffices to prove:

$$(*) \qquad U_1 \neq U_2 \in T' \;\Rightarrow\; U_1 \cap \Omega_{\mathrm{ip}} \neq U_2 \cap \Omega_{\mathrm{ip}}.$$

Choose $x \in (U_1 \backslash U_2) \cup (U_2 \backslash U_1)$ with $i(x)$ minimal. As $U_1, U_2 \in T'$, $i(x) = -1$ (i.e., $x \in N_0$) is impossible, so $x \in (N_{i+1} \backslash N_i) \cap \Omega$ for $i = i(x)$. If $x \in \Omega_{\mathrm{ip}}$ we succeed, so assume not, i.e., x is $\bar{N}$-pertinent, so for some small $X \in N_i$ $x \in X$. Hence by the choice of $\bar{N}$: for some $U_1', U_2' \in N_i \cap T$ we have: $U_1' \cap X = U_1 \cap X$, $U_2' \cap X = U_2 \cap X$ so $U_1' \cap X$, $U_2' \cap X \in N_i$ are distinct (as x witness) so there is $x' \in N_i \cap X$, $x' \in U_1' \equiv x' \notin U_2'$; but this implies $x' \in U_1 \equiv x' \notin U_2$, contradicting $i(x)$'s minimality. $\square_{16}$

We define a binary relation $\preccurlyeq$ on Ω_{ip} by:

$$x \preccurlyeq y \;\Leftrightarrow\; \text{for all } U \in B, \text{ if } y \in U \text{ then } x \in U.$$

Claim 17. *The relation $\preccurlyeq$ is clearly reflexive and transitive. It is antisymmetric by Claim 13(3).*

Observation 18. *If $J \subseteq \Omega_{\mathrm{ip}}$ is linearly ordered by $\preccurlyeq$ then J is small.*

Proof. For each $U_1, U_2 \in B$ such that $U_1 \cap J \not\subseteq U_2 \cap J$ choose $y_{U_1,U_2} \in J \cap (U_1 \backslash U_2)$. Let $I = \{y_{U_1,U_2} : U_1, U_2 \in B \;\&\; U_1 \cap J \not\subseteq U_2 \cap J\}$. Clearly $|I| \leqslant \mu$. We claim that I is dense in J (with respect to $\preccurlyeq$, i.e., I has a member in every non-empty interval of J). Suppose that $x, y, z \in J$, $x \prec y \prec z$. By 13(3) we find $U_1, U_2 \in B$ such that $x \in U_1$, $y \notin U_1$, and $y \in U_2$, $z \notin U_2$. Consider $y_{U_2,U_1} \in I$. Easily $x \prec y_{U_2,U_1} \prec z$. Thus if $(x, z) \neq \emptyset$ then $(x, z) \cap I \neq \emptyset$.

Now note that each Dedekind cut of I is a restriction of at most 3 Dedekind cuts of J (and the restriction of a Dedekind cut of J to I is a Dedekind cut of I). For this suppose that Y_1, Y_2, Y_3, Y_4 are lower parts of distinct Dedekind cuts of J with the same restriction to I, w.l.o.g. $Y_1 \subset Y_2 \subset Y_3 \subset Y_4$. For $i = 2, 3, 4$ choose $y_i \in Y_i$ such that $Y_1 \prec y_2$, $Y_2 \prec y_3$ and $Y_3 \prec y_4$. As $(y_2, y_4) \neq \emptyset$ we find $x \in (y_2, y_4) \cap I$. Since $y_2 \prec x$ we get $x \notin Y_1$ and since $x \prec y_4$ we obtain $x \in Y_4$. Consequently x distinguishes the restrictions of cuts determined by Y_1 and Y_4 to I.

To finish the proof of the observation apply observation 11 to I (which has essentially the same number of Dedekind cuts as J). $\square_{18}$

Continuation 19 (of the Proof of Theorem 2). Now it suffices to prove that for each $x \in \Omega_{\text{ip}}$, $i = i(x) > 0$ there is no member y of $\Omega_{\text{ip}} \cap N_i$ such that x, y are $\preccurlyeq$-incomparable.

[Why? Then we can divide Ω_{ip} to μ sets such that any two in the same part are $\preccurlyeq$-comparable contradicting 16 + 18 and 8. How? By defining a function $h: \Omega_{\text{ip}} \to \mu$ such that $h(x) = h(y) \Rightarrow x \preccurlyeq y \vee y \preccurlyeq x$. We define $h \restriction (\Omega_{\text{ip}} \cap N_i)$ by induction on i, in the induction step let $N_{i+1} \setminus N_i = \{x_{i,\varepsilon} : \varepsilon < \mu\}$. Choose $h(x_{i,\varepsilon})$ by induction on ε: for each ε there are $\leqslant |\varepsilon| < \mu$ forbidden values so we can carryout the definition.]

So assume this fails, so we have: for some $x \in \Omega_{\text{ip}}$, $i = i(x) > 0$ there is $y_0 \in N_i \cap \Omega_{\text{ip}}$ which is $\preccurlyeq$-incomparable with x; so there are $U_0, V_0 \in B$ such that $x \in V_0$, $x \notin U_0$, $y_0 \in U_0$, $y_0 \notin V_0$. Now $U^* = \bigcup \{U \in T: y_0 \notin U\}$ is in $T \cap N_i$ and $x \in U^*$ (as V_0 witnesses it) but by 13(2) we know that U^* is small, so it contradicts "$x \in \Omega_{\text{ip}}$". This finishes the proof of Theorem 2. $\square_2$

Concluding Remarks 20. Condition (b) of Theorem 2 holds easily for $\mu = \lambda$. Still it may look restrictive, and the author was tempted to try to eliminate it (on such set-theoretic conditions see [16, §6]). But instead of working "honestly" on this the author for this purpose proved (see [19]) that it follows from ZFC, and therefore can be omitted, hence

Conclusion 21 (Main). *If λ is strong limit, $\operatorname{cf} \lambda = \aleph_0$, and T a topology with base B, $|T| > |B| \geqslant \lambda$, then $|T| \geqslant 2^\lambda$ and the conclusion of* 2(2) *holds.*

Theorem 22. (1) *Under the assumptions of Theorem* 2 *(or* 21*), if the topology T is of size $\geqslant 2^\lambda$ then there are distinct $x_\eta \in \Omega$ for $\eta \in \bigcup_{n<\omega} \prod_{l<n} \mu_l$ such that letting $Z = \{x_\eta: \eta \in \bigcup_{n<\omega} \prod_{l<n} \mu_l\}$ one of the following occurs:*

(a) *there are $U_\eta \in T$ (i.e., open) for $\eta \in \prod_{l<\omega} \mu_l$ such that:*

$$U_\eta \cap Z = \{x_\nu \in Z : (\exists n < \lg(\nu))(\nu \restriction n = \eta \restriction n \,\&\, \nu(n) < \eta(n))\};$$

(b) *there are $U_\eta \in T$ for $\eta \in \prod_{l<\omega} \mu_l$ such that:*

$$U_\eta \cap Z = \{x_\nu \in Z : (\exists n < \lg(\nu))(\nu \restriction n = \eta \restriction n \,\&\, \nu(n) > \eta(n))\};$$

(c) *there are $U_\eta \in T$ for $\eta \in \prod_{l<w} \mu_l$ such that:*

$$U_\eta \cap Z = \{x_\nu \in Z : \neg \nu \lhd \eta\}.$$

(2) *If in addition $\lambda = \aleph_0$ then we get*

($\oplus$) *there are distinct $x_q \in \Omega$ for $q \in \mathbb{Q}$ (the rationals) such that for every real r, for some (open) set $U \in T$*

$$U \cap \{x_q : q \in \mathbb{Q}\} = \{x_q : q \in \mathbb{Q},\, q < r\}.$$

Observation 23. *Suppose that there are distinct $x_\eta \in \Omega$ (for $\eta \in \bigcup_{n\in\omega}\prod_{l<n}\mu_l$) such that one of the following occurs:*

(d) *there are $U_\eta \in T$ for $\eta \in \prod_{l<\omega}\mu_l$ such that:*

$$U_\eta \cap Z = \{x_\nu \in Z : \nu = \rho^\wedge\langle\zeta\rangle \,\&\, [\neg\, \rho \triangleleft \eta \text{ or } \rho \triangleleft \eta \,\&\, \eta(\lg(\rho)) = \zeta]\};$$

(e) *there are $U_\eta \in T$ for $\eta \in \prod_{l<\omega}\mu_l$ such that:*

$$U_\eta \cap Z = \{x_\nu \in Z : \nu = \rho^\wedge\langle\zeta\rangle \,\&\, [\neg\, \rho \triangleleft \eta \text{ or } \rho \triangleleft \eta \,\&\, \eta(\lg(\rho)) < \zeta]\};$$

(f) *there are $U_\eta \in T$ for $\eta \in \prod_{l<\omega}\mu_l$ such that:*

$$U_\eta \cap Z = \{x_\nu \in Z : \nu = \rho^\wedge\langle\zeta\rangle \,\&\, [\neg\, \rho \triangleleft \eta \text{ or } \rho \triangleleft \eta \,\&\, \eta(\lg(\rho)) > \zeta]\}.$$

Then for some distinct $x'_\nu \in \Omega$ ($\nu \in \bigcup_{n\in\omega}$) the clause (c) *of Theorem* 22 *holds.*

Proof. Let U_η (for $\eta \in \prod_{l\in\omega}\mu_l$) be given by one of the clauses. For $\nu \in \prod_{l<n}\mu_l$, $n \in \omega$ let $g(\nu) \in \prod_{l<2n}\mu_l$ be such that $g(\nu)(2l) = 0$, $g(\nu)(2l+1) = \nu(l)$ and for $\eta \in \prod_{l\in\omega}\mu_l$ let $g(\eta) = \bigcup_{l<\omega} g(\eta \restriction l)$ (we assume that $\mu_l < \mu_{l+1}$). Next define points $x'_\nu \in \Omega$ and open sets U'_η as

$$U'_\eta = U_{g(\eta)}, \qquad x'_\nu = \begin{cases} x_{g(\nu)^\wedge\langle 1\rangle} & \text{if we are in clause (d),} \\ x_{g(\nu)^\wedge\langle 0\rangle} & \text{if we are in clauses (e), (f).} \end{cases}$$

Then x'_ν, U'_η exemplify clause (c) of Theorem 22. $\square_{23}$

Proof 24 (of 22 for the case $\lambda = \aleph_0$). It suffices to prove 22(2), as ($\oplus$) implies (a). Let $\mu = \lambda^+$ (no connection to μ of Theorem 2). By Theorem 2(2) and 21 w.l.o.g. $|\Omega| = \lambda$, $|B| \leqslant \lambda$. Let $\mathscr{I} = \{Z \subseteq \Omega : |\{U \cap Z : U \in T\}| < \mu\}$; again it is a proper ideal on Ω (but not necessarily even $\aleph_1$-complete). Let $P = \{(U, V) : U \subseteq V$ are from T, $V \setminus U \notin \mathscr{I}\}$. Clearly $P \neq \emptyset$ (as $(\emptyset, \Omega) \in P$); if for every $(U_0, U_1) \in P$ there is U such that (U_0, U), (U, U_1) are in P then we can easily get clause ($\oplus$). So by renaming w.l.o.g.

$(*)_1$ $\quad (\forall V \in T)(V \in \mathscr{I} \text{ or } \Omega \setminus V \in \mathscr{I})$.

We try to choose by induction on $n < \omega$, (x_n, U_n) such that

(a) $\quad x_n \in U_n \in T$,
(b) $\quad x_n \notin \bigcup_{l<n} U_l$,
(c) $\quad U_n \in \mathscr{I}$ and $x_l \notin U_n$ for $l < n$,
(d) $\quad |\{V \in T : (\forall l \leqslant n)(x_l \notin V)\}| \geqslant \mu$.

If we succeed, $\{U \cap \{x_n : n < \omega\} : U \in T\}$ includes all subsets of the infinite set $\{x_n : n < \omega\}$, which is much more than required (in particular ($\oplus$) holds).

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Suppose we have defined (x_n, U_n) for $n < m$ and that there is no (x_m, U_m) satisfying (a)–(d). This means that if $x \in U \in T \cap \mathscr{I}$, $(\forall n < m)(x_n \notin U)$ and $x \notin \bigcup_{n<m} U_n$ then

$(*)_2$ $$|\{V \in T: (\forall n < m)(x_n \notin V) \text{ and } x \notin V\}| < \mu.$$

Let $U^* = \bigcup \{U \in T \cap \mathscr{I}: (\forall n < m)(x_n \notin U)\}$. As $|\Omega| < \mu = \operatorname{cf} \mu$ we get

$(*)_3$ $$\left|\left\{V \in T: (\forall n < m)(x_n \notin V) \,\&\, U^* \setminus \left(V \cup \bigcup_{n<m} U_n\right) \neq \emptyset\right\}\right| < \mu.$$

Suppose that $U^* \notin \mathscr{I}$. Then, by $(*)_1$, $\Omega \setminus U^* \in \mathscr{I}$ (as U^* is open). Since (by clause (c)) $\bigcup_{n<m} U_n \in \mathscr{I}$ we can find an open set U such that $(\forall n < m)(x_n \notin U)$ and

$$\mu \leqslant \left|\left\{V \in T: V \cap \left(\bigcup_{n<m} U_n \cup (\Omega \setminus U^*)\right) = U \cap \left(\bigcup_{n<m} U_n \cup (\Omega \setminus U^*)\right)\right\}\right|$$

(this is possible by (d)). But if $V \cap (\bigcup_{n<m} U_n \cup (\Omega \setminus U^*)) = U \cap (\bigcup_{n<m} U_n \cup (\Omega \setminus U^*))$, $V \neq U \cup U^*$, then $U^* \setminus (V \cup \bigcup_{n<m} U_n) \neq \emptyset$, $(\forall n < m)(x_n \notin V)$. This contradicts $(*)_3$. Thus $U^* \in \mathscr{I}$. Hence (by (d)) we have

$(*)_4$ $$\mu \leqslant |\{V \in T: V \setminus U^* \neq \emptyset \,\&\, (\forall n < m)(x_n \notin V)\}|.$$

Since $|B| < \mu$ we find $V_0 \in B$ such that $V_0 \setminus U^* \neq \emptyset$, $(\forall n < m)(x_n \notin V_0)$ and $\mu \leqslant |\{V \in T: V_0 \subseteq V\}|$. The last condition implies that $\Omega \setminus V_0 \notin \mathscr{I}$ and hence $V_0 \in \mathscr{I}$ (by $(*)_1$). By the definition of U^* we conclude $V_0 \subseteq U^*$, a contradiction, thus proving 22 (when $\lambda = \aleph_0$). $\square_{24}$

Proof 25 (of 22 when $\lambda > \aleph_0$). By Theorems 2, 21 w.l.o.g. $|\Omega| = |B| = \lambda$. Let $\mathscr{I} = \{A \subseteq \Omega: |\{U \cap A: U \in T\}| \leqslant \lambda\}$; it is an ideal. Let $\mathscr{I}^+ = \mathscr{P}(\Omega) \setminus \mathscr{I}$ and let μ_n be as in Theorem 2.

Observation 26. *It is enough to prove*

($\otimes_1$) *for every $Y \in \mathscr{I}^+$ and n we can find a sequence $\bar{U} = \langle U_\zeta: \zeta < \mu_n \rangle$ of open subsets of Ω such that one of the following occurs:*

(a) *$\bar{U}$ increasing, $Y \cap U_{\zeta+1} \setminus U_\zeta \in \mathscr{I}^+$;*

(b) *$\bar{U}$ decreasing, $Y \cap U_\zeta \setminus U_{\zeta+1} \in \mathscr{I}^+$;*

(c) *$Y \cap U_\zeta \setminus \bigcup_{\varepsilon \neq \zeta} U_\varepsilon \in \mathscr{I}^+$;*

(d) *for some $\langle V_\zeta, y_\zeta: \zeta < \mu_n \rangle$ we have $Y \cap (\bigcap_{\zeta<\mu_n} U_\zeta \setminus \bigcup_{\zeta<\mu_n} V_\zeta) \in \mathscr{I}^+$, V_ζ's and U_ζ's are open, $V_\zeta \subseteq U_\zeta$, $y_\zeta \in Y$ are pairwise distinct and*

$(*)$ $$U_\zeta \cap \{y_\varepsilon: \varepsilon < \mu_n\} = V_\zeta \cap \{y_\varepsilon: \varepsilon < \mu_n\} = \{y_\varepsilon: \varepsilon \leqslant \zeta\};$$

(e) *like* (d) *but*

$(*)$ $$U_\eta \cap \{y_\varepsilon: \varepsilon < \mu_n\} = V_\zeta \cap \{y_\varepsilon: \varepsilon < \mu_n\} = \{y_\varepsilon: \zeta \leqslant \varepsilon < \mu_n\};$$

(f) *like* (d) *but*

$(*)''$ $$U_\zeta \cap \{y_\varepsilon: \varepsilon < \mu_n\} = V_\zeta \cap \{y_\varepsilon: \varepsilon < \mu_n\} = \{y_\zeta\};$$

(g) *like* (d) *but*

$(*)'''$ $\quad U_\zeta \cap \{y_\varepsilon : \varepsilon < \mu_n\} = V_\zeta \cap \{y_\varepsilon : \varepsilon < \mu_n\} = \{y_\varepsilon : \varepsilon < \mu_n, \varepsilon \neq \zeta\}$;

(h) *there are* V_ζ, y_ζ *for* $\zeta < \mu_n$ *such that* $V_\zeta \subseteq U_\zeta$ *are open,* $y_\zeta \in Y$ *are pairwise distinct,* $(U_\zeta \backslash V_\zeta) \cap \bigcap_{\xi \neq \zeta} V_\xi \in \mathscr{I}^+$ *and*

$(**)$ $\quad U_\zeta \cap \{y_\varepsilon : \varepsilon < \mu_n\} = V_\zeta \cap \{y_\varepsilon : \varepsilon < \mu_n\} = \{y_\varepsilon : \varepsilon < \zeta\}$;

(i) *like* (h) *but*

$(**)'$ $\quad U_\zeta \cap \{y_\varepsilon : \varepsilon < \mu_n\} = V_\zeta \cap \{y_\varepsilon : \varepsilon < \mu_n\} = \{y_\varepsilon : \zeta \leqslant \varepsilon < \mu_n\}$;

(j) *like* (h) *but*

$(**)''$ $\quad U_\zeta \cap \{y_\varepsilon : \varepsilon < \mu_n\} = V_\zeta \cap \{y_\varepsilon : \varepsilon < \mu_n\} = \{y_\zeta\}$;

(k) *like* (h) *but*

$(**)'''$ $\quad U_\zeta \cap \{y_\varepsilon : \varepsilon < \mu_n\} = V_\zeta \cap \{y_\varepsilon : \varepsilon < \mu_n\} = \{y_\varepsilon : \zeta \neq \varepsilon, \varepsilon < \mu_n\}$.

Proof. First note that if $n < m < \omega$, $Y_1 \subseteq Y_0$, Y_1, $Y_0 \in \mathscr{I}^+$ and one of the cases (a)–(k) of $(\otimes_1)$ occurs for Y_1, m then the same case holds for Y_0, n. Consequently, $(\otimes_1)$ implies that for each $Y \in \mathscr{I}^+$ one of (a)–(k) occurs for Y, n for every $n \in \omega$. Moreover, if $(\otimes_1)$ then for some $x \in \{\mathrm{a,b,c,d,e,f g,h,i,j,k}\}$ and $Y_0 \in \mathscr{I}^+$ we have

$(*)$ for every $Y_1 \subseteq Y_0$ from $\mathscr{I}^+$ and $n \in \omega$ case (x) holds.

If $x = \mathrm{a}$, clause (a) of 22(1) holds. For this we inductively define open sets V_η, V_η^- for $\eta \in \bigcup_{n\in\omega} \prod_{l<n} \mu_l$ such that for $\eta \in \prod_{l<n} \mu_l$, $\zeta < \mu_n$:

1. $V_\eta^- \subseteq V_\eta$, $\quad (V_\eta \backslash V_\eta^-) \cap Y_0 \in \mathscr{I}^+$, $\quad (V_{\eta^\wedge\langle\zeta+1\rangle} \backslash V_{\eta^\wedge\langle\zeta\rangle}) \cap Y_0 \in \mathscr{I}^+$;
2. if $\xi < \mu_{n+1}$ then $\quad V_\eta \subseteq V_{\eta^\wedge\langle\zeta\rangle} \subseteq V_{\eta^\wedge\langle\zeta,\xi\rangle} \subseteq V^-_{\eta^\wedge\langle\zeta+1\rangle}$.

Let $\langle U_\zeta : \zeta < \mu_0 \rangle$ be the increasing sequence of open sets given by (a) for Y_0, $n = 0$. Put $V_{\langle\zeta\rangle} = U_{2\zeta+1}$, $V^-_{\langle\zeta\rangle} = U_{2\zeta}$ for $\zeta < \mu_0$. Suppose we have defined V_η, V_η^- for $\lg(\eta) \leqslant m$. Given $\eta \in \prod_{l<m-1} \mu_l$, $\zeta < \mu_{m-1}$. Apply (a) for $(V^-_{\eta^\wedge\langle\zeta+1\rangle} \backslash V_{\eta^\wedge\langle\zeta\rangle}) \cap Y_0$ and $n = m$ to get a sequence $\langle U_\xi : \xi < \mu_m \rangle$. Put

$$V_{\eta^\wedge\langle\zeta,\xi\rangle} = (U_{2\xi+1} \cap V^-_{\eta^\wedge\langle\zeta+1\rangle}) \cup V_{\eta^\wedge\langle\zeta\rangle},$$

$$V^-_{\eta^\wedge\langle\zeta,\xi\rangle} = (U_{2\xi} \cap V^-_{\eta^\wedge\langle\zeta+1\rangle}) \cup V_{\eta^\wedge\langle\zeta\rangle}.$$

Next for each $\eta^\wedge\langle\zeta\rangle \in \bigcup_{n\in\omega} \prod_{l<n} \mu_l$ choose $x_{\eta^\wedge\langle\zeta\rangle} \in V_{\eta^\wedge\langle\zeta+1\rangle} \backslash V^-_{\eta^\wedge\langle\zeta+1\rangle}) \cap Y_0 \backslash \{x_{\eta\restriction l} : l \leqslant \lg(\eta)\}$. As the last sets are pairwise disjoint we get that x_η's are pairwise distinct. Moreover, if we put $U_\eta = \bigcup_{n\in\omega} V_{\eta\restriction n}$ (for $\eta \in \prod_{n\in\omega} \mu_l$) then we have

$$U_\eta \cap \left\{ x_\nu : \nu \in \bigcup_{n\in\omega} \prod_{l<n} \mu_l \right\} = \{x_\nu : (\exists n < \lg(\nu))(\nu \restriction n = \eta \restriction n \,\&\, \nu(n) < \eta(n))\}.$$

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Similarly one can show that if $x = \mathrm{b}$, clause (b) of 22(1) holds and if $x = \mathrm{c}$ then we can get a discrete set of cardinality λ hence all clauses 22(1) hold.

Suppose now that $x = \mathrm{d}$. By the induction of n we choose Y_n, $\langle U_{n,\zeta}, V_{n,\zeta}, y_{n,\zeta}: \zeta < \mu_n \rangle$:

$$Y_0 = Y \quad (\in \mathscr{I}^+),$$

$U_{n,\zeta}$, $V_{n,\zeta}$, $y_{n,\zeta}$ (for $\zeta < \mu_n$) are given by (d) for Y_n,

$$Y_{n+1} = Y_n \cap \bigcap_{\zeta < \mu_n} U_{n,\zeta} \setminus \bigcup_{\zeta < \mu_n} V_{n,\zeta} \in \mathscr{I}^+.$$

For $\eta \in \prod_{l \leqslant n} \mu_l$ $(n \in \omega)$ we let

$$W'_\eta = V_{n,\eta(n)} \cap \bigcap_{m<n} U_{m,\eta(m)}.$$

As $V_{n,\eta(n)} \cap \{ y_{n,\zeta} : \zeta < \mu_n \} = \{ y_{n,\zeta} : \zeta \leqslant \eta(n) \}$ and $\{ y_{n,\zeta} : \zeta < \mu_n \} \subseteq Y_n \subseteq Y_{m+1} \subseteq U_{m,\eta(m)}$ (for $m < n$) we get

$$W'_\eta \cap \{ y_{n,\zeta} : \zeta < \mu_n \} = \{ y_{n,\zeta} : \zeta \leqslant \eta(n) \},$$

$$W'_\eta \cap \{ y_{m,\zeta} : \zeta < \mu_m \} \subseteq U_{m,\eta(m)} \cap \{ y_{m,\zeta} : \zeta < \mu_m \}$$
$$\subseteq \{ y_{m,\zeta} : \zeta \leqslant \eta(m) \} \quad (\text{for } m < n).$$

Now for $\eta \in \prod_{n<\omega} \mu_n$ we define $W_\eta = \bigcup_{l<\omega} W'_{\eta \restriction l}$. Then for each n, $W_\eta \cap \{ y_{n,\zeta} : \zeta < \mu_n \} = \{ y_{n,\zeta} : \zeta \leqslant \eta(n) \}$. By renaming this implies clause (a) of 22(1). [For $\eta \in \prod_{l \leqslant n} \mu_l$ let $x_\eta = y_{n+1, \gamma(\eta)+1}$, where $\gamma(\eta) = \mu_n^n \times \eta(0) + \mu_n^{n-1} \times \eta(1) + \mu_n^{n-2} \times \eta(2) + \cdots + \mu_n^1 \times \eta(n-1) + \eta(n)$. Note: μ_n^l is the lth ordinal power of μ_n. For $\eta \in \prod_{l<\omega} \mu_l$ let $\bar{\gamma}(\eta) = \langle \gamma(\eta \restriction 1), \gamma(\eta \restriction 2), \ldots \rangle$ and let $U_\eta = W_{\bar{\gamma}(\eta)}$.]

For $x = \mathrm{e}$ we similarly get clause (b) of 22(1). For $x = \mathrm{f}$ we similarly get a discrete set of cardinality λ so all clauses of 22(1) hold. The case $x = \mathrm{g}$ corresponds to the clause (c) of 22(1).

Suppose now that $x = \mathrm{h}$. By induction on n we define Y_η, U_η, V_η and x_η for $\eta \in \prod_{l \leqslant n} \mu_l$:

$$Y_{\langle\, \rangle} = Y,$$

$U_{\eta^\wedge\langle\zeta\rangle}$, $V_{\eta^\wedge\langle\zeta\rangle}$, $x_{\eta^\wedge\langle\zeta\rangle}$ are U_ζ, V_ζ, y_ζ given by the clause (h) for Y_η, μ_{n+1},

$$Y_{\eta^\wedge\langle\zeta\rangle} = (U_{\eta^\wedge\langle\zeta\rangle} \setminus V_{\eta^\wedge\langle\zeta\rangle}) \cap \bigcap_{\xi \neq \zeta} V_{\eta^\wedge\langle\xi\rangle}.$$

For $\eta \in \prod_{n<\omega} \mu_l$ put $U'_\eta = \bigcup_{l<\omega} V_{\eta \restriction l}$. Then

$$U'_\eta \cap \left\{ x_\nu : \nu \in \bigcup_{n<\omega} \prod_{l<n} \mu_l \right\}$$
$$= \{ x_\nu : \nu = \rho^\wedge \langle \zeta \rangle \,\&\, [\neg \rho \lhd \eta \text{ or } \rho \lhd \eta \,\&\, \eta(\lg(\rho)) < \zeta] \}$$

witnessing case (e) of 23, hence by Observation 23, case (c) of 22(1) holds.

If $x = \mathrm{i}$ then we similarly get case (f) of 23 and if $x = \mathrm{j}$ we get (d) of 23, hence case (c) of 22. Lastly $x = \mathrm{k}$ implies the case (c) of 22(1). $\square_{26}$

Claim 27. *If $\kappa < \lambda$, $\langle Z_\zeta : \zeta < \kappa \rangle$ is a partition of Ω or just $\bigcup_{\zeta \in \kappa} Z_\zeta \in \mathscr{I}^+$ then for some countable $w^* \subseteq \kappa$, for every infinite $w \subseteq w^*$, $\bigcup_{\zeta \in w} Z_\zeta \notin \mathscr{I}$.*

Proof. W.l.o.g. $\Omega = \bigcup_{\zeta < \kappa} Z_\zeta$. Otherwise there are $\mathscr{P} \subseteq [\kappa]^{\aleph_0}$ and $\langle T_w : w \in \mathscr{P} \rangle$, $T_w \subseteq T$, $|T_w| \leqslant \lambda$ such that for every $w^* \in [\kappa]^{\aleph_0}$ and $U \in T$, for some $w \subseteq w^*$, $w \in \mathscr{P}$ and $V \in T_w$ we have $U \cap (\bigcup_{\zeta \in w} Z_\zeta) = V \cap (\bigcup_{\zeta \in w} Z_\zeta)$. Let $\{U_\zeta : \zeta < \lambda\}$ list $\bigcup \{T_w : w \in \mathscr{P}\}$ (note that since $\kappa < \lambda$ also $|[\kappa]^{\leqslant \aleph_0}| = \kappa^{\aleph_0} < \lambda$). We claim that there is $U \in T$ such that for every $\xi < \lambda$ there are $\alpha, \beta \in \Omega$ for which:

(a) $\alpha \in U \Leftrightarrow \beta \notin U$,

(b) $(\forall \zeta < \xi)(\alpha \in U_\zeta \Leftrightarrow \beta \in U_\zeta)$,

(c) $(\forall \varepsilon < \kappa)(\alpha \in Z_\varepsilon \Leftrightarrow \beta \in Z_\varepsilon)$.

Indeed, to find such U consider equivalence relations E_ξ (for $\xi < \lambda$) determined by (b) and (c), i.e., for $\alpha, \beta \in \Omega$:

$$\alpha E_\xi \beta \text{ iff } (\forall \zeta < \xi)(\alpha \in U_\zeta \Leftrightarrow \beta \in U_\zeta) \text{ and } (\forall \varepsilon < \kappa)(\alpha \in Z_\varepsilon \Leftrightarrow \beta \in Z_\varepsilon).$$

The relation E_ξ has $\leqslant 2^{|\xi| + \kappa} < \lambda$ equivalence classes. Consequently for each $\xi < \lambda$

$$|\{V \in T : V \text{ is a union of } E_\xi\text{-equivalence classes}\}| < \lambda.$$

As $|T| > \lambda$ we find a non-empty open set U which for no $\xi < \lambda$ is a union of E_ξ-equivalence classes. This U is as needed.

Now let (α_n, β_n) be a pair (α, β) satisfying (a)–(c) for $\xi = \lambda_n$ and let $\{\alpha_n, \beta_n\} \subseteq Z_{\zeta_n}$. Then $w^* = \{\zeta_n : n < \omega\}$, U contradict the choice of $\mathscr{P}$ and $\langle T_w : w \in \mathscr{P} \rangle$. $\square_{27}$

Proof 28 (*of* $(\otimes_1)$ *of* 26). For notational simplicity we assume that $Y = \Omega$. Let $B = \bigcup_{n < \omega} B_n$, $|B_n| < \lambda$, $\emptyset \in B_0$ and $B_n \subseteq B_{n+1}$.

As in the proof of Claim 5 *w.l.o.g. for every* $x \neq y$ from Ω we have

$$|\{U \in T : x \in U \Leftrightarrow y \notin U\}| > \lambda.$$

Let $y_\zeta \in \Omega$ for $\zeta < \mu_{n+6}$ be pairwise distinct. For each $\zeta < \xi < \mu_{n+6}$ there is $\varepsilon = \varepsilon(\zeta, \xi) \in \{\zeta, \xi\}$ such that $T^0_{\zeta,\xi} \stackrel{\text{def}}{=} \{U \in T : \{y_\zeta, y_\xi\} \cap U = \{y_\varepsilon\}\}$ has cardinality $> \lambda$. For each $U \in T^0_{\zeta,\xi}$ there is $V[U] \in B$, $y_\varepsilon \in V[U] \subseteq U$. As $|B| \leqslant \lambda$ for some $V^*_{\zeta,\xi} \in B$ we have that the set

$$T^1_{\zeta,\xi} = \{U \in T : \{y_\zeta, y_\xi\} \cap U = \{y_\varepsilon\} \text{ and } y_\varepsilon \in V^*_{\zeta,\xi} \subseteq U\}$$

has cardinality $> \lambda$. For $U \in T$ let f_U, g_U be functions such that:

1. $f_U : \mu_{n+6} \to \omega$, $\quad g_U : \mu_{n+6} \to B$,
2. $g_U(\varepsilon) = \emptyset$ iff $y_\varepsilon \notin U$,
3. if $y_\varepsilon \in U$ then $y_\varepsilon \in g_U(\varepsilon) \subseteq U$,
4. $f_U(\varepsilon) = \min\{n \in \omega : g_U(\varepsilon) \in B_n\}$.

For each $\zeta < \xi < \mu_{n+6}$ we find $f_{\zeta,\xi}: \mu_{n+6} \to \omega$ such that the set

$$T^2_{\zeta,\xi} = \{U \in T^1_{\zeta,\xi}: f_U = f_{\zeta,\xi}\}$$

has cardinality $> \lambda$. By the Erdös–Rado theorem we may assume that for each $\zeta < \xi < \mu_{n+5}$, $\varepsilon < \mu_{n+5}$ the value of $f_{\zeta,\xi}(\varepsilon)$ depends on relations between ζ, ξ and ε only. Consequently for some $n^* < \omega$, if $\varepsilon < \mu_{n+5}$, $U \in T^2_{\zeta,\xi}$, $\zeta < \xi < \mu_{n+5}$ then $g_U(\varepsilon) \in B_{n^*}$. As $|B_{n^*}| < \lambda$ we find (for each $\zeta < \xi < \mu_{n+5}$) a function $g_{\zeta,\xi}: \mu_{n+6} \to B_{n^*}$ such that the set

$$T^3_{\zeta,\xi} = \{U \in T^2_{\zeta,\xi}: g_U = g_{\zeta,\xi}\}$$

is of size $> \lambda$. Let

$$U_{\zeta,\xi} = \bigcup T^3_{\zeta,\xi}, \qquad V_{\zeta,\xi} = \bigcup_{\varepsilon < \mu_{n+5}} g_{\zeta,\xi}(\varepsilon).$$

Clearly

$$(*) \qquad V_{\zeta,\xi} \subseteq U_{\zeta,\xi}, \qquad U_{\zeta,\xi} \setminus V_{\zeta,\xi} \notin \mathscr{I},$$
$$U_{\zeta,\xi} \cap \{y_\zeta, y_\xi\} = V_{\zeta,\xi} \cap \{y_\zeta, y_\xi\} = \{y_{\varepsilon(\zeta,\xi)}(\zeta,\xi)\} \quad \text{and}$$
$$(**) \qquad U_{\zeta,\xi} \cap \{y_\delta: \delta < \mu_{n+5}\} = V_{\zeta,\xi} \cap \{y_\delta: \delta < \mu_{n+5}\}.$$

Let $T_1 = \{V_{\zeta,\xi}, U_{\zeta,\xi}: \zeta < \xi < \mu_{n+5}\}$, so $|T_1| < \lambda$. Define a two-place relation E_{T_1} on Ω:

$$x \; E_{T_1} \, y \text{ iff } (\forall U \in T_1)(x \in U \;\Rightarrow\; y \in U).$$

Clearly E_{T_1} is an equivalence relation with $\leqslant 2^{|T_1|} < \lambda$ equivalence classes. Hence by Claim 27 for each $\zeta < \xi < \mu_{n+5}$, for some ω-sequence of E_{T_1}-equivalence classes $\langle A_{\zeta,\xi,n}: n < \omega\rangle$ we have:

$$A_{\zeta,\xi,n} \subseteq U_{\zeta,\xi} \setminus V_{\zeta,\xi} \quad \text{and for each infinite } w \subseteq \omega, \quad \bigcup_{n \in w} A_{\zeta,\xi,n} \notin \mathscr{I}.$$

By the Erdös–Rado theorem, w.l.o.g. for $\zeta_1 < \zeta_2 < \mu_{n+4}$, ξ_1, $\xi_2 < \mu_{n+4}$ the truth values of "$\varepsilon(\zeta_1, \zeta_2) = \zeta_1$", "$y_{\xi_1} \in V_{\zeta_1,\zeta_2}$", "$y_{\xi_1} \in U_{\zeta_1,\zeta_2}$", "$A_{\zeta_1,\zeta_2,n} \subseteq U_{\xi_1,\xi_2}$", "$A_{\zeta_1,\zeta_2,n} \subseteq V_{\xi_1,\xi_2}$", "$A_{\zeta_1,\zeta_2,n} = A_{\zeta_1,\zeta_2,m}$", "$A_{\zeta_1,\zeta_2,n} = A_{\xi_1,\xi_2,m}$" depend just on the order and equalities among ζ_1, ζ_2, ξ_1, ξ_2 (and of course n, m).

As each infinite union $\bigcup_{n \in \omega} A_{\zeta,\xi,n}$ is in $\mathscr{I}^+$, w.l.o.g. those truth values also do not depend on n (for the last one we mean "$A_{\zeta_1,\zeta_2,n} = A_{\xi_1,\xi_2,n}$"). Note: if $A_{1,2,n} = A_{3,4,m}$ then $A_{1,2,n} = A_{3,4,n} = A_{1,2,m}$.

Now, $A_{\zeta,\xi,n}$ is either included in U_{ξ_1,ξ_2} or is disjoint from it (uniformly for n); similarly for V_{ξ_1,ξ_2}.

Case A: $A_{3,4,n} \cap U_{1,2} = \emptyset$. Let $U'_\zeta = \bigcup_{\xi \leqslant \zeta} U_{2\xi,2\xi+1}$. Then $\langle U'_\zeta: \zeta < \mu_n\rangle$ is an increasing sequence of open sets and $\bigcup_{n \in \omega} A_{2\zeta+2,2\zeta+3,n} \subseteq U'_{\zeta+1} \setminus U'_\zeta$, which witnesses that the last set is in $\mathscr{I}^+$. Thus we get clause (a).

Case B: $A_{1,2,n} \cap U_{3,4} = \emptyset$. Let $U'_\zeta = \bigcup_{\zeta \leqslant \xi < \mu_n} U_{2\xi,2\xi+1}$. Then $\langle U'_\zeta: \zeta < \mu_n\rangle$ is a decreasing sequence of open sets and $\bigcup_{n \in \omega} A_{2\zeta,2\zeta+1,n} \subseteq U'_\zeta \setminus U'_{\zeta+1}$. Consequently we get clause (b).

Thus we have to consider the case

$$A_{1,2,n} \subseteq U_{3,4} \quad \text{and} \quad A_{3,4,n} \subseteq U_{1,2}$$

only. So we assume this.

Case C: $A_{1,2,n} \cap V_{3,4} = \emptyset$, $A_{3,4,n} \cap V_{1,2} = \emptyset$. Let $U'_\zeta = U_{2\zeta,2\zeta+1}$, $V'_\zeta = V_{2\zeta,2\zeta+1}$.

Subcase C1: $y_1 \in U_{3,4}$, $y_5 \in U_{3,4}$. Then let y'_ζ be the unique member of $\{y_{2\zeta}, y_{2\zeta+1}\} \setminus \{y_{\varepsilon(2\zeta,2\zeta+1)}\}$. By (**) we easily get that $\langle U'_\zeta, V'_\zeta, y'_\zeta : \zeta < \mu_n \rangle$ witnesses the clause (g).

Subcase C2: $y_1 \notin U_{3,4}$ *or* $y_5 \notin U_{3,4}$. Then we put $y'_\zeta = y_{\varepsilon(2\zeta,2\zeta+1)}$ and we get one of the cases (d), (e) or (f).

Case D: $A_{1,2,n} \subseteq V_{3,4}$, $A_{3,4,n} \cap V_{1,2} = \emptyset$. We let $U'_\zeta = \bigcup \{V_{2\xi,2\xi+1} : \xi \leqslant \zeta\}$. Thus U'_ζ increases with ζ and $U'_{\zeta+1} \setminus U'_\zeta$ includes $\bigcup_{n\in\omega} A_{2\zeta,2\zeta+1,n}$. Thus clause (a) holds.

Case E: $A_{1,2,n} \cap V_{3,4} = \emptyset$, $A_{3,4,n} \subseteq V_{1,2}$. Let $U'_\zeta = \bigcup \{V_{2\xi,2\xi+1} : \xi \geqslant \zeta\}$. Then U'_ζ decreases with ζ and the clause (b) holds.

Case F: $A_{1,2,n} \subseteq V_{3,4}$, $A_{3,4,n} \subseteq V_{1,2}$. Let $U'_\zeta = U_{2\zeta,2\zeta+1}$, $V'_\zeta = V_{2\zeta+2\zeta+1}$. If $y_1, y_5 \in U_{3,4}$ then we put $y'_\zeta \in \{y_{2\zeta}, y_{2\zeta+1}\} \setminus \{y_{\varepsilon(2\zeta,2\zeta+1)}\}$ and we get case (k). Otherwise we put $y'_\zeta = y_{\varepsilon(2\zeta,2\zeta+1)}$ and we obtain one of the cases (h), (i) or (j). $\square_{22}$

Concluding Remarks 29. (1) *Assume that a topology T on Ω is given with a base B, $|B| \leqslant \lambda$, and that λ, $\langle \mu_n : n \in \omega \rangle$ are as before (μ_n regular for simplicity). If*

(*) $\quad x_\nu \in \Omega$ *for* $\nu \in \bigcup_{n\in\omega} \prod_{l<n} \mu_l$ *and* $U_\eta \in T$ *for* $\eta \in \prod_{n\in\omega} \mu_n$;

(**) *if* $n < \omega$, $\nu \in \prod_{l<n} \mu_l$ *and* $\eta \in \prod_{l<\omega} \mu_l$ *then for some* k,

$$(\forall \eta')\left(\eta' \in \prod_{l<\omega} \mu_l \,\&\, \eta' \restriction k = \eta \restriction k \;\Rightarrow\; U_{\eta'} \cap \{x_\nu\} = U_\eta \cap \{x_\nu\}\right),$$

hold, then we can find $S \subseteq \bigcup_{n<\omega} \prod_{l<n} \mu_l$ and $\langle U^*_{\eta,\nu} : \eta, \nu \in \prod_{l<n} \mu_l \cap S$ for some $n\rangle$ and $\langle U^*_\eta : \eta \in \lim S\rangle$ (where $\lim S = \{\eta \in \prod_{l<\omega} \mu_l : (\forall l < \omega)(\eta \restriction l \in S)\}$) such that

(a) $\langle\ \rangle \in S$, *S is closed under initial segments and*

$$\eta \in S \,\&\, n = \lg \eta \;\Rightarrow\; (\exists \alpha)(\eta\char`^\langle\alpha\rangle \in S)$$

and for some infinite $w \subseteq \omega$, for every $n < \omega$ and $\eta \in \lim S$ we have:

$$n \in w \;\Leftrightarrow\; (\exists^{\geqslant 2} \alpha < \mu_n)(\eta\char`^\langle\alpha\rangle \in S) \;\Leftrightarrow\; (\exists^{\mu_n} \alpha < \mu_n)(\eta\char`^\langle\alpha\rangle \in S);$$

(b) *if $\rho, \nu \in \prod_{l<n} \mu_l \cap S$ and $\nu \triangleleft \eta \in S \cap \prod_{l<\omega} \mu_l$ then*

$$U^*_\eta \cap \{x_\rho\} = U^*_{\nu,\eta} \cap \{x_\rho\};$$

(c) *for $\eta \in \lim S$,*

$$U^*_\eta \cap \{x_\rho : \rho \in S\} = U_\eta \cap \{x_\rho : \rho \in S\};$$

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(d) $U^*_{\eta,\nu} \in B$ *for* $\eta, \nu \in \left(\prod_{l<n} \mu_l\right) \cap S$, $n < \omega$;

(e) $U^*_\eta = \bigcup \left\{ U^*_{\eta \restriction n, \nu} : n < \omega,\ \nu \in \left(\prod_{l<n} \mu_l\right) \cap S \right\}$ *for* $\eta \in \lim S$.

(2) *So in Theorem* 22, *the case* (c) *can be further described.*

(3) *We can consider basic forms for any analytic family of subsets of* λ (*then we have more cases; as in* 23 *and* $(\otimes_1)$ *of* 26).

References

[1] A. Hajnal and I. Juhasz, Some remarks on a property of topological cardinal functions, Acta Math. Acad. Sci. Hungar. 20 (1969) 25–37.

[2] A. Hajnal and I. Juhasz, On the number of open sets, Ann. Univ. Sci. Budapest 16 (1973) 99–102.

[3] I. Juhasz, Cardinal functions in topology, Math. Center Tracts, Amsterdam, 1971.

[4] I. Juhasz, Cardinal functions in topology—ten years later. Math. Center Tracts, Amsterdam, 1980.

[5] I. Juhasz and S. Shelah, How large can a hereditary separable or hereditary Lindelof space be, Israel J. Math. 53 (1986) 355–364. [JuSh 231].

[6] K. Kunen and J. Roitman, Attaining the spread of cardinals of cofinality ω, Pacific J. Math. 70 (1977) 199–205.

[7] C. Kuratowski, Topologie I (Państwowe Wydawnictwo Naukowe, Warszawa, 1958).

[8] J. Roitman, Attaining the spread at cardinals which are not strong limit, Pacific J. Math. 57 (1975) 545–551.

[9] M. Rubin and S. Shelah, Combinatorial problems on trees: partitions. Δ-systems and large free subsets, Ann. Pure Appl. Logic 33 (1987) 43–82. [RuSh 117]

[10] S. Shelah, On cardinal invariants in topology, Gen. Topology Appl. 7 (1977) 251–259. [Sh 36]

[11] S. Shelah, Remarks on Boolean algebras, Algebra Universalis 11 (1980) 77–89. [Sh 92]

[12] S. Shelah, Canonization theorems and applications, J. Symbolic Logic 46 (1981) 345–353. [Sh 95]

[13] S. Shelah, Remarks on the number of ideals of Boolean algebras and open sets of a topology, Lecture Notes in Math. 1182 (Springer, Berlin, 1982) 151–187. [Sh 233].

[14] S. Shelah, Number of pairwise non-elementarily embeddable models, J. Symbolic Logic 54 (1989) 1431–1455. [Sh 262]

[15] S. Shelah, $\aleph_{\omega+1}$ has a Jonsson Algebra, in: Cardinal Arithmetic (Oxford Univ. Press, Oxford), Ch. III. [Sh 355]

[16] S. Shelah, Advances in Cardinal Arithmetic, Proceedings of the Conference in Banff, Alberta, April 1991. [Sh 420]

[17] S. Shelah, Further cardinal Arithmetic, Israel J. Math., accepted. [Sh 430]

[18] S. Shelah, Cardinalities of countably based topologies, Israel J. Math., 83 (1993) 369–374. [Sh 454]

[19] S. Shelah, The generalized continuum hypothesis revisited, Preprint. [Sh 460]

[20] S. Shelah, Remarks on general topology (1/78), Preprint. [Sh C1]

[21] S. Shelah, If $\diamondsuit_{\aleph_1}$ + "there is an $\aleph_1$-Kurepa tree with κ branches" then some B.A. of power $\aleph_1$ has λ filters and $\lambda^{\aleph_0}$-ultrafilters, Mimeographed notes from Madison, Fall 77. [Sh E6]

[22] S. Shelah, P-points, $\beta(\omega)$ and other results in general topology, Notices AMS 25 (1978) A-365. [Sh G.1]