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CLASSIFICATION THEORY FOR NON-ELEMENTARY CLASSES I: THE NUMBER OF UNCOUNTABLE MODELS OF $\psi \in L_{\omega_1,\omega}$. PART B

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ABSTRACT
We continue here Part A, and the main results are proved here. This part deals with n-dimensional diagrams of models.

§3. Good systems

In this section we shall introduce the diagrams over which we shall define in the next section the generalized amalgamation properties. We shall have ordered sets I which will be used as index sets of diagrams; for each $s \in I$, there corresponds a model M_s such that $s \leqq t$ (in the order of I) implies $M_s \subseteq M_t$. We shall work with ordered sets I such that, for every $s, t \in I$, $s \cap t$ — the infimum of $\{s, t\}$ — is defined; an additional requirement on our diagrams is $M_{s\cap t} = M_s \cap M_t$. Such diagrams will be called I-systems and defined in Definition 3.2(1). In this section, the reader should think of I as a power set of $n = \{0, 1, \cdots, n-1\}$ and the order is the inclusion. We shall usually work with particular kinds of systems. They are introduced in Definition 3.3. The main theorem in this section is the generalized symmetry lemma, which will help to prove the goodness of systems.

NOTATION 3.1. (1) Let $\mathcal{P}(S)$ be the family of subsets of S, $\mathcal{P}^-(S) = \mathcal{P}(S) - \{S\}$. We use usually $\mathcal{P}(n)$, remembering $n = \{0, 1, \cdots, n-1\}$.

(2) We use I mainly to denote a hereditary subset of some $\mathcal{P}(n)$, but

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generally it denotes a partially ordered set, finite if not indicated otherwise, such that

(i) for any $s, t \in I$ there is a largest lower bound $s \cap t$, and if s, t has an upper bound then they have a least upper bound $s \cup t$;

(ii) for each $s \in I$ there is a natural number $n = |s|_I$ and a mapping $g = g_s = g_{s,I}$ from $\mathcal{P}(n)$ into I, $g(n) = s$, g commutes with $\cup$ and $\cap$, and for each $t \in I$, $s \not\leqq t$, for some $u \in \mathcal{P}^-(n)$, $s \cap t \leqq g_s(u)$;

(iii) $n(I) = \max_{s \in I} |s|_I$ exists.

Notice each linearly ordered set I is as required, and $n(I) = 1$.

(3) I_1 is a submodel of I_2, $(I_1 \subseteq I_2)$ if $s \in I_1 \Rightarrow s \in I_2$, the intersection, order and g_s are the restriction, and union is the same when it exists in I_1.

(4) I_1 is an initial segment of I_2 if $I_1 \subseteq I_2$, and $s_1 \in I_2$, $s_2 < s_1$ implies $s_2 \in I_1$.

(5) $I_1 \times I_2$ is partially ordered by $(s_1, s_2) \leqq (t_1, t_2)$ iff $s_1 \leqq t_1$, and $s_2 \leqq t_2$ (it is as required). $I^{\alpha,\beta}$ is $I \times \{\alpha, \beta\}$ where $\alpha < \beta$; notice that $I^{\alpha,\beta}$ is isomorphic to $\mathcal{P}(n+1)$ if I is isomorphic to $\mathcal{P}(n)$. Naturally for $I_1 \times I_2$ we can have

$$|\langle s_1, s_2\rangle|_{I_1 \times I_2} = |s_1|_{I_1} + |s_2|_{I_2} \text{ and } g_{\langle s_1, s_2\rangle}(u) = \langle g_{s_1}(u \cap |s_1|_{I_1}), g_{s_2}(\{l : |s_1|_{I_1} + l \in u\})\rangle.$$

DEFINITION 3.2. (1) An I-system $\mathcal{S}$ is an indexed set M_s $(s \in I)$ such that:

(i) $\bigcup_{s \in I} M_s$ is an atomic set;

(ii) $s \leqq t$ implies $M_s \subseteq M_t$;

(iii) $M_{s \cap t} = M_s \cap M_t$.

We let $A_s = \bigcup_{t < s} M_t$ even if $s \notin I$, but I is an initial segment of I_1, and $s \in I_1$, $(\forall t \in I_1)\,[t < s \Rightarrow t \in I]$.

When it isn't self-evident, we write $M_s^{\mathcal{S}}$, $A_s^{\mathcal{S}}$.

(2) We call $\mathcal{S}$ a (λ, I)-system if $\|M_s\| = \lambda$ for each $s \in I$.

DEFINITION 3.3. (1) The system is called *full* if for every s, M_s is weakly full over A_s.

(2) The system is called *good* if for each enumeration $\bar{s} = \langle s(i) : i < j\rangle$ of I (always without repetitions, and such that $s(i_1) <_I s(i_2) \Rightarrow i_1 < i_2$), it is good for $\bar{s}$, which means (for not necessarily finite I)† that for each $l < j$:

(i) $A_{s(l)}$ is good;

(ii) for every $\bar{b}_m \in M_{s(m)}$ for $m \leqq l$, we can find $\bar{b}'_m \in M_{s(m) \cap s(l)}$ such that

(a) $s(m) \leqq s(l) \Rightarrow \bar{b}'_m = \bar{b}_m$,

(b) $\bar{b}_0 {}^\wedge \bar{b}_1 {}^\wedge \cdots$ and $\bar{b}'_0 {}^\wedge \bar{b}_1 {}^\wedge \cdots$ realizes the same type;

(iii) the triple $A_{s(l)}$, $M_{s(l)}$, $\bigcup_{m < l} M_{s(m)}$ is in stable amalgamation.

(3) We say the system is weakly good if for some $t^* \in I$, for each enumeration

† But $\bigcup_m \bar{b}_m$ below should be finite.

Sh:87b

$\bar{s}$ of I in which t^* is the last element, the conditions of (2) hold for each l, $l+1<j$ and there are such enumerations. Speaking on enumeration of I for weakly good systems we mean t^* is the last.

(4) We say "fully good system" instead of "full and good system", and "weakly fully good system" means "full and weakly good system".

Claim 3.4. *(1) If J is a submodel of I, $\mathcal{S}^0=\{M_s : s\in I\}$ a [full] [good] I-system,* then *$\mathcal{S}^1=\{M_s : s\in J\}$ is a [full] [good] J-system, and for $s\in J$, $A_s^{\mathcal{S}^0}=A_s^{\mathcal{S}^1}$.*

(2) If $\langle s(0),\cdots,s(k-1)\rangle$ is an enumeration of I, then for each l, $\{s(0),\cdots,s(l-1)\}$ is an initial segment of I.

(3) If $\mathcal{S}=\{M_s : s\in I\}$ is a good system, $\langle s(0),\cdots,s(k-1)\rangle$ an enumeration, then for each l, $(A_{s(l)},\bigcup_{i<l}M_{s(i)})$ satisfies the Tarski–Vaught condition, hence for each $t\in I$, $(A_t,\bigcup_{t\not\leq s}M_s)$ satisfies the Tarski–Vaught condition.

(4) In Definition 3.3(2), part (iii) follows by (i) and (ii).

(5) In Definition 3.3(2), we can replace (ii), (iii) by (ii)$^-$, (iv) where (ii)$^-$ is like (ii) with $\bar{b}_l=\varnothing$ and (iv) says: for $\bar{b}\in A_{s(l)}$, tp$(\bar{b},\bigcup_{m<l}M_{s(m)})$ does not split over some finite $B\subseteq A_{s(l)}$.

Proof. (1) By Notation 3.1(3) $g_s^I=g_s^J$, hence $|s|_I=|s|_J$, and let $n=|s|_I$. By Notation 3.1(2)(ii),

$$A_s^{\mathcal{S}^0}=\bigcup_{\substack{t<s\\ t\in I}}A_t^{\mathcal{S}}=\bigcup\{A_t^s : t=g_s^I(u),u\in\mathcal{P}^-(n)\}$$

$$=\bigcup\{A_t^s : t=g_s^J(u),u\in\mathcal{P}^-(n)\}=\bigcup_{\substack{t<s\\ t\in J}}A_t^{\mathcal{S}^1}=A_s^{\mathcal{S}^1}.$$

(2) Easy.

(3) By Definition 3.3(2)(iii) for the first phrase, and by choosing an appropriate enumeration for the second.

(4) So suppose $\mathcal{S}$ is an I-system, $\bar{s}=\langle s(i): i<j\rangle$ an enumeration of I, and (i) and (ii) from Definition 3.3(2) are satisfied for a given $l<j$. We should prove that $A_{s(l)}$, $M_{s(l)}$, $\bigcup_{m<l}M_{s(m)}$ is in stable amalgamation.

Now $A_{s(l)}$ is a good set by Definition 3.3(2)(i), and $(A_{s(l)},\bigcup_{m<l}M_{s(m)})$ satisfies the Tarski–Vaught condition by Definition 3.3(2)(ii). The part left to show is that, for any $\bar{b}\in M_{s(l)}$, tp$(\bar{b},\bigcup_{m<l}M_{s(m)})$ does not split over some finite $B\subseteq A_{s(l)}$.

By Lemma 2.2(1), as $A_{s(l)}$ is a good set, for some finite $B\subseteq A_{s(l)}$, tp$(\bar{b},A_{s(l)})$ does not split over B, and we shall show that even tp$(\bar{b},\bigcup_{m<l}M_{s(m)})$ does not

split over B, thus finishing. So let $\bar{c}_1, \bar{c}_2 \in \bigcup_{m<l} M_{s(m)}$, $\operatorname{tp}(\bar{c}_1, B) = \operatorname{tp}(\bar{c}_2, B)$ and it suffices to prove that $\operatorname{tp}(\bar{c}_1, B \cup \bar{b}) = \operatorname{tp}(\bar{c}_2, B \cup \bar{b})$. We can easily find $\bar{b}_m \in M_{s(m)}$ for $m \leqq l$ such that $\bar{b}_l = \bar{b}$, $B \subseteq \bigcup \{\bar{b}_m : s(m) < s(l)\}$, and $\bar{c}_1 {}^\wedge \bar{c}_2 \subseteq \bigcup_{m<l} \bar{b}_m$. By Definition 3.3(2)(ii) there is an elementary mapping f, $\operatorname{Dom} f = \bigcup_{m \leqq l} \bar{b}_m$, $f(\bar{b}_m) = \bar{b}'_m \in M_{s(m) \cap s(l)}$, such that $s(m) \leqq s(l)$ implies $f(\bar{b}_m) = \bar{b}_m$. Clearly $f \restriction (B \cup \bar{b}) = \mathrm{id}$, hence $\operatorname{tp}(\bar{c}_i, B \cup \bar{b}) = \operatorname{tp}(f(\bar{c}_i), B \cup \bar{b})$. Hence

$$\operatorname{tp}(f(\bar{c}_1), B) = \operatorname{tp}(\bar{c}_1, B) = \operatorname{tp}(\bar{c}_2, B) = \operatorname{tp}(f(\bar{c}_2), B).$$

As $\bar{c}_i \subseteq \bigcup_{m<l} M_{s(m)}$, $f(\bar{c}_i) \subseteq \bigcup_{m<l} M_{s(m) \cap s(l)} = A_{s(l)}$, and as $\operatorname{tp}(\bar{b}, A_{s(l)})$ does not split over B, by the previous sentence $\operatorname{tp}(f(\bar{c}_1), B \cup \bar{b}) = \operatorname{tp}(f(\bar{c}_2), N \cup \bar{b})$. But as $\operatorname{tp}(\bar{c}_i, B \cup \bar{b}) = \operatorname{tp}(f(\bar{c}_i), B \cup \bar{b})$, clearly $\operatorname{tp}(\bar{c}_1, B \cup \bar{b}) = \operatorname{tp}(\bar{c}_2, B \cup \bar{b})$, as required.

(5) Clearly (ii)$^-$, (iv) follow from (i), (ii), (iii). Now assume (i), (ii)$^-$, (iv); by (4) it suffices to prove (ii). So we are given $\bar{b}_m$ $(m \leqq l)$ as there. By (iv) we can find a finite $B \subseteq A_{s(l)}$ such that $\operatorname{tp}(\bar{b}_l, A_{s(l)})$ does not split over B. Now w.l.o.g. $B \subseteq \bigcup \{\bar{b}_m : s(m) < s(l)\}$. Now we define $\bar{b}'_m$ $(m < l)$ as guaranteed by the weaker (ii), and let $\bar{b}'_l = \bar{b}_l$.

THE GENERALIZED SYMMETRY LEMMA 3.5. *For finite systems in the definitions of good systems and of weakly good systems, we can replace "for every enumeration $\bar{s}$" by "for some enumeration $\bar{s}$".*

PROOF. It suffices to prove that *if* our system is good for $\bar{s} = \langle s(0), \cdots, s(k-1) \rangle$ *then* our system is good for $\bar{t}$, where $\bar{t} = \langle t(0), \cdots, t(k-1) \rangle$ is such that for some $l^* : s(i) = t(i)$ for $i \neq l^*$, $l^* + 1$, and $t(l^*) = s(l^* + 1)$, $t(l^* + 1) = s(l^*)$.

We can also ignore the case of "weakly good" (check the definition).

We now check that, for each $l < k$, the conditions (from Definition 3.3(2)) for the system to be good for $\bar{t}$ hold. For $l \neq l^*$, $l^* + 1$ the conditions are the same as the corresponding conditions for $\bar{s}$. For $l = l^*$ the conditions are just easier than the corresponding conditions for $\bar{s}$. So we have to deal with $l = l^* + 1$.

As for condition (i), it is the same as condition (i) for $\bar{s}$ and l^*. For condition (ii)$^-$ we prove a more general claim (apply Claim 3.6(2) for $\{M_{t(i)} : i \leqq l^*\}$, $h(t(i)) = t(i) \cap t(l^*)$ which is a good system by Claim 3.4(1); then we finish by Claim 3.4(5) if we prove (iv)).

Now for every $\bar{b} \in M_{t(l^*+1)}$, $\bar{c} \in M_{t(l^*)}$, remembering that $\bar{s}$ is a good enumeration, we know that letting $D = \bigcup_{i<l^*} M_{t(i)}$, $\operatorname{tp}(\bar{b}, D)$ is the stationarization of $\operatorname{tp}(\bar{b}, B)$ for some finite $B \subseteq A_{t(l^*+1)} \subseteq D$ and $\operatorname{tp}(\bar{c}, D \cup M_{t(l^*+1)})$ is the stationarization of $\operatorname{tp}(\bar{c}, C)$ for some finite $C \subseteq A_{t(l^*)} \subseteq D$. Hence also $\operatorname{tp}(\bar{c}, D \cup \bar{b})$ is the stationarization of $\operatorname{tp}(\bar{c}, C)$. By the symmetry property, 1.4(1)(c), also

$\mathrm{tp}(\bar{b}, D \cup \bar{c})$ is the stationarization of $\mathrm{tp}(\bar{b}, B)$. As this holds for every $\bar{c} \in M_{t(I^*)}$ also $\mathrm{tp}(\bar{b}, D \cup M_{t(I^*)})$ is the stationarization of $\mathrm{tp}(\bar{b}, B)$. By 1.4(1)(b) we finish.

Claim 3.6. *Suppose the I-system $\{M_s : s \in I\}$ is good for the enumeration $\bar{s} = \langle s(0), \cdots, s(k-1)\rangle$, $J \subseteq I$ an initial segment, h a function from I to J, $h \restriction J =$ the identity, h is order ($\leqq$) preserving.*

Let $A_J = \bigcup_{s \in J} A_s$. Then:

(1) *The pair $(A_J, \bigcup_{s \in I} M_s)$ satisfies the Tarski–Vaught condition.*

(2) *For every $\bar{b}_m \in M_{s(m)}$ $(m < k)$ there are $\bar{b}'_m \in M_{h(s(m))}$ such that: $s(m) \in J$ implies $\bar{b}'_{s(m)} = \bar{b}_{s(m)}$, and*

$$\mathrm{tp}(\bar{b}_0 {}^\wedge \bar{b}_1 {}^\wedge \cdots, \emptyset) = \mathrm{tp}(\bar{b}'_0 {}^\wedge \bar{b}'_1 {}^\wedge \cdots, \emptyset).$$

Remark. If I is the initial segment of $I^* = I \cup \{t\}$, $t \notin I$, $J = \{s \in I : s < t\}$, $h(s) = s \cap t$ are O.K.

Proof. (1) Follows by Claim 3.6(2).

(2) We can find sequences $\bar{a}_m$, $\bar{c}_m$, $\bar{d}_m$ for $m < k$ such that

(i) $\bar{a}_m \subseteq \bigcup_{i<m, s(i)<s(m)} (\bar{c}_i {}^\wedge \bar{d}_i)$, and $\bar{a}_m \subseteq A_{s(m)}$ and $\bar{b}_m \subseteq \bar{c}_m {}^\wedge \bar{d}_m \subseteq M_{s(m)}$,

(ii) $\mathrm{tp}(\bar{c}_i, \bar{a}_i) \vdash \mathrm{tp}(\bar{c}_i, A_{s(i)})$ hence $\mathrm{tp}(\bar{c}_i, \bar{a}_i) \vdash \mathrm{tp}(\bar{c}_i, \bigcup_{l<i} M_{s(l)})$;

(iii) $\mathrm{tp}(\bar{d}_i, A_{s(i)} \cup \bar{c}_i)$ is a stationarization of $\mathrm{tp}(\bar{d}_i, \bar{c}_i)$, hence so is $\mathrm{tp}(\bar{d}_i, \bigcup_{l<i} M_{s(l)} \cup \bar{c}_i)$.

The existence of $\bar{a}_m$, $\bar{c}_m$, $\bar{d}_m$ $(m < k)$ is proved as follows. We prove by downward induction on $j \leqq k$ that there are $\bar{a}^j_m$, $\bar{c}^j_m$, $\bar{d}^j_m$ $(m \leqq k)$ such that (i) holds for every $m < k$ and (ii), (iii) hold for every i, $j \leqq i < k$.

For $j = k$ let $\bar{c}^k_m = \bar{b}^k_m$, $\bar{a}^k_m = \bar{d}_m = \langle\ \rangle$ (the empty sequence), and clearly they are as required.

Suppose we have defined for $j+1$ and we want to define for j. For $i < k$, $s(i) \not\leqq s(j)$ let $\bar{a}^j_i = \bar{a}^{j+1}_i$, $\bar{c}^j_i = \bar{c}^{j+1}_i$, $\bar{d}^j_i = \bar{d}^{j+1}_i$.

Now $A_{s(j)}$ is good (by hypothesis that, for some enumeration, (i) of Definition 3.3(2) holds), and $A_{s(j)} \subseteq M_{s(j)}$, $\bar{c}^{j+1}_j {}^\wedge \bar{d}^{j+1}_j \subseteq M_{s(j)}$, so $\mathrm{tp}(\bar{c}^{j+1}_j {}^\wedge \bar{d}^{j+1}_j, A_{s(j)}) \in D_{A_{s(j)}}$. Hence by Lemma 2.2(1) there are $\bar{a}^*_j \in A_{s(j)}$, $\bar{c}^*_j \in M_{s(j)}$, $\bar{d}^*_j \in M_{s(j)}$ such that:

(a) $\bar{c}^{j+1}_j {}^\wedge \bar{d}^{j+1}_j \subseteq \bar{c}^*_j {}^\wedge \bar{d}^*_j$,

(b) $\mathrm{tp}(\bar{c}^*_j, \bar{a}^*_j) \vdash \mathrm{tp}(\bar{c}^*_j, A_{s(j)})$,

(c) $\mathrm{tp}(\bar{d}^*_j, A_{s(j)} \cup \bar{c}^*_j)$ is stationarization of $\mathrm{tp}(\bar{d}^*_j, \bar{c}^*_j)$.

We now let $\bar{c}^j_j = \bar{c}^*_j$, $\bar{d}^j_j = \bar{d}^*_j$ and $\bar{a}^j_j = \bar{a}^*_j$, and for $i < k$, $s(i) < s(j)$ (hence $i < j$) we let $\bar{a}^j_i = \bar{a}^{j+1}_i$, $\bar{c}^j_i = \bar{c}^{j+1}_i$ and $\bar{d}^j_i = \bar{d}^{j+1}_i \cup (M_{s(i)} \cap \bar{a}^*_j)$ (that is, the ranges of the sequences satisfy those conditions). Clearly we have defined $\bar{a}^j_i$, $\bar{c}^j_i$, $\bar{d}^j_i$ for all i's.

Why does $\bar{a}^j_m$, $\bar{c}^j_m$, $\bar{d}^j_m$ $(m < \omega)$ satisfy the induction hypothesis?

(i) Obvious.

(ii) For $i \neq j$ it is obvious; for $i = j$, by Definition 3.3(2)(iii) (which by assumption holds for the enumeration $\bar{s}$), $(A_{s(i)}, \bigcup_{l<i} M_{s(l)})$ satisfy the Tarski–Vaught condition, so by (b) above it follows.

(iii) For $i \neq j$ it is obvious. For $i = j$ remember (c) above; as $A_{s(l)}$, $M_{s(j)}$, $\bigcup_{l<j} M_{s(l)}$ is in stable amalgamation and Lemma 2.10(4), the second phrase, follows too.

From the above it follows that:

(ii)′ $\operatorname{tp}(\bar{c}_i, \bigcup\{\bar{c}_l \frown \bar{d}_l : s(l) \leqq s(i), l < i\}) \vdash \operatorname{tp}(\bar{c}_i, \bigcup_{l<i} \bar{c}_l \frown \bar{d}_l)$;

(iii)′ $\operatorname{tp}(\bar{d}_i, \bigcup_{l<i}(\bar{c}_l \frown \bar{d}_l) \cup \bar{c}_i)$ is a stationarization of $\operatorname{tp}(\bar{d}_i, \bar{c}_i)$.

So we have defined $\bar{a}_m$, $\bar{c}_m$, $\bar{d}_m$ as required.

Now we define by induction on i an elementary mapping f_i, increasing with i, whose domain is $\bigcup_{l<i}(\bar{c}_l \frown \bar{d}_l)$ and $f_i(\bar{c}_l \frown \bar{d}_l) \in M_{h(s(l))}$ and $f_i(\bar{c}_l \frown \bar{d}_l) = \bar{c}_l \frown \bar{d}_l$ if $s(l) \in J$.

For $i = 0$ the conditions hold, so let us define f_{i+1} assuming f_i has been defined.

Case I: $s(i) \notin J$

Clearly $\bigcup\{f_i(\bar{c}_l \frown \bar{d}_l) : s(l) \leqq s(i), l < i\} \subseteq M_{h(s(i))}$, hence we can find $\bar{c}'_i \in M_{h(s(i))}$ such that if we extend $g_i = f_i \restriction \{\bar{c}_l \frown \bar{d}_l : l < i, s(l) \leqq s(i)\}$ by $\bar{c}_i \to \bar{c}'_i$ we get an elementary mapping g'_i. By (ii)′, $f_i \cup g'_i$ is an elementary mapping and we call it f'_i. Now $\operatorname{tp}(\bar{d}_i, \bar{c}_i)$ is stationary, hence so is $g'_i(\operatorname{tp}(\bar{d}_i, \bar{c}_i))$. Now choose a finite $B^i \subseteq M_{h(s(i))}$, such that

$$p = \operatorname{tp}(f_i[\bar{c}_0 \frown \bar{d}_0 \frown \cdots \frown \bar{c}_{i-1} \frown \bar{d}_{i-1}], M_{h(s(i))})$$

and $p \restriction B^i$ has the same rank, $\bar{c}'_i \subseteq B^i$, and choose a $\bar{d}'_i \in M_{h(s(i))}$ which realizes the stationarization of $g'_i(\operatorname{tp}(\bar{d}_i, \bar{c}_i))$ over B_i. By the symmetry lemma (Theorem 1.4(1)(c)), $\operatorname{tp}(\bar{d}'_i, f_i[\bar{c}_0 \frown \bar{d}_0 \frown \cdots \frown \bar{c}_{i-1} \frown \bar{d}_{i-1}] \cup \bar{c}'_i)$ is the stationarization of $g'_i(\operatorname{tp}(\bar{d}_i, \bar{c}_i))$, hence $f'_i \cup \{\bar{d}_i \to \bar{d}'_i\}$ is an elementary mapping, and it is as required.

Case II: $s(i) \in J$

In this case we have to define $f_{i+1}(\bar{c}_{i+1} \frown \bar{d}_{i+1}) = \bar{c}_{i+1} \frown \bar{d}_{i+1}$, and have to prove this is an elementary mapping. Remembering f_i are the identity mappings on $\bigcup\{\bar{c}_l \frown \bar{d}_l : l < i, s(l) \leqq s(i)\}$, as J is an initial segment there is no new point here.

So we have proved Claim 3.6, hence Lemma 3.5.

CLAIM 3.7. *Let (for simplicity) I be finite, $\mathcal{S} = \{M_s : s \in I\}$ an I-system, then $\mathcal{S}$ is good* iff *the following are satisfied*:

(α) *each* $A_{s(m)}$ *is good (this does not depend on the enumeration)*;

(β) *if* J *is an initial segment of* I, *then for some* $s \in J$, *for no* $t \in J$, $s < t$, *and for every* $\bar{b} \in M_s$, $\mathrm{tp}(\bar{b}, \bigcup_{t \in J, t \neq s} M_t)$ *does not split over some finite subset of* A_s.

PROOF. We prove, by induction on k, that if J is an initial segment of I, $|J| = k$, then $\{M_s : s \in J\}$ is a good system, assuming (α) and (β).

Clearly this suffices and for $k = 0$ it is trivial.

So let $|J| = k + 1$, let $s \in J$ be as mentioned in (β), and let $\langle s(m) : m \leqq k \rangle$ be an enumeration of J such that $s(k) = s$. Let us check the condition from Definition 3.3(2). For $l < k$ they follow by the induction hypothesis, so let $l = k$.

(i) This is guaranteed by (α) of Claim 3.7.

(ii) By the induction hypothesis, $\{M_{s(m)} : m < k\}$ is good. Suppose $\bar{b}_m \in M_{s(m)}$ ($m \leqq k$) are given and we should find $\bar{b}'_m \in M_{s(m) \cap s(k)}$ as required there. By condition (β) (in the claim) w.l.o.g. $\mathrm{tp}(\bar{b}_k, \bigcup \{M_{s(m)} : m < k\})$ does not split over $\bigcup \{\bar{b}_m : s(m) < s(k)\}$. As $\langle M_{s(m)} : m < k \rangle$ is good, we can apply Claim 3.6 to $J - \{s(k)\}$, $\{s(m) : s(m) < s(k)\}$ and $h : h(s(m)) = s(m) \cap s(k)$ standing for I, J, h, resp. and get $\langle \bar{b}'_m : m < k \rangle$; letting $\bar{b}'_k = \bar{b}_k$ we finish (ii). By Claim 3.4(4), (iii) follows.

So we finish one direction; the other direction is easy.

§4. The generalized amalgamation properties

In this section, we explicate the scheme we have described in the Introduction, in [18], of reducing the analysis of a model M (which we have or we want to construct) to amalgamation properties for n-dimensional cubes (for uniqueness or existence). Definition 4.1, which introduces an appropriate family of properties, is the main definition of the section and possibly of the paper. We concentrate on fully good $(\lambda, \mathcal{P}^-(n))$-systems, as under reasonable conditions they are unique, so the uniqueness and non-uniqueness properties become complementary (but we shall also get the conclusions about all good systems, i.e., the "strong" properties).

Theorem 4.5 is the main theorem of the section; it says how to decompose a (λ, I)-system $\{M_s : s \in I\}$ to $\{M_s^\alpha : s \in I\}$, $\alpha < \lambda$, and how good and full are the systems $\{M_s^\alpha : s \in I\}$ and $\{M_{\langle s,l \rangle}^{\alpha,\beta} : s \in I, l \in \{0, 1\}\}$ where $M_{\langle s,l \rangle}^{\alpha,\beta}$ is M_s^α if $l = 0$, M_s^β if $l = 1$. This lays the ground for proving uniqueness and building amalgamation (thus proving existence or non-uniqueness) by induction on the cardinality.

DEFINITION 4.1. (1) K satisfies (or has) the (λ, n)-goodness property if for every fully good $(\lambda, \mathcal{P}^-(n))$-system $\mathcal{S}$, $A_n^{\mathcal{S}}$ is a good set.

(2) K satisfies (or has) the (λ, n)-existence property if every fully good $(\lambda, \mathcal{P}^{-}(n))$-system $\mathcal{S}$ can be completed to a weakly good, full $(\lambda, \mathcal{P}(n))$-system, i.e., we can find an M weakly full over $A_n^{\mathcal{S}}$, $\|M\| = \lambda$.

(3) K satisfies the (λ, n)-uniqueness property provided that: if $\mathcal{S}$ is a fully good $(\lambda, \mathcal{P}^{-}(n))$-system, and for $l = 0, 1$ $A_n^{\mathcal{S}} \subseteq M_l$, $\|M_l\| = \lambda$, M_l weakly full over $A_n^{\mathcal{S}}$, then M_0, M_1 are isomorphic over $A_n^{\mathcal{S}}$.

(4) K satisfies the (λ, n)-non-uniqueness property provided that, for any fully good $(\lambda, \mathcal{P}^{-}(n))$-system, there are M_l $(l = 0, 1)$ weakly full over $A_n^{\mathcal{S}}$, $\|M_l\| = \lambda$ and there are no M weakly full over $A_n^{\mathcal{S}}$ and elementary embeddings $f_l : M_l \to M$, $f_0 \restriction A_n^{\mathcal{S}} = f_1 \restriction A_n^{\mathcal{S}} = \text{id}$.

(5) The meaning of abbreviations, such as "$(<\lambda, \leqq n)$-non-uniqueness property", is clear.

(6) For each of those properties the meaning of "the strong (λ, n)-x property" means we do not demand the systems to be full (but in (3), M_l still is weakly full over A_n).

Remark. Note that it may occur that some of those properties hold vacuously, as no good $(\lambda, \mathcal{P}^{-}(n))$-system exists.

Lemma 4.2. (1) *The strong $(\aleph_0, n)$-existence property always holds.*

(2) *The following properties hold*: [*strong*] $(\aleph_0, n)$-*goodness*, [*strong*] $(\aleph_0, n)$-*existence*, [*strong*] $(\aleph_0, n)$-*uniqueness for* $n = 0, 1$.

(3) *The* [*strong*] $(\aleph_0, n)$-*uniqueness property is equivalent to the* [*strong*] $(\aleph_0, n)$-*goodness property.*

(4) *If the* (λ, n)-*goodness property holds,* $\mathcal{S}$ *is a full weakly good* $(\lambda, \mathcal{P}(n))$-*system,* then *it is good* (*hence fully good*).

(5) *If the strong* (λ, n)-*goodness property holds, we can waive the fullness.*

(6) *If the* $(\lambda, 1)$-*existence property holds, then the* (λ, n)-*existence property is equivalent to*: *for every fully good* $(\lambda, \mathcal{P}^{-}(n))$-*system* $\mathcal{S}$ *there is* M, $A_n^{\mathcal{S}} \subseteq M$.

Proof. (1) Let $\mathcal{S}$ be an $(\aleph_0, \mathcal{P}^{-}(n))$-system, then $A_n^{\mathcal{S}}$ is a countable atomic set, hence for some countable M, $A_n^{\mathcal{S}} \subseteq M$.

(2) See Theorem 1.4 and Lemma 2.2.

(3) We have already proven "good implies unique" (in Lemma 2.2). For the other direction, uniqueness implies, by (1), the existence of a universal model over the relevant set (which is countable), hence the set is good by Claims 2.3 and 2.4.

(4) Check the definition.

(5) Obvious.

(6) Easy.

Sh:87b

THEOREM 4.3. (1) *Suppose*

(a) I_1 *is an initial segment of* I_2, $n(I_2) \leqq n^*$.

(b) $\mathcal{S}^1 = \{M_s : s \in I_1\}$ *is a fully good* (λ, I_1)*-system.*

(c) K *satisfies the* $(\lambda, \leqq n^*)$*-existence and* $(\lambda, \leqq n^*)$*-goodness properties.*

Then *we can complete* $\mathcal{S}^1$ *to a fully good* (λ, I_2)*-system.*

(2) *If, in addition, the* $(\lambda, \leqq n^*)$*-uniqueness property holds, then the completion above is unique up to isomorphism over* $\mathcal{S}^1$.

(3) *In part* (1), *in* (b) *and in the conclusion, we can omit the "fully" if in* (c) *we demand the strong properties.*

PROOF. (1) Let $\bar{s}_l = \{s(i) : i < k_l\}$ be an enumeration of I_l for $l = 1, 2$ (possible as I_1 is an initial segment of I_2). We now define $M_{s(i)}$ for $k_1 \leqq i < k_2$ by induction on i. By Lemma 3.5 it suffices to prove $\{M_{s(i)} : i < k_2\}$ is good for the enumeration $\bar{s}_2$, so by Definition 3.3(2) we have to define $M_{s(i)}$ so that conditions (i), (ii), (iii) there hold and $M_{s(i)}$ is weakly full over $A_{s(i)}$, assuming $\{M_{s(j)} : j < i\}$ is a fully good system, $\|M_{s(j)}\| = \lambda$. By Notation 3.1(2)(ii) there is an appropriate $g : \mathcal{P}^-(n) \to I_2$, $n = |s|_{I_2}$. By Lemma 3.5, $\{M_{g(s)} : s \in \mathcal{P}^-(n)\}$ is a fully good $(\lambda, \mathcal{P}^-(n))$-system; so by the (λ, n)-existence property, there is M, full over $\bigcup \{M_{g(s)} : s \in \mathcal{P}^-(n)\} = A_{s(i)}$, $\|M\| = \lambda$, and by the (λ, n)-goodness condition $A_{s(i)}$ is good (i.e., condition (i)). If we let $M_{s(i)} = M$ the fullness condition holds. Now by Claim 3.6, $(A_{s(i)}, \bigcup_{j<i} M_{s(j)})$ satisfies the Tarski–Vaught condition, hence by Lemma 2.10(2) we can find an elementary mapping f, $f \restriction A_{s(i)} = \mathrm{id}$, and $A_{s(i)}$, $f(M_{s(i)})$, $\bigcup_{j<i} M_{s(j)}$ is in stable amalgamation (i.e., condition (iii)), so $f(M)$ is the desired $M_{s(i)}$ by the proof of Claim 3.7.

(2),(3) Easy.

CONCLUSION 4.4. Suppose the $(\lambda, < n)$-existence and $(\lambda, < n)$-goodness properties hold. Then there is a fully good $(\lambda, \mathcal{P}^-(n))$-system. It is unique (up to isomorphism) if the $(\lambda, < n)$-uniqueness property holds.

THEOREM 4.5. (1) *If A is an atomic set, we can find countably many functions from A to A, such that, if $B \subseteq A$ is closed under those functions, then*

(a) (B, A) *satisfies the Tarski–Vaught condition,*

(b) A *is good iff* B *is good.*

(2) *If A, B, C is in stable amalgamation we can find countably many functions on $A \cup B \cup C$ such that, if $A^* \subseteq A \cup B \cup C$ is closed under them,* then $A \cap A^*$, $B \cap A^*$, C *is in stable amalgamation, and so is* A', $B \cap A^*$, C' *if* $A \cap A^* \subseteq A' \subseteq A$, $C' \subseteq C$ *and* A' *is closed under the functions.*

(3) *Suppose I is finite, $\mathcal{S}$ is a (λ, I)-system. Let κ be a regular cardinal big enough so that $\mathcal{S} \in H(\kappa)$ (the family of sets of hereditary power $< \kappa$),* $\aleph_0^\alpha <$

$(H(\kappa),\in)$ $(\alpha<\delta_0)$ *increasing, continuous,* $\mathcal{S}\in N_0^\alpha$, $N_0^\alpha\in N_0^{\alpha+1}$, $\|N_0^{\alpha+1}\|=\|N_0^\alpha\|\leqq\mu$, $N_0=\bigcup_{\alpha<\delta_0}N_0^\alpha$, $\|N_0\|=\mu$, $M_{\langle s,0\rangle}=M_s\cap N_0$ *for* $s\in I$ $(\delta_0\leqq\mu<\lambda)$. *Then*

(a) $\mathcal{S}^0=\{M_{\langle s,0\rangle}:\langle s,0\rangle\in I\times\{0\}\}$ *is a* (μ,I)*-system*;

(b) *if* $\mathcal{S}$ *is good*, then $\mathcal{S}^0$ *is good*;

(c) *if* $\mathcal{S}$ *is full*, $\operatorname{cf}\delta_0=\mu$, then $\mathcal{S}^0$ *is full*;

(d) *if* δ_0 *is divisible by* μ, *and the* $(\aleph_0,\leqq n(I)+1)$*-goodness property holds and* $\mathcal{S}$ *is full*, then $\mathcal{S}^0$ *is full.*

(4) *Suppose* I, $\mathcal{S}$, N_0^α $(\alpha<\delta_0)$, N_0, $M_{\langle s,0\rangle}$ *are as in* (3), *and* N_1^α $(\alpha<\delta_1)$, N_1, $M_{\langle s,1\rangle}$ *satisfy similar conditions, and* $N_1^0=N_0$. *Let* $I^*=I\times\{0,1\}$ (*see Notation* 3.1) *and* $M_{\langle s,l\rangle}=M_s\cap N_l$ *for* $s\in I$, $\mathcal{S}^*=\{M_{\langle s,l\rangle}:\langle s,l\rangle\in I^*\}$. *Then*

(a) $\mathcal{S}^*$ *is a* (μ,I^*)*-system*;

(b) *if for all* $t\in I$, *the* $(\aleph_0,n(t)+1)$*-goodness property holds and* $\mathcal{S}$ *is fully good*, then $\mathcal{S}^*$ *is fully good.*

(5) *Suppose* I *is finite*, $\mathcal{S}$ *a* (λ,I)*-system*, $N_0\in N_1$, $N_0<N_1<(H(\kappa),\in)$, $\|N_l\|=\mu$, $M_{\langle s,l\rangle}=M_s\cap N_l$ *for* $s\in I$, $l\in\{0,1\}$, *and* $\mathcal{S}^*=\{M_{\langle s,l\rangle}:\langle s,l\rangle\in I\times 2\}$.

(a) *If the* $(\aleph_0,n(I)+1)$*-goodness* [*the strong* $(\aleph_0,n(I)+1)$*-goodness*] *property holds, and* $\mathcal{S}$ *is fully good* [*is good*], then $\mathcal{S}^*$ *is good.*

(b) *If in* (a) *we replace* $n(I)+1$ *by* $n(I)$, *and* $|\{t\in I:n(t)=n(I)\}|=1$ *then we get "*$\mathcal{S}^*$ *is weakly good".*

PROOF. (1) We define for every n and formula $\varphi=\varphi(\bar x,\bar y)$, $l(\bar y)=n$ some n-place functions F_l^φ $(l<\omega)$ such that:

(a) if $\bar a=\langle a_0,\cdots,a_{n-1}\rangle\in A$, $\models(\exists\bar x)\varphi(\bar x,\bar a)$, and there are ψ and $\bar b$ such that $\models(\forall\bar x)[\psi(\bar x,\bar b)\to\varphi(\bar x,\bar a)]$, $\models(\exists\bar x)\psi(\bar x,\bar b)$, and $\psi(\bar x,\bar b)$ isolates a complete type over A, then for some ψ this holds for $\bar b=\langle F_0^\varphi(\bar a),F_2^\varphi(\bar a),F_4^\varphi(\bar a),\cdots\rangle$;

(b) if for some $\bar b\in A$, $\models\varphi[\bar b,\bar a]$, then $\bar b=\langle F_1^\varphi(\bar a),F_3^\varphi(\bar a),F_5^\varphi(\bar a),\cdots\rangle$ satisfies this too.

It is trivial to define the F_l^φ such that they will be sufficient. If A is not good, trivially also B is not good.[†]

(2) Similar proof (using the characterization of D_A in Lemma 2.2(1)).

(3),(4),(5) We shall prove (3), (4) and (5) simultaneously by induction on $|I|$, and fixing I we prove them in the following order: 3(a), 3(b), 3(c), 4(b), 5(a), 5(b), 3(d), and 4(a) (e.g., the proof of 3(d) depends on the fact that (5) holds). However, the proofs will be written in the usual order.

[†] Provided that:

(c) if $\exists\bar x\varphi(\bar x,\bar a)$, $\bar a\in A$, $\{\varphi(\bar x,\bar a)\}$ has no isolated extensions over A, then for some such φ, $\bar a$, for every $\bar b$ and ψ, $\bar a\subseteq\{F_l^\psi(\bar b):l<\omega\}$.

Sh:87b

(3)(a) Easy.

(3)(b) Easy by Theorem 4.5(1)(b), 4.5(2), and a similar argument for (ii) of Definition 3.3(2).

(3)(c) Easy by Lemma 2.12(2)(ii).

(3)(d) Let $\alpha < \delta_0$, and $\mathcal{S}^{0,\alpha} = \{M^{0,\alpha}_{\langle s,l\rangle} : \langle s,l\rangle \in I \times 2\}$ be defined by $M^{0,\alpha}_{\langle s,0\rangle} = M_s \cap N^\alpha_0$, $M^{0,\alpha}_{\langle s,1\rangle} = M_s \cap N^{\alpha+1}_0$. So by (5) it is a good system, and clearly it is $\aleph_0$-full. So for each $s \in I$, for every α, $A^{\mathcal{S}}_s \cap N^\alpha_0$, $M_s \cap N^\alpha_0$, $A^{\mathcal{S}}_s \cap N^{\alpha+1}_0$ is in stable amalgamation and $(M_s \cap N^\alpha_0) \cup (A^{\mathcal{S}}_s \cap N^{\alpha+1}_0)$ is a good set and (as $N^\alpha_0 \in N^{\alpha+1}_0$, $N^\alpha_0 < N^{\alpha+1}_0 < (H(\kappa), \in)$) for every finite $B \subseteq M_s \cap N^{\alpha+1}_0$, and stationary $p \in S^m(B)$, the stationarization of p over $B \cup (M_s \cap N^\alpha_0) \cup (A^{\mathcal{S}}_s \cap N^{\alpha+1}_0)$ is realized in $M_s \cap N^{\alpha+1}_0$ (remember M_s was λ-full over $A^{\mathcal{S}}_s$, $\lambda > \mu$). So by Lemma 2.12(2)(iii) we finish.

(4)(a) Easy.

(4)(b) First combine (3)(d) and (5)(a).

(5)(a) We prove it using Claim 3.7. In our notation we refer to the system $\mathcal{S}^*$ unless specified otherwise.

We first check condition (α), i.e., that $A_{\langle s,l\rangle}$ is good. For $l = 0$ it follows by (3)(b). For $l = 1$, by the induction hypothesis for $n(I)$ applied to (5)(a), $\{M_{\langle t,1\rangle} : t < s, t \in I\}$ is a good system, $\aleph_0$-full if $\mathcal{S}$ was $\aleph_0$-full. What occurs when we add to it $M_{\langle t,0\rangle}$? By the choice of N_0, N_1 and as $\mathcal{S}$ was a good system, $\bigcup_{t<s} M_{\langle t,0\rangle}$, $M_{\langle s,0\rangle}$, $\bigcup_{t<s} M_{\langle t,1\rangle}$ is in stable amalgamation, and if $\mathcal{S}$ is $\aleph_0$-full then $M_{\langle s,0\rangle}$ is $\aleph_0$-full over $\bigcup_{t<s} M_{\langle t,0\rangle}$. As $\mathcal{S}$ was a good system by Theorem 4.5(1), $\bigcup_{t<s} M_{\langle t,0\rangle}$ is a good set. So by Claim 3.7 easily $\{M_{\langle t,m\rangle} : \langle t,m\rangle < \langle s,1\rangle\}$ is a good system, and it is $\aleph_0$-full if $\mathcal{S}$ is λ-full.

If the strong $(\aleph_0, n(s)+1)$-goodness property holds, then by Theorem 4.5(1), (2), (3) clearly the strong $(\mu, n(s)+1)$-goodness property holds, hence $\bigcup \{M_{\langle s,m\rangle}; \langle t,m\rangle < \langle s,1\rangle\}$ is a good set, a required conclusion.

If the strong $(\aleph_0, n(s)+1)$-goodness property fails, by the hypothesis of (5)(a), the $(\aleph_0, n(s)+1)$-goodness property holds and $\{M_{\langle t,m\rangle} : \langle t,m\rangle < \langle s,1\rangle\}$ is an $\aleph_0$-fully good (I,μ)-system. If $\bigcup \{M_{\langle s,m\rangle} : \langle t,m\rangle < \langle s,1\rangle\}$ is not good, then by Theorem 4.5(1), (2) we can contradict the $(\aleph_0, n(s)+1)$-goodness property.† So we finish proving (α) from Claim 3.7.

Now we shall prove (β) from the fact that Claim 3.7 holds, thus finishing the proof of Theorem 4.5(5)(a). So let $J \subseteq I \times 2$ be an initial segment.

Case a: There is $\langle s,0\rangle \in J$ such that $\langle s,1\rangle \notin J$

Choose $t_* \in I$ such that $s \leqq t_*$, $\langle t_*,0\rangle \in J$ but $t_* < t \in I \Rightarrow \langle t,0\rangle \notin J$;

† We use the uniqueness of such an $(\aleph_0, \mathcal{P}^-(n(s)+1))$-system.

this is possible as I, J are finite. Now $\langle t_*,1\rangle \notin J$ (as $\langle s,1\rangle \leqq \langle t_*,1\rangle, \langle s,1\rangle \notin J$). We shall show that $\langle t_*,0\rangle \in J$ can serve as the required element of J in Claim 3.7(β); that is, for every $\bar b \in M_{\langle t_*,0\rangle}$ for some finite $B \subseteq A_{\langle t_*,0\rangle}$, $\mathrm{tp}(\bar b, \bigcup\{M_{\langle s,l\rangle} : \langle s,l\rangle \in J - \{\langle t_*,0\rangle\}\})$ does not split over B.

Let $J(0) = \{s : \langle s,0\rangle \in J\} = \{s : (\exists l)\langle s,l\rangle \in J\}$. Let $\bar b \in M_{\langle t_*,0\rangle}$, then $\bar b \in M^{\mathscr{S}}_{t_*}$, so there is a finite $B \subseteq A^{\mathscr{S}}_{t_*}$ such that $\mathrm{tp}(\bar b, \bigcup_{s\in J(0), s\neq t_*} M^{\mathscr{S}}_s)$ does not split over B (B exists as $\mathscr{S}$ is a good system). As $\mathscr{S}$, $\bar b \in N_0$, we can choose $B \subseteq N_0$. So $B \subseteq A^{\mathscr{S}}_{t_*} \cap N_0 = A^{\mathscr{S}^*}_{\langle t_*,0\rangle}$ and

$$\mathrm{tp}(\bar b, \bigcup\{M_{\langle s,l\rangle} : \langle s,l\rangle \in J - \{\langle t,1\rangle\}\}) \subseteq \mathrm{tp}(\bar b, \bigcup\{M_s : s \in J(0) - \{t_*\}\}),$$

hence it does not split over B.

Case b: Not case a.

Now let $I(0) = \{s : \langle s,0\rangle \in J\}$, so if case a fails $J = I(0)\times 2$, and let $I(1) = \{s \in I(0)$: for no $t \in I(0)$, $s < t\}$, so as I is finite, $I(1)$ is not empty. Choose $t_0 \in I(1)$ and let $I(2) = I(0) - \{t_0\}$, $J(1) = J - \{\langle t_0,1\rangle\}$; so by the induction hypothesis (on $|I|$), $\{M_{\langle s,l\rangle} : \langle s,l\rangle \in I(2)\times 2\}$ is a good system.

Let $\bar b \in M_{\langle t_0,1\rangle}$, and we shall prove that

(*) for some finite $B \subseteq A_{\langle t_0,1\rangle}$, $\mathrm{tp}(\bar b, \bigcup_{\langle t,l\rangle \in J(1)} M_{\langle t,l\rangle})$ does not split over B.

As we have proved condition (α) of Claim 3.7, $A_{\langle t_0,1\rangle}$ is a good set, hence we can find a finite $B \subseteq A_{\langle t_0,1\rangle}$ such that $\mathrm{tp}(\bar b, A_{\langle t_0,1\rangle})$ does not split over B.

Now we have to prove that $\mathrm{tp}(\bar b, \bigcup\{M_{\langle s,l\rangle} : \langle s,l\rangle \in J - \{\langle t_0,1\rangle\}\})$ does not split over the finite $B \subseteq A_{\langle t_0,1\rangle}$. *So let* $\bar c_0 \in M_{\langle t_0,0\rangle}$, $\bar c_1 \in M_{\langle t_0,0\rangle}$, $\bar d_0$, $\bar d_1 \in \bigcup\{M_{\langle s,l\rangle} : s \in I(0) - \{t_0\}, l = 0,1\}$, $\mathrm{tp}(\bar c_0 {}^\frown \bar d_0, B) = \mathrm{tp}(\bar c_1 {}^\frown \bar d_1, B)$, and we should prove $\mathrm{tp}(\bar c_0 {}^\frown \bar d_0, B \cup \bar b) = \mathrm{tp}(\bar c_1 {}^\frown \bar d_1, B \cup \bar b)$, and this suffices. (Note that any sequence $\bar a \in \bigcup\{M_{\langle s,l\rangle} : \langle s,l\rangle \in J - \{\langle t_0,1\rangle\}\}$ can be represented by $\bar c_0 {}^\frown \bar d_0$, as mentioned here, and that not necessarily even $l(\bar c_0) = l(\bar c_1)$.)[†]

Choose $\bar b^* \in M_{\langle t_0,1\rangle}$, $B \cup \bar b {}^\frown \bar c_0 {}^\frown \bar c_1 \subseteq \bar b^*$, such that $\mathrm{tp}(\bar b^*, A^{\mathscr{S}}_{t_0})$ does not split over $\bar b^* \cap A^{\mathscr{S}}_{t_0} = \bar b^* \cap A_{\langle t_0,1\rangle}$ (possible as $\bar b {}^\frown \bar c_0 {}^\frown \bar c_1 \in N_1 < (H(\lambda), \in)$, $\mathscr{S}$ a good system). Clearly we can find $\bar d'_0, \bar d'_1 \in \bigcup_{s<t_0} M_s = A^{\mathscr{S}}_{t_0}$ such that

$$\mathrm{tp}(\bar d_0 {}^\frown \bar d_1, \bar b^* \cap A^{\mathscr{S}}_{t_0}) = \mathrm{tp}(\bar d'_0 {}^\frown \bar d'_1, \bar b^* \cap A^{\mathscr{S}}_{t_0})$$

(because $\mathscr{S}$ is a good system, hence $(A^{\mathscr{S}}_{t_0}, \bigcup\{M_s : s \in I(0), s \neq t_0\})$ satisfies the

[†] Really there is no need that two sequences from $M_{\langle t_0,0\rangle} \cup \bigcup\{M_{\langle s,l\rangle} : s \in I(0) - \{t_0\}, l = 0,1\}$, which realize the same type over B, will have such representations. But the change does not affect the proof.

Sh:87b

Tarski–Vaught condition). Moreover, we can assume $\bar{d}'_0{}^\wedge \bar{d}'_1 \in N_1$, because $\bar{b}^* \in N_1 < (H(\lambda), \in)$, and $M_s \in N_1$ for every s. So $\bar{d}'_1{}^\wedge \bar{d}'_1 \in \bigcup_{s<t_0} M_{\langle s,1\rangle}$.

By the choice of $\bar{b}^*$ (and as $\mathcal{S}$ is a good system), $\mathrm{tp}(\bar{b}^*, \bigcup\{M_s : s \in I(0), s \neq t_0\})$ does not split over $\bar{b}^* \cap A^{\mathcal{S}}_{t_0}$, hence we have:

$$(*)_1 \qquad \mathrm{tp}(\bar{d}_0{}^\wedge \bar{d}_1{}^\wedge \bar{b}^*, \emptyset) = \mathrm{tp}(d'_0{}^\wedge \bar{d}'_1{}^\wedge \bar{b}^*, \emptyset).$$

By the choice of $\bar{b}^*$ and by $(*)_1$ it suffices to prove

$$(*)_2 \qquad \mathrm{tp}(\bar{c}_0{}^\wedge \bar{d}'_0, B \cup \bar{b}) = \mathrm{tp}(\bar{c}_1{}^\wedge \bar{d}'_1, B \cup \bar{b}).$$

Similarly, because $\mathrm{tp}(\bar{c}_0{}^\wedge \bar{d}_0, B) = \mathrm{tp}(\bar{c}_1{}^\wedge \bar{d}_1, B)$ and $B \subseteq \bar{b}^*$, the following holds:

$$(*)_3 \qquad \mathrm{tp}(\bar{c}_0{}^\wedge \bar{d}'_0, B) = \mathrm{tp}(\bar{c}_1{}^\wedge \bar{d}'_1, B).$$

But $\bar{c}_0$, $\bar{c}_1 \subseteq M_{\langle t_0,0\rangle}$, $\bar{d}'_0$, $\bar{d}'_1 \subseteq A^{\mathcal{S}}_{t_0} \cap N_1$. So clearly

$$(*)_4 \qquad \bar{c}_0, \bar{c}_1, \bar{d}'_0, \bar{d}'_1 \subseteq M_{\langle t_0,0\rangle} \cup (A^{\mathcal{S}}_{t_0} \cap N_1) = A_{\langle t_0,1\rangle}.$$

By $(*)_2$, $(*)_3$, $(*)_4$, and the definition of splitting, it suffices to prove that $\mathrm{tp}(\bar{b}, A_{\langle t_0,1\rangle})$ does not split over B. But this is how B was chosen. So we finish proving (β), hence (5)(a).

(5)(b) A similar proof to (5)(a).

Conclusion 4.6. Suppose $\mathcal{S} = \{M_s : s \in I\}$ is a [fully] good (λ, I)-system. If the $(<\lambda, n(I)+1)$-goodness property holds, we can define M^α_s $(\alpha < \lambda)$ such that

(a) M^α_s is increasing, continuous, $\| M^\alpha_s \| = \aleph_0 + |\alpha|$, $M_s = \bigcup_{\alpha<\lambda} M^\alpha_s$;

(b) $\mathcal{S}^\alpha = \{M^\alpha_s : s \in I\}$ is a [fully] good $(\aleph_0 + |\alpha|, I)$-system;

(c) for $\alpha < \beta$, $\mathcal{S}^{\alpha,\beta}$ is a [fully] good $(\aleph_0 + |\alpha|, I \times \{\alpha, \beta\})$-system, where $M^{\mathcal{S}^{\alpha,\beta}}_{\langle s,\gamma\rangle} = M^\gamma_s$ $(\gamma = \alpha, \beta)$.

Conclusion 4.7. The $(\aleph_0, n)$-goodness property implies the (λ, n)-goodness property for every λ.

We can replace $\aleph_0$ by any regular $\mu \leqq \lambda$.[†]

Proof. By Theorem 4.5(3)(a), (b), (c), let $\mu = \aleph_0$; if (λ, n)-goodness fails then, by (1)(b), $(\aleph_0, n)$-goodness fails.

§5. Transferring the positive properties among cardinals

The main aim of this section is to prove that excellent K has models in all cardinals, and more generally to present the nice properties of such classes. As a

[†] Similarly for the strong properties.

bonus we get the categoricity theorem (Theorem 5.9) for them: such a class is categorical in all or no uncountable cardinal. The notion "excellent" is parallel in some sense to totally transcendental.

We define K to be excellent if it has the $(\aleph_0, n)$-goodness property for every n. By Conclusion 4.7 the (λ, n)-goodness property holds for every λ; now by Theorems 5.1 and 5.2 also the (λ, n)-existence and (λ, n)-uniqueness properties hold for every λ. Theorems 5.1 and 5.2 are important also for non-excellent classes. They give the positive properties which hold (and are complemented in Theorem 6.1). As for non-excellent classes we want to build many models in $\aleph_{n(K)}$ where $n(K) = \mathrm{Min}\{n$: the $(\aleph_0, n)$-goodness property fails$\}$; we need to be able to build at least one model.

Later we prove that excellent classes have also the strong positive properties, and in fact over any good system there is a primary model, and using it the categoricity theorem will be proved.

THEOREM 5.1. (1) *Suppose* $\lambda > \aleph_0$ *and the* $(< \lambda, \leqq n+1)$-*uniqueness and goodness properties hold. Then the* (λ, n)-*uniqueness property holds.*

(2) *A similar assertion holds for the strong properties.*

REMARK. In fact only the $(< \lambda, n+1)$-uniqueness and $(\aleph_0, n)$-goodness properties are necessary. A similar remark holds for Theorem 5.2.

PROOF. (1) Let $\mathscr{S} = \{M_s : s \in \mathscr{P}^-(n)\}$ be a fully good $(\lambda, \mathscr{P}^-(n))$-system, and assume for $l = 0, 1$ that M^l_n is full over A_n, $\| M^l_n \| = \lambda$. So we can complete $\mathscr{S}$ to a fully good $(\lambda, \mathscr{P}(n))$-system $\mathscr{S}_l$ by letting $M^{\mathscr{S}_l}_n = M^l_n$. Now we apply Conclusion 4.6 to $\mathscr{S}_l$ and get models $M^{l,\alpha}_s$. Checking the proof we see that w.l.o.g. for $s \in \mathscr{P}^-(n)$, $M^{0,\alpha}_s = M^{1,\alpha}_s = M^\alpha_s$. Now we define by induction on $\alpha < \lambda$ an isomorphism f_α from $M^{0,\alpha}_n$ onto $M^{1,\alpha}_n$, f_α increasing, $f_\alpha \restriction A^\alpha_n = \mathrm{id}$. For $\alpha = 0$ we use the $(< \lambda, n)$-uniqueness property, and for α limit we take the union. For $\alpha = \beta + 1$, let g_β be the extension of f_β by the identity mapping on A^α_n, g_β is elementary (as $\mathscr{S}^{\beta,\alpha}_l$ is a fully good $\mathscr{P}^-(n+1)$-system (by Conclusion 4.6), A^β_n, $M^{l,\alpha}_n$, A^α_n is in stable amalgamation, so use Lemma 2.10(1)). So clearly g_β is an isomorphism from $A^{\mathscr{S}^{\beta,\alpha}_0}_{n+1}$ onto $A^{\mathscr{S}^{\beta,\alpha}_1}_{n+1}$ so by the $(< \lambda, n+1)$-uniqueness property, we can extend g_β to an isomorphism f_α from $M^{0,\alpha}_n$ onto $M^{1,\alpha}_n$ as required. Now $f_\lambda = \bigcup_{\alpha<\lambda} f_\alpha$ is as required.

(2) A similar proof.

THEOREM 5.2. (1) *Suppose* $\lambda > \aleph_0$, *and the* $(< \lambda, \leqq n+1)$-*existence and the* $(< \lambda, \leqq n)$-*goodness properties hold. Then the* (λ, n)-*existence property holds.*

(2) *The same holds for the strong properties.*

Sh:87b

PROOF. We concentrate on (1). Let $\mathcal{S} = \{M_s : s \in \mathcal{P}^-(n)\}$ be a $(\lambda, \mathcal{P}^-(n))$-system, and let M_s^α ($\alpha < \lambda, s \in \mathcal{P}^-(n)$) be as in Conclusion 4.6 (for $I = \mathcal{P}^-(n)$, $n(I) = n - 1$), so the $(<\lambda, \leqq n(I)+1)$-goodness property holds.

Define, by induction on $\alpha < \lambda$, a model M_n^α such that:

(a) M_n^α is increasing and continuous (in α),

(b) $A_n^\alpha \subseteq M_n^\alpha$, M_n^α is full over A_n^α, M_n^α has power $\aleph_0 + |\alpha|$,

(c) A_n^α, M_n^α, A_n is in stable amalgamation,

(d) for each $\bar{c} \in M_n^\alpha$, and stationary $p \in D_{\bar{c}}$, the stationarization of p over $M_n^\alpha \cup A_n^{\alpha+1}$ is realized in $M_n^{\alpha+1}$.

For $\alpha = 0$, we use the $(<\lambda, n)$-existence property to define M_n^0, and Lemma 2.10(2) to take care of (c) too. For limit α, we take the union and by Lemmas 2.11(3) and 2.12 the conditions are satisfied.

For $\alpha = \beta + 1$ successor we know by Theorem 4.5(3) that $\mathcal{S}^{\beta,\alpha}$ is a fully good system. We also know M_n^β is full over A_n^β, and A_n^β, M_n^β, A_n^α is in stable amalgamation, remembering that by the $(<\lambda, n)$-goodness property, A_n^α is a good set. Let, for $s \in \mathcal{P}^-(n+1)$, M_s^* be: M_s^β if $s \in \mathcal{P}^-(n)$, M_n^β if $s = n$, and $M_{s-\{n\}}^\alpha$ if $n \in s \in \mathcal{P}^-(n+1)$. So $\{M_s^* : s \in \mathcal{P}^-(n+1)\}$ is a fully good system by Claim 3.7. Hence by using the $(<\lambda, n+1)$-existence property we can define M_n^α to satisfy (a), (b), and also (d) (for this remember we get a weakly full model over $M_n^\beta \cup A_n^\beta$) and by Lemma 2.10(2), to satisfy (c) too. M_n is $\bigcup_{\alpha<\lambda} M_n^\alpha$. M_n is weakly full over A_n by Lemma 2.12(2)(iii). Although formally (d) above is somewhat weaker than what we need there (there, the stationarization of p over $M_n^\alpha \cup A_n$ is realized in $M_n^{\alpha+1}$), by Lemma 2.10(4) this follows.

LEMMA 5.3. *Suppose $n > 1$, the (λ, n)-non-uniqueness property fails, but the (λ, n)-existence, $(\lambda, 1)$-existence and $(\lambda, 1)$-uniqueness properties hold. Then for some fully good $(\lambda, \mathcal{P}^-(n))$-system $\mathcal{S}$, over $A_n^{\mathcal{S}}$ there is a universal model (of power λ).*

PROOF. Let $\mathcal{S} = \{M_s : s \in \mathcal{P}^-(n)\}$ be a fully good $(\lambda, \mathcal{P}^-(n))$-system which exemplifies that the (λ, n)-non-uniqueness property fails. By the (λ, n)-existence property there is at least one N weakly full over A_n, $\|N\| = \lambda$. Clearly N is full over $\emptyset$, hence by the $(\lambda, 1)$-existence property there is N^* weakly full over N, $\|N^*\| = \lambda$. We now show that N^* is universal over A_n.

So let $A_n \subseteq M$, $\|M\| = \lambda$, then (by the choice of $\mathcal{S}$) there are M^* weakly full over A_n, $N < M^*$ and an elementary embedding f_0, of N^* over A_n into M^*. We can w.l.o.g. assume $\|M^*\| = \lambda$. Clearly M^* is full over $\emptyset$, hence by the $(\lambda, 1)$-existence property there is M^{**} weakly full over M^*, $\|M^{**}\| = \lambda$, and so M^{**} is full over N. So by the $(\lambda, 1)$-uniqueness property there is an isomorphism

g from M^{**} onto N^* over N. So $gf_0 : M \to N^*$ exemplifies M can be elementarily embedded into N^* over A_n, so N^* is universal over A_n.

DEFINITION 5.4. A class of models K is excellent if for every n it has the $(\aleph_0, n)$-goodness, $(\aleph_0, n)$-existence and $(\aleph_0, n)$-uniqueness properties.

We note

CLAIM 5.5. *Let K be our class from Theorem* 1.1.

(1) *K is excellent iff for each n, it has the $(\aleph_0, n)$-goodness property.*

(2) *If K is not excellent, then, for some $n = n(K) \geqq 2$, K has the $(\aleph_0, < n)$-goodness, $(\aleph_0, \leqq n)$-existence and $(\aleph_0, < n)$-uniqueness and $(\aleph_0, n)$-non-uniqueness properties.*

PROOF. (1) By Lemma 4.2(1) K has the $(\aleph_0, n)$-existence properties for each n. As for $(\aleph_0, n)$-uniqueness, use Lemma 4.2(3).

(2) Let $n = n(K)$ be the first n such that the $(\aleph_0, n)$-goodness property fails.

By Claim 5.5(1), $n(K)$ exists (i.e. $< \omega$), and by Lemma 4.2(2), $n \geqq 2$. By the choice of n, the $(\aleph_0, < n)$-goodness property holds, so by Lemma 4.2(1), (3) the $(\aleph_0, \leqq n)$-existence property and the $(\aleph_0, < n)$-uniqueness property hold, but the $(\aleph_0, n)$-uniqueness property fails.

However, we need something stronger: that the $(\aleph_0, n)$-non-uniqueness property holds, which is the main point. As the $(\aleph_0, < n)$-uniqueness property holds, by 4.3(1) and (2) there is, up to isomorphism, a unique $(\aleph_0, \mathcal{P}^-(n))$-system $\mathcal{S} = \{M_s : s \in \mathcal{P}^-(n)\}$ which is fully good. So if the $(\aleph_0, n)$-non-uniqueness property fails, it fails for $\mathcal{S}$, hence by Lemma 5.3, over $A_n^{\mathcal{S}}$ there is a universal model, hence by Claims 2.3 and 2.4, $A_n^{\mathcal{S}}$ is good; so by the uniqueness of $\mathcal{S}$ the $(\aleph_0, n)$-goodness property holds, which contradicts the choice of n.

REMARK. The following is an open problem: Does there exist a sentence $\psi \in L_{\omega_1,\omega}$ which satisfies our assumptions, i.e., $I(\aleph_1, \psi) < 2^{\aleph_1}$, and its class K (from Theorem 1.1) is not excellent? We believe the answer is yes.

THEOREM 5.6. *If K is excellent, then*

(1) *it has the (λ, n)-goodness, (λ, n)-existence and (λ, n)-uniqueness properties for each n,*

(2) *K has one and only one (up to isomorphism) full model in λ, for each λ.*

PROOF. (1) By induction on λ. For $\lambda = \aleph_0$ this is the hypothesis. For $\lambda > \aleph_0$, use Conclusion 4.7 for goodness, Theorem 5.2 for existence and Theorem 5.1 for uniqueness.

(2) By the $(\lambda, 0)$-existence and $(\lambda, 0)$-uniqueness properties.

LEMMA 5.7. *Suppose the* $(\aleph_0, \leqq n)$*-goodness property holds. Then the strong* $(\aleph_0, \leqq n)$*-goodness, strong* $(\aleph_0, \leqq n)$*-existence and strong* $(\aleph_0, \leqq n)$*-uniqueness properties hold.*

PROOF. We prove by induction on $m \leqq n$ that the strong $(\aleph_0, m)$ properties hold. For $m = 0, 1$ there is no problem; so suppose $m \geqq 2$ and we have proved for each $m' < m$. By Lemma 4.2(1) the strong $(\aleph_0, m)$-existence property holds and the strong $(\aleph_0, m)$-uniqueness property follows from the strong $(\aleph_0, m)$-goodness property (Lemma 4.2(3)) so we have to prove only the last.

So let $\mathcal{S} = \{M_s : s \in \mathcal{P}^-(m)\}$ be a $(\aleph_0, \mathcal{P}^-(m))$-good system (not necessarily full, of course). Let $k(\mathcal{S})$ be the number of $l < m$ such that for some s, $l \in s$ and M_s is not full over A_s. We prove A_m is good by induction on $k(\mathcal{S})$. If $k(\mathcal{S})$ is zero, $\mathcal{S}$ is a fully good $(\aleph_0, \mathcal{P}^-(m))$-system, so our hypothesis implies this. So assume we have proved the statement for $k < k(\mathcal{S})$. As $k(\mathcal{S}) > 0$, there are l_0, s, $l_0 \in s$, and M_s is not full over A_s; w.l.o.g. $l_0 = m - 1$. Now we define I as $\mathcal{P}^-(m) \cup \{t \cup \{m\} : m - 1 \in t \in \mathcal{P}^-(m)\}$ (it is easy to check I is as required). Let $\langle s(i) : i < 2^m - 1\rangle$ be an enumeration of $\mathcal{P}^-(m)$, and $\langle s(i) : i < i^*\rangle$, $i^* = 2^m - 1 + (2^{m-1} - 1)$, be an enumeration of I. Now we define countable $M_{s(i)}$ so that $\{M_{s(j)} : j < i^*\}$ is a good system. For $i < 2^m - 1$ they are defined. Suppose we have defined for $j < i$, and we want firstly to define $g = g_{s(i)}$. Let $s(i) = \{i_0, \cdots, i_{\alpha-1}, m - 1, m\}$ (so $|s(i)| = \alpha + 2 \leqq m$), and $|s(i)|_I = \alpha + 1$, and for $t \in \mathcal{P}^-(\alpha + 1)$, $g(t) = \{i_\beta : \beta \in t, \beta < \alpha\} \cup \{m : \alpha \in t\} \cup \{m - 1\}$.[†]

Now $\{M_{g(t)} : t \in \mathcal{P}^-(\alpha + 1)\}$ is a good $(\aleph_0, \mathcal{P}^-(\alpha + 1))$-system, and $\alpha + 1 < \alpha + 2 = |s(i)| < m + 1$ so $\alpha + 1 < m$, therefore by the induction hypothesis there is $M_{s(i)}$, full over $A_{s(i)} = \bigcup \{M_{g(t)} : t \in \mathcal{P}^-(\alpha + 1)\}$, such that $A_{s(i)}$, $M_{s(i)}$, $\bigcup_{j<i} M_{s(j)}$ is in stable amalgamation.

Let $J = \{s \in I : m - 1 \in s \Rightarrow m \in s\}$; it is easy to check $\mathcal{S}' = \{M_s : s \in J\}$ is also a good $(\aleph_0, J)$-system, J is isomorphic to $\mathcal{P}^-(m)$, and $k(\mathcal{S}') < k(\mathcal{S})$. So $\bigcup_{s\in J} M_s$ is a good set but trivially it is equal to $\bigcup_{s\in I} M_s$. It is also clear that for any N, $A_n \subseteq N$, we can embed $\bigcup_{s\in I} M_s$ into N' over A_n for some N', $N < N'$. [By 3.5 $(A^{\mathcal{S}}_m, \bigcup_{s\in I} M_s)$ satisfies the Tarski–Vaught condition, so apply 2.10(2), so w.l.o.g. $N' \cup \bigcup_{s\in I} M_s$ is an atomic set; as it is countable, we finish.] Hence every $p \in D_{A^{\mathcal{S}}_n}$ has the form $q \restriction A^{\mathcal{S}}_n$, $q \in D_{\cup_{s\in I} M_s}$, so $D_{A^{\mathcal{S}}_n}$ is countable.

This implies A_n is good, as required.

CONCLUSION 5.8. Suppose K is excellent.

(1) The strong (λ, n)-goodness, strong (λ, n)-existence, and strong (λ, n)-uniqueness properties hold for every n, λ.

[†] So $m \in g(t)$ iff $\alpha \in t$.

(2) Over every good (λ, I)-system over $\bigcup_{s\in I} M_s$ there is a primary model (I finite).

(3) For every fully good (λ, I)-system $\{M_s : s \in I\}$ over $\bigcup_{s\in I} M_s$ there is an F^m-constructible model M which is full over it, $\| M \| = \lambda$.

PROOF. (1) Similar to the proof of Theorem 5.6(1) using Lemma 5.7.

(2) Prove by induction on λ.

(3) Prove by induction on λ.

THEOREM 5.9. *Suppose K is excellent. If K is categorical in at least one uncountable cardinal,* then *K is categorical in every uncountable cardinal.*

PROOF. Suppose $\lambda > \aleph_0$ is the first such that K is not categorical in λ. By Theorem 5.6(2) there is a model $M \in K$ of power λ which is not full. By the definition, there is a stationary $p \in S^m(B)$, $B \subseteq M$ finite, and $A \subseteq M$, $B \subseteq A$, $|A| < \lambda$, such that the stationarization of p over A (call it q) is not realized in M. Let $|A| \leqq \mu < \lambda$. Now we can define an increasing continuous sequence of $M_i < M$ $(i < \mu^+)$, $\| M_i \| = \mu$, $A \subseteq M_0$, $M_i \neq M_{i+1}$. For each i choose $a_i \in M_{i+1} - M_i$, $\bar{b}_i \in M_i$, $\mathrm{tp}(a_i, M_i)$ is the stationarization of $\mathrm{tp}(a_i, \bar{b}_i)$.

Without loss of generality there exist $\bar{b}_{i_0}$, a_{i_0} and $r = \mathrm{tp}(a_{i_0}, \bar{b}_{i_0})$ such that $S = \{i < \mu^+ : \mathrm{tp}(a_i, \bar{b}_i) = r\}$ is a stationary set.

[Let $S' = \{\delta < \mu^+ : \delta$ limit ordinal$\}$; this set is closed unbounded. Now we define a function $f : S' \to \mu^+$,

$$f(i) = \min\{j : \mathrm{tp}(a_i, M_i) \text{ is the stationarization of } \mathrm{tp}(a_i, M_j)\}.$$

By Theorem 1.4(1)(a) there exists a finite set $\bar{b}_i \subseteq M_i$ such that $\mathrm{tp}(a_i, M_i)$ is the stationarization of $\mathrm{tp}(a_i, \bar{b}_i)$; since $i \in S'$ there exists $j < i$ such that this set is included in M_j. Hence f is a regressive function so we apply Fodor's theorem and get $S \subseteq S'$, a stationary set of ordinals $< \mu^+$, a finite sequence $\bar{b}_{i_0} \in M_{i_0}$, and a type r over $\bar{b}_{i_0}$ such that $S = \{i : \bar{b}_i = \bar{b}_{i_0}$ and $\mathrm{tp}(a_i, \bar{b}_i) = r\}$. Hence $S = \{i : \bar{b}_i = \bar{b}$, $\mathrm{tp}(a_i, \bar{b}_i) = r\}$ is stationary.] And w.l.o.g. $\bar{b} \in M_0$, $\omega \subseteq S$. We can easily find countable $N_n < M_n$ $(n < \omega)$, $N_{n+1} \cap M_n = N_n$, $a_n \in N_{n+1}$, $\bar{b}_{i_0} \in N_0$, $B \subseteq N_0$ and every N_n does not realize the type $q^* = q \restriction (A \cap N_0)$.

Let $\chi > \aleph_0$ be any cardinal. Now we define by induction on $i < \chi$, N_i, a_i, such that: $\| N_i \| = |i| + \aleph_0$, $j < i \Rightarrow N_j < N_i$, $a_i \in N_{i+1}$ realizes the stationarization of r over N_i, and for $i \geqq \omega$, N_{i+1} is primary over $N_i \cup \{a_i\}$.

It is quite easy to prove that N_χ does not realize q^*. [By 1.4(4), $\bigcup_i N_i$ is constructible (see 1.5) over $N_\omega \cup \{a_i : \omega \leqq i < \chi\}$: just combine the constructions of the N_{i+1}. So for every $\bar{c} \in \bigcup_i N_i$ for some n, m and $i_n > \cdots > i_0 \geqq \omega$, $\mathrm{tp}(\bar{c}, N_m \cup \{a_{i_l} : l \leqq n\})$ is isolated hence for some $\bar{c}' \in N_\omega$,

$$\operatorname{tp}(\bar{c}^1\langle a_{i_0}, \cdots, a_{i_n}\rangle, N_m) = \operatorname{tp}(\bar{c}'^\wedge\langle a_{m+0}, \cdots, a_{m+n}\rangle, N_m),$$

hence $\bar{c}$ does not realize q.]

CLAIM 5.10. *If $N \cup \{a\}$ is atomic, then over it there is a primary model.*

PROOF. By induction on $\lambda = \|N\|$.

For $\lambda = \aleph_0$ we know the result (as $N \cup \{a\}$ is a good set by Lemma 2.2(2), and apply Lemma 2.2(0)).

For $\lambda > \aleph_0$, let N_i $(i < \lambda)$ be increasing continuous, $\|N_i\| = |i| + \aleph_0 < \lambda$, $N = \bigcup_{i<\lambda} N_i$, and $\operatorname{tp}(a, N)$ does not split over some finite $B \subseteq N_0$. Now we define by induction on $i < \lambda$ a model M_i, such that: M_i is increasing, continuous, $a \in M_0$, $M_i \cap N = N_i$ and N_i, N, M_i is in stable amalgamation, and M_{i+1} is primary over $M_i \cup N_{i+1}$. $\|M_i\| = |i| + \aleph_0$.

For $i = 0$ use the case $\lambda = \aleph_0$, and note that as M_0 is constructible over $N_0 \cup \{a\}$, for every $\bar{c} \in M_0$, $\operatorname{tp}(\bar{c}, N_0 \cup \{a\}) \vdash \operatorname{tp}(\bar{c}, N \cup \{a\})$ as the first is isolated. So by Lemma 2.10(2), N_0, N, M_0 is in stable amalgamation.

For i limit, $M_i = \bigcup_{j<i} M_j$, and by Lemma 2.11(3), N_i, N, M_i is in stable amalgamation.

For $i = j + 1$, M_i primary over $M_j \cup N_i$ exists by Conclusion 5.8(2), and as for the case $i = 0$, N_i, N, M_i is in stable amalgamation. It is easy to check $\bigcup_{i<\lambda} M_i$ is as required.

REMARK. We can also deal with I-systems which are not (λ, I)-systems, but there is no real need.

§6. Transferring the negative properties up

Up to now we have dealt almost exclusively with the "structure", "positive", "algebraic" part of the theory. Now we shall deal with the "non-structure" part parallel to ch. VIII in [14]; but here failure of amalgamation and related properties replace instability requirements. Hence this part has a combinational character. So here we assume K is not excellent, and want to prove mainly that K has many non-isomorphic models in some cardinality. Lemma 5.5 has done some of the work, saying that $2 \leqq n(k) < \omega$; K has the $(\aleph_0, < n(k))$-goodness, $(\aleph_0, \leqq n(k))$-existence, $(\aleph_0, < n(k))$-uniqueness and $(\aleph_0, n(k))$-non-uniqueness properties. As the $(\aleph_{n(k)-1}, 0)$-uniqueness property follows, we naturally expect to build many non-isomorphic models in $\aleph_{n(k)}$. For this we first (Theorem 6.1) prove that the $(\mu, n+1)$-non-uniqueness property implies the (μ^+, n)-non-uniqueness property (under reasonable assumptions) using the weak diamond (equivalently

$2^{\mu} < 2^{\mu^+}$). This fits nicely with Theorems 5.1 and 5.2, where the positive properties were transferred up. We may expect to arrive at $\aleph_{n(k)-1}$ and then imitate the proof of Theorem 1.3, but we are stuck at $\lambda =^{\text{def}} \aleph_{n(k)-2}$. In Lemma 6.2(1), (2) we sum up what we obtain: (λ, l)-uniqueness, (λ, m)-existence for $l = 0, 1$, $m = 0, 1, 2$, and $(\lambda, 2)$-non-uniqueness.

In Lemma 6.2(3) we try a more direct attack assuming the λ^+-amalgamation property fails for full models (this is stronger than "the $(\lambda^+, 2)$-existence property fails" as even amalgamations which are not stable are forbidden). Immediately, the proof of Theorem 1.3 gives quite many models, but we do not have the parallel of "for every N there is N', $N < N'$ (both countable) such that the set of types over N' realized is maximal". We circumvent this by a partition to two cases imitating an answer to the existence of such N': we ask whether (for a full model of power λ^+) for every $N < M$ there is M', extending M such that any two extensions of M' can be amalgamated over N. The proof is suitable for more general contexts, as discussed in [17].

By Lemma 6.2 we can assume the λ^+-amalgamation properties for full models, and in Lemma 6.3, using again the weak diamond, we use it to strengthen the $(\lambda, 2)$-non-uniqueness property.

Theorem 6.4 is the main work — we use our knowledge on models of power λ to build many non-isomorphic models of power λ^{++}, using the weak diamond on λ^+ as well as on λ^{++}. To help the reader the proof is preceded by a long motivation of the way the proof goes, and then we give a proof from a stronger set theoretic assumption: diamonds on λ^+ and λ^{++}.

Unfortunately, in Theorem 6.4 we do not prove $I(\lambda^{++}, K) = 2^{\lambda^{++}}$ just that there are many (e.g., $I(\lambda^{++}, K)^{\aleph_0} = 2^{\lambda^{++}}$) (and more). In Theorem 6.14 we prove $I(\lambda^{++}, K) = 2^{\lambda^{++}}$ if the ideal of non-small subsets of λ^+ is not λ^{++}-saturated. In both cases, a failure of our hypothesis seems a non-trivial set theoretic problem, involving large cardinals.

THEOREM 6.1. *Suppose $n \geqq 2$, μ regular, $2^{\mu} < 2^{\mu^+}$, and the $(\mu, \leqq n)$-goodness, $(\mu, \leqq n+1)$-existence, $(\mu, \leqq n)$-uniqueness and $(\mu, n+1)$-non-uniqueness properties hold.* Then *the (μ^+, n)-non-uniqueness property holds but the $(\mu^+, < n)$-goodness, $(\mu^+, \leqq n)$-existence and $(\mu^+, < n)$-uniqueness hold.*

PROOF. Some parts of the conclusion follow by Conclusion 4.7, and Theorems 5.1 and 5.2, so we have to deal with the (μ^+, n)-non-uniqueness property only.

We let $\lambda = \mu^+$, and we first almost repeat the proof of Theorem 5.2. Let $\mathcal{S} = \{M_s : s \in \mathcal{P}^-(n)\}$, M_s^{α} $(\alpha < \lambda)$ be as there, and we can assume $\| M_s^{\alpha} \| = \mu$ for each $\alpha < \lambda$, $s \in \mathcal{P}^-(n)$. W.l.o.g. $\bigcup_s |M_s^\alpha|$ is the set of odd ordinals $< \mu(1+\alpha)$.

We now define by induction on $\alpha \leqq \lambda$, for each $\eta \in {}^\alpha 2$, a model M_n^η such that:

I:

(a) M_n^η is increasing (i.e., for $\beta < l(\eta)$, $M_n^{\eta\restriction\beta} < M_n^\eta$) and continuous (i.e., for limit α, $M_n^\eta = \bigcup_{\beta<\alpha} M_n^{\eta\restriction\beta}$ when $l(\eta) = \alpha$), and $|M_n^\eta| = \mu(1 + l(\eta))$.

(b) M_n^η is full over $A_n^{l(\eta)}$.

(c) $A_n^{l(\eta)}$, M_n^η, A_n is in stable amalgamation.

(d) For each $\bar{c} \in M_n^{\eta\restriction\beta}$, $\alpha = \beta + 1$, $\eta \in {}^\alpha 2$, and stationary $p \in D_{\bar{c}}$, the stationarization of p over $M_n^{\eta\restriction\beta} \cup A_n^\alpha$, is realized in M_n^η.

II:

(a) For $\eta \in {}^\beta 2$ $(\beta < \lambda)$ there is no model M, $M_n^\eta \cup A_n^{\beta+1} \subseteq M$, and elementary mappings f_l from $M_n^{\eta\wedge\langle l\rangle}$ into M, $f_l \restriction (M_n^\eta \cup A_n^{\beta+1}) = \mathrm{id}$ (for $l = 0, 1$).

We define by induction on α. For $\alpha = 0$, α limit, the proof is as in Theorem 5.2. For $\alpha = \beta + 1$, $\eta \in {}^\beta 2$, again $\mathcal{S}^{\beta,\alpha}$ is a fully good system, and if we add M_n^η we get (after renaming) a fully good $(\mu, \mathcal{P}^-(n+1))$-system. Now we use the $(\mu, n+1)$-non-uniqueness property† to get, for $l = 0, 1$, $M_n^{\eta\wedge\langle l\rangle}$ weakly full over $M_n^\eta \cup A_n^\alpha$, such that there are no M weakly full over $M_n^\eta \cup A_n^\alpha$, and elementary embeddings of M_l into M over $M_n^\eta \cup A_n^\alpha$.

Now we show that over A_n there is no universal model. As all fully good $(\lambda, \mathcal{P}^-(n))$-systems are isomorphic (see Conclusion 4.4), by Lemma 5.3 this implies that the (λ, n)-non-uniqueness property holds.

Let N be universal over A_n (so $\|N\| = \lambda$). Then for each $\eta \in {}^\lambda 2$ there is an elementary embedding f_η of M_η into N over A_n. By [3] (see here Theorem 1.3, and [3] 6.1, 7.1) there are $\eta, \nu \in {}^\lambda 2$, $\alpha < \lambda$, $\eta \restriction \alpha = \nu \restriction \alpha$, $\eta(\alpha) = 0$, $\nu(\alpha) = 1$, and $\alpha = \mu\alpha$, $f_\eta \restriction M_n^{\eta\restriction\alpha} = f_\nu \restriction M_n^{\nu\restriction\alpha}$, so we get a contradiction to condition II(a).

Lemma 6.2. *Suppose K is not excellent, and $\lambda = \aleph_{n(K)-2}$* (see Claim 5.5(2)) *and $2^{\aleph_l}$ $(l \leqq n(K) - 2)$ is strictly increasing. Then:*

(1) *K satisfies the (λ, l)-existence and (λ, m)-uniqueness properties for $l = 0, 1, 2$, $m = 0, 1$ but satisfies the $(\lambda, 2)$-non-uniqueness property.*

(2) *K satisfies the $(\lambda^+, 0)$-uniqueness, $(\lambda^+, 0)$-existence and $(\lambda^+, 1)$-existence properties, hence has a full model in λ^{++}.*

(3) *K has the λ^+-amalgamation property for full models provided that $2^{\lambda^+} < 2^{\lambda^{++}}$, and $I(\lambda^{++}, K) < 2^{\lambda^{++}}$ (i.e., if M_l are full models of power λ^+, $M_0 < M_1, M_2$,* then *there is a full model M, $M_0 < M$ and elementary embeddings of M_1, M_2 into M over M_0).*

† Remembering that there is, up to isomorphism, a unique such system.

PROOF. (1) By Claim 5.5(2) and Theorem 6.1.

(2) Easy.

(3) Our proof splits into two cases.

Case I: There are full models $N < M$ in K of cardinality λ^+, such that for every $M' \in K$, *if* $\|M'\| = \lambda^+$ and $M < M'$ *then* there are models M_l ($l = 0, 1$), $M' < M_l \in K$, $\|M_l\| = \lambda^+$, such that M_0, M_1 cannot be amalgamated over N (i.e., there is no N', $N < N' \in K$ and elementary embeddings $f_l : M_l \to N'$, $f_l \restriction N = \mathrm{id}$).

In this case we define by induction on $\alpha < \lambda^{++}$ for each $\eta \in {}^{\alpha}2$ a model M_η such that:

(a) M_η has cardinality λ^+, $\beta < l(\eta) \Rightarrow M_{\eta\restriction\beta} < M_\eta$, and for limit $l(\eta)$, $M_\eta = \bigcup_{\beta<l(\eta)} M_{\eta\restriction\beta}$, and $M_{\langle\ \rangle} = M$ ($\langle\ \rangle$ is the unique sequence of length zero);

(b) for each η, $M_{\eta\hat{}\langle 0\rangle}$, $M_{\eta\hat{}\langle 1\rangle}$ cannot be amalgamated over N;

(c) $M_{\eta\hat{}\langle l\rangle}$ is full over M_η.

There is no problem in the definition.

Now for each $\eta \in {}^{\lambda^{++}}2$ let $M_\eta = \bigcup_{\alpha<\lambda^{++}} M_{\eta\restriction\alpha}$ so each such M_η is a full model of cardinality λ^{++}. If $\eta \neq \nu \in {}^{\lambda^{++}}2$, then M_η, M_ν are not isomorphic over N (if $f : M_\eta \to M_\nu$ is an elementary embedding, $f \restriction N = \mathrm{id}$, let α be minimal such that $\eta(\alpha) \neq \nu(\alpha)$; then, letting $\delta = \eta \restriction \alpha$, $M_{\delta\hat{}\langle 0\rangle}$, $M_{\delta\hat{}\langle 1\rangle}$ could be amalgamated over N by using f, contradicting (b)). Now as $2^{\lambda^{++}} > 2^{\lambda^+}$, $I(\lambda^{++}, K) = 2^{\lambda^{++}}$ (easy, by [14] VIII 1.3).

Case II: Not case I, so, for every full model M of cardinality λ^+ there are M_l, $l = 0, 1$, $M < M_l$ full of cardinality λ^+ such that they cannot be amalgamated over M and for every M^l, $l = 0, 1$, $M_0 < M^l$ of power λ^+, M^0 and M^1 can be amalgamated over M; and the same requirement for every M^l, $M_1 < M^l$. Now by induction on $\alpha < \lambda^{++}$, for $\eta \in {}^{\alpha}2$ define M_η (the fullness at limit stages is preserved by Lemma 2.12(2)):

(1) M_η full, and $|M_\eta| = \lambda^+(1 + l(\eta))$;

(2) $\nu \triangleleft \eta \Rightarrow M_\nu < M_\eta$;

(3) for $l(\eta)$ limit, $M_\eta = \bigcup_{\alpha<l(\eta)} M_{\eta\restriction\alpha}$.

If $l(\eta) = \beta + 1$, choose $M_{\eta\restriction\beta\hat{}\langle l\rangle}$, $l = 0, 1$ such that they cannot be amalgamated over $M_{\eta\restriction\beta}$ and if $M_{\eta\restriction\beta\hat{}\langle 0\rangle} < M^l$, $l = 0, 1$, are of power λ^+, then M^0 and M^1 can be amalgamated over $M_{\eta\restriction\beta}$; similarly for $M_{\eta\restriction\beta\hat{}\langle 1\rangle}$.

For $\eta \in {}^{\lambda^{++}}2$ define $M_\eta = \bigcup_{\alpha<\lambda^{++}} M_{\eta\restriction\alpha}$.

The construction is trivial, using that in λ^+ up to isomorphism there is a unique full model.

By $2^{\lambda^+} < 2^{\lambda^{++}}$ and [3] 3.2, applying Ulam's theorem, let $\{S_\alpha : \alpha < \lambda^{++}\}$ be a partition of λ^{++} to non-small subsets. Assume $\delta < \lambda^{++}$ such that $\lambda^+\delta = \delta$, let η,

$\nu \in {}^{\delta}2$, and $h : M_\eta \to M_\nu$ be an elementary embedding. Now define a function F: $F(\eta, \nu, h) = 1$ if the diagram

$$\begin{array}{lll} M_{\eta^\wedge\langle 0\rangle} & & \\ \cup\!\shortmid & & \\ M_\eta & \xrightarrow[h]{} & M_{\nu^\wedge\langle 0\rangle} \end{array}$$

can be amalgamated; otherwise $F(\eta, \nu, h) = 0$. By the weak diamond there exists $\rho_\alpha \in {}^{\lambda^{++}}2$ for $\alpha < \lambda^{++}$ such that for each $\alpha < \lambda^{++}$ for every $\eta \in {}^{\lambda^{++}}2$, $\nu \in {}^{\lambda^{++}\geqq}2$, $h : \lambda^{++} \to \lambda^{++}$, the set $\{\delta \in S_\alpha : F(\eta \restriction \delta,\ \nu \restriction \delta,\ h \restriction \delta) = \rho_\alpha(\delta)\}$ is stationary.

For each $I \subseteq \lambda^{++}$ define $\eta_I \in {}^{\lambda^{++}}2$ such that $\eta_I(i) = \rho_\alpha(i)$ provided $i \in S_\alpha$, $\alpha \in I$, otherwise $\eta_I(i) = 0$; notice that it is well defined since $\{S_\alpha : \alpha < \lambda^{++}\}$ are a partition.

We are going to prove that for every $I, J \subseteq \lambda^{++}$, $I \neq J \Rightarrow M_{\eta_I} \not\cong M_{\eta_J}$. Let $I \not\subseteq J$ and assume there exists $h : M_{\eta_I} \to M_{\eta_J}$, an elementary embedding, and choose $\gamma \in I - J$. The set $C = \{\delta < \lambda^{++} : h \restriction \delta$ is a function into δ, $\lambda^+\delta = \delta\}$ is closed, unbounded; denote

$$S'_\gamma = \{\delta \in S_\gamma : F(\eta_I \restriction \delta,\ \eta_J \restriction \delta,\ h \restriction \delta) = \rho_\gamma(\delta)\} \cap C.$$

Choose $\delta \in S'_\gamma$ and denote $\eta = \eta_I \restriction \delta$, $\nu = \eta_J \restriction \delta$. From the definition of η_J it is clear that $\eta_J(\delta) = 0$. Now we shall check the two possibilities:

If $\eta_I(\delta) = 0$: then $\rho_\gamma(\delta) = 0$ and since $\delta \in S'_\gamma$ we have $F(\eta, \nu, h \restriction \delta) = 0$, but by the choice of h we know that the diagram

$$\begin{array}{lll} M_{\eta^\wedge\langle 0\rangle} & & \\ \cup\!\shortmid & & \\ M_\eta & \xrightarrow[h\restriction\delta]{} & M_{\nu^\wedge\langle 0\rangle} \end{array}$$

can be amalgamated. This contradicts the definition of function F.

If $\eta_I(\delta) = 1$: then $\rho_\gamma(\delta) = 1$ and also $F(\eta, \nu, h \restriction \delta) = 1$. Using F's definition, the diagram

$$\begin{array}{lll} M_{\eta^\wedge\langle 0\rangle} & & \\ \cup\!\shortmid & & \\ M_\eta & \xrightarrow[h\restriction\delta]{} & M_{\nu^\wedge\langle 0\rangle} \end{array}$$

can be amalgamated.

On the other hand, the diagram

$$\begin{array}{ccc} M_{\eta\hat{}\langle 1\rangle} & & \\ \cup\!\!| & & \\ M_\eta & \xrightarrow[h\restriction\delta]{} & M_{\nu\hat{}\langle 0\rangle} \end{array}$$

can be amalgamated. By the diagram before the last, M_η is embeddable into M_ν (by h) and there is a model M_1, $M_{\nu\hat{}\langle 0\rangle} < M_1$ of power λ^+ such that $M_{\eta\hat{}\langle 0\rangle}$ is embeddable into M_1 by some $h_1 \supseteq h \restriction \delta$. By the choice of h there is an ordinal $\delta + 1 < \gamma < \lambda^{++}$, denote $M_2 = M_{\eta,l\gamma}$, s.t. $M_{\eta\hat{}\langle 1\rangle}$ is embeddable into M_2 by a mapping extending $h \restriction \delta$. Now by our choices $M_{\nu\hat{}\langle 0\rangle} < M_l$ for $l = 0, 1$, therefore, by the construction, M_0, M_1 can be amalgamated over M_ν and this contradicts the assumption that $M_{\eta\hat{}\langle l\rangle}$, $l = 0, 1$ cannot be amalgamated over M_η.

LEMMA 6.3. *Suppose the conclusions of Lemma 6.2 hold and $2^\lambda < 2^{\lambda^+}$. Suppose further that $M_0 < M_1 < M_1'$, $M_0 < M_2$ and M_0, M_2, M_1' is in stable amalgamation, M_2, M_1 are full over M_0, M_0 is full over $\emptyset$, M_1' is full over M_1, and each has cardinality λ.* Then *for every N, $\lambda = \| N \|$, $M_1 < N$, $M_2 < N$, M_1, N, M_1' are in stable amalgamation, there are N^1, N^2 of power λ such that $|M_1'| \cup |N| \subseteq N^l$, but N^1, N^2 cannot be amalgamated over $|M_1'| \cup M_2$.*

REMARK. This is a strengthening of the $(\lambda, 2)$-non-uniqueness property.

PROOF. Suppose not, and N is a counterexample, and w.l.o.g. N is weakly full over $|M_1| \cup |M_2|$. Notice that by the $(\lambda, 1)$-uniqueness property, this N is a counterexample for all possible M_1'. Now we define by induction on $\alpha < \lambda^+$ models M_α^0, M_α^1 such that

(a) $\| M_\alpha^l \| = \lambda$, M_α^l is full, M_α^1 is full over M_α^0, $M_0^0 = M$, $M_0^1 = M_1$,

(b) for limit α, $M_\alpha^l = \bigcup_{\beta<\alpha} M_\beta^l$,

(c) the quadruples $(M_\alpha^0, M_\alpha^1, M_{\alpha+1}^0, M_{\alpha+1}^1)$ and (M_0, M_1, M_2, N) are isomorphic (note the order; it has significance).

Let $M_*^0 = \bigcup_{\alpha<\lambda^+} M_\alpha^0$, $M_*^1 = \bigcup_{\alpha<\lambda^+} M_\alpha^1$, and w.l.o.g. $|M_\alpha^0| = \{3i : i < \lambda(1+\alpha)\}$, $|M_\alpha^1| = |M_\alpha^0| \cup \{3i+1 : i < \lambda(1+\alpha)\}$. Clearly M_*^0 is full and M_*^1 is full over M_*^0 and M_α^0, M_*^0, M_α^1 is in stable amalgamation for each α.

Now we define by induction on $\alpha < \lambda^+$, for each $\eta \in {}^\alpha 2$, a model M_η with universe $\{3i, 3i+2 : i < \lambda(1+\alpha)\}$ such that

(i) for $\beta < l(\eta)$, $M_{\eta\restriction\beta} < M_\eta$,

(ii) M_η is full over $M_{l(\eta)}^0$ and $M_{l(\eta)}^0$, M_η, M_*^0 are in stable amalgamation,

(iii) $M_{\eta\hat{}\langle 0\rangle}$, $M_{\eta\hat{}\langle 1\rangle}$ cannot be amalgamated over $|M_\eta| \cup |M_{l(\eta)+1}^0|$.

This is provable by the $(\lambda, 2)$-non-uniqueness property.

Let, for $\eta \in {}^{\lambda^+}2$, $M_\eta = \bigcup_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$. As by Lemma 6.2(3) K has the λ^+-amalgamation property for full models, for each $\eta \in {}^{\lambda^+}2$, M^0_*, M^1_*, M_η can be amalgamated, so there are N_η, f_η, $|N_\eta| = \lambda^+$, $M^1_* < N_\eta$, and f_η is an elementary embedding of M_η into N_η over M^0_*.

Suppose for some α, η, ν the following hold:

(α) $\eta\restriction\alpha = \nu\restriction\alpha$, $\eta(\alpha)=0$, $\nu(\alpha)=1$, $\alpha<\lambda^+$, $\alpha = \lambda\alpha$;

(β) $N_\alpha =^{\mathrm{df}} N_\eta\restriction\alpha = N_\nu\restriction\alpha < N_\eta$, N_ν and M^1_α, M^1_*, N_α is in stable amalgamation;

(γ) $f_\eta\restriction M_{\eta\restriction\alpha} = f_\nu\restriction M_{\eta\restriction\alpha}$ is into N_α.

We can choose $\beta = \lambda\beta < \lambda^+$, such that $|N_\alpha|\cup|M^1_{\alpha+1}| \subseteq N_\eta\restriction\beta < N_\eta$, $|N_\alpha|\cup|M^1_{\alpha+1}| \subseteq N_\nu\restriction\beta < N_\nu$ and $f_\eta\restriction M_{\eta\restriction(\alpha+1)}$ is into $N_\eta\restriction\beta$ and $f_\nu\restriction M_{\nu\restriction(\alpha+1)}$ is into $N_\nu\restriction\beta$. Now by (c) and the choice of M_0, M_1, M_2 and N, the models $N_\eta\restriction\beta$, $N_\nu\restriction\beta$ can be amalgamated over $|N_\alpha|\cup|M^0_{\alpha+1}|$ (using, say, g_η, g_ν), but then $M_{\eta\restriction(\alpha+1)}$, $M_{\nu\restriction(\alpha+1)}$ can be amalgamated over $|M_{\eta\restriction\alpha}|\cup|M^0_{\alpha+1}|$ (by $g_\eta f_\eta$ and $g_\nu f_\nu$) contradicting (iii). But there are such α, η, ν by [3] 7.1 (see §1 here).

THEOREM 6.4. *Assume the conclusions of Lemma* 6.2 *hold, and also* $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$. *Then* $I(\lambda^{++},K) \geqq \mu(n(k))$.[†]

PROOF. We divide the proof into cases, according to set-theoretic possibilities; in fact, not all cases are necessary but as our aim is also to exemplify the techniques, we do not economize. In most cases we get $2^{\lambda^{++}}$ models, no one elementarily embeddable into another, but we do not get it always.

EXPLANATION. The natural plan to build $2^{\lambda^{++}}$ non-isomorphic models of cardinality λ^{++}, is to build a tree of full models M^η ($\eta \in {}^{\lambda^{++}\geqq}2 = \bigcup_{\alpha\leqq\lambda^{++}} {}^\alpha 2$) such that $|M^\eta| = \lambda^+(1+l(\eta))$, $\eta \triangleleft \nu \Rightarrow M^\eta < M^\nu$, by induction on $l(\eta)$, and then to find many non-isomorphic M^η ($\eta \in {}^{\lambda^{++}}2$), maybe imitating the proof of Theorem 1.3.

It is natural to make $M^{\eta\hat{}\langle 0\rangle}$, $M^{\eta\hat{}\langle 1\rangle}$ not isomorphic over M^η, but this of course is not sufficient as K has the λ^+-amalgamation property for full models, so it may well occur that for some ν, $\eta\hat{}\langle 1\rangle \triangleleft \nu$, and $M^{\eta\hat{}\langle 0\rangle}$ can be elementarily embedded in M^ν, and even that for some ν^l, $\eta\hat{}\langle l\rangle \triangleleft \nu^l \in {}^{\lambda^{++}}2$, M^{ν^0}, M^{ν^1} are isomorphic over M^η. We shall try to prevent the first possibility (hence the second).

The natural way to define $M^{\nu\hat{}\langle l\rangle}$ seems to be the following: we let $M^\nu =$

[†] We define, for $2 \leqq n < \omega$, $\mu(n)$ as $\bigcup\{\chi^+ : \mathrm{Unif}(\aleph_n, \chi, 2^{\aleph_{n-1}}, 2^{\aleph_{n-1}})\text{ fail}\}$. On the definition of Unif see [16] XIV 1.1 and (2) of the notation after it (on p. 463). By [16] XIV 1.10, p. 467 if $\chi < 2^{\aleph_n}$, $\mathrm{Unif}(\aleph_n, \chi, 2^{\aleph_{n-1}}, 2^{\aleph_{n-1}})$ holds, *then* $\chi^{\aleph_0} = 2^{\aleph_n}$. Moreover, there is a family of 2^λ subsets of χ, each of power $\aleph_n$, the intersection of any two finite ones.

We stipulate $\mu(1) = 2^{\aleph_1}$.

$\bigcup_{\beta<\lambda^+} M^\nu_\beta$, $\|M^\nu_\beta\| = \lambda$, $\beta < \gamma \Rightarrow M^\nu_\beta < M^\nu_\gamma$, $M^\nu_\beta = \bigcup_{\gamma<\beta} M^\nu_\gamma$ for limit β (for any ν). So M^η, M^η_β are given, and we want to define $M^{\eta\hat{}\langle 1\rangle}$. We do it by defining $M^{\eta\hat{}\langle 1\rangle}_\beta$ by induction on β, such that $M^{\eta\hat{}\langle 1\rangle}_\beta \cap M^\eta = M^\eta_\beta$, and M^η_β, $M^{\eta\hat{}\langle 1\rangle}_\beta$, $M^\eta_{\beta+1}$ is in stable amalgamation. As K satisfies the $(\lambda, 2)$-non-uniqueness property, for each $\beta = \gamma + 1$ we have a real choice in defining $M^{\eta\hat{}\langle 1\rangle}_\beta$; we can define it in two ways which cannot be amalgamated over $M^\eta_\beta \cup M^{\eta\hat{}\langle 1\rangle}_\gamma$. So we can define a tree of those possibilities $M^{\eta\hat{}\langle 1\rangle}_\rho$ ($\rho \in {}^{\lambda^+ \geqq}2$) increasing and continuous in ρ; $M^{\eta\hat{}\langle 1\rangle}_\rho \cap M^\eta = M^\eta_{l(\rho)}$ and $M^{\eta\hat{}\langle 1\rangle}_{\rho\hat{}\langle 1\rangle}$, $M^{\eta\hat{}\langle 1\rangle}_{\rho\hat{}\langle 0\rangle}$ cannot be amalgamated over $M^\eta_{l(\rho)+1} \cup M^{\eta\hat{}\langle 1\rangle}_\rho$.

Now if $M^\eta < N$, $\|N\| = \lambda^+$, let $N = \bigcup_{\beta<\lambda^+} N_\beta$, N_β $<$-increasing and continuous, $\|N_\beta\| = \lambda$, so $C = \{\beta : N_\beta \cap M^\eta = M^\eta_\beta\}$ is closed unbounded. By the diamond (or weak diamond), for some $\rho \in {}^{\lambda^+}2$, $M^{\eta\hat{}\langle 1\rangle}_\rho$ cannot be elementarily embedded into N over M^η (like in §1).

However, we are not given specific N; we rather have to choose $\rho \in {}^{\lambda^+}2$ which is good for every $N = M^\nu$, $\eta\hat{}\langle 0\rangle \leqq \nu$, necessarily, most of those M^ν will be constructed *after* $M^{\eta\hat{}\langle 1\rangle}$. So a reasonable solution is to take on ourselves some obligation to be fulfilled by every M^ν, $\eta\hat{}\langle 0\rangle \leqq \nu$. Let us first try to see how (and whether) we can continue $\alpha < \lambda^+$ steps. So let $\eta\hat{}\langle 0\rangle \triangleleft \nu$, $l(\eta) = \xi$, $l(\nu) = \xi + \alpha$. The reasonable assumption is:

$$M^{\nu\restriction(\xi+i)}_\beta \cap M^{\nu\restriction(\xi+j)} = M^{\nu\restriction(\xi+j)}_\beta \qquad \text{for } \beta < \lambda^+, \quad j \leqq i \leqq \alpha.$$

If $\diamondsuit_{\lambda^+}$ holds, we can assume that, defining $\rho(\delta)$, we are given an elementary embedding g_δ of $M^{\eta\hat{}\langle 1\rangle}_{\rho\restriction\delta}$ into M^ν, over M^η_δ (i.e., a "guess" for them). For "most" δ's, g_δ is into M^ν_δ; so by choosing the right $\rho(i)$ we need to know how $M^\eta_{\delta+1}$, M^ν_δ are amalgamated, and a reasonable way to make this known is to predetermine how, for each $i < \alpha$, $M^{\rho\restriction i}_{\delta+1}$, $M^{\rho\restriction(i+1)}_\delta$ are amalgamated (for another reasonable way see case II).

For this, let F be as defined later (in Convention 6.7) and we demand (for $i \leqq \alpha$)

$$(*) \qquad M^{\nu\restriction(\xi+i+1)}_{\delta+1} = F(M^{\nu\restriction(\xi+i)}_\delta, M^{\nu\restriction(\xi+i+1)}_\delta, M^{\nu\restriction(\xi+i)}_{\delta+1}, |M^{\nu\restriction(\xi+i+1)}_{\delta+1}|).$$

Unfortunately, this causes us another problem. Maybe $\nu(\xi+2) = 1$, then trying to define $M^{\nu\restriction(\xi+1)\hat{}\langle 1\rangle}_\nu$ we see that we lack the freedom we have used: we just "promise" to use a specific way for amalgamating (in (*)). So what do we do?

We use Lemma 6.3.

By Lemma 6.3, though we have determined how $M^{\nu\restriction(\xi+1)}_{\delta+1}$, $M^{\nu\restriction(\xi+2)}_\delta$ are amalgamated we still have the free choice to amalgamate $M^{\nu\restriction(\xi+1)}_{\delta+2}$, $M^{\nu\restriction(\xi+2)}_\delta$. What about $M^{\nu\restriction(\xi+i)\hat{}\langle 1\rangle}$? Here we can use the freedom we have for amalgamation $M^{\nu\restriction(\xi+i)}_{\delta+i+1}$, $M^{\nu\restriction(\xi+i+1)}_\delta$. Generally, for each ν we have a function $f_\nu : \lambda^+ \to \lambda^+$ keeping track how far our promise goes.

But all this was when $\alpha < \lambda^+$; but for $\alpha \leqq \lambda^+$, do we go bankrupt having made too many promises? We made them modulo D_{λ^+} (the filter of closed unbounded sets) and succeed in continuing.

DEFINITION 6.5. In our proof let $(\bar{M}, f)$ denote a pair, where $\bar{M} = \langle M_i : i < \lambda^+ \rangle$, M_i increasing and continuous, $\| M_i \| = \lambda$, M_{i+1} full over M_i, and f is a function from λ^+ to λ^+. We let $M_{\lambda^+} = \bigcup_{i<\lambda^+} M_i$. We say $(\bar{M}^1, f^1) \leqq (\bar{M}^2, f^2)$ if:

(a) $\{i < \lambda^+ : f^1(i) \leqq f^2(i)\} \in D_{\lambda^+}$ (= the filter over λ^+ generated by the closed unbounded sets);

(b) for some $S \in D_{\lambda^+}$, for every $\alpha \in S$, and $i < \lambda^+$, if $\alpha \leqq i \leqq \alpha + f^1(\alpha)$ then $M_i^2 \cap M_{\lambda^+}^1 = M_i^1$ and $M_i^1, M_i^2, M_{\lambda^+}^1$ is in stable amalgamation and $M_{\lambda^+}^1 < M_{\lambda^+}^2$.

CLAIM 6.6. (1) $\leqq$ *is a quasi-order* (*among those pairs*) (*i.e., transitive and* $(\bar{M}, f) \leqq (\bar{M}, f)$).

(2) *Any increasing sequence of pairs of length* $< \lambda^{++}$ *has a least upper bound.*

PROOF. (1) Trivial.

(2) W.l.o.g. the sequence is $(\bar{M}^\xi, f^\xi)$, $\xi < \mu$, where μ is a regular cardinal $\leqq \lambda^+$. For $\xi < \zeta < \mu$, let $S_{\xi,\zeta}$ be a closed unbounded subset of λ^+ exemplifying (a) and (b) hold for $(\bar{M}^\xi, f^\xi) \leqq (\bar{M}^\zeta, f^\zeta)$. First assume $\mu \leqq \lambda$, let $S \subseteq \bigcap_{\xi<\zeta<\mu} S_{\xi,\zeta}$ be a closed unbounded subset of λ^+, such that $\alpha \in S$, $\xi < \mu$, $\alpha < \beta \leqq \alpha + f^\xi(\alpha)$ implies $\beta \notin S$. Let $S = \{\alpha_i : i < \lambda^+\}$, α_i increasing. Notice that for every i, $\xi < \zeta < \mu \Rightarrow f^\xi(\alpha_i) \leqq f^\zeta(\alpha_i)$.

Now we define by induction on i an ordinal $\beta_i : \beta_0 = \alpha_0$, for limit δ, $\beta_\delta = \bigcup_{i<\delta} \beta_i$ and $\beta_{i+1} = \beta_i + \sup_{\xi<\mu} f^\xi(\alpha_i)$ if it is $> \beta_i$, and $\beta_i + 1$ otherwise (clearly β_i is strictly increasing). Now we define $M_{\beta_i} = \bigcup_{\xi<\mu} M^\xi_{\alpha_i}$, and if $0 < j < \sup_{\xi<\mu} f^\xi(\alpha_i)$ then for some $\xi_0 < \mu$, $\xi_0 < \xi < \mu \Rightarrow j < f^\xi(\alpha_i)$, and we let $M_{\beta_i+j} = \bigcup_{\xi_0<\xi<\mu} M^\xi_{\alpha_i+j}$. Let $\bar{M} = \langle M_\beta : \beta < \lambda^+ \rangle$, and f be defined by $f(\beta_i) = \sup_{\xi<\mu} f^\xi(\alpha_i)$, $f(\beta_i + j) = 0$ for $0 < j < f(\beta_i)$. Clearly $(\bar{M}, f)$ is as required. The proof for $\mu = \lambda^+$ is similar, using diagonal intersection.

CONVENTION 6.7. Now let F be a function such that:

(a) if $\| M_l \| = \lambda$, $|M_1| \cap |M_2| = |M_0|$, $|M_1| \cup |M_2| \subset A$, each M_l is full, $|A - |M_1| \cup |M_2|| = \lambda$, *then* $F(M_0, M_1, M_2, A)$ is a model M, $|M| = A$, $M_l < M$, M weakly full over $|M_1| \cup |M_2|$, and M_1, M_2 are in stable amalgamation in M;

(b) if for $i = 0, 1$, M_0^i, M_1^i, M_2^i, A^i are as in (a), f is a one-to-one function from $|M_1^0| \cup |M_2^0|$ onto $|M_1^1| \cup |M_2^1|$, and $f \upharpoonright M_l^0$ is an isomorphism from M_l^0 onto M_l^1 for $l = 0, 1, 2$, then we can extend f to an isomorphism from $F(M_0^0, M_1^0, M_2^0, A^0)$ onto $F(M_0^1, M_1^1, M_2^1, A^1)$.

DEFINITION 6.8. $(\bar{M},f)\leqq_F(M',f')$ if (a), (b) from Definition 6.5 hold, and

(c) there is a sequence $(\bar{M}^\xi,f^\xi)$, $\xi\leqq\zeta$, where $\zeta<\lambda^{++}$, increasing (by $\leqq$) and continuous (see Claim 6.6(2)), $(\bar{M}^0,f^0)=(\bar{M},f)$, $(\bar{M}^\zeta,f^\zeta)=(\bar{M}',f')$ such that for every $\xi<\zeta$ for a closed unbounded set of α's for each β,

$$\alpha\leqq\beta<f^\xi(\alpha),\qquad M^{\xi+1}_{\beta+1}=F(M^\xi_\beta,M^{\xi+1}_\beta,M^\xi_{\beta+1},|M^{\xi+1}_{\beta+1}|).$$

If in (c) $\zeta=1$, we say $(\bar{M}',f')$ is an immediate F-successor of $(\bar{M},f)$.

CLAIM 6.9. (1) $\leqq_F$ *is a quasi-order.*

(2) *For* $\leqq_F$, *any increasing sequence of length* $<\lambda^{++}$ *has a least upper bound.*

PROOF. Similar to Claim 6.6.

CLAIM 6.10. *Suppose* $(\bar{M},f)\leqq_F(\bar{M}',f')$ *and this is exemplified by* $(\bar{M}^\xi,f^\xi)$, $\xi\leqq\zeta<\lambda^{++}$.

Let $M^\zeta_{\lambda^+}=\bigcup_{i<\lambda^+}N_i$, N_i *increasing and continuous,* $\|N_i\|=\lambda$. *Let* $\zeta+1=\bigcup_{i<\lambda^+}A_i$, A_i *increasing and continuous and* $|A_i|\leqq\lambda$. *Then for a closed unbounded set of* $i<\lambda^+$ *the following holds:*

(*) $0,\ \zeta\in A_i$, $[\xi\in A_i\equiv\xi+1\in A_i]$, $N_i=\bigcup_{\xi\in A_i}M^\xi_i$; *and* $\xi_0<\xi_1\in A_i$, $i\leqq j<i+f(i)$ *implies* $M^{\xi_0}_j=M^{\xi_1}_j\cap M^{\xi_0}_{\lambda^+}$ *and* $M^{\xi_0}_j$, $M^{\xi_1}_j$, $M^{\xi_0}_{\lambda^+}$ *is in stable amalgamation; and for* j, $i\leqq j<i+f(i)$,

$$M^{\xi+1}_{j+1}=F(M^\xi_j,M^{\xi+1}_j,M^\xi_{j+1},|M^{\xi+1}_{j+1}|).$$

PROOF. Easy.

So we have now finished the preliminaries and start to prove Theorem 6.4.

Special Case: $\diamondsuit_{\lambda^+}$ and $\diamondsuit_{\lambda^{++}}$

REMARK. By Gregory [5], this holds in our context if G.C.H. $+\ \lambda\neq\aleph_0$.[†]

The proof is quite similar to [15], so we do it in a somewhat informal way. We define by induction on $\alpha<\lambda^{++}$ pairs $(\bar{M}^\eta,f^\eta)$ $(\eta\in{}^\alpha2)$ with universe $\lambda^+(1+\alpha)$ (i.e., $|M^\eta_{\lambda^+}|=\lambda^+(1+\alpha)$) such that for $\alpha=\beta+1$, $\eta\in{}^\alpha2$, $(\bar{M}^\eta,f^\eta)$ is an immediate $\leqq_F$-successor of $(\bar{M}^{\eta\restriction\beta},f^{\eta\restriction\beta})$, and for limit α we take the least upper bound (see Claim 6.9(2) above).

In stage α, $\diamondsuit_{\lambda^{++}}$ "gives" us $\eta^\alpha\in{}^\alpha2$, $\nu^\alpha\in{}^\alpha2$, and an elementary embedding g_α from $M^{\eta^\alpha}_{\lambda^+}$ into $M^{\nu^\alpha}_{\lambda^+}$. We look at g_α as an initial segment of some isomorphism from some $\bigcup_{\alpha<\lambda^{++}}M^{\eta\restriction\alpha}_{\lambda^+}$ onto $\bigcup_{\alpha<\lambda^{++}}M^{\nu\restriction\alpha}_{\lambda^+}$, $\eta^\alpha\lhd\eta\in{}^{\lambda^{++}}2$, $\nu^\alpha\lhd\nu\in{}^{\lambda^{++}}2$. We

[†] If λ has cofinality $\aleph_0$ this holds by [15].

define $\bar{M}^{\nu^\alpha\wedge\langle 0\rangle} = \bar{M}^{\nu^\alpha\wedge\langle 1\rangle}$ in any reasonable way, $f_{(i)}^{\nu^\alpha\wedge\langle 1\rangle} = \min\{\gamma : \gamma > f^{\nu^\alpha}(i)$ and g_α maps $M_\gamma^{\eta^\alpha}$ into $M_\gamma^{\nu^\alpha}\}$, and it suffices to find $\bar{N}$, $(M^\eta, f^\eta) \leqq_F (\bar{N}, f^\eta)$, which, for no ν, $\nu^\alpha \lhd \nu \in {}^{\lambda^{++}}2$ has an elementary embedding extending g_α into $M_{\lambda^+}^\nu$. We define N_i $(i < \lambda^+)$ by induction on i, $N_i \cap M_{\lambda^+}^\eta = M_i^\eta$. For each i, such that $(\forall j < i)$ $\{f^\eta(j) < i\}$, $\diamondsuit_{\lambda^+}$ give us as a guess model $\underset{\sim}{M}_\xi^i$ $(\xi < \xi_i < \lambda^+)$, forming an increasing elementary chain, $M_0^i = M_i^{\nu^\alpha}$, so that using the notation of Claim 6.10, letting $(\bar{M}, f) =^{\text{def}} (\bar{M}^{\nu^\alpha}, f^{\nu^\alpha})$, for every possible $(\bar{M}', f')$, $(\bar{M}^{\nu^\alpha}, f^{\nu^\alpha}) \leqq_F (\bar{M}', f')$ for a stationary set of $i < \lambda^+$, $\langle \underset{\sim}{M}_\xi^i : \xi < \xi_i \rangle = \langle M_i^\xi : \xi \in A_i \rangle$ (possible, by the definition of diamond, and a little coding). Now we have no choice in defining N_{i+j}, $j \leqq f^\eta(i)$. But for $N_{f(i)+1}$ we have a free choice which, by Lemma 6.3, is enough to destroy the "guess" $\langle \underset{\sim}{M}_\xi^i : \xi < \xi_i \rangle$ as a candidate for a projection of $(\bar{M}^\nu, f^\nu)$ for which g_α can be extended to an elementary embedding of N_{λ^+} into $M_{\lambda^+}^\nu$.

So we can finish case I.

Convention 6.11. For each pair $(\bar{M}, f)$ (as in Definition 6.6) choose a closed unbounded set $S \subseteq \lambda^+$ such that, if $\alpha \in S$, $\alpha < \beta \leqq \alpha + f(\alpha)$, then $\beta \notin S$.

Let $S = \{\alpha_i : i < \lambda^+\}$ be an enumeration of S. Now we define by induction on $\beta < \lambda^+$, for each $\eta \in {}^\beta 2$, a model M_η such that:

(a) $M_\eta \cap M_{\lambda^+} = M_\beta$; moreover, the triple M_β, M_η, $M_{\beta+1}$ is in stable amalgamation.

(b) M_η is increasing and continuous (i.e., $\eta \lhd \nu$ implies $M_\eta < M_\nu$, and if $\beta = l(\eta)$ is limit, then $M_\eta = \bigcup_{\gamma<\beta} M_{\eta\restriction\gamma}$).

(c) If $\alpha_i \leqq \beta < \alpha_i + f(\alpha_i)$, $\eta \in {}^{\beta+1}2$ then

$$M_\eta = F(M_\beta, M_{\eta\restriction\beta}, M_{\beta+1}, |M_\eta|).$$

(d) If $\alpha_i \leqq \beta \leqq \alpha_i + f(\alpha_i)$, $\eta \in {}^\beta 2$. $\nu \in {}^\beta 2$ and $\eta \restriction \alpha_i = \nu \restriction \alpha_i$, then $M_\eta = M_\nu$.

(e) If $\beta = \alpha_i + f(\alpha_i) + 1$, $\eta \in {}^\beta 2$, $\nu \in {}^\beta 2$, $\eta \restriction \alpha_i = \nu \restriction \alpha_i$ *then* $\eta(\alpha_i) = \nu(\alpha_i)$ implies $M_\eta = M_\nu$, but $\eta(\alpha_i) \neq \nu(\alpha_i)$ implies that M_η, M_ν cannot be amalgamated over $M_{\eta\restriction\alpha_i} \cup M_\beta$.

The main point is (e), and we can do it by the conclusion of Lemma 6.3.

We then let, for $\eta \in {}^{\lambda^+}2$, $M_\eta = \bigcup_{\beta<\lambda^+} M_{\eta\restriction\beta}$, $\bar{M}_\eta = \langle M_{\eta\restriction i} : i < \lambda^+ \rangle$.

Of course, for $(\bar{M}', f')$ we have M'_η, etc.

Claim 6.12. *For any* $(\bar{M}, f)$ *there is* $\eta \in {}^{\lambda^+}2$ *such that for no* $(\bar{M}', f')$, $(\bar{M}, f+1) \leqq_F (\bar{M}', f')$, *can we embed* M_η *into* M'_{λ^+} *over* M_{λ^+} (*we use Convention* 6.11 *and* $(f+1)$ $(i) = f(i) + 1$, and we are assuming $2^\lambda < 2^{\lambda^+}$).

Proof. Left to the reader.

Claim 6.13. *We can define for every* $\alpha < \lambda^{++}$, $\nu \in {}^\alpha 2$ *a pair* $(\bar{M}^\nu, f^\nu)$ *such* *that*:

(a) $(\bar{M}^\nu, f^\nu)$ *is increasing and continuous, i.e.,* $\eta \triangleleft \nu$ *implies* $(\bar{M}^\eta, f^\eta) \leqq (\bar{M}^\nu, f^\nu)$ *and even* $(\bar{M}^\eta, f^\eta) \leqq_F (\bar{M}^\nu, f^\nu)$;

(b) $M_{\lambda^+}^{\eta^\wedge\langle 1\rangle}$ *cannot be elementarily embedded into* N *over* $M_{\lambda^+}^\eta$, *if* $(\bar{M}^{\eta^\wedge\langle 0\rangle}, f^{\eta^\wedge\langle 0\rangle}) \leqq_F (\bar{N}, f)$, $N = N_{\lambda^+}$, *so in particular if* $N_{\lambda^+} = M_{\lambda^+}^\nu$, $\eta^\wedge\langle 0\rangle \trianglelefteq \nu$;

(c) $|M_{\lambda^+}^\eta| = \lambda^+(1 + l(\eta))$.

Notation. For $\eta \in {}^{\lambda^{++}}2$, $M^\eta = \bigcup_{\alpha<\lambda^{++}} M_{\lambda^+}^{\eta\restriction\alpha}$.

Proof. We define by induction on $\alpha < \lambda^{++}$.

For $\alpha = 0$: $M_{\lambda^+}^{\langle\,\rangle}$ is any full model of power λ^+, and we define $M_i^{\langle\,\rangle}$ $(i < \lambda^+)$ such that $\| M_i^{\langle\,\rangle} \| = \lambda$, $i < j \Rightarrow M_i^{\langle\,\rangle} < M_j^{\langle\,\rangle}$, $M_i^{\langle\,\rangle}$ is increasing continuous $(i < \lambda^+)$ and $M_{\lambda^+} = \bigcup_{i<\lambda^+} M_i$. So $\bar{M}^{\langle\,\rangle} = \langle M_i^{\langle\,\rangle} : i < \lambda \rangle$ and $f^{\langle\,\rangle}$ is the function with domain λ^+ and constant value zero.

For α *limit*: use Claim 6.9(2).

For $\alpha = \beta + 1$ *successor*: use Lemma 6.12, i.e., for $\nu \in {}^\beta 2$, we let $M^{\nu^\wedge\langle 1\rangle}$ be $\bar{M}_\eta$ for the η we get from Claim 6.12 for $(\bar{M}, f) = (\bar{M}^\nu, f^\nu)$, $f^{\nu^\wedge\langle 1\rangle} = f^\nu$, and $(\bar{M}^{\nu^\wedge\langle 0\rangle}, f^{\nu^\wedge\langle 0\rangle})$ is arbitrary except that it is an immediate F-successor of $(\bar{M}^\nu, f^\nu + 1)$ and $f^{\nu^\wedge\langle 0\rangle} = f^\nu + 1$.

Now we can prove something on the number of non-isomorphic models.

General Case: $I(\lambda^{++}, K) > \chi$ if $\mathrm{Unif}(\lambda^{++}, \langle \chi, \lambda^{++} \rangle, \lambda^{++})$

Proof. Suppose $\{M_i : i < \mu\}$ is a list of all models from K of power λ^{++} (up to isomorphism), and w.l.o.g. $|M_i| = \lambda^{++}$. So for each $\eta \in {}^{\lambda^{++}}2$ there is $i_\eta < \mu$ and an isomorphism f_η of M^η (from Claim 6.13) into M_{i_η}.

By [16] def. XIV 1.1 (p. 463), there are $\alpha < \lambda^{++}$, $\eta \in {}^{\lambda^{++}}2$, $\nu \in {}^{\lambda^{++}}2$, such that $\alpha = \lambda^+\alpha$ (so $|M_{\lambda^+}^{\eta\restriction\alpha}| = |M_{\lambda^+}^{\nu\restriction\alpha}| = \alpha$), $\eta \restriction \alpha = \nu \restriction \alpha$, $i_\eta = i_\nu$, $f_\eta \restriction \alpha = f_\nu \restriction \alpha$ and $\eta(\alpha) = 0$, $\nu(\alpha) = 1$. Hence $f_\eta^{-1}(f_\nu \restriction M_{\lambda^+}^{\nu\restriction(\alpha+1)})$ is an elementary embedding of $M_{\lambda^+}^{\nu\restriction(\alpha+1)} = M_{\lambda^+}^{(\nu\restriction\alpha)^\wedge\langle 1\rangle}$ into M^η, which is the identity on $M_{\lambda^+}^{\nu\restriction\alpha} = M_{\lambda^+}^{\eta\restriction\alpha}$; and by cardinality considerations it is into M^ρ for some $\eta \restriction (\alpha + 1) \triangleleft \rho \triangleleft \eta$, $l(\rho) < \lambda^{++}$, so $(\eta \restriction \alpha)^\wedge\langle 0\rangle \leqq \rho$, contradicting (b) of Claim 6.13.

(1) Why cannot we have $2^{\lambda^{++}}$ as in Theorem 1.3? We are missing the "M_l realizes a maximal set of distinct types". From another point of view the isomorphism from M^η onto M^ν, for many $\alpha < \lambda^{++}$, takes $M_{\lambda^+}^{\eta\restriction\alpha}$ onto $M_{\lambda^+}^{\nu\restriction\alpha}$, but it is mixing the levels too much, i.e., if $f : \lambda^+ \to \lambda^+$ is defined by $f(i) = \min\{j : i < j$, g maps $M^{\eta\restriction\alpha}$ into $M_j^{\nu\restriction\alpha}\}$, it may be too big, i.e., bigger than $f^{\eta\restriction\alpha}$.

(2) Originally, we stated here that $I(\lambda^{++}, K)^{\aleph_0} = 2^{\lambda^{++}}$, without elaborating a proof of the set theoretic theorem we use. However, we were urged to explain this (by Rami Grossberg) and the explanation grew to section 1 of [16] XIV.

THEOREM 6.14. *Suppose that the conclusion of Lemma 6.2 holds, and that the ideal of small subsets of λ^+ is not λ^{++}-saturated [i.e., there are non-small $S_i^* \subseteq \lambda^+$ ($i < \lambda^{++}$) such that the intersection of any two is small].*

Then $I(\lambda^{++}, K) = 2^{\lambda^{++}}$.

REMARK. The hypothesis is very weak. For instance, its negation implies the existence of inner models with large cardinals, and the smallness of $\{\alpha < \lambda^+, \operatorname{cf} \alpha < \lambda\}$ when λ is regular (which occurs when $\lambda = \aleph_{n(k)-2}$).

PROOF. Let $\langle S_i^* : i < \lambda^{++} \rangle$ be non-small subsets of λ^+ [$i \neq j \Rightarrow S_i^* \cap S_j^*$ is small]. As the ideal of small subsets of λ^+ is normal, w.l.o.g. [$i \neq j \Rightarrow |S_i^* \cap S_j^*| < \lambda^+$]. Our point of departure from the proof of Theorem 6.4 is replacing the use of Definitions 6.5 and 6.8 by

DEFINITION 6.15. (1) In our proof let $\bar{M}$ denote a sequence such that

$$\bar{M} = \langle M_i : i < \lambda^+ \rangle, \qquad M_i \text{ increasing continuous.}$$

M_i is a full model of power λ, and M_{i+1} is full over M_i. We let $M_{\lambda^+} = \bigcup_{i<\lambda^+} M_i$.

(2) We say $\bar{M}^1 \leqq^* M^2$ if for a closed unbounded set of $\alpha < \lambda^+$ the following hold: $M_\alpha^2 \cap M_{\lambda^+}^1 = M_\alpha^1$, $M_{\alpha+1}^2 \cap M_{\lambda^+}^1 = M_{\alpha+1}^1$, and the triples $M_\alpha^1, M_\alpha^2, M_{\lambda^+}^1$ and $M_{\alpha+1}^1, M_{\alpha+1}^2, M_{\lambda^+}^1$ are in stable amalgamation.

(3) For $S \subseteq \lambda^+$ we say $\bar{M} \leqq_S^* \bar{M}'$ if there is a sequence $\langle \bar{M}^\xi : \xi \leqq \zeta \rangle$ where $\zeta < \lambda^{++}$, $\bar{M}^0 = \bar{M}$, $\bar{M}' = \bar{M}_\zeta$, the sequence is increasing and continuous, and for every ξ, for a closed unbounded set of $\alpha < \lambda^+$,

$$\alpha \in S \Rightarrow M_{\alpha+1}^{\xi+1} = F(M_\alpha^\xi, M_\alpha^{\xi+1}, M_{\alpha+1}^\xi, |M_{\alpha+1}^{\xi+1}|).$$

If in (3) $\zeta = 1$, we say $\bar{M}'$ is an immediate $\leqq_S^*$-successor of $\bar{M}$.

REMARK. So the definition of $\leqq_S^*$ depends on the choice of the S.

CLAIM 6.16. (1) *$\leqq^*$ is a partial quasi-order.*

(2) *Any $\leqq^*$-increasing and continuous sequence of length $\zeta < \lambda^{++}$ has a least upper bound.*

(3) *If $S_1 \subseteq S_2 \subseteq \lambda^+$ or even $S_1 - S_2$ is not stationary, then $\bar{M} \leqq_{S_2}^* \bar{M}^1 \Rightarrow \bar{M} \leqq_{S_1}^* \bar{M}^1 \Rightarrow \bar{M} \leqq^* \bar{M}^1$.*

(4) *If $\langle \bar{M}^\xi : \xi < \zeta \rangle$ is increasing and continuous by $\leqq_S^*$, $\xi < \lambda^{++}$, then its $\leqq^*$-least upper bound is also its $\leqq_S^*$-least upper bound.*

PROOF. Similar to the proofs of Claims 6.6 and 6.9.

CLAIM 6.17. *We can define for $\alpha < \lambda^{++}$, $\nu \in {}^\alpha 2$, a sequence $\bar{M}^\nu$ such that:*

(1) *for $\eta \trianglelefteq \nu$, $\bar{M}^\eta \leqq^* \bar{M}^\nu$;*

(2) $|M^{\eta}_{\lambda^+}| = \lambda^+(1 + l(\eta))$;

(3) *if* $\alpha = l(\nu)$ *is limit,* $\bar{M}^{\nu}$ *is least upper bound of* $\langle \bar{M}^{\nu \restriction \beta} : \beta < \alpha \rangle$;

(4) *if* $\alpha = l(\nu) = \beta + 1$ *then* $\bar{M}^{\nu \restriction \beta} \leqq^*_{(\lambda^+ - S^*_\beta)} M^{\nu}$;

(5) *if* $\beta < l(\nu)$ *then* $\bar{M}^{\nu \restriction (\beta+1)} \leqq^*_{(\lambda^+ - S^*_\beta)} M^{\nu}$;

(6) *if* $\alpha = l(\nu)$, *then there is no* $\bar{M}^*$ *such that* $\bar{M}^{\nu \wedge \langle 1 \rangle} \leqq^*_{(\lambda^+ - S^*_\alpha)} \bar{M}^*$ *and* $M^{\nu \wedge \langle 0 \rangle}_{\lambda^+}$ *can be elementarily embedded into* $M^*_{\lambda^+}$ *over* $M^{\langle\rangle}_{\lambda^+}$.

Proof. Straightforward, the main proof being that for ζ, A_i ($i < \lambda^+$) as in Theorem 6.1, for a closed unbounded set of $i < \lambda^+$, there is at most one $\xi \in A_i$ such that $i \in S^*$ [this holds as $|S^*_\xi \cap S^*_\zeta| \leqq \lambda$ for $\xi \neq \zeta$].

The proof of the theorem is now straightforward: for $\eta \neq \nu \in {}^{\lambda^{++}}2$. The models $\bigcup_{\alpha < \lambda^{++}} M^{\eta \restriction \alpha}_{\lambda^+}$ and $\bigcup_{\alpha < \lambda^{++}} M^{\nu \restriction \alpha}_{\lambda^+}$ are not isomorphic over M_{λ^+}, so we finish as in Lemma 6.2(3), case I.

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