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A Hanf number for saturation and omission: the superstable case

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Suppose $\mathbf{t} = (T, T_1, p)$ is a triple of two theories in vocabularies $\tau \subset \tau_1$ with cardinality λ, $T \subseteq T_1$ and a τ_1-type p over the empty set that is consistent with T_1. We consider the Hanf number for the property "there is a model M_1 of T_1 which omits p, but $M_1 \restriction \tau$ is saturated". In [2], we showed that this Hanf number is essentially equal to the Löwenheim number of second order logic. In this paper, we show that if T is superstable, then the Hanf number is less than $\beth_{(2^{(2^\lambda)^+})^+}$.

1 Introduction

Let $\mathcal{K}$ be a family of classes of structures. An element $\mathbf{K} \in \mathcal{K}$ is *bounded* if it does not have arbitrarily large members. Hanf observed that if κ is the supremum of the maxima of the sizes of structures in the bounded $\mathbf{K} \in \mathcal{K}$, then any $\mathbf{K}$ that has a member of cardinality at least κ has arbitrarily large models [4]. In many cases this bound κ can be calculated (e.g., for a countable first order theory, it is $\aleph_0$ by the upwards Löwenheim-Skolem theorem). In this paper we call a Hanf number for a family $\mathcal{K}$ of classes *calculable* if it is bounded by a function that can be computed by an arithmetic function in ZFC (cf. Definition 1.1) and if not it is *incalculable*.

The following definition is more abstract than needed for this paper but we include it for comparison with other works where other Hanf functions are shown to be not calculable.

Definition 1.1 A function f (a class-function from cardinals to cardinals) is *strongly calculable* if f can (provably in ZFC) be defined in terms of cardinal addition, multiplication, exponentiation, and iteration of the $\beth$ function. A function f is *calculable* if it is (provably in ZFC) eventually dominated by a strongly calculable function. If not, it is *incalculable*.

We extend our work on Newełski's question from [6] about calculating the Hanf number of the following property:

Definition 1.2 Let $\tau \subset \tau_1$ be two vocabularies and let T and T_1 be theories in the vocabulary τ and τ_1, respectively. Let λ be a cardinal.

(1) If $\mathbf{t} = (T, T_1, p) = (T_\mathbf{t}, T_{1,\mathbf{t}}, p_\mathbf{t})$ be a triple such that $|\tau_1| \leq \lambda$, $T \subseteq T_1$ and p is a τ_1-type over the empty set consistent with T_1, then we say $M_1 \models \mathbf{t}$ if M_1 is a model of T_1 which omits p, but $M_1 \restriction \tau$ is saturated.
(2) Let $\mathbf{N}_\lambda$ be the set of $\mathbf{t}$ as in (1) with $|\tau_1| = \lambda$. By $\mathrm{H}(\mathbf{N}_\lambda)$ we denote the *Hanf number of* $\mathbf{N}_\lambda$, i.e., the least cardinal κ such that if $\mathbf{t} \in \mathbf{N}_\lambda$ has a model of cardinality κ, it has arbitrarily large models.[1]
(3) The *Hanf number of a logic* $\mathcal{L}$ (e.g., $\mathcal{L}_{\kappa^+,\kappa}$) is the least cardinal μ such that if an $\mathcal{L}$-sentence has a model in cardinal μ, then it has arbitrarily large models.

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[1] Thus, 'there is an M with cardinality κ such that $M \models \mathbf{t}$ and $\mathbf{t} \in \mathbf{N}_\lambda$' replaces the notation in [2], '$P^\lambda_N(\mathbf{K_t}, \kappa)$ holds'.

Under mild set theoretic hypotheses, we showed in [2] that $\mathrm{H}(\mathbf{N}_\lambda)$ essentially equals the Löwenheim number of second order logic, which is incalculable. In § 2 we restrict the question by requiring that the theory T be superstable; the number is then easily calculable in terms of Beth numbers.

The phenomenon that stability considerations can greatly lower Hanf number estimates was earlier explored in [5]. Work in preparation extends the current context to strictly stable theories.

Much of this paper depends on a standard way of translating between sentences in languages of the form $\mathcal{L}_{\lambda,\omega}(\tau)$ and first order theories in an expanded vocabulary τ that omit a family of types. This translation dates back to [3]; a short explanation of the process appears in [1, Chapter 6.1]. [7, Chapter VII.5] is an essential reference for this paper. In those references, these (equivalent) Hanf numbers of sentences and associated pair of a family of types and theory are calculated using the 'well-ordering number of a class'. We begin with a slight rewording of [7, Definition VII.5.1], using language from [3].

Definition 1.3

1. The Morley number $\mu(\lambda,\kappa)$ is the least cardinal μ such that if a first order theory T in a vocabulary of cardinality λ has a model in cardinality μ which omits a family of κ types over the empty set, it has arbitrarily large such models.
2. The well-ordering number $\delta(\lambda,\kappa)$ is the least ordinal α such that if a first order theory T in a vocabulary τ of cardinality λ, which includes a symbol $<$ has a model which omits a family κ types over the empty set and $<$ is well ordering of type α, then there is such a model where $<$ is not a well-order.

The connection between these two notions is discussed in [7, § VII].

Fact 1.4

1. *If* $\kappa > 0$, $\mu(\lambda,\kappa) = \beth_{\delta(\lambda,\kappa)}$.
2. *For every infinite cardinal* ϑ, $\mathrm{H}(\mathcal{L}_{\vartheta^+,\omega}) \leq \mu(\vartheta,1) < \beth_{(2^\vartheta)^+}$.

Proof. Item 1 is [7, VII.5.4]. Recall that Lopez-Escobar and Chang (cf., e.g., [3]) showed how to code sentences of $\mathcal{L}_{\lambda^+,\omega}$ as first order theories omitting types. More strongly (as in the proof of [7, Theorem VII.5.1.4]) one can code by omitting a single type. That $\mathrm{H}(\mathcal{L}_{\lambda^+,\omega}) < \beth_{(2^\lambda)^+}$ is now clear from [7, Theorems VII.5.4 & VII.5.5.7]. □

2 Computing $\mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda)$

2.1 Introduction

We study the following notions in this section.

Definition 2.1 Let $\mathbf{N}^{\mathrm{ss}}_\lambda$ denote the set[2] of $\mathbf{t}$ with $|\tau_1| = \lambda$ with the additional requirement that $T_\mathbf{t}$ is a superstable theory. Now we have the natural notion of the Hanf number, $\mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda)$ for this set: If $\mathbf{t} \in \mathbf{N}^{\mathrm{ss}}_\lambda$ has a model of cardinality $\geq \mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda)$, it has arbitrarily large models.

We will prove the following theorem:

Theorem 2.2

$$\mathrm{H}(\mathcal{L}_{\lambda^+,\omega}) < \beth_{(2^\lambda)^+} < \mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda) < \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\omega}) < \beth_{(2^{(2^\lambda)^+})^+}.$$

The first and fourth of these inequalities are immediate from Fact 1.4.2 taking ϑ first as λ and then as 2^λ.

In § 2.2, we give a rather involved proof that $\beth_{(2^\lambda)^+}$ is strictly less than $\mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda)$; together with the first inequality, this implies immediately that $\mathrm{H}(\mathcal{L}_{\lambda^+,\omega}) < \mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda)$. Note that less than or equal, $\mathrm{H}(\mathcal{L}_{\lambda^+,\omega}) \leq \mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda)$ is

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[2] Technically, this is not a set since a vocabulary is a sequence of relation symbols and we could use different names for the symbols; this pedantry can be avoided in at least two ways: restrict the symbols to come from a specified set; return to Tarski's convention of discussing not vocabularies but similarity types, the equivalence classes of enumerated vocabularies such that the ith symbol has arity n_i.

straightforward. Just set $\mathbf{t}$ as (T_0, T_1, p) where T_0 is pure equality and (T_1, p) encode a given sentence $\psi \in \mathcal{L}_{\lambda^+,\omega}$. Then T_0 is superstable and every model is saturated, so we have the desired interpretation.

The second and third inequalities are in §§ 2.2 and 2.3, respectively.

2.2 The second inequality

As noted the first inequality in Theorem 2.2 is standard. Thus by showing in Theorem 2.4, the second inequality appearing in Theorem 2.2 we will have $\mathrm{H}(\mathcal{L}_{\lambda^+,\omega}) < \mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda)$ and in fact

Theorem 2.3 $\mathrm{H}(\mathcal{L}_{\lambda^+,\omega}) < \beth_{(2^\lambda)^+} < \mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda)$.

The proof of the second inequality requires the construction of two triples $\mathbf{t}_1, \mathbf{t}_2$. The first characterizes 2^λ; the second $\beth_{(2^\lambda)^+}$.

Fix $\mathbf{L}_\lambda$ as the set of constructible sets of hereditary cardinality less than λ. In fact, any transitive model with cardinality λ of a very weak set theory would suffice.

Theorem 2.4 *There is a* $\mathbf{t}_1 = (T, T_1, p) \in \mathbf{N}^{\mathrm{ss}}_\lambda$ *with* $|\tau(T_1)| \le \lambda$ *such that:*

1. *There is an* $M \models \mathbf{t}_2$ *with cardinality* $\beth_{(2^\lambda)^+}$
2. *but there is no* $M \models \mathbf{t}_2$ *with cardinality greater than* $\beth_{(2^\lambda)^+}$.

Proof. We first introduce $\mathbf{t}_1 \in \mathbf{N}^{\mathrm{ss}}_\lambda$ and prove several properties of it.

In the first stage, we define T to be the prototypic superstable theory with λ-independent unary predicates, $P_{1,t}$ (i.e., in the vocabulary τ). The set P_2 will have cardinality λ when the type p is omitted. Then E_3 will be an extensional relation on $P_2 \times P_3$ so that P_3 has cardinality at most 2^λ. The function F maps the universe into P_3 while respecting the $P_{1,t}$ (and $\neg P_{1,t}$) and so that for any d, $F(d)$ codes via E_3 the τ-type of d. Thus saturation with respect to the $P_{1,t}$ guarantees that P_3 has cardinality exactly 2^λ.

Now we begin the formal development: τ contains unary predicates $P_{1,t}$ for $t \in \mathbf{L}_\lambda$; let T assert any Boolean combination of the $P_{1,t}$ is consistent. Let $\tau_1 = \tau \cup \{c_t : t \in \mathbf{L}_\lambda\} \cup \{P_2, P_3, E_2, E_3, F, F_1\}$. Here the P_i and E_i are unary and binary relations while F is a unary function and F_1 is a binary function. The various names of the symbols are chosen to keep the names different and not for any evocative purpose.

In the following definition, clauses 1) through 4) set the scene; clauses 5) through 7) are the crux of proving Lemmas 2.7 and 2.8; the other clauses are preparation for the proof of Lemma 2.11. □

Definition 2.5 Let T_1 be the τ_1-theory such that for any τ_1-structure M, $M \models T_1$ iff

1. $M{\restriction}\tau \models T$;
2. $\langle c_t^M : t \in \mathbf{L}_\lambda\rangle$ are pairwise distinct elements of M;
3. $c_t^M \in P_2^M$ for $t \in \mathbf{L}_\lambda$;
4. $E_2^M \subseteq P_2^M \times P_2^M$ and (P_2^M, E_2^M) is a model of $\mathrm{Th}(\mathbf{L}_\lambda, \varepsilon)$;
5. F^M is a function from M onto P_3^M such that $M \models (\forall x)[P_{1,t}(x) \leftrightarrow P_{1,t}(F(x))]$ for every $t \in \mathbf{L}_\lambda$;
6. (extensionality) $E_3^M \subseteq P_2^M \times P_3^M$ satisfies
$$(\forall x_1)(\forall x_2)[P_3(x_1) \wedge P_3(x_2) \to (\exists y)[P_2(y) \wedge (yE_3x_1 \leftrightarrow \neg yE_3x_2)])$$
(so, we know that $b_1 \neq b_2$ implies $A^2_{M,b_1} \neq A^2_{M,b_2}$ where $A^2_{M,b} := \{a \in P_2^M : aE_3^M b\}$);
7. for every $d \in M$:
$$d \in P^M_{1,t} \leftrightarrow c^M_t \in A^2_{M,F^M(d)}$$
(i.e., $c_t^M E_3 F^M(d)$);
8. if $t_1, t_2 \in \mathbf{L}_\lambda$, then $M \models c_{t_1} E_2 c_{t_2}$ if and only if $\mathbf{L}_\lambda \models t_1 \in t_2$;
9. for every $d \in P_3^M$, $\langle F_1(e, d) : e \in M\rangle$ is a 1-1 function from M into $\{f \in M : F^M(f) = d\}$.

The required τ_1-type p to complete the definition of $\mathbf{t}$ is
$$p(x) = \{P_2(x)\} \cup \{x \neq c_t : t \in \mathbf{L}_\lambda\}.$$

Definition 2.6 We call a model M of T_1 *standard* when

1. $c_t^M = t$ for $t \in \mathbf{L}_\lambda$;
2. $P_2^M = \{t : t \in \mathbf{L}_\lambda\}$;
3. for every $X \subseteq \mathbf{L}_\lambda$, there is $b \in P_3^M$ such that $A^2_{M,b} = X$ (note that b is unique by Definition 2.5.6).

Lemma 2.7 *If M is a standard model of T_1 then M omits p and $M \restriction \tau$ is saturated. That is, $M \models \mathbf{t}$.*

Proof. Condition 2) asserts p is omitted. A saturated model M of T is one where for each $X \subseteq \mathbf{L}_\lambda$, $q_X(x) = \bigwedge_{t \in \mathbf{L}_\lambda} P_{1,t}(x)^{t \in X}$ is realized $|M|$ times. Clauses 1) and 3) of Definition 2.6 guarantee there is a $b_X \in P_3^M$ such that $X = A^2_{M,b_X}$. By clause 7) of Definition 2.5 any element of $(F^M)^{-1}(b_X)$ realizes q_X. Finally, Condition 8) of Definition 2.5 implies $|(F^M)^{-1}(b_X)| = |M|$. □

Lemma 2.8 *If $M \models \mathbf{t}_1$ ($\mathbf{t}_1 = (T, T_1, p)$) then M is (isomorphic to) a standard model of T_1.*

Proof. Since M omits p, $P_2^M = \{c_t^M : t \in \mathbf{L}\}$. The map $g : P_2^M \to \mathbf{L}_\lambda$ is a well-defined isomorphism from (P_2^M, E_2^M) by condition 9 of Definition 2.5.

Finally, condition 3) of Definition 2.6 holds because the saturation provides a realization d_X of $q_X(x) = \bigwedge_{t \in \mathbf{L}_\lambda} P_{1,t}(x)^{t \in X}$. But then by condition 7) of Definition 2.5, $A^2_{M,F^M(d_X)} = \{c_t^M \in P_2^M : c_t^M E_3^M F^M(d_X)\} = X$ as required. □

Now we introduce a second triple, $\mathbf{t}_2 = (T, T_2, p) \in \mathbf{N}^{\mathrm{ss}}_\lambda$, which will have models up to but no larger than $\beth_{(2^\lambda)^+}$.

We add a new predicate P_4 which will be linearly (indeed well) ordered by $<_4$ (say as $\{a_\alpha : \alpha < \beta\}$). The well-ordering is obtained by first requiring that every non-empty definable subset of P_4 has a least element and then showing every countable subset is definable (using a set theoretic structure imposed on P_2). A function G_4 projects the universe onto P_4. The predicate R will code the subsets of $(G_4)^{-1}(a_\alpha)$ by elements of $(G_4)^{-1}(a_{\alpha+1})$. An induction then bounds the cardinality of any model of $\mathbf{t}_2$.

Definition 2.9 The signature τ_2 expands τ_1 by adding $<_4, P_4, R, G_4, G_5$ where G_4 is unary and G_5 is binary. Let T_2 be the τ_2-theory such that for any τ_1-structure M, $M \models T_2$ iff

1. $M \restriction \tau_1 \models T_1$;
2. $<_4$ is a linear order of P_4^M satisfying the first order theory of well-orderings;
3. G_4 is a function from M onto P_4^M;
4. if $c R^M d$ then $G_4(c) <_4^M G_4(d)$;
5. if $d_1 \neq d_2$ then for some $d \in M$, $d R^M d_1 \equiv \neg d R^M d_2$;
6. G_5^M is a partial function from $P_4^M \times P_4^M$ to P_3^M. If $d \in P^4_M$ then and $d_1 <_4^M d_2 <_4^M d$ then $G_5(d_1, d) \neq G_5(d_2, d)$ (so every proper initial segment of P_4^M has cardinality $\leq |P_3^M|$);
7. for any $\varphi(x, \mathbf{y}) \in L(\tau_2)$ and $\mathbf{d}$ in M with the same length as $\mathbf{y}$, $\{a \in P_4^M : M \models \varphi(a, \mathbf{d})\}$ is either empty or has a first element.

Observe that $\mathbf{t}_2 = (T, T_2, p) \in \mathbf{N}^{\mathrm{ss}}_\lambda$.

Lemma 2.10 *There is an $M \models \mathbf{t}_2$ of cardinality $\beth_{(2^\lambda)^+}$.*

Proof. We define a τ_2-model as follows: The universe of M is $\mathbf{V}_{(2^\lambda)^+}$ where $\mathbf{V}_\alpha$ is the αth stage in the cumulative hierarchy. Furthermore,

$$c_t^M = t \text{ for } t \in \mathbf{L}_\lambda,$$

$$P_2^M = \mathbf{L}_\lambda = \{c_t^M : t \in \mathbf{L}_\lambda\},$$

$$E_2^M = \varepsilon \restriction P_2^M,$$

$$P^3_M = \wp(\mathbf{L}_\lambda),$$

$$E_3^M = \{(t, s) : t \in \mathbf{L}_\lambda, s \in \wp(\mathbf{L}_\lambda), t \in s\},$$

$$P^M_{1,t} = \bigcup\{Y_s : s \in \wp(\mathbf{L}_\lambda) \wedge t \in s\},$$

$$P^M_4 = (2^\lambda)^+, \text{ and}$$

$$R^M = \varepsilon{\restriction}V_{(2^\lambda)^+}.$$

Let $\langle Y_s : s \in \wp(\lambda)\rangle$ be a partition of $|M| = V_{(2^\lambda)^+}$ such that each Y_s has cardinality $||M||$ and $s \in Y_s$ (This implies $Y_s \cap \wp(\mathbf{L}_\lambda) = \{s\}$). Furthermore, The function F^M maps M to $\wp(\mathbf{L}_\lambda) = P^3_M$: for $d \in Y_s$, we have $F(d) = s$. We choose $F^M_1 : M \times M \to M$ as 1-1 as function that maps $M \times Y_s$ into Y_s and let $<^M_4$ be the natural order $\varepsilon{\restriction}(2^\lambda)^+$ on P^M_4. We let G^M_5 be any binary function from P^M_4 into P^M_3 such that if $d_1 <^M_4 d_2 <^M_4 d$ then $G_5(d_1, d) \neq G_5(d_2, d)$, and finally G^M_4 maps M to P^M_4 by $G^M_4(a)$ is the least α such that $a \in V_{\alpha+1}$.

We have defined M to satisfy T_2; it omits p by clause 2) and 3). And conditions 5) and 6) show $M{\restriction}\tau_1$ is a standard model of T_1. So by Lemma 2.7, $M{\restriction}\tau$ is saturated and $M \models \mathbf{t}_1$. □

Now we show $\mathbf{t}_2$ has no model of cardinality greater than $\beth_{(2^\lambda)^+}$.

Lemma 2.11 *If $M \models \mathbf{t}_2$, $|M| \le \beth_{(2^\lambda)^+}$.*

Proof. Since $M \models \mathbf{t}_2$, $M \models \mathbf{t}_1$ so by Lemma 2.8, without loss of generality, $M{\restriction}\tau_1$ is standard.
The proof of Lemma 2.11 is easy from the next two claims. □

Claim 2.12 *If $d_n <^4_M d_{n-1} <^4_M d$ for $n < \omega$, there is $\varphi(x, \mathbf{y}) \in L(\tau_2)$ and $\mathbf{a} \in M$ with same length as $\mathbf{y}$ such that $\{b : M \models \varphi(b, \mathbf{a})\} = \{d_n : n < \omega\}$.*

Proof. Let $D = \{d_n : n < \omega\}$ and writing $G^M_5(d, d_n)$ as b_n, let $B = \{b_n : n < \omega\}$. Our goal is to show that D is τ_2-definable. Letting $g(x)$ denote the function $G^M_5(d, x)$, $D = g^{-1}(B)$ and g is τ_2-definable. So it suffices to show B is definable.

Let X_n denote $A^2_{M,b_n} = \{a \in M : P^M_2(a) \wedge aE^M_2 b_n\}$ and set $X = \bigcup_{n<\omega}\{n\} \times X_n$. Now X_n is a τ_2-definable subset of P^M_2, so X is definable in (P^M_2, E^M_2) using the set theoretic operations. And $b \in B$ if and only $(n, b)) \in X$ so B is τ_2-definable. □

Claim 2.13 *The structure $(P^M_4, <^M_4)$ is well-ordered of order type at most $(2^\lambda)^+$.*

Proof. By Lemma 2.12 the range of any infinite descending sequence is τ_2-definable. But then by clause 7 of Definition 2.9, it has a least element.

Since $M{\restriction}\tau_1$ is standard, $|P^M_3| = 2^\lambda$. Then condition 6) of Definition 2.9 implies the order type of $(P^M_4, <^M_4)$ is at most $(2^\lambda)^+$. So we can write $(P^M_4, <^M_4)$ as $\langle a_\alpha : \alpha < \beta\rangle$ for some $\beta \le (2^\lambda)^+$.

To complete the proof, we can show by induction that $|\{a \in M : G^M_4(a) < a_\alpha\}| \le \beth_\alpha(\lambda)$. Condition 4) of Definition 2.9 shows that for any $d \in M$ with $G_4(d) = a_\alpha$, if bRd, then $G^M_4(b) < a_\alpha$. So with respect to R, d codes a subset of $(G^M_4)^{-1}(a_\alpha)$. Since R is extensional by Condition 4) of Definition 2.9, the $|(G^M_4)^{-1}(a_\alpha| \le \beth_{\alpha+1}(\lambda)$ and we finish by induction.

Theorem 2.4 is immediate from Lemmas 2.10 and 2.11. □

The Hanf number $\mathcal{L}_{\lambda^+,\omega}$ can consistently be less than $\beth_{(2^\lambda)^+}$. Cf. [8] and [7, Chapter VII.5].

2.3 The third inequality

The previous section completed the proof of the second inequality in Theorem 2.2; we pass to the third.

Lemma 2.14 $\mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda) < \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\omega})$.

We first show $\mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda) \le \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\omega})$ by constructing a map from $\mathbf{t} \in \mathbf{N}^{\mathrm{ss}}_\lambda$ to $\psi_{\mathbf{t}} \in \mathcal{L}_{(2^\lambda)^+,\omega}$; this construction depends heavily on the superstability hypothesis. Then we use some observations on Hanf numbers to show the inequality is strict: $\mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda) < \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\omega})$.

Lemma 2.15 *For each $\mathbf{t} = (T, T_1, p) \in \mathbf{N}^{\mathrm{ss}}_\lambda$, there is a τ_2 extending τ_1 with $|\tau_1| = |\tau_2| = \lambda$ and a $\psi \in \mathcal{L}_{(2^\lambda)^+,\omega}$ such that* $\mathrm{spec}(\mathbf{t}) = \mathrm{spec}(\psi)$.

Proof. In preparation consider a fixed saturated model M of cardinality $2^{|T|}$ of T.

To form τ_2, we add to τ_1 constants $\langle c_\alpha : \alpha < (2^\lambda)\rangle$ and as described below a unary predicate P and $2n + 1$-ary functions H_n and function symbols $G_{n,m}$ indexing maps from N to N^m by $n + m$ tuples. The type p will be $\{P(x)\} \cup \{x \neq c_\alpha : \alpha < 2^\lambda\}$.

We write $\mathbf{y}_1 \equiv_{\mathbf{x}_1,\mathbf{x}_2} \mathbf{y}_2$ if for every $\varphi(\mathbf{v}, \mathbf{w})$, $\varphi(\mathbf{x}_1, \mathbf{y}_1) \leftrightarrow \varphi(\mathbf{x}_2, \mathbf{y}_2)$. Furthermore, $\mathbb{F}(\mathbf{x})$ is the collection (for $i < 2^\lambda$) of m-ary finite equivalence relations $E_i(\mathbf{x}; \mathbf{y}, \mathbf{z})$ over $\mathbf{x}$. $\square$

Definition 2.16

1. Recall that a model N is $\mathrm{F}^{\mathrm{a}}_{\kappa(T)}$-saturated (also called a-saturated and ε-saturated) if each strong type over a set of size less than $\kappa(T)$ is realized. For superstable theories $\mathrm{F}^{\mathrm{a}}_{\kappa(T)}$-saturated is just $\mathrm{F}^{\mathrm{a}}_{\aleph_0}$-saturated (each strong type over a finite set is realized).
2. A model N is strongly ω-homogeneous, if any two finite sequences that realize the same type over the empty set are automorphic in N.

Fact 2.17 *If a model M of a stable theory is $\mathrm{F}^{\mathrm{a}}_{\kappa(T)}$-saturated and for each set of infinite indiscernibles $\mathbf{I}$ in M there is an equivalent set of indiscernibles $\mathbf{I}'$ in M that has cardinality $|M|$, then M is saturated.* [7, III.3.10.2]

Notation 2.18 Now let $\psi_{\mathbf{t}} \in \mathcal{L}_{(2^\lambda)^+,\omega}(\tau_1)$ assert of a model N:

1. The specified $p = p_{\mathbf{t}}$ is omitted;
2. P^M satisfies the complete $\tau \cup \{c_\alpha : \alpha < 2^\lambda\}$-diagram of M, the saturated model of cardinality $2^{|T|}$ specified at the beginning of the proof;
3. $N \restriction \tau$ is strongly ω-homogeneous (add $2n + 1$-ary functions H_n satisfying if $\mathbf{a} \equiv \mathbf{b}$, $(\lambda z)H_n(\mathbf{a}, \mathbf{b}, z)$ is a τ-automorphism taking $\mathbf{a}$ to $\mathbf{b}$; this is expressible since having the same type over the empty set is expressible in $\mathcal{L}_{(2^\lambda)^+,\omega}(\tau)$);
4. for each $n < \omega, m < \omega$ there is an $(n + m + 1)$-ary function G (into m-tuples) such that G witnesses that for any n-tuple $\mathbf{a}$ and m-tuple $\mathbf{b}$, if $\operatorname{stp}(\mathbf{b}/\mathbf{a})$ is realized infinitely often then it is realized $|N|$-times. Formally, N satisfies:

$$(\forall \mathbf{x}\mathbf{z}) \left[\bigwedge_{n<\omega} ((\exists^{\geq n} \mathbf{y}) \bigwedge_{E_i(\mathbf{x};\mathbf{y},\mathbf{z}) \in \mathbb{F}(\mathbf{x})} E_i(\mathbf{x}, \mathbf{z}, \mathbf{y})) \to (\forall w) \bigwedge_{E_i(\mathbf{x};\mathbf{y},\mathbf{z}) \in \mathbb{F}(\mathbf{x})} E_i(\mathbf{x}, \mathbf{z}, G(\mathbf{x}, \mathbf{z}, w)) \right]$$

where for every $\mathbf{x}, \mathbf{z}$ $\lambda w G(\mathbf{x}, \mathbf{z}, w)$ is a 1-1 map from N into N^m.

We shall now prove $\mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda) \leq \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\omega})$: Suppose $\mathbf{t} \in \mathbf{N}^{\mathrm{ss}}_\lambda$, that $\psi_{\mathbf{t}}$ is as specified in Notation 2.18, and $N \models \psi_{\mathbf{t}}$. Since $|N| = |N \restriction \tau_1|$, it suffices to show $N \restriction \tau_1 \models \mathbf{t}$. Clearly N omits $p_{\mathbf{t}}$ and $(N \restriction P^N) \restriction \tau$ is superstable; in particular it is an elementary extension of the $\mathrm{F}^{\mathrm{a}}_{\aleph_0}$-saturated model M. We must show $N \restriction \tau$ is saturated.

But N is strongly ω-homogeneous by Notation 2.18.2. So each consistent strong type p over an n-element sequence $\mathbf{a} \in N$ is realized by $H_n^{-1}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ where $\mathbf{b} \in M$ satisfies $\mathbf{a} \equiv \mathbf{b}$ and $\mathbf{c} \models H_n(\mathbf{a}, \mathbf{b}, q)$ (where the H_n transforms a strong type over $\mathbf{a}$ to one over $\mathbf{b}$ in the natural manner). Thus, N is $\mathrm{F}^{\mathrm{a}}_{\aleph_0}$-saturated so we may apply Fact 2.17.

Every infinite indiscernible set J in $N \restriction \tau$ is based on a finite $\mathbf{d}$. That is, there is a strong type p_J over $\mathbf{d}$ such that J contains infinitely many realizations of p. Now the conditions on G of Notation 2.18.4 guarantee that p_J is realized $|N|$ times in N as required.

Now we strengthen the inequality $\mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda) \leq \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\omega})$ to a strict one.

Claim 2.19 $\mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda) < \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\omega})$.

Proof. [7, Theorems VII.5.4 and VII.5.5.1] show for any μ, $\operatorname{cf}(\mathrm{H}(\mathcal{L}_{\mu^+,\omega})) \geq \mu^+$; in particular, $\operatorname{cf}(\mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\omega}) \geq (2^\lambda)^+$. But there are at most 2^λ classes in $\mathbf{N}^{\mathrm{ss}}_\lambda$ and Lemma 2.15 implies that the supremum of the spec of each is less than $\mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\omega})$. Thus, $\mathrm{H}(\mathbf{N}^{\mathrm{ss}}_\lambda) < \mathrm{H}(\mathcal{L}_{(2^\lambda)^+,\omega})$. $\square$

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