Sh:73

Annals of Mathematical Logic 14 (1978) 73–87.

MODELS WITH SECOND ORDER PROPERTIES II. TREES WITH NO UNDEFINED BRANCHES

Saharon SHELAH*

Institute of Mathematics, The Hebrew University, Jerusalem, Israel

Received 9 March 1977

We prove several theorems of the form: a first order theory T has a model M (sometimes with additional conditions) such that (some) trees defined in M, have no branches except those defined in it. We have some applications e.g. an example for compact logic L(Q), where in $L_{\omega_1,\omega}(Q)$ well-ordering is definable.

Introduction

In the introduction we first explain the results of Keisler and Kunen from [6], then explain our results, and then show some applications.

Let T be a (complete first order) theory with a tree (called in [6] a ranked tree), i.e. $<_1$ is a patial order, in which $\{x : x \leqslant_1 y\}$ is totally ordered for every y, R a rank function whose range is ordered by $\leqslant_2$. Now by [5, Theorem B, p. 340], if $\diamondsuit_{\aleph_1}$, T is countable, and T satisfies some schemes (which are equivalent to: every countable model of T has an end extension with respect to the tree, i.e. no new elements are smaller than an old one), *then* T has an $\aleph_1$-like model (i.e., in the tree $\{x : R(x) \leqslant_2 y\}$ is countable for each y) such that every ω_1-branch is definable.

The results are generalized to κ^+-like trees, when $\kappa = \kappa^{<\kappa}$, and $V = L$ (see [5, pp. 346–347] with some inaccuracies) provided that Chang condition is satisfied (see Definition 7) (it is satisfied by set theories).

Let us explain our results (they are actually somewhat more general).

By Theorem 6 the existence of models with $\aleph_1$-like tree is absolute; so if we have a proof with $\diamondsuit_{\aleph_1}$ we can prove without it. In Theorem 8 we reprove the result on κ^+-like trees under the right hypothesis. In Theorem 12 we prove that for λ regular, $\lambda^+ \geqslant |T|$, T not necessarily countable, T has a model of cardinality λ^+ in which no tree has an undefined branch.

Some of the results will be imporved in [11]. Let us turn to applications.

Application A. We can eliminate $\diamondsuit_{\aleph_1}$ from most results of [6, §3], e.g. any consistent theory extending ZF, has an $\aleph_1$-like model in which all possible classes are definable.

* The author wants to thank the United-States–Israel Binational Science Foundation for partially supporting his research by Grant No. 1110.

Application B. If every model of T has an end extension (by $T, <$ is an order of the universe), and T has Skolem functions, *then* some model of T has a definable end extension (see [13, 2.21] for a proof).

Application C. For every model N of ZFC, we can define naturally its number theory model $P(N)$. It was asked which models M are $P(N)$ for some $N \models$ZFC, and shown that they should satisfy the consequence of ZFC (in the language of number theory) and, be recursively saturated (see Barwise and Schlipf [1], Theorem 2). For countable M those necessary conditions are sufficient. Kaufman deals with the problem of whether the necessary conditions are sufficient in general, and gives a negative answer under $\diamondsuit_{\aleph_1}$ by showing:

(1) In every model $P(N)$, the tree of functions from a "natural number" to $\{0, 1\}$ (i.e. subsets) has an undefined branch.

(2) (Imitating [5, Theorem B].) There is a recursively saturated model satisfying the above mentioned conditions, with no undefined branch.

We can eliminate $\diamondsuit_{\aleph_1}$ by Theorem 6 (as being recursively saturated can be expressed in $L_{\omega_1,\omega}$) or deduce (2) from Theorem 12(2) (let $\lambda = (2^{\aleph_0})^+$, $\kappa = \aleph_1$, so we get an $\aleph_1$-saturated counterexemple).

Application D. A compact logic stronger than first-order for countable models exists.

Friedman asked for the existence of a logic which is compact but stronger than first order logic even for countable models. In [9] we give such an example by a generalize quantifier: an order has a Dedekined cut whose cofinaties belongs to $\{\omega, \kappa\}$, when κ is weakly compact, or using the diamond. So we still do not have an example in ZFC.

Now we can introduce the quantifier $(Q_{Br}x, y)\langle\varphi_1(x, y), \varphi_2(x, y), \varphi_3(x, y)\rangle$ which means the triple of formulas is a tree, which has no branch.

(D1) Now $L(Q_{Br})$ is compact by Theorem 12.

(D2) Let us show $L(Q_{Br})$ is stronger than first order logic for countable models. For each ordinal α let us define a model $M_\alpha = (\omega \cup S_\alpha, \not<, R)$ where S is the set of decreasing sequences of ordinals $<\alpha$, $\not<$: the relation of being an initial segment on S_α, $R: S_\alpha \to \omega$ is a partial function giving the length of the sequence. Let S'_α be $\{\langle\alpha_0, \ldots, \alpha_{k-1}\rangle : k < \omega$ and for some $k(0)$ $\alpha = \alpha_0 = \cdots = \alpha_{k(0)-1} > \alpha_{k(0)} > \alpha_{k(0)+1} > \cdots > \alpha_{k-1}\}$, and M'_α be defined accordingly. For every large enough $\alpha < \omega_1$. M_α, M'_α are elementarily equivalent in first order logic but not in $L(Q_{Br})$.

Application E. We can answer the following question of J. Stavi: whether if for a generalized quantifier Q, L(Q) is compact then in $L_{\omega_1,\omega}(Q)$ we cannot define well-ordering (as a pseudo-elementary class.) We give a counterexample. The same holds for the ω-logic of L(Q).

We use Q_{Br}, of course. We define $N_\alpha = ((\alpha+1) \cup S_\alpha, \not<, R, D, P, <)$ where S_α, $\not<$, R are as above, $P = \omega$, $<$ orders the ordinals and $D: S_\alpha \to \alpha$ a partial function,

$D(\langle\alpha_0, \ldots, \alpha_{k-1}\rangle) = \alpha_{k-1}$ if $k \geq 1$, and $D(\langle\,\rangle) = \alpha$. If $\psi \in L_{\omega_1,\omega}(Q)$ says the obvious properties of N_α (including ω is standard, and some first order properties) $\{N_\alpha: \alpha$ ordinal$\}$ will be the class of models of ψ (up to isomorphism).

Application F. Replacing diamond by a case of the GCH.

This appears in [3], and we hope it is a pattern. We describe it quite schematically. So suppose we can construct in V using $\diamondsuit_{\aleph_1}$, such a model M with universe ω_1, such that there is no $S \subseteq M$, $P(S, M)$. So we define inductively M_α, $|M_\alpha| = \omega(1+\alpha)$, and in stage α kill some approximation $S_\alpha \subseteq M_\alpha$, and this will be sufficient. Suppose we do similarly using CH, now we try to kill each $S \subseteq M_\alpha$ in its turn but this may come in some $\beta > \alpha$: Now sometimes if S does not appear in M_β we can do it. Those which appear form a tree, so if it has no ω_1-branch we shall finish.

We announced results (from here) in [8, 10].

Notation. $|M|$ is the universe of the model M, $|A|$ the cardinality of A, for $\bar{a} \in M$, and formula $\varphi(x, \bar{y})$ $\varphi(M, \bar{a}) = \{b \in |M|: M \models \varphi[b, \bar{a}]\}$.

Let us write $\bar{a} \in A$ instead $\bar{a} = \langle a_0, \ldots\rangle$, $a_0, \ldots \in A$.

Definition 1. In a (first order) language L an L-tree $t = t(\bar{z})$ is a triple of formulas of L. $\langle \leq_1(\bar{u}, \bar{v}; \bar{z}), \leq_2(\bar{u}, \bar{v}; \bar{z}), R(\bar{u}, \bar{v}; \bar{z})\rangle$ such that for each L-model M and $\bar{c} \in |M|$.

(i) $\leq_1(\bar{u}, \bar{v}; \bar{c})$ and $\leq_2(\bar{u}, \bar{v}; \bar{c})$ are transitive, hence, letting $D_l(t(\bar{c}), M) = \{\bar{v} \in |M|;\ (\exists \bar{u}) \leq_l(\bar{u}, \bar{v}; \bar{c})\}$, $\leq_l(\bar{u}, \bar{v}; \bar{c}) \wedge \leq_l(\bar{v}, \bar{u}; \bar{c})$ is an equivalence relation on $D_l(t(\bar{c}), M)$, denoted by $E_l^{t(c)}$. So naturally $\leq_l(\bar{u}, \bar{v}; \bar{c})$ induce a partial order on the set of $E_l^{t(\bar{c})}$-equivalence classes $D_l^*(t(\bar{c}), M)$, and we denote it by $\leq_l^{t(\bar{c})}$.

(ii) $\leq_2^{t(c)}$ is a total order of $D_2^*(t(\bar{c}), M)$ and for every $\bar{a} \in D_1(t(\bar{c}), M)$, $\{\bar{b}/E_1^{t(\bar{c})}: M \models \leq_1(\bar{b}, \bar{a}, \bar{c})\}$ is totally ordered by $\leq_1^{t(\bar{c})}$.

(iii) $R(\bar{u}, \bar{v}; \bar{z})$ is a rank function from $D_1^*(t(\bar{c}), M)$ onto $D_2^*(t(\bar{c}), M)$, i.e. the (universal closure) of the following formulas are satisfied by M:

$$R(\bar{u}, \bar{v}; \bar{c}) \to \bar{u} \in D_1(t(\bar{c}), M) \wedge \bar{v} \in D_2(t(\bar{c}), M),$$

$$R(\bar{u}, \bar{v}_1; \bar{c}) \wedge R(\bar{u}, \bar{v}_2; \bar{c}) \to \bar{v}_1 E_2^{t(c)} \bar{v}_2,$$

$$R(\bar{u}, \bar{v}; \bar{c}) \wedge \bar{u} E_1^{t(\bar{c})} \bar{u}_1 \wedge R(\bar{u}_1, \bar{v}_1; \bar{c}) \to \bar{v} E_2^{t(\bar{c})} \bar{v}_1$$

$$(\forall \bar{v} \in D_2(t(\bar{c}), M))[(\exists \bar{u} \in D_1(t(\bar{c}), M)) \wedge \exists \bar{v}_1 \in D_2(t(\bar{c}), M)]\ [R(\bar{u}, \bar{v}_1, \bar{c}) \wedge \bar{v}_1 E_2^{t(\bar{c})} \bar{v}],$$

$$R(\bar{u}_1, \bar{v}_1; \bar{c}) \wedge R(\bar{u}_2, \bar{v}_2; \bar{c}) \wedge \leq_1(\bar{u}_1, \bar{u}_2; \bar{c}) \to \leq_2(\bar{v}_1, \bar{v}_2; \bar{c}),$$

$$R(\bar{u}_1, \bar{v}, \bar{c}) \wedge R(\bar{u}_2, \bar{v}; \bar{c}) \wedge \leq_1(\bar{u}_1, \bar{u}_2; \bar{c}) \to \leq_1(\bar{u}_2, \bar{u}_1; \bar{c}).$$

This function will be denoted by $R^{t(\bar{c})}$.

Remark. When no confusion arises we omit M, $(t(\bar{c}))$, in the above notations, e.g. write D_1^*.

Definition 2. Let $t^M(\bar{c})$ be the interpretation of the tree in M for $\bar{z} \to \bar{c}$.

Definition 3. (1) A branch of the tree $t^M(\bar{c})$ is a set $B \subseteq D_1^*(t(\bar{c}), M)$ such that: (i) B is (totally) ordered by $\leqslant_1^{t(\bar{c})}$; (ii) $\bar{a}/E_1^{t(\bar{c})} \in B$ and $\bar{b} \leqslant_1^{t(\bar{c})} \bar{a} \to \bar{b}/E_1^{t(\bar{c})} \in B$; (iii) $\{R(\bar{a}/E_1^{t(\bar{c})}): \bar{a}/E_1^{t(\bar{c})} \in B\}$ is an unbounded subset of $D_2^*(t(\bar{c}), M)$.

(2) A branch is definable if there is a formula of L, with parameters from M, defining it in M (i.e. defining $\bar{a} \in M \cdot \bar{a}/E_1^{t(\bar{c})} \in B$). Notice that a branch is definable iff it has a cofinal definable subset; if in $D_2^*(t(\bar{c}), M)$ there is a last element every branch is definable.

Remark 4. (1) We will not gain generality by dealing with $t(\bar{z})$ which for some $\bar{c} \in |M|$ define a tree, as this is a first order demand $\psi = \psi(\bar{z})$ on $\bar{z}$ and we can replace $t(\bar{z})$ by $\langle \psi \to \leqslant_1, \psi \to \leqslant_2, R \rangle$, so when $M \vDash \neg\psi[\bar{c}]$ we get a trivial tree.

Convension 5. (2) For notational simplicity, we shall deal only with trees $t(\bar{z})$ satisfying: $\bar{u}$, $\bar{v}$ are u, v, the equivalence relations $E_l^{t(\bar{c})}$ are the equality on their domains and $D_l(t(\bar{c}), M)$ $(l = 1, 2)$ are disjoint (so we can write $\leqslant$ instead $\leqslant_l$) and remember the function R is onto.

This has two justifications:

(i) The proofs are almost the same (only in Theorem 10 we have to deal also with the number of equivalence classes too).

(ii) We can replace each model M by M^{eq} as in [12, III §6], and use the theorems here on it.

We call a tree $t^M(\bar{c})$ trivial if in $D_2(t(\bar{c}), M)$ there is a last element (by $\leqslant_2^{t(\bar{c})}$).

(3) If $a \in D_1(t(\bar{c}), M)$ $b \leqslant_2^{t(\bar{c})} R^{t(\bar{c})}(a)$, we assume there is an element $a \mid b \in D_1(t(\bar{c}), M)$, $a \mid b \leqslant_1^{t(\bar{c})} a$, and $R^{t(\bar{c})}(a \mid b) = b$ (of course, it is unique) (again, we do not lose generality).

Theorem 6. *For every $\psi \in L_{\omega_1,\omega}(Q)$ (Q denotes the quantifier "there are uncountably many, L countable) the answer to*

(*) *ψ has a model M of cardinality $\aleph_1$ such that: for any tree $t_n^M(\bar{c})$ (first order) if $D_2(t(\bar{c}), M)$ has cofinality $\aleph_1$ (by $\leqslant_2^{t(\bar{c})}$), then all the branches of the tree are definable;*

is absolute i.e. there is a suitable completeness theorem.

More concretely, we construct from ψ a sentence $\psi^* \in L^*_{\omega_1,\omega}(Q)$. ($L^*$ some countable extension of L, Q the quantifier "there are uncountably many"), such that (*) holds iff ψ^* has a model (the latter is absolute by Keisler[5]).

Remark. (1) We can of course, make $\psi^* \in L^*_{\omega_1,\omega}$.

(2) We could add also the quantifier $Q^{cf}_{\aleph_0}$ (the cofinality of an order is $\aleph_0$).

Proof. Let ψ^* be the coniunction of the following sentences (part 7) is the main point)

(1) $\psi \wedge (Qx)(x = x)$.

(2) $\neg(\mathrm{Q}x)P(x)$, $(\exists x)P(x)$, and $\leqslant^0$ orders P in a dense order with no first nor last element.

(3) (the sentences saying that) $\leqslant^*$ is a total ordering of the model, each initial segment is countable, there is a first element, each element has a successor, and each element, is for some n the n-successor of a limit element or the first element.

(4) For each $<^*$-limit element a, $\{b : b <^* a\}$ is (the universe of) an elementary submodel (in first order language L).

(5) Let $t_n(\bar{x})$ $(n < \omega)$ enumerate the trees in L. For every $n < \omega$, and $<^*$-limit element a, and $\bar{c} \subseteq \{b : b <^* a\}$, for each $b \in D_2(t_n(\bar{c}))$, $b <^* a$, there is $b_1 \in D_2(t_n(\bar{c}))$, $b <_2^{t_n(\bar{c})} b_1$, and $b <^* b_1 <^* a$ provided that $P_n(\bar{c})$.

(6) For each $t_n(\bar{c})$, either $G_n(x; \bar{c})$ is a function from P into an unbounded subset of $D_2(t_n(\bar{c}))$, and $\neg P_n(\bar{c})$, *or* $G_n(x; \bar{c})$ is a function (from all the model) into $D_2(t_n(\bar{c}))$, its range is unbounded (by $<_2^{t_n(\bar{c})}$) and $x <^* y \Rightarrow G_n(x; \bar{c}) <_2^{t_n(\bar{c})} G_n(y; \bar{c})$ and for a limit a, $\{x\} \cup \bar{c} \subseteq \{b : b <^* a\} \Rightarrow G_n(x; \bar{c}) \leqslant^* a$, and $P_n(\bar{c})$.

(7) For each n and $\bar{c}$, the (partial) function $H_n(x; \bar{c})$ has domain $\{a \in D_1(t(\bar{c})) : R^{t(\bar{c})}(a)$ is the range of $G_n(x; \bar{c})\}$, is into P, and $a \leqslant_1^{t(\bar{c})} b$ implies $H_n(a; \bar{c}) \leqslant^0 H_n(b, \bar{c})$ (when both are defined). Now, in this context for $a \neq b$, $H_n(a, \bar{c}) = H_n(b, \bar{c})$ *iff* letting a^* be the $\leqslant^*$-last limit element $a^* \leqslant^* a$, there is a formula $\varphi(x; \bar{y}) \in \mathrm{L}$, and parameter $\bar{d} \in \{a' : a' <^* a^*\}$, such that:

$$(**) \qquad \{x : x <^* b, \varphi(x; \bar{d})\} = \{x \in D_1(t(\bar{c})) : x <^* b,\ x \leqslant_1^{t(\bar{c})} b\}.$$

Remark. Notice that by the conditions of $\varphi(x, \bar{d})$, it defines an initial segment of a branch of $t_n^M(\bar{c})$. Notice also that if $b^* \leqslant^* b$ is limit, $\bar{d}^1 \in \{x : x <^* b^*\}$, and $\{x : x <^* b,\ \varphi^1(x, \bar{d}^1)\} = \{x \in D_1(t(\bar{c}))_n : x <^* b,\ x \leqslant_1^{t(\bar{c})} b\}$ then $\varphi^1(x, \bar{d}^1) \equiv \varphi(x; \bar{d})$ (remember $\{x : x <^* b^*\}$ is (the universe of) an L-elementary submodel).

Remark. Part (7) says that the trees are similar to special Aronszajn trees, except the existence of definable branches.

Now we come to the proof.

The "if" part. Let M^* be a model of ψ^*, and M be its L-reduct, and we shall prove that M exemplifies (*). By (1) M is an uncountable model of ψ. So let $\bar{c} \in |M|$, $n > \omega$. If $M^* \models \neg P_n(\bar{c})$, then $D_1(t(\bar{c}), M)$ has cofinality $\aleph_0$ (with $\leqslant_2^{t(\bar{c})}$) (proved by (6); P is countable by (2)) hence the demand in (*) holds trivially.

So, we can assume $M^* \models P_n(\bar{c})$, and suppose $B \subseteq D_2(t(\bar{c}), M)$ is a branch, and let $B_1 = \{b \in B : R^{t(\bar{c})}(b)$ is in the range of $G_n(x; \bar{c})\}$, so B_1 is unbounded in B, and by (7), $b_1, b_2 \in B_1$, $b_1 \leqslant_1^{t(\bar{c})} b_2 \Rightarrow H_n(b_1, \bar{c}) \leqslant^0 H_n(b_2; \bar{c})$. As P^{M^*} is countable B_1 is $\aleph_1$-like, there are $b^* \in B_1$ and $d^* \in P^{M^*}$, such that $b^* \leqslant_1^{t(\bar{c})} b \in B_1 \Rightarrow H_n(b, \bar{c}) = d^*$. We apply the second phrase in (7), so letting b^+ be the last $\leqslant^*$ limit element $\leqslant^* b^*$; assuming $\bar{c} \subseteq \{x; x \leqslant^* b^+\}$. For every $b \in B_1$, $b^* \leqslant_1^{t(\bar{c})} b$, there are $\varphi_b \in \mathrm{L}$ and $\bar{d}_b \in \{c : c <^* b^+\}$ such that $\{x : x <^* b,\ \varphi_b(x, \bar{d}_b)\} = \{x : x <^* b,\ x \leqslant_1^{t(\bar{c})} b\}$

As $(|M|, \leqslant^*)$ is $\aleph_1$-like, some $\varphi, \bar{d}$ are $\varphi_b, \bar{d}_b$ for $\aleph_1$ b's, so $\varphi(x, \bar{d})$ define the branch B (in fact, each $\varphi_b(x, \bar{d}_b)$ is sufficient, and the cardinality consideration is unnecessary).

The "only if" part. Suppose M is a model of ψ of cardinality $\aleph_1$ in which every tree with $\mathrm{cf}(D_2(t(\bar{c}), M)) = \aleph_1$ does not have undefinable branches. We can find an increasing and continuous sequence $M_\alpha (\alpha < \omega_1)$ of countable elementary sub-models of M such that if $\bar{c} \in |M|$ the cofinality of $D_2(t(\bar{c}), M)$ (as ordered by $\leqslant_2^{t(\bar{c})}$), is $\aleph_1$, then there is $a \in D_2(t(\bar{c}), M_{\alpha+1})$ which is an upper bound of $D_2(t(\bar{c}), M_\alpha)$ (by $\leqslant_2^{t(\bar{c})}$).

By change of names we can assume $|M_\alpha| = \omega(1+\alpha)$.

Now we want to expand M to a model N satisfying (1)–(7). We let $P^N = \omega$, where $\leqslant^{*N}$ denotes the natural order on ω_1, $\leqslant^0$ denotes an order of ω of order type of the rationals, $P_n(\bar{c})$ holds iff $(D_2(t_n(\bar{c})), \leqslant_2^{t_n(\bar{c})})$, has cofinality ω_1 if $\rceil P_n(\bar{c})$ let $G_n(x, \bar{c})$ be a function from ω onto an unbounded subset of $(D_2(t_n(\bar{c})), \leqslant_2^{t(c)})$; if $P_n(\bar{c})$ let $G_n(x, \bar{c})$ be a function from ω_1 into $D_2(t_n(\bar{c}))$: we define by induction on x, such that $G_n(x, \bar{c})$ is bigger by $\leqslant_2^{t_n(\bar{c})}$ than any $y <^* x$ (possible as if x is a successor), then $G_n(x, \bar{c})$ should be $< x + \omega$ and bigger by $\leqslant_2^{t_n(\bar{c})}$ then $G_n(x-1, \bar{c})$, and possibly $x-1$, if x is limit, use the choice of the M_α's).

For notational simplicity we assume we have just one tree t_0 (i.e. $t_n = t_0$ and $\bar{z}$ is empty). Let $D_l = D_l(t, M)$, etc; and D_2^t = the range of G_0; $D_1^t = \{a \in D_1 : R^t(a) \in D_2^t\}$.

Now we shall define a generic extension V^1 of the universe (of sets) V, such that $\aleph_1^{V^1} = \aleph_1^V$, and in V^1 the required function H_0 exists; this will finish the proof.

The set P of forcing conditions is the set of functions h from finite subsets of D_1^t into $\mathbb{Q}$ such that

(I) h does not contradict the demands of H_0 in (7). When h is regarded as a part of H_0.

(II) if $a \in \mathrm{Dom}\, h$, and a^* is the first limit element, such that for some $\varphi \in \mathrm{L}$, $\bar{d} \in \{x : x < a^*\}$,

$$\{x : x <^* a,\ \varphi(x, \bar{d})\} = \{x : x <^* a,\ x \leqslant_1^t a\}$$

and $a^* < a$, and b is the $<^*$-minimal element such that $b \in D_1^t$, $a^* \leqslant b$ and $b \leqslant_1^t a$, *then* $b \in \mathrm{Dom}\, h$.

We order P by inclusion. We shall choose a generic $P' \subseteq P$, and let $H_0 = \bigcup\{h : h \in P'\}$. This will finish the proof provided we prove

Fact A: $\{h \in P$: a is in the domain of $h\}$ is dense in P for every $a \in D_1^t$.

Fact B: P satisfies the countable chain condition.

Proof of Fact B. Suppose $\{h_\alpha : \alpha < \omega_1\} \subseteq P$ are pairwise contradictory. w.l.o.g $|\mathrm{Dom}\, h_\alpha| = n$ for each $\alpha < \omega_1$. We prove the assertion by induction on n let $\mathrm{Dom}\ h_\alpha = \{a_0^\alpha, \ldots, a_{n-1}^\alpha\}$, $h_\alpha(a_n^\alpha) = q_n^\alpha \in P^N$; and as P^N is countable, w.l.o.g $q_m^\alpha = q_m$.

Sh:73

If $n=1$, necessarily every two a_0^α's are $\leqslant_1$-comparable, and distinct, so by (6) $\{a \in D_1^t$: for some α, $a \leqslant_1 a_0^\alpha\}$ is a branch of t^M, hence definable by some $\varphi(y, \bar{d})$ and $\bar{d} \in M_\beta$ for some β; hence for every large enough α $h_\alpha \cup h_{\alpha+1} \in P$.

So suppose $n>1$.

Notice D_2^t has (by $\leqslant_2^t$) order-type ω_1. By standard technique and renaming we can assume that for some m, k:

(α) for $1<m$ $R^t(a_1^\alpha)=R^t(a_1^0)$, and, there is a strictly increasing sequence (by $<^*$ and $\leqslant_2^t$) $\gamma_\alpha \in D_2^t(\alpha<\omega_1)$ such that $R^t(a_l^\alpha)<\gamma_0$ for $l<m$ and $\gamma_\alpha \leqslant R(a_l^\alpha)<\gamma_{\alpha+1}$ for $m \leqslant l<n$;

(β) for $l<k$, $a_l^\alpha=a_l^0$, and $\{a_l^\alpha : k \leqslant l<n\} \cup \{a_l^0 : l<k\}$ are pairwise distinct.

For $\alpha<\beta<\omega_1$, $h_\alpha \cup h_\beta \notin P$; clearly it is a well-defined function satisfying (II). Hence (I) fails, so there are $p(\alpha, \beta)<n$, $q(\alpha, \beta)<n$ such that $a_{p(\alpha, \beta)}^\alpha$, $a_{q(\alpha, \beta)}^\beta$ exemplify it.

As h_α, $h_\beta \in P$ and as $q_l^\alpha=q_l$, clearly $p(\alpha, \beta)$, $q(\alpha, \beta) \geqslant m$; and by (7) it is also clear that $a_{p(\alpha, \beta)}^\alpha$, $a_{q(\alpha, \beta)}^\beta$ are $\leqslant_1^t$-comparable.

Let D be uniform filter over ω_1. So for each α there are $p(\alpha)<n$, $q(\alpha)<n$ and $s(\alpha) \in\{0,1\}$ such that $S_\alpha=\{\beta<\omega_1 : p(\alpha, \beta)=p(\alpha),\ q(\alpha, \beta)=q(\alpha)$, and $a_{p(\alpha, \beta)}^\alpha \leqslant_1^t a_{q(\alpha, \beta)}^\beta$ iff $s(\alpha)=1\} \in D$. Similarly there are p, q and $s \in\{0,1\}$ such that

$$S=\{\alpha<\omega_1 : p(\alpha)=p,\ q(\alpha)=q,\ s(\alpha)=s\} \in D .$$

Suppose first $s=0$; so when $\alpha \in S$, $s(\alpha)=0$, and by Definition 1 (ii), $\{a_{q(\alpha)}^\beta : \beta \in S_\alpha\}$ are pairwise $\leqslant_1^t$-comparable, so necessarily they are in some branch, and (as they are pairwise distinct) form an unbounded set, but $a_{p(\alpha)}^\alpha$ is such a bound, contradiction.

So assume $s=1$, so for every $\alpha, \gamma \in S$, choose $\beta \in S_\alpha \cap S_\gamma \in D$, so $a_p^\alpha \leqslant_1^t a_q^\beta$, $a_p^\gamma \leqslant_1^t a_\alpha^\beta$, so again by Definition 1(ii) a_p^α, a_p^γ are $\leqslant_1^t$-comparable; so $\{a_p^\alpha : \alpha \in S\}$ is unbounded in some branch, which is necessarily definable by some $\varphi(x, \bar{d})$. Let $a^* \in M$ be limit (by $\leqslant^*$) $\bar{d} \in\{x : x<a^*\}$; and w.l.o.g. for every α and $l \geqslant m$, $a^*<a_l^\alpha$.

So necessarily for some $l<m$ $a_l^0 \leqslant_1^t a_p^\alpha$ for every $\alpha \in S$ so for $\beta \in S \cap S_\alpha$ $a<\beta$, $h_\alpha \cup h_\beta \notin P$, implies $h_\alpha \notin P$, contradiction.

Proof of fact A is left to the reader.

Definition 7. (1) A set W of pairs of formulas (in some fixed L) $\langle\varphi(x, \bar{y}); \psi(\bar{y})\rangle$ is called a set of cardinality witnesses.

(2) A (λ, W)-model is a model M such that $\langle\varphi(x, \bar{y}), \psi(\bar{y})\rangle \in W$, $M \vDash \psi[\bar{a}]$ implies $|\varphi(M, \bar{a})|=|\{b \in|M| : M \vDash \varphi[b, \bar{a}]\}|<\lambda$. Let $M<_W N$ mean $M<N$, and $M \vDash \psi[\bar{a}]$ implies $\varphi(N, a) \subseteq|M|$).

(3) The set W is T-closed if:

(i) If $\psi(\bar{y})=(\exists! x) \varphi(x ; \bar{y})$ (($\exists! x$) means there is a unique x such that . . .), *then* $\langle\varphi(x ; \bar{y}), \psi(\bar{y})\rangle \in W$.

(ii) If $\langle\varphi_l(x, \bar{y}_l), \psi_l(\bar{y})\rangle \in W$ for $l<n$ and $\varphi(x, \bar{y})=\varphi(x, \bar{y}_0, \ldots, \bar{y}_{n-1})= \bigvee_{l<n} \varphi_l(x ; \bar{y}_l)$, $\psi(\bar{y})=\bigwedge_{l<n} \psi_l(\bar{y}_l)$, *then* $\langle\varphi(x ; \bar{y}), \psi(\bar{y})\rangle \in W$.

(iii) Suppose $\langle \varphi_1(x; y, \bar{z}), \psi_1(y, \bar{z})\rangle \in W$ and $\langle \varphi_2(x, \bar{z}), \psi_2(\bar{z})\rangle \in W$ and T implies:

$$\varphi(x, \bar{z}) \to (\exists y)(\varphi_1(x; y, \bar{z}) \wedge \psi_1(y, \bar{z})),$$
$$(\exists x)\varphi_1(x; y, \bar{z}) \to \varphi_2(y, \bar{z}),$$
$$\psi(\bar{z}) \to \psi_2(\bar{z}).$$

Then $\langle \varphi(x, \bar{z}), \psi(\bar{z})\rangle \in W$. (This means that the union of a "small" family of "small" sets is "small".)

(4) The T-closure of W, $\mathrm{cl}_T(W)$ is the smallest T-closed W^1, $W \subseteq W^1$, (clearly it always exists.

(5) The formula $\varphi(x, \bar{a})$ is called W-small (in M) if $M \models \psi[\bar{a}]$ for some ψ satisfying $\langle \varphi(x, \bar{y}), \psi(\bar{y})\rangle \in \mathrm{cl}_T(W)$. A type p in M is small if $\bigwedge q$ is W-small for some finite $q \subseteq p$.

Definition 8. (T, W) satisfies the Chang condition if whenever $M_i (i < \alpha)$ is increasing by $<_W$, each M_i is λ-compact, then every W-small 1-type over $\bigcap_{i<\alpha} M_i$ of cardinality $<\lambda$ is realized.

Lemma 9. (1) *$M <_W N$ iff for every W-small formula $\varphi(x; \bar{a})$ in M, $\varphi(N, \bar{a}) \subseteq |M|$.* (2) *Suppose for every $\langle \varphi(x, \bar{y}), \psi(\bar{y})\rangle \in W$, there is $\theta(x, \bar{z})$ such that: if M is a model of T, $M \models \psi[\bar{a}]$, and $n < \omega$, $\bar{b}_1, \ldots, b_n \in \varphi(M, \bar{a})$ then for some $\bar{c} \in \psi_\varphi(M, \bar{a})(\psi_\varphi(x, \bar{a})$ small), $\theta(M, \bar{c}) = \{b_1, \ldots, b_n\}$. Then (T, W) satisfies Chang condition.* (3) *Suppose we work in $\mathfrak{C}^{eq}$ (see 12, III, §6] i.e. there are names for equivalence classes) and if $\langle \varphi(x, \bar{y}), \psi(\bar{y})\rangle \in W$, $x_1 E^\theta_{\bar{y}} x_2 \overset{\mathrm{df}}{=} (\forall z_1 \cdots)[\bigwedge_l \varphi(z, \bar{y}) \to \theta(x_1, z, \ldots) \equiv \theta(x_2, z_1, \ldots)]$, then $\langle (\exists x_1)(x = x_1/E^\theta_{\bar{y}}), \psi(\bar{y})\rangle \in W$. Then Chang condition holds.*

Proof. (1) Trivial.

(2) Is contained in the proof of Chang's two cardinal theorem, see e.g. [3].

(3) Not hard to prove.

Definition 10. Let T be a complete first order theory. W a set of cardinality witnesses in $L(T)$. For a model M of T, if $<(x, y, \bar{c})$ orders $\{x : (\exists y) <(x, y, \bar{c})\}$, we say $<(x, y, \bar{c})$ has W-small cofinality if for some W-small formula $\varphi(x, \bar{b})$ in M and $\theta(x, y, \bar{a})$, M satisfies

$$(\exists y) <(x, y, \bar{c}) \to (\exists z)\theta(x, z, \bar{a}),$$
$$\theta(x, z, \bar{a}) \to \varphi(z, \bar{b}),$$
$$\varphi(z, \bar{b}) \to (\exists x)((\exists y) <(x, y, \bar{c}) \wedge (\forall x_1)[<(x, x_1, \bar{c}) \to \neg\theta(x_1, z, \bar{a})].$$

Theorem 10. *Assume $\lambda = \lambda^{<\lambda}$ and $V = L$. Suppose T is a complete first order theory in $L = L(T)$ $|T| \leq \lambda$ W a set of witnesses for L, $W = \mathrm{cl}_T(W)$, and (T, W) satisfies Chang condition. Then T has a model M satisfying*

(i) *$\|M\| = \lambda^+$, M is λ-compact.*

(ii) $|\varphi(M, \bar{a})| \leq \lambda$ *iff* $\varphi(x, \bar{a})$ *is W-small.*

(iii) *if* $<(x, y, \bar{a})$ *orders* $\{x \in M : M \models (\exists y) < (x, y, \bar{a})\}$, *then this order has cofinality* $\leq \lambda$ *iff* $<(x, y, \bar{a})$ *has a W-small cofinality.*

(iv) *if* $t(\bar{c}) = t(\bar{c}, M)$ *is a w.s tree in M, and* $<_2^{t(c)}$ *has cofinality* λ^+, *then* $t(\bar{c})$ *has no undefinable branch.*

Remark. By [11, 2.2] Chang's conditions can be omitted.

Proof. As $V = L$ by Jensen [4] for $\delta < \lambda^+$, cf. $\delta = \lambda$ there are sets $S_\delta \subseteq \delta$, finite sequences $\bar{c}_\delta \in \delta$ and trees t_δ (in L) such that for any $S \subseteq \lambda^+$, $\bar{c} \in \lambda^+$, t, $\{\delta < \lambda^+ : \text{cf } \delta = \lambda, \bar{c}_\delta = \bar{c}, t_\delta = t, S \cap \delta = S_\delta\}$ is stationary, hence non-empty. This is the only use of $V = L$.

Now we define by induction on $\alpha < \lambda^+$ models M_α, and types p_α such that

(A) M_α has universe $\lambda(1 + \alpha)$ and is λ-compact for a successor;

(B) for $\beta < \alpha$, $M_\beta <_W M_\alpha$, if $\bar{a} \in M_\beta$, $\varphi(x, \bar{a})$ is not W-small, then $\varphi(M_\alpha, \bar{a}) \neq \varphi(M_\beta, \bar{a})$, and if $<(x, y, \bar{c})$ orders $\{x \in M_\beta : M_\beta = (\exists y) < (x, y, \bar{c})\}$, and does not have W-small cofinality, then for some $a \in M_\alpha - M_\beta$ for every $b \in M_\beta$, $M_\alpha = <(b, a, \bar{c}) \wedge (\exists y) < (a, y, \bar{c})$;

(C) M_α omits p_β for $\beta < \alpha$;

(D) p_α has the form $\{a <_1^{t_\alpha(\bar{c}_\alpha)} x : a \in S_\alpha\}$ when S_α is a branch of $t_\alpha(\bar{c}_\alpha, M_\alpha)$ which is not defined in M_α, p_α is contradictory otherwise (or not defined at all, if you want).

There are no difficulties: note that the types we omit has the property: for every $<\lambda$ formulas in the type there is a formula in the type implying all of them (see [2]).

Conclusion 11. If T is countable, W a set of cardinality witnesses, $\lambda = \aleph_0$, the conclusion of 10 holds without assuming $V = L$.

Proof. By 10 and 16.

Theorem 12. (1) Suppose λ is regular $\lambda^+ \geq |T|$, T a complete first order theory. Then T has a model of cardinality λ^+, in which no tree $t_r(\bar{c})$ has an undefined branch.

(2) In the model above we can demand every order defined in M with no last element, has cofinality λ^+, and if $\kappa = \max\{\kappa : \lambda^{<\kappa} \leq \lambda^+, \kappa \leq \lambda\}$, M is κ-saturated.

Proof. (1) For simiplicity we assume $\lambda \geq |T| + \aleph_1$. We choose pairwise disjoint stationary sets $S_{t(\bar{c})} \subseteq \{\delta < \lambda^+ : \text{cf } \delta = \lambda\}$ for t a tree (in L) $\bar{c} \in \lambda^+$ such that $\delta \in S_{t(\bar{c})}$ implies $\bar{c} \in \delta$. We define by induction on $\alpha < \lambda^+$ models M_α with universe $\lambda(1 + \alpha)$ such that

(A) for $\beta < \alpha$, $M_\beta < M_\alpha$;

(B) if $\delta \in S_{t(\bar{c})}$ and in $D_2^*(t(\bar{c}), M_\delta)$ there is no last element by $\leq_2^{t(\bar{c})}$, then for some $a \in D_2^*(t(\bar{c}), M_{\delta+1})$, $M_{\delta+1} \models b <^{t(\bar{c})} a$ for every $b \in D_2^*(t(\bar{c}), M_\delta)$;

(C) if $\delta<\alpha$, δ a limit point of $S_{t(\bar{c})}$, $a\in D_1^*(t(\bar{c}),M_\alpha)$, and $B=\{b\in D_1^*(t(\bar{c}),M_\alpha):M_\delta\models b\leqslant_1^{t(c)}a\}$ is a branch of $D_1(t(\bar{c}),M_\delta)$, then it is definable in M_δ.

Now $M=\bigcup_{\alpha<\lambda^+}M_\alpha$ is as required; clearly $\|M\|=\lambda^+$, suppose B is an undefined branch of some tree $t(\bar{c},M)$, $\leqslant_2^{t(\bar{c},M)}$ has cofinality λ^+, then $S=\{\alpha<\lambda^+:(M_\alpha,B\cap|M_\alpha|)<(M,B),\ \bar{c}\in M_\alpha\}$ is closed and unbounded, so some $\alpha\in S$ is in $S_{t(\bar{c})}$ and even a limit point of it, contradiction by (B) and (C).

For $\alpha=0$, α limit there are no problems. For $\alpha+1$ cf $\alpha\neq\lambda$ or $\alpha\in S_{t(c)}$, and $\leqslant_2^{t(\bar{c})}$ has a last element (in M_α) the proof is just easier than in the remaining case.

However, in order to carry the proof, we have also to choose for each $\delta\in S_{t(\bar{c})}$ which is a limit point of $S_{t(\bar{c})}$, elements $a^\delta_{\alpha,n}$ $(\alpha<\lambda,\ n<\omega)$ in $M_{\delta+1}$ and demand

(D) if $k<\omega$, $\delta(0)<\cdots<\delta(k)<\alpha$ and $\{a^{\delta(l)}_{\alpha,n}:\alpha<\lambda,\ n<\omega\}$ are defined for $l\leqslant k$, *then* for every $\bar{c}\in M_\alpha$ $\{\alpha:$ for each $l\leqslant k$ $\{a^{\delta(l)}_{\alpha,n}:n<\omega\}$ is an indiscernible set over $\bar{c}\cup\{a^{\delta(i)}_{\alpha,n}:i<l,\ n<\omega\}\}$ belongs to D_λ (= includes a closed unbounded subset of λ);

(E) each $a^\delta_{\alpha,n}$ belongs to $D_2(t(\bar{c}),M_{\delta+1})$, $a^\delta_{\alpha,0}\in M_\delta$, and $a<_2^{t(\bar{c})}a^\delta_{\alpha,1}$ for each $a\in D_2(t(\bar{c}),M_\delta)$. So we assume M_δ is defined, $\delta\in S_{t(c)}$, $\leqslant_2^{t(c)}$ has no last element (below everything is modulo $CD(M_\alpha)$).

We define by induction sets $\Gamma_i\ |i<\lambda|$ of formulas such that

(a) Γ_i is a set of formulas from $L(T)$ with parameters from M_α, and only the variables $\{x_j:j<i\}$ appear in it;

(b) $\Gamma_i^*=\Gamma_i\cup\{x_j\neq a:j<i\ a\in M_\alpha\}\cup\{b\leqslant_2^{t(c)}x_\beta:\beta$ even, $b\in D_2(t(\bar{c}),M_\alpha)\}$ is finitely satisfiable in M_α;

(c) Γ_i increase with i, and $|\Gamma_i|<\lambda$.

Our intention is that if the assignment $x_i\to c_i$ satisfies $\Gamma^*=\bigcup_{i<\lambda}\Gamma_i^*$ in some N, then $M_\alpha<N$, and the submodel of N with universe $|M_\alpha|\cup\{c_i:i<\lambda\}$ is suitable to be $M_{\alpha+1}$ (so c_0 shows B holds). We can take care of the requirements (D) (E) in stages $\omega^2\alpha+n(n<\omega)$ as in [11, §1, §2] so we concentrate on the rest. We have naturally λ assignments, so we can fulfil all of their in λ steps, provided we can do it for each one. We assume below w.l.o.g., i is odd.

(I) *Completeness (of Γ^*):* So Γ_i is given, φ a formula in $\{x_j:j<i\}$ and we want that $\varphi\in\Gamma_{i+1}$ or $\neg\varphi\in\Gamma_{i+1}$.

If $\Gamma_i\cup\{\varphi\}$ is as required for Γ_{i+1} we finish. Otherwise as (a), (c) cannot fail, (b) fails, so $\Gamma_i^*\cup\{a\neq x_i:a\in M_\alpha\}\cup\{\varphi\}$ is inconsistent (= not finitely satisfiable in M_α). Now Γ_i^* is consistent hence $\Gamma_i^*\cup\{a\neq x_i:a\in M_\alpha\}$ is consistent hence $\Gamma_i^*\cup\{a\neq x_i:a\in M_\alpha\}\cup\{\neg\varphi\}$ is consistent, hence $\Gamma_{i+1}=\Gamma_i\cup\{\neg\varphi\}$ is as required.

(II) $M_\alpha<N$: For this suppose $(\exists x)\varphi(x,\bar{y},\bar{b})$ is a formula, $\bar{y}\subseteq\{x_j:j<i\}$. By (I) we can assume $(\exists x)\varphi(x,\bar{y},\bar{b})\in\Gamma_i$ and we first try to define Γ_{i+1} as $\Gamma_i\cup\{\varphi(x_i,\bar{y},\bar{b})\}$, so we again can assume (b) fails. So $\Gamma_i^*\cup\{\varphi(x_i;\bar{y},\bar{b})\}\cup\{a\neq x_i:a\in M_\alpha\}$ is inconsistent, so $\Gamma_i^*,\ \varphi(x_i,\bar{y},\bar{b})\vdash\bigvee_{l=1}^n a_l=x_i$ for some $n<\omega$, $a_l\in M_\alpha$. If for some l, $\Gamma_i^*\cup\{\varphi(a_l,\bar{y},\bar{b})\}$ is consistent. Let it be Γ_{i+1}, otherwise we get: Γ_i^* is consistent, $(\exists x)\varphi(x,\bar{y},\bar{b})\in\Gamma_i\subseteq\Gamma_i^*$, $\Gamma_i^*,\ \varphi(x_i,\bar{y},\bar{b})\models\bigvee_{l=1}^n a_l=x_i$ and $\Gamma_i^*\vdash\neg\varphi(a_l,\bar{y},\bar{b})$, for each $l\leqslant n$ an easy contradiction.

(III) (C) (from the list of demands on M_α) is satisfied we can assume i is odd.

Sh:73

So suppose $\delta \leq \alpha$; δ a limit point of $S_{s(\bar{d})}$ (here $s(\bar{d})$ is a tree) $\rho < i$ and we want that in the end, $\{a \in M_\delta : a \leq_1^{s(\bar{d})} c_\rho\}$ will not be an undefined branch of $s(\bar{d})$ in M_δ.

As δ is a limit of $S_{s(\bar{d})}$, by (B) we can choose $b_\xi \in D_2(s(\bar{d}), M_\delta)$ $(\xi < \lambda)$ strictly increasing and unbounded by $\leq_2^{s(\bar{d})}$ there. For each ξ, we try to let Γ_{i+1} be $\Gamma_i \cup \{\psi_\xi\}$, where $\psi_\xi = \psi_\xi(x_\rho, x_i, b_\xi) = [x_i \leq_1^{s(\bar{d})} x_\rho \wedge R^{s(\bar{d})}(x_i) = b_\xi] \vee R^{s(\bar{d})}(x_\rho) \leq b_\xi]$ and let $\psi = \psi(x, z, y)$.

If this Γ_{i+1} is suitable, we finish, ($c_i \notin M_\alpha$ in the end) otherwise $\Gamma_i^* \cup \{\psi_\xi\} \cup \{a \neq x_i : a \in M_\alpha\}$ is inconsistent, hence has a finite inconsistent subset. Assuming w.l.o.g. Γ_i is closed under conjunctions, and remembering $\leq_2^{t(\bar{c})}$ is an order, we can assume this finite subset has the form $\Gamma_i^\xi = (\{\varphi_\xi\} \cup \{a_l^\xi \neq x_{i(m)} : l, m < n(\xi) \cup \{a_\xi \leq_2^{t(\bar{c})} x_{i(m)} : i(m) \text{ even}\} \cup \{\psi_\xi\}) \cup (\{a_l^\xi \neq x_i : l < n(\xi)\})$ where $\{a_l^\xi : l < n(\xi)\} \subseteq M_\alpha$, $\varphi_\xi \in \Gamma_i$, $a_\xi \in D_2(t(\bar{c}), M_\alpha)$, $\varphi_\xi(x_{i(0)}, \ldots, x_{i(n(\xi)-1)})$ and we can assume $0 = i(0)$, $i(1)$, $i(2), \ldots, i(n(\xi)-1) < i$, and let $i(n(\xi)) = i$, and $i(1) = \rho$. Now the number of possible $\langle \varphi_\xi, n(\xi) \rangle$ is $|T| + \aleph_0 < \lambda$, λ regular, so by renaming we can assume $\varphi_\xi = \varphi$, $n(\xi) = n$, and let $n(*) = n + 1$. let

$$\theta(z, y) = z \in D_1(s(\bar{d})) \wedge y \in D_2(s(\bar{d})) \wedge R^{s(\bar{d})}(z) = y \wedge (\forall v)(\exists \cdots x_m^r \cdots)_{m<n, r<n(*)}$$
$$\Big[\bigwedge \{x_{m(1)}^{r(1)} \neq x_{m(2)}^{r(2)} : m(1), m(2) < n,\ r(1) < r(2) < n(*)\}$$
$$\wedge \bigwedge_{r<n(*)} \varphi(x_0^r, x_1^r, \ldots, x_{n-1}^r) \wedge \bigwedge_{r<n(*)} \Big[v \in D_2(t(\bar{c})) \rightarrow \bigwedge_{\substack{i(m) \\ \text{even}}} v \leq_2^{t(\bar{c})} x_{i(m)}^r \Big]$$
$$\wedge \bigwedge_{r<n(*)} \psi(x_1^r, z, y) \Big].$$

Now

Fact α: For each $\xi < \lambda$, $M_\alpha \models (\exists^{\leq n} z)\theta(z, b_\xi)$.

For suppose we have distinct $z_k \in M_\alpha (k \leq n(*))$ $M_\alpha \models \theta[z_k, b_\xi]$, for at least one $k = k(0)$, $z_{k(0)} \notin \{a_l^\xi : l < n\}$. Looking at the definition of θ, for this $k(0)$ we choose as v the element $a_\xi \in D_2(t(\bar{c}), M_\alpha)$ (see above) and we get that some x_m^r satisfies the long conjunction. Again, as the sets $\{x_m^r : m < n\}$ for $r < n(*)$ are pairwise disjoint, at least one of them is disjoint to $\{a_l^\xi : l < n\}$, say for $r = r(0)$. But now we get Γ_i^ξ is consistent ($x_m^{r(0)}$ $(m < n)$, $z_{k(0)}$ stand for $x_{i(m)} (m < n)$, x_i resp.) note $i(1) = \rho$). Contradiction.

Fact β: If $M_\alpha < N$, $\xi < \lambda$, the assignment $x_j \rightarrow c_j$ $(j < i)$ satisfies Γ_i^*, *then* $\neg R^{s(\bar{d})}(c_\rho) <^{s(\bar{d})} b_\xi$ and element c satisfying $N \models R^{s(\bar{d})}(c) = b_\xi \wedge c \leq_1^{s(\bar{d})} c_\rho$, belongs to $\theta(M_\alpha, b_\xi)$.

Note that when $b_\xi \leq_2^{s(\bar{d})} R^{s(\bar{d})}(c_\rho)$ such $c \in N$ exists and is unique (by "what" is a tree here). If $c \notin M_\alpha$ the assignment $x_{i(n)} \rightarrow c_{i(m)}$ $(m < n) x_i \rightarrow c$ shows Γ_i^ξ is consistent contradiction. So suppose $c \in M_\alpha$, and we show $M \models \theta[c, b_\xi]$. The first three conjunctions hold easily, so let $v \in M_\alpha$ be arbitrary. We now choose $x_m^r (m < n)$ in M_α, by induction on r, so that the relevant parts of the conjunction hold. If we have defined up to r, in N there are suitable elements $(c_{i(0)}, \ldots)$ so as $M_\alpha < N$ there are in M_α.

Fact γ: If $M_\alpha \models \theta(z, y) \wedge z_1 \leq_1^{s(\bar{d})} z \wedge R^{s(\bar{d})}(z_1) = y_1$ then $M_\alpha \models \theta(z_1, y_1)$.

Proof. Remember we assume that $\Gamma_i^* \cup \{\psi_\xi\} \cup \{a \neq x_i : a \in M_\alpha\})$ is inconsistent for all $\xi < \lambda$. ψ_ξ is the disjunction of two formulas one of them is $R^{s(d)}(x) \leq b_\xi$, so $\Gamma_i^* \cup \{R^{s(\bar{d})}(x_p) \leq b_\xi\} \cup \{a \neq x_i : a \in M_\alpha\}$ is inconsistent, as neither of the first two sets in this union contains a formula with x_i we can omit the last set and still have an inconsistent set:

$$\Gamma = \Gamma_i^* \cup \{R^{s(\bar{d})}(x_p) \leq b_\xi\}.$$

Look at the definition of $\theta(y, z)$; by the inconsistency of Γ we can omit there from $\psi(x_1^r, z, y)$ the second disjunct, i.e. replace $\psi(x_1^r, z, y)$ by $z \leq x_i^r \wedge R^{s(\bar{d})}(z) = y$. Now it is clear that also z_1, y_1 will satisfy (the revised form of) θ.

Now we almost finish, let $\theta_1(z, y)$ be $z \in D_1(s(\bar{d})) \wedge y \in D_2(s(\bar{d})) \wedge R^{s(\bar{d})}(z) = y \wedge (\exists y_1, z_1)\ (\theta(z_1, y_1) \wedge z \leq_1^{s(\bar{d})} z_1 \wedge (\exists^{\leq n} x)\theta(x, y_1)$. It is easy to check that $\theta(z, b_\xi)$, $\theta_1(z, b_\xi)$ are equivalent, θ_1 satisfies facts α, β, γ, and $M_\alpha \models (\forall y)(\exists^{\leq n} z)\theta_1(z, y)$. By fact $\gamma\{z : (\exists y)\theta(z, y)\}$ form a subtree of $D_1(s(\bar{d}), M_\delta)$ and by fact β it suffices to prove each branch of its intersection with M_δ is definable in M_δ. As the number of branches in $\leq n$ this seems very reasonable.

By (D) the set $A = \{b \in M : b \leq_2^{s(d)} b_\xi$ for some $\xi < \lambda\}$ is not definable in M_α, $\alpha > \delta$. The important part of (D) is that it assures there are many $a_{\beta,0}^\delta$, $a_{\beta,1}^\delta$, one of them in A and the other outside A and they belong to an indiscernible set. Assume A is definable in M_α for $\alpha > \delta$, let us say by $\varphi(x, \bar{c})$, by (D) there is a set $\{a_{\beta,n}^\delta : n < \omega\}$ of indiscernibles over $\bar{c}$ (and other parameters), $a_{\beta,0}^\delta \in A$, $a_{\beta,1}^\delta \notin A$, so $\varphi(a_{\beta,0}^\delta, \bar{c})$ but $\neg\varphi(a_{\beta,2}^\delta, \bar{c})$ a contradiction.

(The other parts of (D) and (E) are needed to enable to keep the indiscernibility of these sets through all the construction, the way it is done and shown in [11].) For $\alpha = \delta$ let

$$\theta^*(z) = (\exists y)[\theta_1(z, y) \wedge \psi(y) \wedge (\forall y_1)(\exists z_1)\quad [y \leq_2^{s(\bar{d})} y_1 \wedge \psi(y_1) \rightarrow \theta_1(z_1, y_1) \wedge z \leq_1^{s(\bar{d})} z_1]].$$

It is easy to check that for some $y_0 \in \psi(M_\alpha)$, $\{z \in M_\alpha : M_\alpha \models \theta^*(z) \wedge R^{s(\bar{d})}(z) \geq y_0\}$, ordered by $\leq_1^{s(d)}$ is just the union of $\leq n$ chains, so each of them is definable in M_α, and they are the branches.

To be more concrete: Note that in $\{x : \theta^*(x)\}$ there are no $n+1$ pairwise incomparable elements (if there were such elements we could find for big enough ξ continuations of them of height b_ξ and so have $n+1$ z's satisfying $\theta(z, b_\xi)$ contradicting fact α). Let us define by induction finite sets A_l, $x \in A_l$ implies $\theta^*(x)$ and there are no incomparable y, z satisfying θ^* so that $x \leq_1^{s(\bar{d})} y$ and $x \leq_1^{s(\bar{d})} z$, and the elements of A_l are pairwise incomparable. We let $A_0 = \emptyset$, if A is defined we check if there is an x incomparable with all elements of A_l and satisfying $\theta^*(x)$, if there is no such x, $A_{l+1}=A_l$ if there is such x we check if it has y_1, z_1 as above, for each of them we check if it has such incomparable continuations, after no more than $n+1$ such steps we get to an element without such splitting (otherwise we have more than n incomparable elements satisfying θ^*) *and we add this element to* A_l to get A_{l+1}. Clearly $A_{n+2}=A_{n+1}$ let i be $\{x_l : l<l_0\}$, for each $l<l_0$ the element of $\{y:\theta^*(y)\wedge x_l<y\}$ are pairwise compatible, and this is the form of every branch so every branch is definable.

For $\alpha>\delta$ choose an element $y_0\in M_\alpha$, $M_\alpha\vDash b_\xi\leqslant_1^{s(\bar d)}y_0$ for every $\xi<\lambda$ such that we can get a maximal k satisfying: there are distinct $z_0,\ldots,z_k\in\theta(M_\alpha,b_\xi)$ and $z^0,\ldots,z^{k-1}\in\theta(M_\alpha,y_0)$ such that $M_\alpha\vDash z_l\leqslant_1^{s(\bar d)}z^l$ for $l<k$. So the z^l "induce" the branches, and by the induction hypothesis we finish.

We leave to the reader the proof of (2) and the case $\lambda<|T|+\aleph_1$ (the compactness totally makes no problem $\lambda<|T|$—we let $T=\bigcup_{\alpha+\lambda^+}T_\alpha$, M_α a model of T_α, for $\lambda=\aleph_0$ the uniformation of n_ξ is made when possible, and after each M_α is defined, we see that for many times we could have uniformized.

The following shows that we do not have to go through the set-theoretic considerations of $\diamondsuit_{\aleph_1}$ and forcing, we give less details and less generality.

Lemma 13. *Suppose we are given countable sets $A_n (n<\omega)$ such that $A_n\subseteq A_{n+1}$, $A_n\neq A_{n+1}$ and countably many relations $R_i(i<i_0)$ on $A_\omega=\bigcup_{n<\omega}A_n$, functions $F_i(i<i_1)$ from A_ω to A_ω, and functions $f_i(i<i_2)$ from A_ω to a set C ($i_0,i_1,i_2<\omega$) such that*

(*) *If $\varphi(\bar x_0,\ldots,\bar x^{n-1})$ is a conjunction of atomic formulas (i.e. $R_i(y_1,\ldots)$, $F_i(y_1,\ldots)=y$, $f_i(y_1,\ldots)=c$ where $c\in C$) and $\bar a_l\in(A_{l+1}-A_l)$ and $k<n$, and $\vDash\varphi[\bar a_0,\ldots,\bar a_{n-1}]$ then there are $n\leqslant m(k)<m(k+1)<,\ldots,$ and $\bar b_l\in(A_{m(l+1)}-A_{m(l)})$ for $k\leqslant l<n$ such that $\vDash\varphi[\bar a_0,\ldots,\bar a_{k-1},\bar b_k,\ldots,\bar b_{n-1}]$*

Then we can find countable sets $A_\alpha(\omega<\alpha<\omega_1)$ such that $A_\alpha(a<\omega_1)$ is increasing, $A_{\alpha+1}\neq A_\alpha$, and extend the definition of R_i, F_i, f_i to $A_{\omega_1}=\bigcup_{\alpha<\omega_1}A_\alpha$ (so R_i/A_ω is the "old" R_i) such that, for every limit $\delta\leqslant\omega_1$:

(a) *if $\varphi(\bar x_0,\ldots,\bar x_{n-1})$ is a conjunction of atomic formulas $0=\alpha(0)<\alpha(1)<\cdots<\alpha(n)<\delta$, $\bar a_l\in(A_{\alpha(l+1)}-A_{\alpha(l)})$, and $k\leqslant n$ and $\vDash\varphi[\bar a_0,\ldots,\bar a_{n-1}]$, then for every $\beta\geqslant\alpha(k)$ there are $\beta<\beta(k)<\beta(k+1)<\cdots<\beta(n-1)<\delta$ and $\bar b_l\in(A_{\beta(l+1)}-A_{\beta(l)})$ (for $k\leqslant l<n$) such that $\vDash\varphi[\bar a_0,\ldots,\bar a_{k-1},\bar b_k,\ldots,\bar b_{n-1}]$.*

(b) *if $\alpha_{(0)}<\cdots<\alpha_{(n)}<\delta$, $\bar a_l\in(A_{\alpha(l+1)}-A_{\alpha(l)})$, $=\varphi[\bar a_0,\ldots,\bar a_{n-1}]$, then there are $\beta(0)<\cdots<\beta(n)<\omega$, $\bar b_l\in(A_{\beta(l+1)}-A_{\beta(l)})$ such that $\vDash\varphi[\bar b_0,\ldots,\bar b_{n-1}]$.*

(c) *If $\varphi(\bar x,\bar y)$ is a disjunction of conjunctions atomic formulas, $A_\omega\vDash(\forall\bar x)(\exists\bar y)\varphi(\bar x,\bar y)$, then $A_{\omega_1}\vDash(\forall\bar x)(\exists\bar y)\varphi(\bar x,\bar y)$, and if $A_n\vDash(\forall\bar x)(\exists\bar y)\varphi(\bar x,\bar y)$ for each n, then $A_\alpha\vDash(\forall\bar x)(\exists\bar y)\varphi(\bar x,\bar y)$ for each α.*

(d) *If φ, $\psi_l(l<\omega)$ are disjunctions of conjunctions of atomic formula, and the following holds for $\alpha=\omega$, then it holds for $\alpha=\omega_1$: if $\bar a_l\in A_{\alpha(l)+1}$, $\alpha(0)<\cdots<\alpha(k)<\alpha$, $A_\alpha\vDash\varphi[\bar a_0,\ldots,\bar a_k]$, then there are $l<\omega$, $\bar b_l\in A_{\alpha(l)}$, $\bar c_l\in A_{\alpha(l)+1}$ such that $A_\alpha\vDash\psi_l[\bar a_0,\bar b_0,\bar c_0,\ldots]$.*

Proof. Quite straightforward. We define $\langle A_\alpha : \alpha < \omega(1+\xi)\rangle$ by induction on ξ, so that (a), (b), (c) are satisfied. For $\xi = 0$ $\langle A_\alpha : \alpha < \omega\rangle$ is given, for ξ limit there is nothing to prove. For $\xi = \rho + 1$, we define by induction on the formulas, $\varphi_n(\bar{x}_0^n, \ldots, \bar{x}_n^n, \bar{c}_n)$, which are conjunctions of atomic formulas, increasing with n i.e. φ_{n+1} includes φ_n, $\bar{x}_l^n = \langle x_l^0, \ldots, x_l^n\rangle$, $\bar{c}_n \in A_{\beta(n)}$, and there are $\beta(n) = \alpha(0) < \alpha(1) < \ldots$, $\bar{a}_l \in (A_{\alpha(l+1)} - A_{\alpha(l)}) \models \varphi_n[\bar{a}_0, \ldots, \bar{c}_n]$. We define them so that by essentially $A_{\omega(1+\rho)+n}$ is $A_{\omega(1+\rho)}$ together with $\{x_l^m : m < \omega,\ l < n\}$.

Theorem 14. *Let M be a model with a countable language L. t^M is an $\aleph_1$-like tree (i.e. $D_2(t, M)$ is $\aleph_1$-like by $\leq_2^1$, and for every $a \in D_2(t, M)$, $\{b : R(b) = a\}$ is countable), then there is a model N, elementary equivalent to M in $L(Q)$, and the tree t^N has no undefinable branches.*

Proof. We work in a variant of $L(Q)$: $M \models (Q\bar{x})\varphi(\bar{x}, \bar{a})$ means that there are $\aleph_1$ pairwise disjoint sequences $\bar{b}_i$ such that $M \models \varphi(\bar{b}_i, a)$ (this can be expressed in $L(Q)$). W.l.o.g. each formula of this logic is equivalence in M to a predicate. So we can assume every formula is equivalent to an atomic formula. We choose elementlary submodels M_n of M, $M_n < M_{n+1}$ such that: (for every formula φ)

(1) If $\bar{a} \in M_n$, $M \models \rceil(Qx)\varphi(x, \bar{a})$, then $\varphi(M, \bar{a}) \subseteq |M_n|$.

(2) If $\bar{a} \in M_n$ and $M \models (Q\bar{x})\varphi(\bar{x}, \bar{a})$, then for some $\bar{b} \in (|M_{n+1}| - |M_n|)$, $M \models \varphi[\bar{b}, \bar{a}]$.

We choose $A_n = |M_n|$, $C = Q$ (the rationals) with the natural R_i's (from the model), and clearly $(*)$ from 13 holds for formulas without f_i's. We shall choose $c_n \in D_2(t, M_{n+1}) - |M_n|$ and define a function f from $\{a \in D_1(t, M_\omega) : (\exists n)R(a) = c_n\}$ to Q, such that: if $R(a_1) = c_n$, $R(a_2) = c_m$, $a_1 <_1^t a_2$, then $f(a_1) \leq f(a_2)$, and $f(a_1) = f(a_2)$ iff for some $\bar{b} \in |M_{n+1}|$, and φ, $\varphi(x, \bar{b})$ defines a branch of the tree, and $a_1, a_2 \in \varphi(M_\omega, \bar{b})$. The point is to do it such that $(*)$ will still hold.

We do it in ω stages, in the nth stage only $c_0, \ldots, c_{n-1}$ where chosen, and f is defined for finitely many instances, $f(a)$ defined implies $R(a) \in \{c_0, \ldots, c_{n-1}\}$. We have to assure $(*)$ from 13 holds (encoding the relations by functions to $\{0, 1\}$) and this is not difficult. Now we use Theorem 13. We get a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ and let N be $\bigcup_{\alpha<\omega_1} A_\alpha$. If there were an undefined branch $\langle a_i : i < \omega_1\rangle$ in N then $\langle f(a_i) : i < \omega_1\rangle$ would be an increasing sequence of length ω_1 in the rationals, a contradiction. Using 13 it is not hard to see that N satisfies the other demands as well.

References

[1] J. Barwise and J. Schlipf, On recursively saturated models of arithmetic, in D. H. Saracino and V. B. Weispfenning, eds., Model Theory and Algebra, Memorial to Robinson, Lecture Notes in Math. (Springer, Berlin, 1975) 42–55.

[2] C.C. Chang, A note on two cardinal problems, Proc. Am. Math. Soc. 16 (1970) 1148–1155.

[3] C.C. Chang and H.J. Keisler, Model theory (North-Holland, Amsterdam, 1973).

[4] R.J. Jensen, The fine structure of the constructible universe, Ann. Math. Logic 4(1972) 229–308.
[5] H.J. Keisler, Logic with the quantifier "there exists uncountably many", Ann. Math. Logic 1 (1970) 1–94.
[6] H.J. Keisler, Models with tree structures, Symp. in honour of Tarski for his 60th birthday. Berkeley 1971, Proc. Symp. in Pure Math. Vol. XXV (Am. Math. Soc., Providence, RI, 1974) 331–349.
[7] M. Rubin and S. Shelah, On linearly ordered models and their end extensions, Am. Math. Soc. Notices 22 (1975) A-646.
[8] M. Rubin and S. Shelah, On the elementary equivalence of automorphism groups of Boolean Algebras, Downward Skolem–Löwenheim theorems and compactness of related quantifiers, J. Symbolic Logic, to appear.
[9] S. Shelah, Generalized quantifiers and compact logic, Trans. Am. Math. Soc. 204 (1975) 342–364.
[10] S. Shelah, Some remarks in model theory, Am. Math. Soc. 23 (1976) A-284.
[11] S. Shelah, Models with second order properties I, Ann. Math. Logic, 14 (1978) 57–72.
[12 S. Shelah, Classification theory (North-Holland, Amsterdam) to appear.
[13] S. Shelah, End extensions and number of non-isomorphic models, J. Symbolic Logic, to appear.