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Partition Theorems from Creatures and Idempotent Ultrafilters*

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Abstract. We show a general scheme of Ramsey-type results for partitions of countable sets of finite functions, where "one piece is big" is interpreted in the language originating in creature forcing. The heart of our proofs follows Glazer's proof of the Hindman Theorem, so we prove the existence of idempotent ultrafilters with respect to suitable operation. Then we deduce partition theorems related to creature forcings.

Keywords: norms on possibilities, partition theorems, ultrafilters

1. Introduction

A typical partition theorem asserts that if a set with some structure is divided into some number of "nice" pieces, then one of the pieces is large from the point of view of the structure under considerations. Sometimes, the underlying structure is complicated and it is not immediately visible that the arguments in hands involve a partition theorem. Such is the case with many forcing arguments. For instance, the proofs of propernes of some forcing notions built according to the scheme of *norms on possibilities* have in their hearts partition theorems stating that at some situations a homogeneous tree and/or a sequence of creatures determining a condition can be found (see, e.g., Rosłanowski and Shelah [10, 11], Rosłanowski, Shelah, and Spinas [12], Kellner and Shelah [7, 8]). A more explicit connection of partition theorems with forcing arguments is given in Shelah and Zapletal [13].

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The present paper is a contribution to the Ramsey theory in the context of finitary creature forcing. We are motivated by earlier papers and notions concerning *norms on possibilities*, but we do not look at possible forcing consequences. The common form of our results here is as follows. If a certain family of partial finite functions is divided into finitely many pieces, then one of the pieces contains all partial functions determined by an object ("a pure candidate") that can be interpreted as a forcing condition if we look at the setting from the point of view of the creature forcing. Sets of partial functions determined by a pure candidate might be considered as "large" sets.

Our main proofs follow the celebrated Glazer's proof of the Hindman Theorem, which reduced the problem to the existence of a relevant ultrafilter on ω in ZFC. Those arguments were presented by Comfort in [3, Theorem 10.3, p. 451] with [3, Lemma 10.1, p. 449] as a crucial step (stated here in Lemma 3.7). The arguments of Section 3 of our paper really resemble Glazer's proof. In that section we deal with the easier case of omittory-like creatures (*loose FFCC pairs* of Definition 2.2(2)) and in the proof of the main result Conclusion 3.10 we use an ultrafilter idempotent with respect to operation $\oplus$ (defined in Definition 3.4). The third section deals with the case of *tight FFCC pairs* of Definition 2.2(4). Here, we consider partitions of some sets of partial functions all of which have domains being essentially intervals of integers starting with some fixed $n < \omega$. While the general scheme of the arguments follows the pattern of Section 3, they are slightly more complicated as they involve *sequences of ultrafilters* and operations on them. As an application of this method, in Theorem 4.9 we give a new proof of the partition theorem by Carlson and Simpson [2, Theorem 6.3]. The next section presents a variation of Section 4: Under weaker assumptions on the involved FFCC pairs we get a weaker, yet still interesting partition theorem. Possible applications of this weaker version include a special case of the partition theorem by Goldstern and Shelah [4] (see Corollary 5.9). These results motivate Section 5, where we develop the parallel of the very weak bigness for candidates with "limsup" demand on the norms.

Our paper is self-contained and all "creature terminology" needed is introduced in Section 2. We also give there several examples of creating pairs to which our results may be applied.

Notation: We use standard set-theoretic notation.

- An integer n is the set $\{0,1,\ldots,n-1\}$ of all integers smaller than n, and the set of all integers is called ω. For integers $n<m$, the interval $[n,m)$ denotes the set of all *integers* smaller than m and greater than or equal to n.
- All sequences will be indexed by natural numbers and a sequence of objects is typically denoted by a bar above a letter with the convention that $\bar{x} = \langle x_i : i < y\rangle$, $y \leq \omega$.
- For a set X, the family of all subsets of X is denoted by $\mathcal{P}(X)$. The domain of a function f is called $\mathrm{dom}(f)$.
- An ideal J on ω is a family of subsets of ω such that

  (i) all finite subsets of ω belong to J but $\omega \notin J$, and
  (ii) if $A \subseteq B \in J$, then $A \in J$ and if $A, B \in J$ then $A \cup B \in J$.

For an ideal J, the family of all subsets of ω that do not belong to J is denoted by J^+, and the filter dual to J is called J^c. Thus J^c consists of all sets $A \subseteq \omega$ for which $\omega \setminus A \in J$.

2. Partial Creatures

We use the context and notation of Rosłanowski and Shelah [10], but below we recall all the required definitions and concepts.

Since we are interested in Ramsey-type theorems and ultrafilters on a countable set of partial functions, we will use pure candidates rather than forcing notions generated by creating pairs. Also, our considerations will be restricted to creating pairs which are forgetful, smooth ([10, 1.2.5]), monotonic ([10, 5.2.3]), strongly finitary ([10, 1.1.3, 3.3.4]), and in some cases omittory-like ([10, 2.1.1]). Therefore, we will reformulate our definitions for this restricted context (in particular, $\mathbf{val}[t]$ is a set of *partial* functions), thus we slightly depart from the setting of [10].

Context 2.1. In this paper $\mathbf{H}$ is a fixed function defined on ω such that $\mathbf{H}(i)$ is a finite non-empty set for each $i < \omega$. The set of all finite non-empty functions f such that $\mathrm{dom}(f) \subseteq \omega$ and $f(i) \in \mathbf{H}(i)$ $\big($for all $i \in \mathrm{dom}(f)\big)$ will be denoted by $\mathcal{F}_{\mathbf{H}}$.

Definition 2.2. (1) *An FP creature*[†] *for* $\mathbf{H}$ *is a tuple*

$$t = \big(\mathbf{nor}, \mathbf{val}, \mathbf{dis}, m_{\mathrm{dn}}, m_{\mathrm{up}}\big) = \big(\mathbf{nor}[t], \mathbf{val}[t], \mathbf{dis}[t], m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}\big)$$

such that

- $\mathbf{nor}$ *is a non-negative real number,* $\mathbf{dis}$ *is an arbitrary object and* $m^t_{\mathrm{dn}} < m^t_{\mathrm{up}} < \omega$, *and*
- $\mathbf{val}$ *is a non-empty finite subset of* $\mathcal{F}_{\mathbf{H}}$ *such that* $\mathrm{dom}(f) \subseteq \big[m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}\big)$ *for all* $f \in \mathbf{val}$.

(2) *An FFCC pair*[‡] *for* $\mathbf{H}$ *is a pair* (K, Σ) *such that*

(a) *K is a countable family of FP creatures for* $\mathbf{H}$,
(b) *for each* $m < \omega$ *the set* $K_{\leq m} := \big\{t \in K\colon m^t_{\mathrm{up}} \leq m\big\}$ *is finite and the set* $K_{\geq m} := \big\{t \in K\colon m^t_{\mathrm{dn}} \geq m \;\&\; \mathbf{nor}[t] \geq m\big\}$ *is infinite,*
(c) *Σ is a function with the domain* $\mathrm{dom}(\Sigma)$ *included in the set*

$$\big\{(t_0, \ldots, t_n)\colon n < \omega,\ t_\ell \in K \text{ and } m^{t_\ell}_{\mathrm{up}} \leq m^{t_{\ell+1}}_{\mathrm{dn}} \text{ for } \ell < n\big\}$$

and the range included in $\mathcal{P}(K) \setminus \{\emptyset\}$,
(d) *if* $t \in \Sigma(t_0, \ldots, t_n)$ *then* ($t \in K$ *and*) $m^{t_0}_{\mathrm{dn}} = m^t_{\mathrm{dn}} < m^t_{\mathrm{up}} = m^{t_n}_{\mathrm{up}}$,
(e) $t \in \Sigma(t)$ (*for each* $t \in K$) *and*
(f) *if* $t \in \Sigma(t_0, \ldots, t_n)$ *and* $f \in \mathbf{val}[t]$, *then*

$$\mathrm{dom}(f) \subseteq \bigcup \big\{\big[m^{t_\ell}_{\mathrm{dn}}, m^{t_\ell}_{\mathrm{up}}\big) : \ell \leq n\big\}$$

and $f \restriction \big[m^{t_\ell}_{\mathrm{dn}}, m^{t_\ell}_{\mathrm{up}}\big) \in \mathbf{val}[t_\ell] \cup \{\emptyset\}$ *for* $\ell \leq n$, *and*

[†] FP stands for **F**orgetful **P**artial creature.

[‡] FFCC stands for *smooth* **F**orgetful *monotonic strongly* **F**initary **C**reature **C**reating pair.

(g) *if* $\bar{t}_0,\ldots,\bar{t}_n \in \mathrm{dom}(\Sigma)$ *and* $\bar{t} = \bar{t}_0{}^\frown\cdots{}^\frown\bar{t}_n \in \mathrm{dom}(\Sigma)$, *then*

$$\bigcup\big\{\Sigma(s_0,\ldots,s_n)\colon s_\ell \in \Sigma(\bar{t}_\ell) \textit{ for } \ell \le n\big\} \subseteq \Sigma(\bar{t}).$$

(3) *An FFCC pair* (K,Σ) *is* loose *if*

($\mathrm{c}^{\mathrm{loose}}$) *the domain of* Σ *is*

$$\mathrm{dom}(\Sigma) = \big\{(t_0,\ldots,t_n)\colon n<\omega,\ t_\ell \in K \textit{ and } m^{t_\ell}_{\mathrm{up}} \le m^{t_{\ell+1}}_{\mathrm{dn}} \textit{ for } \ell < n\big\}.$$

(4) *An FFCC pair* (K,Σ) *is* tight *if*

($\mathrm{c}^{\mathrm{tight}}$) *the domain of* Σ *is*

$$\mathrm{dom}(\Sigma) = \big\{(t_0,\ldots,t_n)\colon n<\omega,\ t_\ell \in K \textit{ and } m^{t_\ell}_{\mathrm{up}} = m^{t_{\ell+1}}_{\mathrm{dn}} \textit{ for } \ell < n\big\},$$

($\mathrm{f}^{\mathrm{tight}}$) *if* $t \in \Sigma(t_0,\ldots,t_n)$ *and* $f \in \mathbf{val}[t]$, *then* $f{\restriction}\big[m^{t_\ell}_{\mathrm{dn}}, m^{t_\ell}_{\mathrm{up}}\big) \in \mathbf{val}[t_\ell]$ *for all* $\ell \le n$, *and*

($\mathrm{h}^{\mathrm{tight}}$) *if* $s_0, s_1 \in K$, $m^{s_0}_{\mathrm{up}} = m^{s_1}_{\mathrm{dn}}$, $f_0 \in \mathbf{val}[s_0]$, $f_1 \in \mathbf{val}[s_1]$ *and* $f = f_0 \cup f_1$, *then there is* $s \in \Sigma(s_0,s_1)$ *such that* $f \in \mathbf{val}[s]$.

Definition 2.3. (Cf. [10, Definition 1.2.4]) *Let* (K,Σ) *be an FFCC pair for* $\mathbf{H}$. (*Below, if* $\bar{s} \notin \mathrm{dom}(\Sigma)$ *then we stipulate* $\Sigma(\bar{s}) = \emptyset$.)

(1) *A* pure candidate for (K,Σ) *is a sequence* $\bar{t} = \langle t_n\colon n<\omega\rangle$ *such that* $t_n \in K$, $m^{t_n}_{\mathrm{up}} \le m^{t_{n+1}}_{\mathrm{dn}}$ (*for* $n<\omega$) *and* $\lim\limits_{n\to\infty} \mathbf{nor}[t_n] = \infty$.
A pure candidate $\bar{t}$ *is* tight *if* $m^{t_n}_{\mathrm{up}} = m^{t_{n+1}}_{\mathrm{dn}}$ (*for* $n<\omega$).
The set of all pure candidates for (K,Σ) *is denoted by* $\mathrm{PC}_\infty(K,\Sigma)$ *and the family of all tight pure candidates is called* $\mathrm{PC}^{\mathtt{tt}}_\infty(K,\Sigma)$.

(2) *For pure candidates* $\bar{t}, \bar{s} \in \mathrm{PC}_\infty(K,\Sigma)$ *we write* $\bar{t} \le \bar{s}$ *whenever there is a sequence* $\langle u_n\colon n<\omega\rangle$ *of non-empty finite subsets of* ω *satisfying*

$$\max(u_n) < \min(u_{n+1}) \ \textit{ and } \ \bar{s}_n \in \Sigma(\bar{t}{\restriction}u_n), \quad \textit{for all } n<\omega.$$

(3) *For a pure candidate* $\bar{t} = \langle t_i\colon i<\omega\rangle \in \mathrm{PC}_\infty(K,\Sigma)$ *we define*

(a) $\mathcal{S}(\bar{t}) = \{(t_{i_0},\ldots,t_{i_n})\colon i_0 < \cdots < i_n < \omega \textit{ for some } n<\omega\}$, *and*
(b) $\Sigma'(\bar{t}) = \bigcup\{\Sigma(\bar{s})\colon \bar{s} \in \mathcal{S}(\bar{t})\}$ *and* $\Sigma^{\mathtt{tt}}(\bar{t}) = \bigcup\{\Sigma(t_0,\ldots,t_n)\colon n<\omega\}$,
(c) $\mathrm{pos}(\bar{t}) = \bigcup\{\mathbf{val}[s]\colon s \in \Sigma'(\bar{t})\}$ *and* $\mathrm{pos}^{\mathtt{tt}}(\bar{t}) = \bigcup\{\mathbf{val}[s]\colon s \in \Sigma^{\mathtt{tt}}(\bar{t})\}$,
(d) $\bar{t} \upharpoonleft n = \langle t_{n+k}\colon k<\omega\rangle$.

Remark 2.4. Loose FFCC and *tight* FFCC are the two cases of FFCC pairs treated in this article. The corresponding partition theorems will be slightly different in the two cases, though there is a parallel. In the loose case we will deal with $\Sigma'(\bar{t})$, $\mathrm{pos}(\bar{t})$ and ultrafilters on the latter set. In the tight case we will use $\Sigma^{\mathtt{tt}}(\bar{t})$, $\mathrm{pos}^{\mathtt{tt}}(\bar{t})$ and *sequences* of ultrafilters on $\mathrm{pos}^{\mathtt{tt}}(\bar{t} \upharpoonleft n)$ (for $n<\omega$).

We will require two additional properties from (K,Σ): *Weak bigness* and *weak additivity* (see Definitions 2.5, 2.6). Because of the differences in the treatment of the two cases, there are slight differences in the formulation of these properties, so we have two variants for each: $\mathtt{l}$-*variant* and $\mathtt{t}$-*variant* (where "$\mathtt{l}$" stands for "loose" and "$\mathtt{t}$" stands for "tight", of course).

Plainly, $\mathrm{PC}^{\mathtt{tt}}_\infty(K,\Sigma)\subseteq \mathrm{PC}_\infty(K,\Sigma)$, $\Sigma^{\mathtt{tt}}(\bar t)\subseteq\Sigma'(\bar t)$ and $\mathrm{pos}^{\mathtt{tt}}(\bar t)\subseteq \mathrm{pos}(\bar t)$. Also, if $\bar t\in \mathrm{PC}^{\mathtt{tt}}_\infty(K,\Sigma)$, then $\bar t\restriction n\in \mathrm{PC}^{\mathtt{tt}}_\infty(K,\Sigma)$ for all $n<\omega$.

Definition 2.5. *Let (K,Σ) be an FFCC pair for $\mathbf{H}$ and $\bar t=\langle t_i\colon i<\omega\rangle\in \mathrm{PC}_\infty(K,\Sigma)$.*

(1) *We say that the pair (K,Σ)* has weak $\mathtt{l}$-additivity for the candidate $\bar t$ *if for some increasing $\mathbf{f}\colon\omega\longrightarrow\omega$, for every $m<\omega$ we have*:
if $s_0,s_1\in\Sigma'(\bar t)$, $\mathbf{nor}[s_0]\ge\mathbf{f}(m)$, $m^{s_0}_{\mathrm{dn}}\ge\mathbf{f}(m)$, $\mathbf{nor}[s_1]\ge\mathbf{f}\big(m^{s_0}_{\mathrm{up}}\big)$ and $m^{s_1}_{\mathrm{dn}}\ge\mathbf{f}\big(m^{s_0}_{\mathrm{up}}\big)$, then we can find $s\in\Sigma'(\bar t)$ such that

$$m^s_{\mathrm{dn}}\ge m,\ \mathbf{nor}[s]\ge m,\ \textit{and}\ \mathbf{val}[s]\subseteq\{f\cup g\colon f\in\mathbf{val}[s_0],\ g\in\mathbf{val}[s_1]\}.$$

(2) *The pair (K,Σ)* has weak $\mathtt{t}$-additivity for the candidate $\bar t$ *if for some increasing $\mathbf{f}\colon\omega\longrightarrow\omega$, for every $n,m<\omega$ we have*:
if $s_0\in\Sigma(t_n,\dots,t_k)$, $k\ge n$, $\mathbf{nor}[s_0]\ge\mathbf{f}(n+m)$, $s_1\in\Sigma(t_{k+1},\dots,t_\ell)$, $\mathbf{nor}[s_1]\ge\mathbf{f}(k+m)$ and $\ell>k$, then we can find $s\in\Sigma(t_n,\dots,t_\ell)$ such that $\mathbf{nor}[s]\ge m$ and $\mathbf{val}[s]\subseteq\{f\cup g\colon f\in\mathbf{val}[s_0],\ g\in\mathbf{val}[s_1]\}$.

(3) *The pair (K,Σ)* has $\mathtt{l}$-additivity *if for all $s_0,s_1\in K$ with $\mathbf{nor}[s_0],\mathbf{nor}[s_1]>1$ and $m^{s_0}_{\mathrm{up}}\le m^{s_1}_{\mathrm{dn}}$ there is $s\in\Sigma(s_0,s_1)$ such that*

$$\mathbf{nor}[s]\ge\min\{\mathbf{nor}[s_0],\mathbf{nor}[s_1]\}-1\ \ \textit{and}\ \ \mathbf{val}[s]\subseteq\{f\cup g\colon f\in\mathbf{val}[s_0],\ g\in\mathbf{val}[s_1]\}.$$

The pair (K,Σ) has $\mathtt{t}$-additivity *if for all $s_0,s_1\in K$ with $\mathbf{nor}[s_0],\mathbf{nor}[s_1]>1$ and $m^{s_0}_{\mathrm{up}}=m^{s_1}_{\mathrm{dn}}$ there is $s\in\Sigma(s_0,s_1)$ such that*

$$\mathbf{nor}[s]\ge\min\{\mathbf{nor}[s_0],\mathbf{nor}[s_1]\}-1.$$

We say that (K,Σ) has $\mathtt{t}$-multiadditivity *if for all $s_0,\dots,s_n\in K$ with $m^{s_\ell}_{\mathrm{up}}=m^{s_{\ell+1}}_{\mathrm{dn}}$ (for $\ell<n$) there is $s\in\Sigma(s_0,\dots,s_n)$ such that* $\mathbf{nor}[s]\ge\max\{\mathbf{nor}[s_\ell]\colon\ell\le n\}-1$.

Definition 2.6. *Let (K,Σ) be an FFCC pair for $\mathbf{H}$ and $\bar t=\langle t_i\colon i<\omega\rangle\in \mathrm{PC}_\infty(K,\Sigma)$.*

(1) *We say that the pair (K,Σ)* has weak $\mathtt{l}$-bigness for the candidate $\bar t$ *whenever the following property is satisfied*:

$(\circledast)^{\bar t}_{\mathtt{l}}$ *if $n_1,n_2,n_3<\omega$ and $\mathrm{pos}(\bar t)=\bigcup\{\mathcal{F}_\ell\colon\ell<n_1\}$, then for some $s\in\Sigma'(\bar t)$ and $\ell<n_1$ we have*

$$\mathbf{nor}[s]\ge n_2,\ m^s_{\mathrm{dn}}\ge n_3,\ \ \textit{and}\ \mathbf{val}[s]\subseteq\mathcal{F}_\ell.$$

(2) *We say that the pair (K,Σ)* has weak $\mathtt{t}$-bigness for the candidate $\bar t$ *whenever the following property is satisfied*:

$(\circledast)^{\bar t}_{\mathtt{t}}$ *if $n,\ n_1,\ n_2<\omega$ and $\mathrm{pos}^{\mathtt{tt}}(\bar t\restriction n)=\bigcup\{\mathcal{F}_\ell\colon\ell<n_1\}$, then for some $s\in\Sigma^{\mathtt{tt}}(\bar t\restriction n)$ and $\ell<n_1$ we have*

$$\mathbf{nor}[s]\ge n_2\ \ \textit{and}\ \ \mathbf{val}[s]\subseteq\mathcal{F}_\ell.$$

(3) *We say that the pair* (K,Σ) has bigness *if for every creature* $t \in K$ *with* $\mathbf{nor}[t] > 1$ *and a partition* $\mathbf{val}[t] = F_1 \cup F_2$, *there are* $\ell \in \{1,2\}$ *and* $s \in \Sigma(t)$ *such that* $\mathbf{nor}[s] \geq \mathbf{nor}[t] - 1$ *and* $\mathbf{val}[s] \subseteq F_\ell$.

Definition 2.7. *Let* (K,Σ) *be an FFCC pair for* $\mathbf{H}$.

(1) (K,Σ) *is* simple except omitting *if for every* $(t_0,\ldots,t_n) \in \mathrm{dom}(\Sigma)$ *and* $t \in \Sigma(t_0,\ldots,t_n)$ *for some* $\ell \leq n$ *we have* $\mathbf{val}[t] \subseteq \mathbf{val}[t_\ell]$.
(2) (K,Σ) *is* gluing on a candidate $\bar{t} = \langle t_i \colon i < \omega\rangle \in \mathrm{PC}_\infty(K,\Sigma)$ *if for every* $n, m < \omega$ *there are* $k \geq n$ *and* $s \in \Sigma(t_n,\ldots,t_k)$ *such that* $\mathbf{nor}[s] \geq m$.

The following two observations summarize the basic dependencies between the notions introduced in Definitions 2.5, 2.6 — separately for the two contexts (see Remark 2.4).

Observation 2.8. Assume (K,Σ) is a loose FFCC pair, $\bar{t} \in \mathrm{PC}_\infty(K,\Sigma)$.

(1) If (K,Σ) has bigness ($\mathtt{l}$-additivity, respectively), then it has weak $\mathtt{l}$-bigness (weak $\mathtt{l}$-additivity, respectively) for the candidate $\bar{t}$.
(2) If (K,Σ) has the weak $\mathtt{l}$-bigness for $\bar{t}$, $k < \omega$ and $\mathrm{pos}(\bar{t}) = \bigcup_{\ell<k} \mathcal{F}_\ell$, then for some $\bar{s} \in \mathrm{PC}_\infty(K,\Sigma)$ and $\ell < k$ we have
$$\bar{t} \leq \bar{s} \text{ and } (\forall n < \omega)(\mathbf{val}[s_n] \subseteq \mathcal{F}_\ell).$$
(3) Assume that (K,Σ) has the weak $\mathtt{l}$-bigness property for $\bar{t} \in \mathrm{PC}_\infty(K,\Sigma)$ and it is simple except omitting. Let $k < \omega$ and $\mathrm{pos}(\bar{t}) = \bigcup_{\ell<k} \mathcal{F}_\ell$. Then for some $\bar{s} \geq \bar{t}$ and $\ell < k$ we have $\mathrm{pos}(\bar{s}) \subseteq \mathcal{F}_\ell$.

Observation 2.9. Assume (K,Σ) is a tight FFCC pair, $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K,\Sigma)$.

(1) If (K,Σ) has bigness and is gluing on $\bar{t}$, then it has the weak $\mathtt{t}$-bigness for the candidate $\bar{t}$.
(2) If (K,Σ) has $\mathtt{t}$-additivity, then it has the weak $\mathtt{t}$-additivity for $\bar{t}$.
(3) If (K,Σ) has the $\mathtt{t}$-multiadditivity, then it has the $\mathtt{t}$-additivity and it is gluing on $\bar{t}$.

In the following two sections we will present partition theorems for the loose and then for the tight case. First, let us offer some easy examples to which the theory developed later can be applied. Example 2.13 will be used to give a new proof of the Carlson-Simpson Theorem in 4.9. Examples 2.10 and 2.12 are much easier, but they also can be used to deduce partition theorems from Conclusions 3.10 and 4.8 (we leave the details for the reader).

Example 2.10. Let $\mathbf{H}_1(n) = n+1$ for $n < \omega$ and let K_1 consist of all FP creatures t for $\mathbf{H}_1$ such that

- $\mathbf{dis}[t] = (u,i,A) = (u^t,i^t,A^t)$ where $u \subseteq \left[m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}\right)$, $i \in u$, $\emptyset \neq A \subseteq \mathbf{H}_1(i)$,
- $\mathbf{nor}[t] = \log_2(|A|)$,

- $\mathbf{val}[t] \subseteq \prod_{j\in u} \mathbf{H}_1(j)$ is such that $\{f(i)\colon f \in \mathbf{val}[t]\} = A$.

For $t_0,\dots,t_n \in K_1$ with $m^{t_\ell}_{\mathrm{up}} \le m^{t_{\ell+1}}_{\mathrm{dn}}$, let $\Sigma_1(t_0,\dots,t_n)$ consist of all creatures $t \in K_1$ such that

$$m^t_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}},\ m^t_{\mathrm{up}} = m^{t_n}_{\mathrm{up}},\ u^t = \bigcup_{\ell\le n} u^{t_\ell},\ i^t = i^{t_{\ell^*}},\ A^t \subseteq A^{t_{\ell^*}}, \qquad \text{for some } \ell^* \le n,$$

and $\mathbf{val}[t] \subseteq \big\{f_0 \cup \dots \cup f_n\colon (f_0,\dots,f_n) \in \mathbf{val}[t_0] \times \dots \times \mathbf{val}[t_n]\big\}$.
Also, let Σ^*_1 be Σ_1 restricted to the set of those tuples $(t_0,\dots,t_n)$ for which $m^{t_\ell}_{\mathrm{up}} = m^{t_{\ell+1}}_{\mathrm{dn}}$ (for $\ell < n$). Then

- (K_1,Σ_1) is a loose FFCC pair for $\mathbf{H}_1$ with bigness and $\mathtt{l}$-additivity,
- (K_1,Σ^*_1) is a tight FFCC pair for $\mathbf{H}_1$ with bigness and $\mathtt{t}$-multiadditivity, and it is gluing on every $\bar t \in \mathrm{PC}^{\mathtt{tt}}_\infty(K_1,\Sigma^*_1)$.

Example 2.11. Let $\mathbf{H}_2(n) = 2$ for $n < \omega$ and let K_2 consist of all FP creatures t for $\mathbf{H}_2$ such that

- $\emptyset \ne \mathbf{dis}[t] \subseteq \big[m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}\big)$,
- $\emptyset \ne \mathbf{val}[t] \subseteq {}^{\mathbf{dis}[t]}2$,
- $\mathbf{nor}[t] = \log_2(|\mathbf{val}[t]|)$.

For $t_0,\dots,t_n \in K_2$ with $m^{t_\ell}_{\mathrm{up}} \le m^{t_{\ell+1}}_{\mathrm{dn}}$, let $\Sigma_2(t_0,\dots,t_n)$ consist of all creatures $t \in K_2$ such that

$$m^t_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}},\ m^t_{\mathrm{up}} = m^{t_n}_{\mathrm{up}},\ \mathbf{dis}[t] = \mathbf{dis}[t_{\ell^*}],\ \text{ and } \mathbf{val}[t] \subseteq \mathbf{val}[t_{\ell^*}],\ \text{ for some } \ell^* \le n.$$

Then (K_2,Σ_2) is a loose FFCC pair for $\mathbf{H}_2$ which is simple except omitting and has bigness.

Example 2.12. Let $\mathbf{H}$ be as in Context 2.1 and let K_3 consist of all FP creatures t for $\mathbf{H}$ such that

- $\emptyset \ne \mathbf{dis}[t] \subseteq \big[m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}\big)$,
- $\mathbf{val}[t] \subseteq \big\{f \in \mathcal{F}_{\mathbf{H}}\colon \mathbf{dis}[t] \subseteq \mathrm{dom}(f) \subseteq \big[m^t_{\mathrm{dn}}, m^t_{\mathrm{up}}\big)\big\}$ satisfies

$$\left(\forall g \in \prod_{i\in\mathbf{dis}[t]} \mathbf{H}(i)\right)(\exists f \in \mathbf{val}[t])(g \subseteq f),$$

- $\mathbf{nor}[t] = \log_3(|\mathbf{dis}[t]|)$.

For $t_0,\dots,t_n \in K_3$ with $m^{t_\ell}_{\mathrm{up}} \le m^{t_{\ell+1}}_{\mathrm{dn}}$, let $\Sigma_3(t_0,\dots,t_n)$ consist of all creatures $t \in K_3$ such that

- $m^t_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$, $m^t_{\mathrm{up}} = m^{t_n}_{\mathrm{up}}$, $\mathbf{dis}[t] \subseteq \bigcup_{\ell\le n} \mathbf{dis}[t_\ell]$, and
- if $f \in \mathbf{val}[t]$, then $\mathrm{dom}(f) \subseteq \bigcup\big\{\big[m^{t_\ell}_{\mathrm{dn}}, m^{t_\ell}_{\mathrm{up}}\big) : \ell \le n\big\}$ and $f\restriction\big[m^{t_\ell}_{\mathrm{dn}}, m^{t_\ell}_{\mathrm{up}}\big) \in \mathbf{val}[t_\ell] \cup \{\emptyset\}$ for all $\ell \le n$.

Also, for $t_0,\ldots,t_n \in K_3$ with $m^{t_\ell}_{\rm up} = m^{t_{\ell+1}}_{\rm dn}$ let $\Sigma^*_3(t_0,\ldots,t_n)$ consist of all creatures $t \in K_3$ such that

- $m^t_{\rm dn} = m^{t_0}_{\rm dn}$, $m^t_{\rm up} = m^{t_n}_{\rm up}$, $\mathbf{dis}[t] \subseteq \bigcup\limits_{\ell\leq n} \mathbf{dis}[t_\ell]$, and
- if $f \in \mathbf{val}[t]$, then $\mathrm{dom}(f) \subseteq \bigcup\big\{\big[m^{t_\ell}_{\rm dn}, m^{t_\ell}_{\rm up}\big) : \ell \leq n\big\}$ and $f{\restriction}\big[m^{t_\ell}_{\rm dn}, m^{t_\ell}_{\rm up}\big) \in \mathbf{val}[t_\ell]$ for all $\ell \leq n$.

Then

- (K_3,Σ_3) is a loose FFCC pair for $\mathbf{H}$ with bigness and $\mathtt{l}$-additivity,
- (K_3,Σ^*_3) is a tight FFCC pair for $\mathbf{H}$ with bigness and $\mathtt{t}$-multiadditivity and it is gluing on every $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K_3,\Sigma^*_3)$.

Example 2.13. Let $N > 0$ and $\mathbf{H}_N(n) = N$. Let K_N consist of all FP creatures t for $\mathbf{H}_N$ such that

- $\mathbf{dis}[t] = (X_t,\varphi_t)$, where $X_t \subsetneq \big[m^t_{\rm dn}, m^t_{\rm up}\big)$, and $\varphi_t : X_t \longrightarrow N$,
- $\mathbf{nor}[t] = m^t_{\rm up}$,
- $\mathbf{val}[t] = \big\{f \in {}^{[m^t_{\rm dn},m^t_{\rm up})}N \colon \varphi_t \subseteq f$ and f is constant on $\big[m^t_{\rm dn}, m^t_{\rm up}\big) \setminus X_t\big\}$.

For $t_0,\ldots,t_n \in K_N$ with $m^{t_\ell}_{\rm up} = m^{t_{\ell+1}}_{\rm dn}$ (for $\ell < n$) we let $\Sigma_N(t_0,\ldots,t_n)$ consist of all creatures $t \in K_N$ such that

- $m^t_{\rm dn} = m^{t_0}_{\rm dn}$, $m^t_{\rm up} = m^{t_0}_{\rm up}$, $X_{t_0} \cup\cdots\cup X_{t_n} \subseteq X_t$,
- for each $\ell \leq n$,
  either $X_t \cap \big[m^{t_\ell}_{\rm dn}, m^{t_\ell}_{\rm up}\big) = X_{t_\ell}$ and $\varphi_t{\restriction}\big[m^{t_\ell}_{\rm dn}, m^{t_\ell}_{\rm up}\big) = \varphi_{t_\ell}$,
  or $\big[m^{t_\ell}_{\rm dn}, m^{t_\ell}_{\rm up}\big) \subseteq X_t$ and $\varphi_t{\restriction}\big[m^{t_\ell}_{\rm dn}, m^{t_\ell}_{\rm up}\big) \in \mathbf{val}[t_\ell]$.

Then

(i) (K_N,Σ_N) is a tight FFCC pair for $\mathbf{H}_N$,
(ii) it has the $\mathtt{t}$-multiadditivity and
(iii) it has the weak $\mathtt{t}$-bigness and is gluing for every candidate $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K,\Sigma)$.

Proof. (i) All demands in Definition 2.2(2) and (4) are easy to verify. For instance, to check Definition 2.2(4)($\mathrm{h}^{\rm tight}$) note that:
If $s_0, s_1 \in K_N$, $m^{s_0}_{\rm up} = m^{s_1}_{\rm dn}$, $f_\ell \in \mathbf{val}[s_\ell]$ (for $\ell = 0,1$) and $s \in K_N$ is such that

$$m^s_{\rm dn} = m^{s_0}_{\rm dn},\ m^s_{\rm up} = m^{s_1}_{\rm up},\ X_s = X_{s_0} \cup \big[m^{s_1}_{\rm dn}, m^{s_1}_{\rm up}\big),\ \varphi_s{\restriction}X_{s_0} = \varphi_{s_0},\ \varphi_s{\restriction}\big[m^{s_1}_{\rm dn}, m^{s_1}_{\rm up}\big) = f_1,$$

then $s \in \Sigma_N(s_0,s_1)$ and $f_0 \cup f_1 \in \mathbf{val}[s]$.

(ii) The s constructed as in (i) above for s_0, s_1 will witness the $\mathtt{t}$-additivity as well. In an analogous way we also show the multiadditivity.

(iii) Let $\bar{t} = \langle t_i \colon i < \omega\rangle \in \mathrm{PC}^{\mathtt{tt}}_\infty(K_N,\Sigma_N)$. Suppose that $n, n_1, n_2 < \omega$ and that $\mathrm{pos}^{\mathtt{tt}}(\bar{t}\restriction n) = \bigcup\{\mathcal{F}_\ell \colon \ell < n_1\}$. By the Hales-Jewett theorem (see [5]) there is $k > n_2$ such that for any partition of ${}^kN$ into n_1 parts there is a combinatorial line included in one of the parts. Then we easily find $s \in \Sigma_N(t_0,\ldots,t_{k-1})$ such that $\mathbf{val}[s] \subseteq \mathcal{F}_\ell$ for some $\ell < n_1$. Necessarily, $\mathbf{nor}[s] \geq k-1 \geq n_2$. This proves the weak $\mathtt{t}$-bigness for $\bar{t}$. Similarly to (ii) we may argue that (K_N,Σ_N) is gluing on $\bar{t}$. ∎

3. Ultrafilters on Loose Possibilities

Here we introduce ultrafilters on the (countable) set $\mathcal{F}_{\mathbf{H}}$ (see Context 2.1) which contain sets large from the point of view of pure candidates for a loose FFCC pair. Then we use them to derive a partition theorem for this case.

Definition 3.1. *Let (K,Σ) be a loose FFCC pair for* **H**.

(1) *For a pure candidate $\bar{t}\in \mathrm{PC}_\infty(K,\Sigma)$, we define (for $\bar{t}\restriction n$, Σ' and* pos *remember Definition 2.3*(3)):

- $\mathcal{A}^0_{\bar{t}}=\{\mathrm{pos}(\bar{t}\restriction n)\colon n<\omega\}$,
- $\mathcal{A}^1_{\bar{t}}$ *is the collection of all sets $A\subseteq\mathcal{F}_{\mathbf{H}}$ such that for some $N<\omega$ we have*

$$\big(\forall s\in\Sigma'(\bar{t})\big)\big(\mathbf{nor}[s]\geq N\ \&\ m^s_{\mathrm{dn}}\geq N\quad\Rightarrow\quad \mathbf{val}[s]\cap A\neq\emptyset\big),$$

- $\mathcal{A}^2_{\bar{t}}$ *is the collection of all sets $A\subseteq\mathcal{F}_{\mathbf{H}}$ such that for some $N<\omega$ we have*

$$(\forall\bar{t}_1\geq\bar{t})\,(\exists\bar{t}_2\geq\bar{t}_1)(\forall s\in\Sigma'(\bar{t}_2))(\mathbf{nor}[s]\geq N\quad\Rightarrow\quad\mathbf{val}[s]\cap A\neq\emptyset\big).$$

(2) *For $\ell<3$ we let $\mathrm{uf}^\ell_{\bar{t}}(K,\Sigma)$ be the family of all ultrafilters D on $\mathcal{F}_{\mathbf{H}}$ such that $\mathcal{A}^\ell_{\bar{t}}\subseteq D$. We also set (for $\ell<3$)*

$$\mathrm{uf}_\ell(K,\Sigma)\stackrel{\mathrm{def}}{=}\bigcup\big\{\mathrm{uf}^\ell_{\bar{t}}(K,\Sigma)\colon \bar{t}\in\mathrm{PC}_\infty(K,\Sigma)\big\}.$$

Proposition 3.2. *Let (K,Σ) be a loose FFCC pair for* **H**, *$\bar{t}\in\mathrm{PC}_\infty(K,\Sigma)$.*

(1) $\mathcal{A}^0_{\bar{t}}\subseteq\mathcal{A}^1_{\bar{t}}\subseteq\mathcal{A}^2_{\bar{t}}$ *and hence also* $\mathrm{uf}^2_{\bar{t}}(K,\Sigma)\subseteq\mathrm{uf}^1_{\bar{t}}(K,\Sigma)\subseteq\mathrm{uf}^0_{\bar{t}}(K,\Sigma)$.
(2) $\mathrm{uf}^0_{\bar{t}}(K,\Sigma)\neq\emptyset$.
(3) *If (K,Σ) has the weak* `1`*-bigness for each $\bar{t}'\geq\bar{t}$, then $\mathrm{uf}^2_{\bar{t}}(K,\Sigma)$ is not empty.*
(4) *If (K,Σ) has the weak* `1`*-bigness for $\bar{t}$, then $\mathrm{uf}^1_{\bar{t}}(K,\Sigma)\neq\emptyset$.*
(5) *Assume* the Continuum Hypothesis. *Suppose that (K,Σ) is simple except omitting (see Definition 2.7*(1)) *and has the weak* `1`*-bigness on every candidate $\bar{t}\in\mathrm{PC}_\infty(K,\Sigma)$. Then there is $D\in\mathrm{uf}^2_{\bar{t}}(K,\Sigma)$ such that*

$$\big(\forall A\in D\big)\big(\exists\bar{t}\in\mathrm{PC}_\infty(K,\Sigma)\big)\big(\mathrm{pos}(\bar{t})\in D\ \&\ \mathrm{pos}(\bar{t})\subseteq A\big).$$

Proof. (2) Note that $\mathcal{A}^0_{\bar{t}}$ has the finite intersection property (fip).

(3) It is enough to show that, assuming (K,Σ) has the weak `1`-bigness for all $\bar{t}'\geq\bar{t}$, $\mathcal{A}^2_{\bar{t}}$ has fip. So suppose that for $\ell<k$ we are given a set $A_\ell\in\mathcal{A}^2_{\bar{t}}$ and let $N_\ell<\omega$ be such that

$$(*)_\ell\quad(\forall\bar{t}_1\geq\bar{t})(\exists\bar{t}_2\geq\bar{t}_1)(\forall s\in\Sigma'(\bar{t}_2))(\mathbf{nor}[s]\geq N_\ell\ \Rightarrow\ \mathbf{val}[s]\cap A_\ell\neq\emptyset\big).$$

Let $N=\max\{N_\ell\colon \ell<k\}$. Then we may choose $\bar{t}'\geq\bar{t}$ such that

$$(*)\quad(\forall s\in\Sigma'(\bar{t}'))(\mathbf{nor}[s]\geq N\ \Rightarrow\ (\forall\ell<k)(\mathbf{val}[s]\cap A_\ell\neq\emptyset)\big).$$

(Why? Just use repeatedly $(*)_\ell$ for $\ell = 0, 1, \ldots, k-1$; remember $\bar{t}' \leq \bar{t}''$ implies $\Sigma'(\bar{t}'') \subseteq \Sigma'(\bar{t}')$.)

For $\eta \in {}^k 2$, set

$$\mathcal{F}_\eta = \big\{ f \in \mathrm{pos}(\bar{t}') \colon (\forall \ell < k)(\eta(\ell) = 1 \ \Leftrightarrow\ f \in A_\ell) \big\}.$$

Then $\mathrm{pos}(\bar{t}') = \bigcup \{\mathcal{F}_\eta : \eta \in {}^k 2\}$ and (K, Σ) has the weak $\mathtt{1}$-bigness for $\bar{t}'$, so we may use Observation 2.8(2) to pick $\eta_0 \in {}^k 2$ and $\bar{s} \geq \bar{t}'$ such that $\mathbf{val}[s_n] \subseteq \mathcal{F}_{\eta_0}$ for all $n < \omega$. Consider $n < \omega$ such that $\mathbf{nor}[s_n] > N$. It follows from $(*)$ that $\mathbf{val}[s_n] \cap A_\ell \neq \emptyset$ for all $\ell < k$. Hence, by the choice of $\bar{s}$, $\eta_0(\ell) = 1$ for all $\ell < k$ and therefore $\emptyset \neq \mathbf{val}[s_n] \subseteq \bigcap\limits_{\ell < k} A_\ell$.

(4) Similarly to (3) above one shows that $\mathcal{A}^1_{\bar{t}}$ has fip.
(5) Assuming CH and using Observation 2.8(3) we may construct a sequence $\langle \bar{t}_\alpha \colon \alpha < \omega_1 \rangle \subseteq \mathrm{PC}_\infty(K, \Sigma)$ such that

- if $\alpha < \beta < \omega_1$ then $(\exists n < \omega)\big(\bar{t}_\alpha \leq (\bar{t}_\beta \restriction n)\big)$,
- if $A \subseteq \mathcal{F}_{\mathbf{H}}$ then for some $\alpha < \omega_1$ we have that either $\mathrm{pos}(\bar{t}_\alpha) \subseteq A$ or $\mathrm{pos}(\bar{t}_\alpha) \cap A = \emptyset$.

(Compare to the proof of [10, 5.3.4].) Then the family

$$\{\mathrm{pos}(\bar{t}_\alpha \restriction n) \colon \alpha < \omega_1 \ \&\ n < \omega\}$$

generates the desired ultrafilter. ∎

Observation 3.3. The sets $\mathrm{uf}^\ell_{\bar{t}}(K, \Sigma)$ (for $\ell < 3$) are closed subsets of the (Hausdorff compact topological space) $\beta_*(\mathcal{F}_{\mathbf{H}})$ of non-principal ultrafilters on $\mathcal{F}_{\mathbf{H}}$. Hence each $\mathrm{uf}^\ell_{\bar{t}}(K, \Sigma)$ itself is a compact Hausdorff space.

Definition 3.4. (1) *For $f \in \mathcal{F}_{\mathbf{H}}$ and $A \subseteq \mathcal{F}_{\mathbf{H}}$ we define*

$$f \oplus A \stackrel{\mathrm{def}}{=} \{g \in \mathcal{F}_{\mathbf{H}} \colon \max(\mathrm{dom}(f)) < \min(\mathrm{dom}(g)) \ \textit{and}\ f \cup g \in A\}.$$

(2) *For $D_1, D_2 \in \mathrm{uf}_0(K, \Sigma)$, we let*

$$D_1 \oplus D_2 \stackrel{\mathrm{def}}{=} \big\{A \subseteq \mathcal{F}_{\mathbf{H}} \colon \{f \in \mathcal{F}_{\mathbf{H}} \colon (f \oplus A) \in D_1\} \in D_2\big\}.$$

Proposition 3.5. (1) *If $A_1, A_2 \subseteq \mathcal{F}_{\mathbf{H}}$ and $f \in \mathcal{F}_{\mathbf{H}}$, then*

$$f \oplus (A_1 \cap A_2) = (f \oplus A_1) \cap (f \oplus A_2) \quad \textit{and}$$
$$\{g \in \mathcal{F}_{\mathbf{H}} \colon \max(\mathrm{dom}(f)) < \min(\mathrm{dom}(g))\} \setminus (f \oplus A_1) = f \oplus (\mathcal{F}_{\mathbf{H}} \setminus A_1).$$

(2) *If $D_1, D_2, D_3 \in \mathrm{uf}_0(K, \Sigma)$, then $D_1 \oplus D_2$ is a non-principal ultrafilter on $\mathcal{F}_{\mathbf{H}}$ and $D_1 \oplus (D_2 \oplus D_3) = (D_1 \oplus D_2) \oplus D_3$.*
(3) *The mapping $\oplus \colon \mathrm{uf}_0(K, \Sigma) \times \mathrm{uf}_0(K, \Sigma) \longrightarrow \beta_*(\mathcal{F}_{\mathbf{H}})$ is right continuous (i.e., for each $D_1 \in \mathrm{uf}_0(K, \Sigma)$ the function $\mathrm{uf}_0(K, \Sigma) \ni D_2 \mapsto D_1 \oplus D_2 \in \beta_*(\mathcal{F}_{\mathbf{H}})$ is continuous).*

Proof. Straightforward, compare with Proposition 4.3. ■

Proposition 3.6. *Assume that a loose FFCC pair (K,Σ) has the weak* $\mathtt{1}$*-additivity (see Definition 2.5(1)) for a candidate $\bar{t}\in \mathrm{PC}_\infty(K,\Sigma)$. If $D_1,D_2\in \mathrm{uf}^1_{\bar{t}}(K,\Sigma)$, then $D_1\oplus D_2\in \mathrm{uf}^1_{\bar{t}}(K,\Sigma)$.*

Proof. Let $\mathbf{f}\colon \omega\to\omega$ witness the weak $\mathtt{1}$-additivity of (K,Σ) for $\bar{t}$, and let $D=D_1\oplus D_2$, $D_1,D_2\in \mathrm{uf}^1_{\bar{t}}(K,\Sigma)$. We already know that D is an ultrafilter on $\mathcal{F}_{\mathbf{H}}$ (by Proposition 3.5(2)), so we only need to show that it includes $\mathcal{A}^1_{\bar{t}}$.

Suppose that $A\in\mathcal{A}^1_{\bar{t}}$ and let $N<\omega$ be such that

$$(*)_1\quad \big(\forall s\in\Sigma'(\bar{t})\big)\big(\mathbf{nor}[s]\geq N\ \&\ m^s_{\mathrm{dn}}\geq N\quad\Rightarrow\quad \mathbf{val}[s]\cap A\neq\emptyset\big).$$

Claim 3.6.1. For every $s\in\Sigma'(\bar{t})$, if $\mathbf{nor}[s]\geq\mathbf{f}(N)$ and $m^s_{\mathrm{dn}}\geq\mathbf{f}(N)$, then $\mathbf{val}[s]\cap\{f\in\mathcal{F}_{\mathbf{H}}\colon f\oplus A\in D_1\}\neq\emptyset$.

Proof of Claim 3.6.1. Suppose $s_0\in\Sigma'(\bar{t})$, $\mathbf{nor}[s_0]\geq\mathbf{f}(N)$, $m^{s_0}_{\mathrm{dn}}\geq\mathbf{f}(N)$. Set

$$B=\bigcup\{f\oplus A\colon f\in\mathbf{val}[s_0]\}.$$

We are going to argue that

$$(*)_2\quad B\in\mathcal{A}^1_{\bar{t}}.$$

So let $M=\mathbf{f}\left(m^{s_0}_{\mathrm{up}}\right)+m^{s_0}_{\mathrm{up}}+1$ and suppose $s_1\in\Sigma(\bar{t})$ is such that $\mathbf{nor}[s_1]\geq M$ and $m^{s_1}_{\mathrm{dn}}\geq M$. Apply the weak additivity and the choice of M to find $s\in\Sigma'(\bar{t})$ such that

$$m^s_{\mathrm{dn}}\geq N,\quad \mathbf{nor}[s]\geq N\quad\text{and}\quad \mathbf{val}[s]\subseteq\{f\cup g\colon f\in\mathbf{val}[s_0]\ \&\ g\in\mathbf{val}[s_1]\}.$$

Then, by $(*)_1$, $\mathbf{val}[s]\cap A\neq\emptyset$ so for some $f\in\mathbf{val}[s_0]$ and $g\in\mathbf{val}[s_1]$ we have $f\cup g\in A$ $\big($and $\max(\mathrm{dom}(f))<m^{s_0}_{\mathrm{up}}<M\leq m^{s_1}_{\mathrm{dn}}\leq\min(\mathrm{dom}(g))\big)$. Thus $g\in(f\oplus A)\cap\mathbf{val}[s_1]\subseteq B\cap\mathbf{val}[s_1]$ and $(*)_2$ follows.

Since $D_1\in\mathrm{uf}^1_{\bar{t}}(K,\Sigma)$ we conclude from $(*)_2$ that $B\in D_1$ and hence (as $\mathbf{val}[s_0]$ is finite) $f\oplus A\in D_1$ for some $f\in\mathbf{val}[s_0]$, as desired. ■

It follows from Claim 3.6.1 that $\{f\in\mathcal{F}_{\mathbf{H}}\colon f\oplus A\in D_1\}\in\mathcal{A}^1_{\bar{t}}$ and hence $\big($as $D_2\in\mathrm{uf}^1_{\bar{t}}(K,\Sigma)\big)$ $\{f\in\mathcal{F}_{\mathbf{H}}\colon f\oplus A\in D_1\}\in D_2$. Consequently, $A\in D_1\oplus D_2$. ■

The following lemma has been known at least since 1950s; see, e.g., [3, Lemma 10.1, p. 449 and p. 452].

Lemma 3.7. *If X is a non-empty compact Hausdorff space, $\odot$ an associative binary operation which is continuous from the right (i.e., for each $p\in X$ the function $q\mapsto p\odot q$ is continuous), then there is a $\odot$-idempotent point $p\in X$ (i.e., $p\odot p=p$).*

Corollary 3.8. *Assume that a loose FFCC pair (K,Σ) has weak* $\mathtt{1}$*-additivity and the weak* $\mathtt{1}$*-bigness for a candidate $\bar{t}\in\mathrm{PC}_\infty(K,\Sigma)$. Then,*

(1) $\mathrm{uf}^1_{\bar{t}}(K,\Sigma)$ *is a non-empty compact Hausdorff space and $\oplus$ is an associative right continuous operation on it.*
(2) *There is $D\in\mathrm{uf}^1_{\bar{t}}(K,\Sigma)$ such that $D=D\oplus D$.*

Proof. (1) By Proposition 3.2(3), Observation 3.3, Proposition 3.5(2) and (3), and Proposition 3.6.

(2) It follows from (1) above that all the assumptions of Lemma 3.7 are satisfied for $\oplus$ and $\mathrm{uf}^1_{\bar t}(K,\Sigma)$, hence its conclusion holds. ∎

Theorem 3.9. *Assume that (K,Σ) is a loose FFCC pair, $\bar t\in \mathrm{PC}_\infty(K,\Sigma)$. Let an ultrafilter $D\in \mathrm{uf}^1_{\bar t}(K,\Sigma)$ be such that $D\oplus D=D$. Then,*

$$\big(\forall A\in D\big)\big(\exists \bar s\geq \bar t\big)\big(\mathrm{pos}(\bar s)\subseteq A\big).$$

Proof. The main ingredient of our argument is given by the following claim.

Claim 3.9.1. Let (K,Σ), $\bar t$ and D be as in the assumptions of Theorem 3.9. Assume $A\in D$ and $n<\omega$. Then there is $s\in\Sigma'(\bar t)$ such that

- $(\bullet)_1$ $\mathbf{val}[s]\subseteq A$, $\mathbf{nor}[s]\geq n$, $m^s_{\rm dn}\geq n$, and
- $(\bullet)_2$ $(\forall f\in \mathbf{val}[s])(f\oplus A\in D)$.

Proof of Claim 3.9.1. Let $A':=\{f\in\mathcal{F}_{\mathbf H}\colon f\oplus A\in D\}$ and $A'':=A\cap A'$. Since $A\in D=D\oplus D$ we know that $A'\in D$ and thus $A''\in D$. Hence $\mathcal{F}_{\mathbf H}\setminus A''\notin \mathcal{A}^1_{\bar t}$ $\big($remember $D\in \mathrm{uf}^1_{\bar t}(K,\Sigma)\big)$. Therefore, there is $s\in\Sigma'(\bar t)$ such that

$$\mathbf{nor}[s]\geq n,\quad m^s_{\rm dn}\geq n,\quad \text{ and }\ \mathbf{val}[s]\cap(\mathcal{F}_{\mathbf H}\setminus A'')=\emptyset.$$

Then $\mathbf{val}[s]\subseteq A$ and for each $f\in\mathbf{val}[s]$ we have $f\oplus A\in D$, as desired. ∎

Now suppose $A\in D$. By induction on n we choose s_n, A_n so that

(a) $A_0=A$, $A_n\in D$, and $A_{n+1}\subseteq A_n$,
(b) $s_n\in\Sigma'(\bar t)$, $\mathbf{nor}(s_n)\geq n$ and $m^{s_n}_{\rm up}\leq m^{s_{n+1}}_{\rm dn}$,
(c) $\mathbf{val}[s_n]\subseteq A_n$,
(d) if $f\in A_{n+1}$, then $m^{s_n}_{\rm up}\leq \min(\mathrm{dom}(f))$,
(e) if $f\in\mathbf{val}[s_n]$, then $A_{n+1}\subseteq f\oplus A_n$.

Suppose we have constructed $s_0,\ldots,s_{n-1}$ and A_n so that demands (a)–(e) are satisfied. Set $N=m^{s_{n-1}}_{\rm up}+n+1$ (if $n=0$ stipulate $m^{s_{n-1}}_{\rm up}=0$) and use Claim 3.9.1 to find $s_n\in\Sigma'(\bar t)$ such that

- $(\bullet)^n_1$ $\mathbf{val}[s_n]\subseteq A_n$, $\mathbf{nor}[s_n]\geq N$, $m^{s_n}_{\rm dn}\geq N$, and
- $(\bullet)^n_2$ $(\forall f\in\mathbf{val}[s_n])(f\oplus A_n\in D)$.

Put

$$A_{n+1}:=A_n\cap\big\{g\in\mathcal{F}_{\mathbf H}\colon m^{s_n}_{\rm up}<\min(\mathrm{dom}(g))\big\}\cap\bigcap_{f\in\mathbf{val}[s_n]} f\oplus A_n.$$

Since $\mathcal{A}^0_{\bar t}\subseteq D$ we know that $\{g\in\mathcal{F}_{\mathbf H}\colon m^{s_n}_{\rm up}<\min(\mathrm{dom}(g))\}\in D$ and since $\mathbf{val}[s_n]$ is finite $\big($and by $(\bullet)^n_2\big)$ also $\bigcap\limits_{f\in\mathbf{val}[s_n]} f\oplus A_n\in D$. Thus $A_{n+1}\in D$. Plainly the other requirements hold too.

After the above construction is carried out we set $\bar s=\langle s_n\colon n<\omega\rangle$. Clearly $\bar s\in \mathrm{PC}_\infty(K,\Sigma)$ and $\bar s\geq\bar t$ (remember clause (b)).

Claim 3.9.2. If $n_0 < \cdots < n_k < \omega$ and $f_\ell \in \mathbf{val}[s_{n_\ell}]$ for $\ell \le k$, then $\bigcup\limits_{\ell \le k} f_\ell \in A_{n_0}$.

Proof of Claim 3.9.2. Induction on k. If $k = 0$ then clause (c) of the choice of s_{n_0} gives the conclusion. For the inductive step suppose the claim holds true for k and let $n_0 < n_1 < \cdots < n_k < n_{k+1}$, $f_\ell \in \mathbf{val}\left[s_{n_\ell}\right]$ (for $\ell \le k+1$). Letting $g = f_1 \cup \cdots \cup f_{k+1}$ we may use the inductive hypothesis to conclude that $g \in A_{n_1}$. By (a) and (e) we know that $A_{n_1} \subseteq A_{n_0+1} \subseteq f_0 \oplus A_{n_0}$, so $g \in f_0 \oplus A_{n_0}$. Hence $f_0 \cup g = f_0 \cup f_1 \cup \cdots \cup f_{k+1} \in A_{n_0}$. ∎

It follows from Claim 3.9.2 that $\mathrm{pos}(\bar{s}) \subseteq A$ (remember (a) above and Definition 2.2(2)(f)). ∎

Conclusion 3.10. Suppose that (K, Σ) is a loose FFCC pair with weak $\mathtt{1}$-bigness and weak $\mathtt{1}$-additivity over $\bar{t} \in \mathrm{PC}_\infty(K, \Sigma)$. Assume also that $\mathrm{pos}(\bar{t})$ is the finite union $\mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n$. Then for some $i \le n$ and $\bar{s} \in \mathrm{PC}_\infty(K, \Sigma)$ we have

$$\mathrm{pos}(\bar{s}) \subseteq \mathcal{F}_i \quad \text{and} \quad \bar{t} \le \bar{s}.$$

Proof. By 3.8, there is $D \in \mathrm{uf}^{1}_{\bar{t}}(K, \Sigma)$ such that $D = D \oplus D$. Clearly for some $i \le n$ we have $\mathcal{F}_i \in D$. By 3.9 there is $\bar{s} \in \mathrm{PC}_\infty(K, \Sigma)$ such that $\bar{t} \le \bar{s}$ and $\mathrm{pos}(\bar{s}) \subseteq \mathcal{F}_i$. ∎

4. Ultrafilters on Tight Possibilities

In this section we carry out for tight FFCC pairs considerations parallel to that from the case of loose FFCC pairs. The main difference now is that we use *sequences* of ultrafilters, but many arguments do not change much.

Definition 4.1. *Let (K, Σ) be a tight FFCC pair for $\mathbf{H}$, $\bar{t} = \langle t_n : n < \omega \rangle \in \mathrm{PC}^{\mathtt{tt}}_\infty(K, \Sigma)$.*

(1) *For $f \in \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$, let $x_f = x^{\bar{t}}_f$ be the unique $m > n$ such that $f \in \mathbf{val}[s]$ for some $s \in \Sigma(t_n, \ldots, t_{m-1})$. (Note Definition 2.2(4)($\mathrm{f}^{\mathrm{tight}}$).)*

(2) *If $f \in \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$, $n < \omega$, $A \subseteq \mathcal{F}_{\mathbf{H}}$, then we set*

$$f \circledast A = f \circledast_{\bar{t}} A = \left\{ g \in \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction x_f) \colon f \cup g \in A \right\}.$$

(3) *We let $\mathrm{suf}_{\bar{t}}(K,\Sigma)$ be the set of all sequences $\bar{D} = \langle D_n : n < \omega \rangle$ such that each D_n is a non-principal ultrafilter on $\mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$.*

(4) *The space $\mathrm{suf}_{\bar{t}}(K,\Sigma)$ is equipped with the (Tichonov) product topology of $\prod\limits_{n<\omega} \beta_*\left(\mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)\right)$. For a sequence $\bar{A} = \langle A_0, \ldots, A_n \rangle$ such that $A_\ell \subseteq \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction \ell)$ (for $\ell \le n$) we set*

$$\mathrm{Nb}_{\bar{A}} = \left\{ \bar{D} \in \mathrm{suf}_{\bar{t}}(K,\Sigma) \colon (\forall \ell \le n)(A_\ell \in D_\ell) \right\}.$$

(5) *For $\bar{D} = \langle D_n : n < \omega \rangle \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$, $n < \omega$ and $A \subseteq \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$, we let*

$$\mathrm{set}^n_{\bar{t}}(A, \bar{D}) = \left\{ f \in \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) : f \circledast A \in D_{x_f} \right\}.$$

(6) *For $\bar{D}^1, \bar{D}^2 \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$, we define $\bar{D}^1 \circledast \bar{D}^2$ to be a sequence $\langle D_n : n < \omega \rangle$ such that for each n*

$$D_n = \left\{A \subseteq \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) : \mathrm{set}^n_{\bar{t}}\left(A, \bar{D}^1\right) \in D^2_n\right\}.$$

Observation 4.2. Let (K, Σ) be a tight FFCC pair for $\mathbf{H}$ and $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K, \Sigma)$. Suppose $f \in \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$, $g \in \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction x_f)$. Then,

(1) $f \cup g \in \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$ (note Definition 2.2(4)($\mathrm{h}^{\mathrm{tight}}$)) and

(2) $(f \cup g) \circledast A = g \circledast (f \circledast A)$ for all $A \subseteq \mathcal{F}_{\mathbf{H}}$.

(3) $\mathrm{suf}_{\bar{t}}(K,\Sigma)$ is a compact Hausdorff topological space. The sets $\mathrm{Nb}_{\bar{A}}$ for $\bar{A} = \langle A_0, \ldots, A_n \rangle$, $A_\ell \subseteq \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction \ell)$, $\ell \le n < \omega$, form a basis of the topology of $\mathrm{suf}_{\bar{t}}(K,\Sigma)$.

Proposition 4.3. *Let (K, Σ) be a tight FFCC pair for $\mathbf{H}$ and $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K, \Sigma)$.*

(1) *If $\bar{D}^1, \bar{D}^2 \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$, then $\bar{D}^1 \circledast \bar{D}^2 \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$.*
(2) *The mapping $\circledast : \mathrm{suf}_{\bar{t}}(K,\Sigma) \times \mathrm{suf}_{\bar{t}}(K,\Sigma) \longrightarrow \mathrm{suf}_{\bar{t}}(K,\Sigma)$ is right continuous.*
(3) *The operation $\circledast$ is associative.*

Proof. (1) Let $\bar{D}^1, \bar{D}^2 \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$, $n < \omega$, and

$$D_n = \left\{A \subseteq \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) : \mathrm{set}^n_{\bar{t}}\left(A, \bar{D}^1\right) \in \bar{D}^2_n\right\}.$$

Let $A, B \subseteq \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$.

(a) If $f \in \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$ and A is finite, then $f \circledast A$ is finite as well, so it does not belong to $D^1_{x_f}$. Consequently, if A is finite then $\mathrm{set}^n_{\bar{t}}\left(A, \bar{D}^1\right) = \emptyset$ and $A \notin D_n$.
(b) $\mathrm{set}^n_{\bar{t}}\left(\mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n), \bar{D}^1\right) = \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) \in D^2_n$ (note Observation 4.2(1)). Thus, $\mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) \in D_n$.

(c) If $A \subseteq B$ then $\mathrm{set}^n_{\bar{t}}\left(A, \bar{D}^1\right) \subseteq \mathrm{set}^n_{\bar{t}}\left(B, \bar{D}^1\right)$, and hence

$$A \subseteq B \;\&\; A \in D_n \;\;\Rightarrow\;\; B \in D_n.$$

(d) $\mathrm{set}^n_{\bar{t}}\left(A \cap B, \bar{D}^1\right) = \mathrm{set}^n_{\bar{t}}\left(A, \bar{D}^1\right) \cap \mathrm{set}^n_{\bar{t}}\left(B, \bar{D}^1\right)$ and hence

$$A, B \in D_n \;\;\Rightarrow\;\; A \cap B \in D_n.$$

(e) $\mathrm{set}^n_{\bar{t}}\left(\mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) \setminus A, \bar{D}^1\right) = \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) \setminus \mathrm{set}^n_{\bar{t}}\left(A, \bar{D}^1\right)$, and hence

$$A \notin D_n \;\;\Rightarrow\;\; \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) \setminus A \in D_n.$$

It follows from (a)–(e) that D_n is a non-principal ultrafilter on $\mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$ and hence clearly $\bar{D}^1 \circledast \bar{D}^2 \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$.

(2) Fix $\bar{D}^1 \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$ and let $\bar{A} = \langle A_\ell : \ell \le n \rangle$, $A_\ell \subseteq \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction \ell)$. For $\ell \le n$ put $B_\ell = \mathrm{set}^\ell_{\bar{t}}\left(A_\ell, \bar{D}^1\right)$ and let $\bar{B} = \langle B_\ell : \ell \le n \rangle$. Then, for each $\bar{D}^2 \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$, we have

$$\bar{D}^1 \circledast \bar{D}^2 \in \mathrm{Nb}_{\bar{A}} \quad \text{if and only if} \quad \bar{D}^2 \in \mathrm{Nb}_{\bar{B}}.$$

(3) Let $\bar{D}^1, \bar{D}^2, \bar{D}^3 \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$. Suppose $n < \omega$, $A \subseteq \mathrm{pos}(\bar{t} \restriction n)$. Then

(i) $A \in \left(\left(\bar{D}^1 \circledast \bar{D}^2\right) \circledast \bar{D}^3\right)_n$ iff $\operatorname{set}^n_{\bar{t}}\left(A, \bar{D}^1 \circledast \bar{D}^2\right) \in D^3_n$ iff $\left\{f \in \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n) \colon f \circledast A \in \left(\bar{D}^1 \circledast \bar{D}^2\right)_{x_f}\right\} \in D^3_n$, and

(ii) $A \in \left(\bar{D}^1 \circledast \left(\bar{D}^2 \circledast \bar{D}^3\right)\right)_n$ iff $\operatorname{set}^n_{\bar{t}}(A, \bar{D}^1) \in \left(\bar{D}^2 \circledast \bar{D}^3\right)_n$ iff $\operatorname{set}^n_{\bar{t}}\left(\operatorname{set}^n_{\bar{t}}\left(A, \bar{D}^1\right), \bar{D}^2\right) \in \bar{D}^3_n$ iff $\left\{f \in \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n) : f \circledast \operatorname{set}^n_{\bar{t}}\left(A, \bar{D}^1\right) \in D^2_{x_f}\right\} \in D^3_n$.

Let us fix $f \in \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n)$ for a moment. Then

$$
\begin{aligned}
& f \circledast A \in \left(\bar{D}^1 \circledast \bar{D}^2\right)_{x_f} \quad \text{iff} \quad \operatorname{set}^{x_f}_{\bar{t}}\left(f \circledast A, \bar{D}^1\right) \in D^2_{x_f} \quad \text{iff} \\
& \left\{g \in \operatorname{pos}^{\mathtt{tt}}\left(\bar{t} \upharpoonright x_f\right) : g \circledast (f \circledast A) \in D^1_{x_g}\right\} \in D^2_{x_f} \quad \text{iff} \\
& \left\{g \in \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright x_f) \colon (f \cup g) \circledast A \in D^1_{x_g}\right\} \in D^2_{x_f} \quad \text{iff} \\
& \left\{g \in \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright x_f) \colon (f \cup g) \in \operatorname{set}^n_{\bar{t}}\left(A, \bar{D}^1\right)\right\} \in D^2_{x_f} \quad \text{iff} \quad f \circledast \operatorname{set}^n_{\bar{t}}\left(A, \bar{D}^1\right) \in D^2_{x_f}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
& \left\{f \in \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n) : f \circledast A \in \left(\bar{D}^1 \circledast \bar{D}^2\right)_{x_f}\right\} \\
& = \left\{f \in \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n) \colon f \circledast \operatorname{set}^n_{\bar{t}}\left(A, \bar{D}^1\right) \in D^2_{x_f}\right\}
\end{aligned}
$$

and (by (i) and (ii)) $A \in \left(\bar{D}^1 \circledast (\bar{D}^2 \circledast \bar{D}^3)\right)_n$ if and only if $A \in \left((\bar{D}^1 \circledast \bar{D}^2) \circledast \bar{D}^3\right)_n$. ∎

Definition 4.4. *Let (K, Σ) be a tight FFCC pair for* $\mathbf{H}$ *and $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K, \Sigma)$.*

(1) *For $n < \omega$, $\mathcal{B}^n_{\bar{t}}$ is the family of all sets $B \subseteq \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n)$ such that for some M we have: if $s \in \Sigma^{\mathtt{tt}}(\bar{t} \upharpoonright n)$ and $\mathbf{nor}[s] \geq M$, then $\mathbf{val}[s] \cap B \neq \emptyset$.*

(2) $\operatorname{suf}^*_{\bar{t}}(K,\Sigma)$ *is the family of all $\bar{D} = \langle D_n \colon n < \omega\rangle \in \operatorname{suf}_{\bar{t}}(K,\Sigma)$ such that $\mathcal{B}^n_{\bar{t}} \subseteq D_n$ for all $n < \omega$.*

Proposition 4.5. *Let (K, Σ) be a tight FFCC pair for* $\mathbf{H}$ *and $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K, \Sigma)$.*

(1) $\operatorname{suf}^*_{\bar{t}}(K,\Sigma)$ *is a closed subset of* $\operatorname{suf}_{\bar{t}}(K,\Sigma)$.

(2) *If (K, Σ) has the weak* $\mathtt{t}$*-bigness for $\bar{t}$, then* $\operatorname{suf}^*_{\bar{t}}(K,\Sigma) \neq \emptyset$.

(3) *If (K, Σ) has the weak* $\mathtt{t}$*-additivity for $\bar{t}$, then* $\operatorname{suf}^*_{\bar{t}}(K,\Sigma)$ *is closed under* $\circledast$.

Proof. (1) Suppose $\bar{D} \in \operatorname{suf}_{\bar{t}}(K,\Sigma) \setminus \operatorname{suf}^*_{\bar{t}}(K,\Sigma)$. Let $n < \omega$ and $B \in \mathcal{B}^n_{\bar{t}}$ be such that $B \notin D_n$. Set $A_n = \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n) \setminus B$ and $A_\ell = \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright \ell)$ for $\ell < n$, and let $\bar{A} = \langle A_0, \ldots, A_n\rangle$. Then $\bar{D} \in \mathrm{Nb}_{\bar{A}} \subseteq \operatorname{suf}_{\bar{t}}(K,\Sigma) \setminus \operatorname{suf}^*_{\bar{t}}(K,\Sigma)$.

(2) It is enough to show that, assuming the weak $\mathtt{t}$-bigness, each family $\mathcal{B}^n_{\bar{t}}$ has fip. To this end suppose that $B_0, \ldots, B_{m-1} \in \mathcal{B}^n_{\bar{t}}$. Pick M_0 such that

$$(*) \quad \big(\forall s \in \Sigma^{\mathtt{tt}}(\bar{t} \upharpoonright n)\big)\big(\forall \ell < m\big)\big(\mathbf{nor}[s] \geq M_0 \ \Rightarrow\ B_\ell \cap \mathbf{val}[s] \neq \emptyset\big).$$

For $\eta \in {}^m 2$ set $C_\eta = \{f \in \operatorname{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n) \colon (\forall \ell < m)\,(f \in B_\ell \ \Leftrightarrow\ \eta(\ell) = 1)\}$. By the weak $\mathtt{t}$-bigness we may choose η and $s \in \Sigma^{\mathtt{tt}}(\bar{t} \upharpoonright n)$ such that $\mathbf{nor}[s] \geq M_0$ and $\mathbf{val}[s] \subseteq C_\eta$. Then $\big(\text{by } (*)\big)$ we also have $\eta(\ell) = 1$ and $\mathbf{val}[s] \subseteq B_\ell$ for all $\ell < m$. Hence $\emptyset \neq \mathbf{val}[s] \subseteq \bigcap\limits_{\ell < m} B_\ell$.

(3) Let an increasing function $\mathbf{f}\colon \omega \longrightarrow \omega$ witness the weat $\mathtt{t}$-additivity of (K,Σ) for $\bar{t}$. Suppose that $\bar{D}^1, \bar{D}^2 \in \mathrm{suf}^*_{\bar{t}}(K,\Sigma)$, $\bar{D} = \bar{D}^1 \circledast \bar{D}^2$. We have to show that for each $n<\omega$, $\mathcal{B}^n_{\bar{t}} \subseteq D_n$ (remember Proposition 4.3(1)). To this end assume that $B \in \mathcal{B}^n_{\bar{t}}$ and let M be such that

$$(\forall s \in \Sigma^{\mathtt{tt}}(\bar{t}\restriction n))\,(\mathbf{nor}[s] \geq M \;\Rightarrow\; \mathbf{val}[s]\cap B \neq \emptyset).$$

Claim 4.5.1. If $s \in \Sigma^{\mathtt{tt}}(\bar{t}\restriction n)$ is such that $\mathbf{nor}[s] \geq \mathbf{f}(n+M)$, then $\mathbf{val}[s] \cap \mathrm{set}^n_{\bar{t}}\left(B, \bar{D}^1\right) \neq \emptyset$.

Proof of Claim 4.5.1. Fix $s_0 \in \Sigma(t_n,\dots,t_{m-1})$ such that $\mathbf{nor}[s_0] \geq \mathbf{f}(n+M)$. Let $A = \bigcup\{f \circledast B\colon f \in \mathbf{val}[s_0]\}$. We claim that

$(\triangle)$ $A \in \mathcal{B}^m_{\bar{t}}$.

(Why? Set $N = \mathbf{f}(m+M)$. Suppose $s_1 \in \Sigma^{\mathtt{tt}}(\bar{t}\restriction m)$ has norm $\mathbf{nor}[s_1] \geq N$. By the weak $\mathtt{t}$-additivity and the choice of N we can find $s \in \Sigma^{\mathtt{tt}}(\bar{t}\restriction n)$ such that $\mathbf{nor}[s] \geq M$ and $\mathbf{val}[s] \subseteq \{f \cup g\colon f \in \mathbf{val}[s_0],\ g \in \mathbf{val}[s_1]\}$. By the choice of M we have $B \cap \mathbf{val}[s] \neq \emptyset$, so for some $f \in \mathbf{val}[s_0]$ and $g \in \mathbf{val}[s_1]$ we have $g \in f \circledast B$. Thus $\mathbf{val}[s_1] \cap A \neq \emptyset$ and we easily conclude that $A \in \mathcal{B}^m_{\bar{t}}$.)

But $\bar{D}^1 \in \mathrm{suf}^*_{\bar{t}}(K,\Sigma)$, so $\mathcal{B}^m_{\bar{t}} \subseteq D^1_m$ and hence, for some $f \in \mathbf{val}[s_0]$, we get $f \circledast B \in D^1_{x_f}$. Then $f \in \mathbf{val}[s_0] \cap \mathrm{set}^n_{\bar{t}}\left(B, \bar{D}^1\right)$. ∎

It follows from Claim 4.5.1 that $\mathrm{set}^n_{\bar{t}}\left(B, \bar{D}^1\right) \in \mathcal{B}^n_{\bar{t}} \subseteq D^2_n$, so $B \in D_n$, as required. ∎

Corollary 4.6. *Assume that (K,Σ) is a tight FFCC pair with the weak $\mathtt{t}$-additivity and the weak $\mathtt{t}$-bigness for $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K,\Sigma)$. Then there is $\bar{D} \in \mathrm{suf}^*_{\bar{t}}(K,\Sigma)$ such that $\bar{D} \circledast \bar{D} = \bar{D}$.*

Proof. By Lemma 3.7, Observation 4.2(3), Proposition 4.3, and Proposition 4.5. ∎

Theorem 4.7. *Assume that (K,Σ) is a tight FFCC pair, $\bar{t} = \langle t_n\colon n<\omega\rangle \in \mathrm{PC}^{\mathtt{tt}}_\infty(K,\Sigma)$. Suppose also that*

(a) $\bar{D} = \langle D_n\colon n<\omega\rangle \in \mathrm{suf}^*_{\bar{t}}(K,\Sigma)$ is such that $\bar{D} \circledast \bar{D} = \bar{D}$, and
(b) $\bar{A} = \langle A_n\colon n<\omega\rangle$ is such that $A_n \in D_n$ for all $n<\omega$.

Then there is $\bar{s} = \langle s_i\colon i<\omega\rangle \in \mathrm{PC}^{\mathtt{tt}}_\infty(K,\Sigma)$ such that $\bar{s} \geq \bar{t}$, $m^{s_0}_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$ and if $i<\omega$, $s_i \in \Sigma^{\mathtt{tt}}(\bar{t}\restriction k)$, then $\mathrm{pos}^{\mathtt{tt}}(\bar{s}\restriction i) \subseteq A_k$.

Proof. Let (K,Σ), $\bar{t}$, $\bar{D}$ and $\bar{A}$ be as in the assumptions. Then, in particular, $\mathcal{B}^k_{\bar{t}} \subseteq D_k$ for all $k<\omega$.

Claim 4.7.1. If $M, k < \omega$ and $B \in D_k$, then there is $s \in \Sigma^{\mathtt{tt}}(\bar{t}\restriction k)$ such that $\mathbf{val}[s] \subseteq B$, $\mathbf{nor}[s] \geq M$ and $(\forall f \in \mathbf{val}[s])\left(f \circledast B \in D_{x_f}\right)$.

Proof of Claim 4.7.1. Since $\bar{D} = \bar{D} \circledast \bar{D}$ and $B \in D_k$, we know that $\mathrm{set}^k_{\bar{t}}(B,\bar{D}) \in D_k$ and thus $B \cap \mathrm{set}^k_{\bar{t}}(B,\bar{D}) \in D_k$. Hence $\mathrm{pos}^{\mathtt{tt}}(\bar{t}\restriction k) \setminus \left(B \cap \mathrm{set}^k_{\bar{t}}(B,\bar{D})\right) \notin \mathcal{B}^k_{\bar{t}}$ and we may find $s \in \Sigma^{\mathtt{tt}}(\bar{t}\restriction k)$ such that $\mathbf{nor}[s] \geq M$ and $\mathbf{val}[s] \subseteq B \cap \mathrm{set}^k_{\bar{t}}(B,\bar{D})$. This s is as required in the assertion of the claim. ∎

Now we choose inductively s_i, B_i, k_i (for $i<\omega$) such that

(i) $B_0 = A_0$, $k_0 = 0$,
(ii) $B_i \in D_{k_i}$, $B_i \subseteq A_{k_i}$, $k_i < k_{i+1} < \omega$, $s_i \in \Sigma\left(t_{k_i}, \ldots, t_{k_{i+1}-1}\right)$,
(iii) $\mathbf{val}[s_i] \subseteq B_i$, $\mathbf{nor}[s_i] \geq i+1$,
(iv) if $f \in \mathbf{val}[s_i]$, then $B_{i+1} \subseteq f \circledast B_i \in D_{k_{i+1}}$.

Clause (i) determines B_0 and k_0. Suppose we have already chosen k_i and $B_i \in D_{k_i}$. By Claim 4.7.1 we may find $k_{i+1} > k_i$ and $s_i \in \Sigma\left(t_{k_i}, \ldots, t_{k_{i+1}-1}\right)$ such that

$$\mathbf{nor}[s_i] \geq i+1, \quad \mathbf{val}[s_i] \subseteq B_i, \quad \text{and} \quad (\forall f \in \mathbf{val}[s_i])\left(f \circledast B_i \in D_{k_{i+1}}\right).$$

We let $B_{i+1} = A_{k_{i+1}} \cap \bigcap \{f \circledast B_i \colon f \in \mathbf{val}[s_i]\} \in D_{k_{i+1}}$. One easily verifies the relevant demands in (ii)–(iv) for s_i, B_{i+1}, k_{i+1}.

After the above construction is carried out, we set $\bar{s} = \langle s_i \colon i<\omega\rangle$. Plainly, $\bar{s} \in \mathrm{PC}^{\mathtt{tt}}_{\infty}(K,\Sigma)$, $\bar{s} \geq \bar{t}$ and $m^{s_0}_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$.

Claim 4.7.2. For each $i, k<\omega$ and $s \in \Sigma(s_i, \ldots, s_{i+k})$ we have $\mathbf{val}[s] \subseteq B_i$.

Proof of Claim 4.7.2. Induction on $k<\omega$. If $k=0$ then the assertion of the claim follows from clause (iii) of the choice of s_i. Assume we have shown the claim for k. Suppose that $s \in \Sigma(s_i, \ldots, s_{i+k}, s_{i+k+1})$, $i<\omega$, and $f \in \mathbf{val}[s]$. Let $f_0 = f{\restriction}\left[m^{s_i}_{\mathrm{dn}}, m^{s_i}_{\mathrm{up}}\right) \in \mathbf{val}[s_i]$ and $f_1 = f{\restriction}\left[m^{s_{i+1}}_{\mathrm{dn}}, m^{s_{i+k+1}}_{\mathrm{up}}\right) \in \mathrm{pos}^{\mathtt{tt}}(\bar{s} \rceil (i+1))$ (remember Definition 2.2(4)($\mathrm{f}^{\mathrm{tight}}$) and Observation 4.2(1)). By the inductive hypothesis we know that $f_1 \in B_{i+1}$, so by clause (iv) of the choice of s_i we get $f_1 \in f_0 \circledast B_i$ and thus $f = f_0 \cup f_1 \in B_i$. ∎

It follows from Claim 4.7.2 that for each $i<\omega$ we have $\mathrm{pos}^{\mathtt{tt}}(\bar{s} \rceil i) \subseteq B_i \subseteq A_{k_i}$, as required. ∎

Conclusion 4.8. Suppose that (K,Σ) is a tight FFCC pair with weak $\mathtt{t}$-bigness and weak $\mathtt{t}$-additivity for $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_{\infty}(K,\Sigma)$.

(a) Assume that, for each $n<\omega$, $k_n<\omega$ and $d_n \colon \mathrm{pos}^{\mathtt{tt}}(\bar{t} \rceil n) \longrightarrow k_n$. Then there is $\bar{s} = \langle s_i \colon i<\omega\rangle \in \mathrm{PC}^{\mathtt{tt}}_{\infty}(K,\Sigma)$ such that $\bar{s} \geq \bar{t}$, $m^{s_0}_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$ and for each $n<\omega$, if n is such that $s_i \in \Sigma^{\mathtt{tt}}(\bar{t} \rceil n)$, then $d_n{\restriction}\mathrm{pos}^{\mathtt{tt}}(\bar{s} \rceil i)$ is constant.
(b) Suppose also that (K,Σ) has $\mathtt{t}$-multiadditivity. Let $d_n \colon \mathrm{pos}^{\mathtt{tt}}(\bar{t} \rceil n) \longrightarrow k$ (for $n<\omega$), $k<\omega$. Then there are $\bar{s} = \langle s_i \colon i<\omega\rangle \in \mathrm{PC}^{\mathtt{tt}}_{\infty}(K,\Sigma)$ and $\ell < k$ such that $\bar{s} \geq \bar{t}$ and for each $i<\omega$, if n is such that $s_i \in \Sigma^{\mathtt{tt}}(\bar{t} \rceil n)$ and $f \in \mathrm{pos}^{\mathtt{tt}}(\bar{s} \rceil i)$, then $d_n(f) = \ell$.

Now we will use Conclusion 4.8 to give a new proof of Carlson-Simpson Theorem. This theorem was used as a crucial lemma in the (inductive) proof of the Dual Ramsey Theorem [2, Theorem 2.2]. Also, as they noted in [2, p. 273], this theorem provides an immediate proof of the Halpern-Läuchli-Laver-Pincus Theorem [6, 9].

Theorem 4.9. (Carlson and Simpson [2, Theorem 6.3]) *Suppose that $0<N<\omega$, $\mathbb{X} = \bigcup\limits_{n<\omega} {}^{n}N$ and $\mathbb{X} = C_0 \cup \cdots \cup C_k$, $k<\omega$. Then there exist a partition $\{Y\} \cup \{Y_i \colon i<\omega\}$ of ω and a function $f \colon Y \longrightarrow N$ such that*

(a) *each* Y_i *is a finite non-empty set,*
(b) *if* $i < j < \omega$ *then* $\max(Y_i) < \min(Y_j)$*,*
(c) *for some* $\ell \le k$*, if* $i < \omega$*,* $g\colon \min(Y_i) \longrightarrow N$*,* $f{\restriction}\min(Y_i) \subseteq g$ *and* $g{\restriction}Y_j$ *is constant for* $j < i$*, then* $g \in C_\ell$*.*

Proof. For $f \in \mathbb{X}$ let $d_0(f) = \min\{\ell \le k\colon f \in C_\ell\}$. Consider the tight FFCC pair (K_N, Σ_N) defined in Example 2.13. It satisfies the assumptions of Conclusion 4.8. Fix any $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K_N, \Sigma_N)$ with $m^{t_0}_{\mathrm{dn}} = 0$ and use Conclusion 4.8(a) to choose $\bar{s} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K_N, \Sigma_N)$ such that $\bar{s} \ge \bar{t}$, $m^{s_0}_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}} = 0$ and $d_0{\restriction}\mathrm{pos}^{\mathtt{tt}}(\bar{s})$ is constant. Set $Y = \bigcup\limits_{i<\omega} X_{s_i}$, $f = \bigcup\limits_{i<\omega} \varphi_{s_i}$ and $Y_i = \big[m^{s_i}_{\mathrm{dn}}, m^{s_i}_{\mathrm{up}}\big) \setminus X_{s_i}$ for $i < \omega$. ∎

5. Very Weak Bigness

The assumptions of Conclusion 4.8 (weak $\mathtt{t}$-bigness and weak $\mathtt{t}$-additivity) are somewhat strong. We will weaken them substantially here, getting weaker but still interesting conclusion.

Definition 5.1. *Let* (K, Σ) *be a tight FFCC pair for* $\mathbf{H}$*,* $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K, \Sigma)$*.*

(1) *For* $n < m < \omega$*, we define*

$$\mathrm{pos}(\bar{t}{\restriction}[n, m)) = \bigcup\{\mathbf{val}[s]\colon s \in \Sigma(t_n, \ldots, t_{m-1})\}$$

and we also keep the convention that $\mathrm{pos}\big(\bar{t}{\restriction}[n, n)\big) = \{\emptyset\}$*.*
[*Note that* $\mathrm{pos}(\bar{t}{\restriction}[n, m)) = \{f_n \cup \cdots \cup f_{m-1}\colon f_\ell \in \mathbf{val}[t_\ell]$ *for* $\ell < m\}$ *(remember* 2.2(4)(f^{tight}) *and* 4.2(1)).]

(2) *We say that* (K, Σ) *has* the very weak $\mathtt{t}$-bigness for $\bar{t}$ *if*

$(\boxtimes)^{\mathrm{vw}}_{\bar{t}}$ *for every* $n, L, M < \omega$ *and a partition* $\mathcal{F}_0 \cup \cdots \cup \mathcal{F}_L = \mathrm{pos}^{\mathtt{tt}}(\bar{t} \mid n)$*, there are* $i_0 = n \le i_1 < i_2 \le i_3$*,* $\ell \le L$ *and* $g_0 \in \mathrm{pos}(\bar{t}{\restriction}[i_0, i_1))$*,* $g_2 \in \mathrm{pos}(\bar{t}{\restriction}[i_2, i_3))$ *and* $s \in \Sigma(t_{i_1}, \ldots, t_{i_2-1})$ *such that*

$$\mathbf{nor}[s] \ge M \quad \text{and} \quad (\forall g_1 \in \mathbf{val}[s])(g_0 \cup g_1 \cup g_2 \in \mathcal{F}_\ell).$$

Observation 5.2. If a tight FFCC pair (K, Σ) has the weak $\mathtt{t}$-bigness for $\bar{t}$, then it has the very weak $\mathtt{t}$-bigness for $\bar{t}$.

Definition 5.3. *Let* (K, Σ) *be a tight FFCC pair for* $\mathbf{H}$ *and* $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K, \Sigma)$*.*

(1) *For* $n < \omega$*,* $\mathcal{C}^n_{\bar{t}}$ *is the family of all sets* $B \subseteq \mathrm{pos}^{\mathtt{tt}}(\bar{t} \mid n)$ *such that for some* M *we have if* $i_0 = n \le i_1 < i_2 \le i_3$*,* $g_0 \in \mathrm{pos}(\bar{t}{\restriction}[i_0, i_1))$*,* $g_2 \in \mathrm{pos}(\bar{t}{\restriction}[i_2, i_3))$ *and* $s \in \Sigma\big(t_{i_1}, \ldots, t_{i_2-1}\big)$*,* $\mathbf{nor}[s] \ge M$*, then* $B \cap \{g_0 \cup g_1 \cup g_2\colon g_1 \in \mathbf{val}[s]\} \ne \emptyset$*.*
(2) $\mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$ *is the family of all* $\bar{D} = \langle D_n\colon n < \omega\rangle \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$ *such that* $\mathcal{C}^n_{\bar{t}} \subseteq D_n$ *for all* $n < \omega$*.*

Proposition 5.4. *Let* (K, Σ) *be a tight FFCC pair for* $\mathbf{H}$ *and* $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_\infty(K, \Sigma)$*.*

(1) $\mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$ *is a closed subset of* $\mathrm{suf}_{\bar{t}}(K,\Sigma)$*,* $\mathrm{suf}^{*}_{\bar{t}}(K,\Sigma) \subseteq \mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$*.*

(2) *If* (K,Σ) *has the very weak* $\mathtt{t}$*-bigness for* $\bar{t}$*, then* $\mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma) \neq \emptyset$.
(3) *If* $\bar{D} \in \mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$*,* $n < \omega$ *and* $B \in \mathcal{C}^n_{\bar{t}}$*, then* $\mathrm{set}^n_{\bar{t}}(B,\bar{D}) = \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$.
(4) $\mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$ *is closed under the operation* $\circledast$ *(defined in 4.1*(6)*).*

Proof. (1) Since in Definition 5.3(1) we allow $i_1 = i_0$ and $i_3 = i_2$ (so $g_0 = g_2 = \emptyset$), we easily see that $\mathcal{C}^n_{\bar{t}} \subseteq \mathcal{B}^n_{\bar{t}}$. Hence $\mathrm{suf}^{*}_{\bar{t}}(K,\Sigma) \subseteq \mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$. The proof that $\mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$ is closed is the same as for Proposition 4.5(1).

(2) Like Proposition 4.5(2).

(3) Let M be such that $B \cap \{g_0 \cup g_1 \cup g_2 \colon g_1 \in \mathbf{val}[s]\} \neq \emptyset$ whenever $g_0 \in \mathrm{pos}(\bar{t}\restriction[n,i_1))$, $g_2 \in \mathrm{pos}(\bar{t}\restriction[i_2,i_3))$, $s \in \Sigma(\bar{t}\restriction[i_1,i_2))$, $\mathbf{nor}[s] \geq M$, $n \leq i_1 < i_2 \leq i_3$. We will show that this M witnesses $f \circledast B \in \mathcal{C}^{x_f}_{\bar{t}}$ for all $f \in \mathrm{pos}^{\mathtt{tt}}(t \restriction n)$.

So suppose that $f \in \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n)$ and $x_f \leq i_1 < i_2 \leq i_3$, $g_0 \in \mathrm{pos}(\bar{t}\restriction[x_f,i_1))$, $s \in \Sigma(\bar{t}\restriction[i_1,i_2))$, $\mathbf{nor}[s] \geq M$ and $g_2 \in \mathrm{pos}(\bar{t}\restriction[i_2,i_3))$. Then $f \cup g_0 \in \mathrm{pos}(\bar{t}\restriction[n,i_1))$ (remember Observation 4.2(1)) and consequently (by the choice of M) $B \cap \big\{(f \cup g_0) \cup g_1 \cup g_2 \colon g_1 \in \mathbf{val}[s]\big\} \neq \emptyset$. Let $g_1^* \in \mathbf{val}[s]$ be such that $f \cup g_0 \cup g_1^* \cup g_2 \in B$. Then $g_0 \cup g_1^* \cup g_2 \in f \circledast B$ witnessing that $(f \circledast B) \cap \big\{g_0 \cup g_1 \cup g_2 \colon g_1 \in \mathbf{val}[s]\big\} \neq \emptyset$.

Since $\mathcal{C}^{x_f}_{\bar{t}} \subseteq D_{x_f}$ we conclude now that $f \circledast B \in D_{x_f}$, so $f \in \mathrm{set}^n_{\bar{t}}(B,\bar{D})$.

(4) Suppose $\bar{D}_1, \bar{D}_2 \in \mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$, $\bar{D} = \bar{D}_1 \circledast \bar{D}_2$. Let $B \in \mathcal{C}^n_{\bar{t}}$, $n < \omega$. By (3) we know that $\mathrm{set}^n_{\bar{t}}(B,\bar{D}_1) = \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) \in D^2_n$ and thus $B \in D_n$. Consequently, $\mathcal{C}^n_{\bar{t}} \subseteq D_n$ for all $n < \omega$, so $\bar{D} \in \mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$. ∎

Corollary 5.5. *Assume that* (K,Σ) *is a tight FFCC pair with the very weak* $\mathtt{t}$*-bigness for* $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_{\infty}(K,\Sigma)$*. Then there is* $\bar{D} \in \mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$ *such that* $\bar{D} \circledast \bar{D} = \bar{D}$.

Theorem 5.6. *Assume that* (K,Σ) *is a tight FFCC pair for* $\mathbf{H}$*,* $\bar{t} = \langle t_n \colon n < \omega\rangle \in \mathrm{PC}^{\mathtt{tt}}_{\infty}(K,\Sigma)$*. Let* $\bar{D} \in \mathrm{suf}^{\diamond}_{\bar{t}}(K,\Sigma)$ *be such that* $\bar{D} \circledast \bar{D} = \bar{D}$ *and suppose that* $A_n \in D_n$ *for* $n < \omega$*. Then there are sequences* $\langle n_i \colon i < \omega\rangle$*,* $\langle g_{3i}, g_{3i+2} \colon i < \omega\rangle$ *and* $\langle s_{3i+1} \colon i < \omega\rangle$ *such that for every* $i < \omega$:

(α) $0 = n_0 \leq n_{3i} \leq n_{3i+1} < n_{3i+2} \leq n_{3i+3} < \omega$,
(β) *if* $j = 3i$ *or* $j = 3i+2$*, then* $g_j \in \mathrm{pos}\left(\bar{t}\restriction\left[n_j,\ldots,n_{j+1}\right)\right)$,
(γ) *if* $j = 3i+1$*, then* $s_j \in \Sigma\big(t_{n_j},\ldots,t_{n_{j+1}-1}\big)$ *and* $\mathbf{nor}[s_j] \geq j$,
(δ) *if* $g_{3\ell+1} \in \mathbf{val}\left[s_{3\ell+1}\right]$ *for* $\ell \in [i,k)$*,* $i < k$*, then* $\bigcup_{j=3i}^{3k-1} g_j \in A_{n_{3i}}$.

Proof. Parallel to Theorem 4.7, just instead of $\mathbf{val}[s_i]$ use $\{g_{i-1} \cup g \cup g_{i+1} \colon g \in \mathbf{val}[s_i]\}$. ∎

Conclusion 5.7. Assume that (K,Σ) is a tight FFCC pair with the very weak $\mathtt{t}$-bigness for $\bar{t} \in \mathrm{PC}^{\mathtt{tt}}_{\infty}(K,\Sigma)$. Suppose that for each $n < \omega$ we are given $k_n < \omega$ and a mapping $d_n \colon \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) \longrightarrow k_n$. Then there are sequences $\langle n_i \colon i < \omega\rangle$, $\langle g_{3i}, g_{3i+2} \colon i < \omega\rangle$, $\langle s_{3i+1} \colon i < \omega\rangle$, and $\langle c_i \colon i < \omega\rangle$ such that for each $i < \omega$:

(α) $0 = n_0 \leq n_{3i} \leq n_{3i+1} < n_{3i+2} \leq n_{3i+3} < \omega$, $c_i \in k_{n_{3i}}$,
(β) if $j = 3i$ or $j = 3i+2$, then $g_j \in \mathrm{pos}(\bar{t}\restriction[n_j,\ldots,n_{j+1}))$,
(γ) if $j = 3i+1$, then $s_j \in \Sigma\big(t_{n_j},\ldots,t_{n_{j+1}-1}\big)$ and $\mathbf{nor}[s_j] \geq j$,

(δ) if $i<k$ and $f\in \mathrm{pos}\big(\bar t\restriction[n_{3i},n_{3k})\big)$ are such that

$$g_{3\ell}\cup g_{3\ell+2}\subseteq f \quad \text{and} \quad f\restriction\big[m^{s_{3\ell+1}}_{\rm dn},m^{s_{3\ell+1}}_{\rm up}\big)\in \mathbf{val}[s_{3\ell+1}], \quad \text{for all } \ell\in[i,k),$$

then $d_{n_{3i}}(f)=c_i$.

Example 5.8. Let $(G,\circ)$ be a finite group. For a function $f\colon S\longrightarrow G$ and $a\in G$ we define $a\circ f\colon S\longrightarrow G$ by $(a\circ f)(x)=a\circ f(x)$ for $x\in S$. Let $\mathbf{H}_G(m)=G$ (for $m<\omega$) and let K_G consist of all FP creatures t for $\mathbf{H}_G$ such that

- $\mathbf{nor}[t]=m^t_{\rm up}$, $\mathbf{dis}[t]=\emptyset$,
- $\mathbf{val}[t]\subseteq {}^{[m^t_{\rm dn},m^t_{\rm up})}G$ is such that $(\forall f\in\mathbf{val}[t])(\forall a\in G)(a\circ f\in\mathbf{val}[t])$.

For $t_0,\ldots,t_n\in K_G$ with $m^{t_{\ell+1}}_{\rm dn}=m^{t_\ell}_{\rm up}$ (for $\ell<n$) we let $\Sigma_G(t_0,\ldots,t_n)$ consist of all creatures $t\in K_G$ such that

- $m^t_{\rm dn}=m^{t_0}_{\rm dn}$, $m^t_{\rm up}=m^{t_n}_{\rm up}$,
- $\mathbf{val}[t]\subseteq\big\{f\in {}^{[m^t_{\rm dn},m^t_{\rm up})}G : (\forall\ell\le n)\big(f\restriction\big[m^{t_\ell}_{\rm dn},m^{t_\ell}_{\rm up}\big)\in\mathbf{val}[t_\ell]\big)\big\}$.

Then

(1) (K_G,Σ_G) is a tight FFCC pair for H_G.
(2) If $|G|=2$, then (K_G,Σ_G) has the very weak $\mathtt{t}$-bigness for every candidate $\bar t\in \mathrm{PC}^{\mathtt{tt}}_\infty(K_G,\Sigma_G)$.

Proof. (1) Straightforward.

(2) Let $G=(\{-1,1\},\cdot)$. Suppose that $\bar t\in\mathrm{PC}^{\mathtt{tt}}_\infty(K_G,\Sigma_G)$ and $\mathrm{pos}^{\mathtt{tt}}(\bar t\restriction n)=\mathcal{F}_0\cup\cdots\cup\mathcal{F}_L$, $n,L,M<\omega$. For future use we will show slightly more than needed for the very weak bigness.

We say that $N\ge n+M$ is ℓ-good (for $\ell\le L$) if

$(\boxdot)_\ell$ there are $j_2\ge j_1>N$, $g_0\in\mathrm{pos}(\bar t\restriction[n,N))$, $g_2\in\mathrm{pos}(\bar t\restriction[j_1,j_2))$ and $s\in\Sigma_G(\bar t\restriction[N,j_1))$ such that $\{g_0\cup g_1\cup g_2\colon g_1\in\mathbf{val}[s]\}\subseteq\mathcal{F}_\ell$.

(Note that if s is as in $(\boxdot)_\ell$, then also $\mathbf{nor}[s]=m^{t_{j_1}}_{\rm dn}\ge j_1>N\ge M$.) We are going to argue that

$(\odot)$ almost every $N\ge n+M$ is ℓ–good for some $\ell\le L$.

So suppose that $(\odot)$ fails and we have an increasing sequence $n+M<N(0)<N(1)<N(2)<\cdots$ such that $N(k)$ is not ℓ-good for any $\ell\le L$ (for all $k<\omega$). Let $m=L+2$ and for each $i\in[n,N(m)]$ fix $f_i\in\mathbf{val}[t_i]$ (note that then $-f_i\in\mathbf{val}[t_i]$ as well). Next, for $j<m$ define

$$h_j=\bigcup_{i=n}^{N(j)-1} f_i\cup\bigcup_{i=N(j)}^{N(m)} -f_i$$

and note that $h_j \in \mathrm{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n)$. For some $\ell \leq L$ and $j < k < m$ we have $h_j, h_k \in \mathcal{F}_\ell$. Set

$$g_0 = \bigcup_{i=n}^{N(j)-1} f_i = h_j \upharpoonright m_{\mathrm{dn}}^{t_{N(j)}} = h_k \upharpoonright m_{\mathrm{dn}}^{t_{N(j)}},$$

$$g_2 = \bigcup_{i=N(k)}^{N(m)} -f_i = h_j \upharpoonright \big[m_{\mathrm{dn}}^{t_{N(k)}}, m_{\mathrm{up}}^{t_{N(m)}}\big) = h_k \upharpoonright \big[m_{\mathrm{dn}}^{t_{N(k)}}, m_{\mathrm{up}}^{t_{N(m)}}\big),$$

and let $s \in \Sigma_G(\bar{t} \upharpoonright [N(j), N(k)))$ be such that

$$\mathbf{val}[s] = \big\{h_j \upharpoonright \big[m_{\mathrm{dn}}^{t_{N(j)}}, m_{\mathrm{dn}}^{t_{N(k)}}\big), h_k \upharpoonright \big[m_{\mathrm{dn}}^{t_{N(j)}}, m_{\mathrm{dn}}^{t_{N(k)}}\big)\big\}.$$

Then $\{g_0 \cup g_1 \cup g_2 \colon g_1 \in \mathbf{val}[s]\} = \{h_j, h_k\} \subseteq \mathcal{F}_\ell$, so g_0, g_2 and s witness $(\boxdot)_\ell$ for $N(j)$, a contradiction. ∎

The following conclusion is a special case of the partition theorem used in Goldstern and Shelah [4] to show that a certain forcing notion preserves a Ramsey ultrafilter (see [4, 3.9, 4.1 and Section 5]).

Corollary 5.9. *Let $\mathbb{Y} = \bigcup_{n<\omega} {}^n\{-1,1\}$. Suppose that $\mathbb{Y} = C_0 \cup \cdots \cup C_L$, $L < \omega$. Then there are a sequence $\langle n_i \colon i < \omega \rangle$, a function $f \colon \omega \longrightarrow \{-1,1\}$, and $\ell < L$ such that*

(a) $0 = n_0 \leq n_{3i} \leq n_{3i+1} < n_{3i+2} \leq n_{3i+3} < \omega$,
(b) *if $g \colon n_{3i} \longrightarrow \{-1,1\}$ for each $j < i$ satisfies*

$$g \upharpoonright \big[n_{3j}, n_{3j+1}\big) \cup g \upharpoonright \big[n_{3j+2}, n_{3j+3}\big) \subseteq f \quad \text{and}$$

$$g \upharpoonright \big[n_{3j+1}, n_{3j+2}\big) \in \big\{f \upharpoonright \big[n_{3j+1}, n_{3j+2}\big), -f \upharpoonright \big[n_{3j+1}, n_{3j+2}\big)\big\},$$

then $g \in C_\ell$.

Proof. By Conclusion 5.7 and Example 5.8. ∎

6. Limsup Candidates

Definition 6.1. *Let (K, Σ) be a tight FFCC pair for* **H** *and J be an ideal on ω.*

(1) A limsup$_J$-candidate *for (K, Σ) is a sequence $\bar{t} = \langle t_n \colon n < \omega \rangle$ such that $t_n \in K$, $m_{\mathrm{up}}^{t_n} = m_{\mathrm{dn}}^{t_{n+1}}$ (for all n) and for each M*

$$\big\{m_{\mathrm{dn}}^{t_n} \colon n < \omega \;\&\; \mathbf{nor}[t_n] > M\big\} \in J^+.$$

The family of all limsup$_J$*-candidates for (K, Σ) is denoted by* $\mathrm{PC}^J_{\mathrm{w}\infty}(K, \Sigma)$.

(2) A finite candidate *for (K, Σ) is a finite sequence $\bar{s} = \langle s_n \colon n < N \rangle$, $N < \omega$, such that $s_n \in K$ and $m_{\mathrm{up}}^{s_n} = m_{\mathrm{dn}}^{s_{n+1}}$ (for $n < N$). The family of all finite candidates is called* $\mathrm{FC}(K, \Sigma)$.

(3) *For* $\bar{s} = \langle s_n \colon n < N \rangle \in \mathrm{FC}(K,\Sigma)$ *and* $M < \omega$ *we set*

$$\mathrm{base}_M(\bar{s}) = \left\{ m^{s_n}_{\mathrm{dn}} \colon n < N \ \&\ \mathbf{nor}[s_n] \geq M \right\}.$$

(4) *Let* $\bar{t}, \bar{t}' \in \mathrm{PC}^J_{\mathrm{w}\infty}(K,\Sigma)$, $\bar{s} \in \mathrm{FC}(K,\Sigma)$. *Then we define* $\bar{t} \upharpoonright n$, $\Sigma^{\mathtt{tt}}(\bar{t})$, $\mathrm{pos}^{\mathtt{tt}}(\bar{t})$, $\bar{t} \leq \bar{t}'$, $\mathrm{pos}(\bar{t}{\restriction}[n,m))$, *and* $\mathrm{pos}(\bar{s})$ *as in the case of tight pure candidates* (*cf. Definitions 2.3 and 5.1*).

(5) *Let* $\bar{t} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K,\Sigma)$. *The family of all finite candidates* $\bar{s} = \langle s_n \colon n < N \rangle \in \mathrm{FC}(K,\Sigma)$ *satisfying*

$$(\forall n < N)(\exists k, \ell)\,(s_n \in \Sigma(\bar{t}{\restriction}[k,\ell))) \quad \textit{and} \quad m^{s_0}_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$$

is denoted by $\Sigma^{\mathrm{seq}}(\bar{t})$.

Definition 6.2. *Let* (K,Σ) *be a tight FFCC pair, and let* J *be an ideal on* ω *and* $\bar{t} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K,\Sigma)$.

(1) *We say that* (K,Σ) has the J-bigness for $\bar{t}$ *if*

$(\otimes)^J_{\bar{t}}$ *for every* $n, L, M < \omega$, *and a partition* $\mathcal{F}_0 \cup \cdots \cup \mathcal{F}_L = \mathrm{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n)$, *there are* $\ell \leq L$ *and a set* $Z \in J^+$ *such that*

$$(\forall z \in Z)\,(\exists \bar{s} \in \Sigma^{\mathrm{seq}}(\bar{t} \upharpoonright n))\,(z \in \mathrm{base}_M(\bar{s}) \ \&\ \mathrm{pos}(\bar{s}) \subseteq \mathcal{F}_\ell).$$

(2) *The pair* (K,Σ) captures singletons (*cf. [10, 2.1.10]*) *if*

$$(\forall t \in K)(\forall f \in \mathbf{val}[t])(\exists s \in \Sigma(t))(\mathbf{val}[s] = \{f\}).$$

(3) *We define* $\mathrm{suf}_{\bar{t}}(K,\Sigma)$ *as in Definition 4.1*(3), $\mathrm{set}^n_{\bar{t}}(A,\bar{D})$ $\big($*for* $A \subseteq \mathrm{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n)$ *and* $\bar{D} \in \mathrm{suf}_{\bar{t}}(K,\Sigma)\big)$ *as in Definition 4.1*(5) *and the operation* $\circledast$ *on* $\mathrm{suf}_{\bar{t}}(K,\Sigma)$ *as in Definition 4.1*(6).

(4) *For* $n < \omega$, $\mathcal{D}^{n,J}_{\bar{t}}$ *is the family of all sets* $B \subseteq \mathrm{pos}^{\mathtt{tt}}(\bar{t} \upharpoonright n)$ *such that for some* $M < \omega$ *and* $Y \in J^c$ *we have if* $\bar{s} \in \Sigma^{\mathrm{seq}}(\bar{t} \upharpoonright n)$ *and* $\mathrm{base}_M(\bar{s}) \cap Y \neq \emptyset$, *then* $B \cap \mathrm{pos}(\bar{s}) \neq \emptyset$.

(5) $\mathrm{suf}^J_{\bar{t}}(K,\Sigma)$ *is the family of all* $\bar{D} = \langle D_n \colon n < \omega \rangle \in \mathrm{suf}_{\bar{t}}(K,\Sigma)$ *such that* $\mathcal{D}^{n,J}_{\bar{t}} \subseteq D_n$ *for all* $n < \omega$.

Remark 6.3. Note that no norms were used in the proofs of Observation 4.2, Proposition 4.3, so those statements are valid for the case of $\bar{t} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K,\Sigma)$ too.

Observation 6.4. (1) Assume that (K,Σ) is a tight FFCC pair with bigness (see Definition 2.6(3)). If (K,Σ) captures singletons or it has the $\mathtt{t}$-multiadditivity (see Definition 2.5(3)), then (K,Σ) has the J-bigness for any $\bar{t} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K,\Sigma)$.

(2) The tight FFCC pairs (K_1,Σ^*_1), (K_3,Σ^*_3), and (K_N,Σ_N) defined in Examples 2.10, 2.12, and 2.13, respectively, have J-bigness on every $\bar{t} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K,\Sigma)$.

Every tight FFCC pair can be extended to a pair capturing singletons while preserving $\mathrm{pos}^{\mathtt{tt}}(\bar{t})$.

Definition 6.5. *Let (K,Σ) be a tight FFCC pair for* $\mathbf{H}$. *Define $K^{\rm sin}$ as the family of all FP creatures t for* $\mathbf{H}$ *such that*

$$\mathbf{dis}[t]=K, \quad \mathbf{nor}[t]=0, \quad \textit{and} \quad |\mathbf{val}[t]|=1.$$

Then we let $K^s=K\cup K^{\rm sin}$ and for $t_0,\dots,t_n\in K^s$ with $m^{t_\ell}_{\rm up}=m^{t_{\ell+1}}_{\rm dn}$ (for $\ell<n$) we set

- *$\Sigma^{\rm sin}(t_0,\dots,t_n)$ consists of all creatures $t\in K^{\rm sin}$ such that $m^t_{\rm dn}=m^{t_0}_{\rm dn}$, $m^t_{\rm up}=m^{t_n}_{\rm up}$ and: If $\mathbf{val}[t]=\{f\}$ then $f\restriction\big[m^{t_\ell}_{\rm dn},m^{t_\ell}_{\rm up}\big)\in\mathbf{val}[t_\ell]$ for all $\ell\le n$;*
- *if $t_0,\dots,t_n\in K$, then $\Sigma^s(t_0,\dots,t_n)=\Sigma(t_0,\dots,t_n)\cup\Sigma^{\rm sin}(t_0,\dots,t_n)$;*
- *if $t_\ell\in K^{\rm sin}$ for some $\ell\le n$, then $\Sigma^s(t_0,\dots,t_n)=\Sigma^{\rm sin}(t_0,\dots,t_n)$.*

Observation 6.6. Let (K,Σ) be a tight FFCC pair for $\mathbf{H}$.

(1) (K^s,Σ^s) is a tight FFCC pair for $\mathbf{H}$ and it captures singletons.
(2) If (K,Σ) has bigness then so does (K^s,Σ^s) and consequently (K^s,Σ^s) has the J-bigness on any $\bar t\in{\rm PC}^J_{\rm w\infty}(K^s,\Sigma^s)$.
(3) If $\bar t\in{\rm PC}^J_{\rm w\infty}(K,\Sigma)$, then $\bar t\in{\rm PC}^J_{\rm w\infty}(K^s,\Sigma^s)$ and ${\rm pos}^{\tt tt}(\bar t)$ with respect to (K,Σ) is the same as ${\rm pos}^{\tt tt}(\bar t)$ with respect to (K^s,Σ^s).

Observation 6.7. Let $G=(\{-1,1\},\cdot)$ and (K_G,Σ_G) be the tight FFCC pair defined in Example 5.8. Suppose that $\bar t\in{\rm PC}^J_{\rm w\infty}(K_G,\Sigma_G)$. Then $\big(K^s_G,\Sigma^s_G\big)$ (sic!) has the J-bigness for $\bar t$.

Proof. Note that ${\rm PC}^J_{\rm w\infty}(K_G,\Sigma_G)\subseteq{\rm PC}^{\tt tt}_\infty(K_G,\Sigma_G)$ and remember $(\odot)$ from the proof of Example 5.8(2). ∎

Proposition 6.8. *Assume that (K,Σ) is a tight FFCC pair for* $\mathbf{H}$, *J is an ideal on ω and $\bar t\in{\rm PC}^J_{\rm w\infty}(K,\Sigma)$.*

(1) ${\rm suf}^J_{\bar t}(K,\Sigma)$ *is a closed subset of the compact Hausdorff topological space* ${\rm suf}_{\bar t}(K,\Sigma)$.
(2) *If (K,Σ) has the J-bigness for $\bar t$, then* ${\rm suf}^J_{\bar t}(K,\Sigma)\neq\emptyset$.
(3) *If $\bar D\in{\rm suf}^J_{\bar t}(K,\Sigma)$, $n<\omega$ and $B\in\mathcal{D}^{n,J}_{\bar t}$, then ${\rm set}^n_{\bar t}(B,\bar D)\in\mathcal{D}^{n,J}_{\bar t}$.*
(4) ${\rm suf}^J_{\bar t}(K,\Sigma)$ *is closed under the operation $\circledast$.*

Proof. (1) Same as Proposition 4.5(1).

(2) Similar to Proposition 4.5(2).

(3) Let $B\in\mathcal{D}^{n,J}_{\bar t}$ be witnessed by $M<\omega$ and $Z\in J^c$. We are going to show that then for each $\bar s\in\Sigma^{\rm seq}(\bar t\restriction n)$ with ${\rm base}_M(\bar s)\cap Z\neq\emptyset$ we have ${\rm pos}(\bar s)\cap{\rm set}^n_{\bar t}(B,\bar D)\neq\emptyset$. So let $\bar s=\langle s_0,\dots,s_k\rangle\in\Sigma^{\rm seq}(\bar t\restriction n)$, ${\rm base}_M(\bar s)\cap Z\neq\emptyset$ and let x be such that $m^{s_k}_{\rm up}=m^{t_x}_{\rm dn}$. Set $A=\bigcup\{f\circledast B\colon f\in{\rm pos}(\bar s)\}$. Suppose that $\bar r\in\Sigma^{\rm seq}(\bar t\restriction x)$. Then $\bar s^\frown\bar r\in\Sigma^{\rm seq}(\bar t\restriction n)$ and ${\rm base}_M(\bar s^\frown\bar r)\supseteq{\rm base}_M(\bar s)$, so ${\rm pos}(\bar s^\frown\bar r)\cap B\neq\emptyset$. Let $g\in{\rm pos}(\bar s^\frown\bar r)\cap B$ and $f_0=g\restriction m^{s_k}_{\rm up}$, $f_1=g\restriction[m^{s_k}_{\rm up},\omega)$. Necessarily, $f_0\in{\rm pos}(\bar s)$, $f_1\in{\rm pos}(\bar r)$ and (as $g=f_0\cup f_1\in B$) $f_1\in f_0\circledast B$. Consequently, $A\cap{\rm pos}(\bar r)\neq\emptyset$. Now we easily conclude that $A\in\mathcal{D}^{x,J}_{\bar t}\subseteq D_x$. Hence for some $f\in{\rm pos}(\bar s)$ we have $f\circledast B\in D_x$, so $f\in{\rm set}^n_{\bar t}(B,\bar D)$.

(4) Follows from (3). ∎

Corollary 6.9. *Assume that (K,Σ) is a tight FFCC pair, J is an ideal on ω and $\bar{t} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K,\Sigma)$. If (K,Σ) has the J-bigness for $\bar{t}$, then there is $\bar{D} \in \mathrm{suf}^J_{\bar{t}}(K,\Sigma)$ such that $\bar{D} \circledast \bar{D} = \bar{D}$.*

Definition 6.10. *Let J be an ideal on ω.*

(1) *A game $\eth_J$ between two players, One and Two, is defined as follows. A play of $\eth_J$ lasts ω steps in which the players construct a sequence $\langle Z_i, k_i \colon i < \omega\rangle$. At a stage i of the play, first One chooses a set $Z_i \in J^+$ and then Two answers with $k_i \in Z_i$. At the end, Two wins the play $\langle Z_i, k_i \colon i < \omega\rangle$ if and only if $\{k_i \colon i < \omega\} \in J^+$.*
(2) *We say that J is* an R-ideal *if player One has no winning strategy in $\eth_J$.*

Remark 6.11. The game $\eth_J$ is attributed to Fred Galvin. He also showed that if J is a maximal ideal on ω, then it is an R-ideal if and only if the dual filter J^c is a Ramsey ultrafilter (cf. [1, Theorem 4.5.3]). Also, the ideal $[\omega]^{<\omega}$ of all finite subsets of ω is an R-ideal.

Theorem 6.12. *Assume that (K,Σ) is a tight FFCC pair, J is an R-ideal on ω and $\bar{t} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K,\Sigma)$. Suppose that $\bar{D} \in \mathrm{suf}^J_{\bar{t}}(K,\Sigma)$ satisfies $\bar{D} \circledast \bar{D} = \bar{D}$ and let $A_n \in D_n$ for $n < \omega$. Then there are $\bar{s} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K,\Sigma)$ and $0 = k(0) < k(1) < k(2) < k(3) < \cdots < \omega$ such that $\bar{t} \le \bar{s}$, $m^{s_0}_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$ and if $i < j$, $\ell < \omega$, $s_{k(i)} \in \Sigma^{\mathtt{tt}}(\bar{t} \restriction \ell)$, then $\mathrm{pos}\big(s_{k(i)}, s_{k(i)+1}, \ldots, s_{k(j)-1}\big) \subseteq A_\ell$.*

Proof. The proof follows the pattern of Theorem 4.7 with the only addition that we need to make sure that at the end $\bar{s} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K,\Sigma)$, so we play a round of $\eth_J$. First,

Claim 6.12.1. Assume $M, \ell < \omega$ and $B \in D_\ell$. Then for some set $Z \in J^+$, for every $x \in Z$, there is $\bar{s} \in \Sigma^{\mathrm{seq}}(\bar{t} \restriction \ell)$ such that

$$x \in \mathrm{base}_M(\bar{s}), \quad \mathrm{pos}(\bar{s}) \subseteq B, \quad \text{and} \quad (\forall f \in \mathrm{pos}(\bar{s}))\big(f \circledast B \in D_{x_f}\big).$$

Proof of Claim 6.12.1. Similar to Claim 4.7.1. Since $\bar{D} \circledast \bar{D} = \bar{D}$ and $B \in D_\ell$, we know that $\mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction \ell) \setminus (B \cap \mathrm{set}^\ell_{\bar{t}}(B, \bar{D})) \notin \mathcal{D}^{\ell,J}_{\bar{t}}$. Therefore, for each $Y \in J^c$ there are $x \in Y$ and $\bar{s} \in \Sigma^{\mathrm{seq}}(\bar{t} \restriction \ell)$ such that $x \in \mathrm{base}_M(\bar{s})$ and $\mathrm{pos}(\bar{s}) \subseteq B \cap \mathrm{set}^\ell_{\bar{t}}(B, \bar{D})$. So the set Z of x as above belongs to J^+. ∎

Consider the following strategy for player One in the game $\eth_J$. During the course of a play, in addition to his innings Z_i, One chooses aside $\ell_i < \omega$, $B_i \in D_{\ell_i}$ and $\bar{s}^i \in \mathrm{FC}(K,\Sigma)$. So suppose that the players have arrived to a stage i of the play and a sequence $\langle Z_j, k_j, \bar{s}^j, \ell_j, B_j \colon j < i\rangle$ has been constructed. Stipulating $\ell_{-1} = 0$ and $B_{-1} = A_0$, One uses Claim 6.12.1 to pick a set $Z_i \subseteq \omega \setminus m^{t_{\ell_{i-1}}}_{\mathrm{dn}}$ such that $Z_i \in J^+$ and for all $x \in Z_i$ there exists $\bar{s} \in \Sigma^{\mathrm{seq}}(\bar{t} \restriction \ell_{i-1})$ with

$$x \in \mathrm{base}_{i+1}(\bar{s}) \;\;\&\;\; \mathrm{pos}(\bar{s}) \subseteq B_{i-1} \;\;\&\;\; (\forall f \in \mathrm{pos}(\bar{s}))\big(f \circledast B_{i-1} \in D_{x_f}\big).$$

The set Z_i is One's inning in $\eth_J$ after which Two picks $k_i \in Z_i$. Now, One chooses $\bar{s}^i \in \Sigma^{\mathrm{seq}}(\bar{t} \restriction \ell_{i-1})$ such that

$(\alpha)_i$ $k_i \in \mathrm{base}_{i+1}(\bar{s}^i)$,

$(\beta)_i$ $\mathrm{pos}(\bar{s}^i) \subseteq B_{i-1}$, and
$(\gamma)_i$ $(\forall f \in \mathrm{pos}(\bar{s}^i))(f \circledast B_{i-1} \in D_{x_f})$.

He also sets

$(\delta)_i$ $\ell_i = x_f$ for all (equivalently: some) $f \in \mathrm{pos}(\bar{s}^i)$, and
$(\varepsilon)_i$ $B_i = A_{\ell_i} \cap \bigcap \{f \circledast B_{i-1} : f \in \mathrm{pos}(\bar{s}^i)\} \in D_{\ell_i}$.

The strategy described above cannot be winning for One, so there is a play $\langle Z_i, k_i : i < \omega \rangle$ in which One follows the strategy, but $\{k_i : i < \omega\} \in J^+$. In the course of this play One constructed aside a sequence $\langle \ell_i, B_i, \bar{s}^i : i < \omega \rangle$ such that $\bar{s}^i \in \Sigma^{\mathrm{seq}}(\bar{t} \restriction \ell_{i-1})$ and conditions $(\alpha)_i$–$(\varepsilon)_i$ hold (where we stipulate $\ell_{-1} = 0$, $B_{-1} = A_0$). Note that $\bar{s}^i{}^\frown \bar{s}^{i+1}{}^\frown \cdots {}^\frown \bar{s}^{i+k} \in \Sigma^{\mathrm{seq}}(\bar{t} \restriction \ell_{i-1})$ for each $i, k < \omega$. Also

$$\bar{s} \stackrel{\mathrm{def}}{=} \bar{s}^0{}^\frown \bar{s}^1{}^\frown \bar{s}^2{}^\frown \cdots \in \mathrm{PC}^J_{\mathrm{w}\infty}(K, \Sigma) \quad \text{and} \quad \bar{s} \geq \bar{t}.$$

Claim 6.12.2. For each $i, k < \omega$, $\mathrm{pos}(\bar{s}^i{}^\frown \bar{s}^{i+1}{}^\frown \cdots {}^\frown \bar{s}^{i+k}) \subseteq B_{i-1} \subseteq A_{\ell_{i-1}}$.

Proof of Claim 6.12.2. Induction on k; fully parallel to Claim 4.7.2. ∎

Now the theorem readily follows. ∎

Conclusion 6.13. Assume that (K, Σ) is a tight FFCC pair, J is an R-ideal on ω and $\bar{t} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K, \Sigma)$. Suppose also that (K, Σ) has J-bigness for $\bar{t}$. For $n < \omega$ let $k_n < \omega$ and let $d_n : \mathrm{pos}^{\mathtt{tt}}(\bar{t} \restriction n) \longrightarrow k_n$. Then there are $\bar{s} \in \mathrm{PC}^J_{\mathrm{w}\infty}(K, \Sigma)$ and $0 = k(0) < k(1) < k(2) < \cdots < \omega$ and $\langle c_i : i < \omega \rangle$ such that

- $\bar{t} \leq \bar{s}$, $m^{s_0}_{\mathrm{dn}} = m^{t_0}_{\mathrm{dn}}$, and
- for each $i, n < \omega$, if $s_{k(i)} \in \Sigma^{\mathtt{tt}}(\bar{t} \restriction n)$, $i < j < \omega$ and $f \in \mathrm{pos}(s_{k(i)}, s_{k(i)+1}, \ldots, s_{k(j)-1})$, then $d_n(f) = c_i$.

Corollary 6.14. *Let $\mathbf{H}^* : \omega \longrightarrow \omega \setminus \{0\}$ be increasing, $\mathbb{Z}^* = \bigcup\limits_{n<\omega} \prod\limits_{i<n} \mathbf{H}^*(i)$ and let J be an R-ideal on ω. Suppose that $\mathbb{Z}^* = C_0 \cup \cdots \cup C_L$, $L < \omega$. Then there are sequences $\langle k_i, n_i : i < \omega \rangle$ and $\langle E_i : i < \omega \rangle$ and $\ell \leq L$ such that*

(a) $0 = n_0 \leq k_0 < n_1 \leq \cdots < n_i \leq k_i \leq n_{i+1} \leq \cdots < \omega$, $\{k_i : i < \omega\} \in J^+$, *and*
(b) *for each $i < \omega$, $\emptyset \neq E_i \subseteq \mathbf{H}^*(i)$, $|E_{k_i}| = i+1$, and*

$$\prod_{j<n_i} E_j \subseteq C_\ell.$$

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