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ON T_3-TOPOLOGICAL SPACE OMITTING MANY CARDINALS

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Abstract

We prove that for every (infinite cardinal) λ there is a T_3-space X with clopen basis, 2^{2^λ} points such that every closed subspace of cardinality $< 2^{2^\lambda}$ has cardinality $< \lambda$.

§0 Introduction

Juhász has asked on the spectrums $c-sp(X) = \{|Y| : Y$ an infinite closed subspace of $X\}$ and $w-sp(X) = \{w(Y) : Y$ a closed subspace of $X\}$. He proved [2] that if X is a compact Hausdorff space, then $|X| > \kappa \Rightarrow c-sp(X) \cap [\kappa, \sum_{\lambda<\kappa} 2^{2^\lambda}] \neq \emptyset$ and $w(X) > \kappa \to w-sp(X) \cap [\kappa, 2^{<\kappa}] \neq \emptyset$. So under GCH the cardinality spectrum of a compact Hausdorff space does not omit two successive regular cardinals, and omits no inaccessible. Of course, the space $\beta(\omega)\backslash\omega$, the space of nonprincipal ultrafilters on ω, satisfies $c-sp(X) = \{\beth_2\}$. Now Juhász and Shelah [3] show that we can omit many singular cardinals, e.g. under GCH for regular $\lambda > \kappa$ there is a compact Hausdorff space X with $c-sp(X) = \{\mu : \mu \leq \lambda, \mathrm{cf}(\mu) \geq \kappa\}$; see more there and in [5]. In fact [3] constructs a Boolean Algebra, so relevant to the parallel problems of Monk [4]. Here we deal with the noncompact case and get a strong existence theorem. Note that trivially for a Hausdorff space $X, |X| \geq \kappa \Rightarrow c-sp(X) \cap [\kappa, 2^{2^\kappa}] \neq \emptyset$, using the closure of any set with κ points, so our result is in this respect best possible.

We prove

THEOREM 0.1. *For every infinite cardinal λ there is a T_3 topological space X, even with clopen basis, with 2^{2^λ} points such that every closed subset with $\geq \lambda$ points has $|X|$ points.*

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In §1 we prove a somewhat weaker theorem but with the main points of the proof present, in §2 we complete the proof of the full theorem.

§1

THEOREM 1.1. *Assume* $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. *Let* $\mu = 2^\lambda, \kappa = \mathrm{Min}\{\kappa : 2^\kappa > \mu\}$. *There is a Hausdorff space* X *with a clopen basis with* $|X| = 2^\kappa$ *such that: if for* $Y \subseteq \lambda$ *is closed and* $|Y| < |X|$ *then* $|Y| < \lambda$.

PROOF. Let $S \subseteq \{\delta < \kappa : \delta \text{ limit}\}$ be stationary. Let $T_\alpha = {}^\alpha\mu$ for $\alpha \leq \kappa$ and let $T = \bigcup_{\alpha \leq \kappa} T_\alpha$. Let $\zeta_\alpha = \cup\{\mu\delta + \mu : \delta \in S \cap (\alpha + 1)\}$ and let $\zeta_{<\alpha} = \cup\{\zeta_\beta : \beta < \alpha\}$.

Stage A: We shall choose sets $u_\zeta \subseteq T_\kappa$ (for $\zeta < \mu \times \kappa$). Those will be clopen sets generating the topology. For each ζ we choose (I_ζ, J_ζ) such that: I_ζ is a $\triangleleft$-antichain of $({}^{\kappa>}\mu, \triangleleft)$ such that for every $\rho \in T_\kappa, (\exists!\alpha)(\rho \restriction \alpha \in I_\zeta)$ and $J_\zeta \subseteq I_\zeta$ and we shall let $u_\zeta = \bigcup_{\nu \in J_\zeta} (T_\kappa)^{[\nu]}$ where $(T_\kappa)^{[\nu]} = \{\rho \in T_\kappa : \nu \triangleleft \rho\}$. Let $I_{\alpha,\zeta} = T_\alpha \cap I_\zeta, J_{\alpha,\zeta} = T_\alpha \cap J_\zeta$ but we shall have $\alpha \notin S \Rightarrow I_{\alpha,\zeta} = \emptyset = J_{\alpha,\zeta}$.

Stage B: Let $Cd : \mu \to {}^{\lambda^+>}(T_{<\kappa})$ be onto such that for every $x \in \mathrm{Rang}(Cd)$ we have $\mathrm{otp}\{\alpha < \mu : Cd(\alpha) = x\} = \mu$.

We say α codes x (by Cd) if $Cd(\alpha) = x$.

Stage C: Definition: For $\delta \leq \kappa$ we call $\bar\eta$ a δ-candidate if

- (a) $\bar\eta = \langle \eta_i : i \leq \lambda \rangle$
- (b) $\eta_i \in T_\delta$
- (c) $(\exists \gamma < \delta)(\bigwedge_{i<j<\lambda} \eta_i \restriction \gamma \neq \eta_j \restriction \gamma)$
- (d) for every odd $\beta < \delta$, we have $Cd(\eta_\lambda(\beta)) = \langle \eta_i \restriction \beta : i \leq \lambda \rangle$
- (e) $\eta_\lambda(0)$ codes $\langle \eta_i \restriction \gamma : i < \lambda \rangle$, where $\gamma = \gamma(\eta \restriction \lambda) = \mathrm{Min}\{\gamma < \delta : i < j < \lambda \Rightarrow \eta_i \restriction \gamma \neq \eta_j \restriction \gamma\}$, it is well defined by clause (c) and
- (f) $\eta_\lambda(0) > \sup\{\eta_i(0) : i < \lambda\}$.

Stage D: Choice: Choose $A_{\xi,\varepsilon} \subseteq \lambda$ for $\xi < \mu \times \kappa, \varepsilon < \lambda$ such that:

$$\xi < \mu \times \kappa \ \& \ \varepsilon_1 < \varepsilon_2 < \lambda \Rightarrow |A_{\xi,\varepsilon_1} \cap A_{\xi,\varepsilon_2}| < \lambda \text{ and even } = \emptyset$$

and

$$\xi_1 < \ldots < \xi_n < \mu \times \kappa, \varepsilon_1 \ldots \varepsilon_{n_1} < \lambda \Rightarrow \bigcap_{\ell=1}^{n} A_{\xi_\ell,\varepsilon_\ell} \text{ is a stationary subset of } \lambda.$$

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Let $\Xi = \{\{(\xi_1,\varepsilon_1),\dots,(\xi_n,\varepsilon_n)\} : \xi_1,\dots,\xi_n < \mu\times\kappa$ is with no repetitions and $\varepsilon_1,\dots,\varepsilon_n < \lambda\}$ and for $x\in\Xi$ let $A_x = \bigcap\limits_{\ell=1}^{n} A_{\xi_\ell,\varepsilon_\ell}$. Let D_0 be a maximal filter on λ extending the club filter such that $x \in \Xi \Rightarrow A_x \neq \emptyset \bmod D_0$.

For $A \subseteq \lambda$ let

$$\mathcal{B}^+(A) = \{x\in\Xi : A\cap A_x = \emptyset \bmod D_0 \text{ but } y \subsetneqq x \Rightarrow A\cap A_y \neq \emptyset \bmod D_0\}$$

$$\mathcal{B}(A) =: \mathcal{B}^+(A)\cup\mathcal{B}^+(\lambda\backslash A).$$

FACT. *$\mathcal{B}(A) =: \mathcal{B}^+(A)\cup\mathcal{B}^+(\lambda\backslash A)$ is predense in Ξ i.e.*

$$(\forall x\subseteq\Xi)(\exists y\in\mathcal{B}(A))(x\cup y\in\Xi).$$

PROOF. If $x\in\Xi$ contradict it then we can add to D_0 the set $\lambda\backslash(A_x\cap A)$ getting D_0'. Now D_0' thus properly extends D_0 otherwise $A_x\cap A = \emptyset \bmod D_0$ hence, let $x'\subseteq x$ be minimal with this property so $x'\in \mathcal{B}^+(A)$ and x by assumption satisfies: $\neg(\exists y\in\Xi)(x\cup y\in\mathcal{B}(A))$ so try $y = x$. For every $z\in\Xi$ we have $A_z\neq\emptyset$ mod D_0.

FACT. *$|\mathcal{B}(A)|\leq\lambda$ for $A\subseteq\lambda$.*

PROOF. Let $\mathbf{B}_0$ be the Boolean Algebra freely generated by $\{x_{\xi,\varepsilon} : \xi < \mu\times\kappa,\varepsilon < \lambda\}$, by Δ-system argument, except $x_{\xi,\varepsilon_1}\cap x_{\xi,\varepsilon_2} = 0$ if $\varepsilon_1\neq\varepsilon_2$; clearly $\mathbf{B}_0$ satisfies λ^+-c.c.

Let $\mathbf{B}^*$ be the completion of $\mathbf{B}_0$. Let f^* be a homomorphism from $\mathcal{P}(\lambda)$ into $\mathbf{B}^*$ such that $C\in D_0\Rightarrow f^*(C) = 1_{\mathbf{B}^*}$ and

$$f(A_{\xi,\varepsilon}) = x_{\xi,\varepsilon}.$$

[Why exists? Look at the Boolean Algebra $\mathcal{P}(\lambda)$ let $I_\lambda = \{A\subseteq\lambda : \lambda\backslash A\in D_0\}$ and $\mathfrak{A}_0 = I_\lambda\cup\{\lambda\backslash A : A\in I_\lambda\}$ is a subalgebra of $\mathcal{P}(\lambda)$, and let $I_\lambda\cup\{A_{\xi,\varepsilon} : \xi\leq\mu\times\kappa,\varepsilon = \lambda\}$ generate a subalgebra $\mathfrak{A}$ of $\mathcal{P}(\lambda)$; it extends $\mathfrak{A}_0$. Let $f_0^* : \mathfrak{A}_0\to\mathbf{B}_0$ be the homomorphism with kernel I_λ. Let f_1^* be the homomorphism from $\mathfrak{A}$ into $\mathbf{B}_0$ extending f_0 such that $f_1^*(A_{\xi,\varepsilon}) = x_{\xi,\varepsilon}$, clearly exists and is onto. Now f_1^* as $\mathbf{B}^*$ is a complete Boolean Algebra can be extended to a homomorphism f_2^* from $\mathcal{P}(\lambda)$ into $\mathbf{B}^*$. Clearly $\mathrm{Ker}(f_2^*) = \mathrm{Ker}(f_2^*) = \mathrm{Ker}(f_0^*) = I_\lambda$ so f_1^* induces an isomorphism from $\mathcal{P}(\lambda)/D_0$ onto $\mathrm{Rang}(f_1^*)\subseteq\mathbf{B}^*$, so the problem translates to $\mathbf{B}^*$. So $\mathbf{B}_0$ satisfies the λ^+-c.c and is a dense subalgebra of $\mathbf{B}^*$ hence of range(f_2^*), so this range is a λ^+-c.c. Boolean Algebra hence $\mathcal{P}(\lambda)/D_0$ satisfies the fact.]

Let $\mathbf{B}_\gamma^*$ be the complete Boolean subalgebra of $\mathbf{B}^*$ generated (as a complete subalgebra) by $\{x_{\xi,\varepsilon} : \xi < \gamma,\varepsilon < \lambda\}$. Clearly $\mathbf{B}^* = \bigcup\limits_{\gamma<\kappa}\mathbf{B}_\gamma^*, \mathbf{B}_\gamma^*$ increasing with γ.

Stage E: We choose by induction on $\delta \in S$ the following

(A) $w_{\delta,\zeta} \subseteq T_\delta$ (for $\zeta < \mu\delta + \mu$) and $J_{\delta,\zeta} \subseteq I_{\gamma,\zeta} \subseteq w_{\delta,\zeta}$

(B) for each δ-candidate $\bar{\eta} = \langle \eta_i : i \le \lambda \rangle$, a uniform filter $D_{\bar{\eta}}$ on λ extending the filter D_0

(C) for each $\nu_1 \ne \nu_2$ in T_δ for some $\zeta < \mu \times \delta + \mu$ we have $\{\nu_1, \nu_2\} \subseteq w_{\delta,\zeta}$ and: $(\exists \delta' \in S \cap (\delta+1))(\nu_1 \in J_{\delta',\zeta}) \equiv (\exists \delta' \in S \cap (\delta+1))(\nu_2 \in J_{\delta',\zeta})$

(D) if $n < \omega, \mu \times \delta + \mu \le \xi_1 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda$ then $\bigcap\limits_{\ell=1}^{n} A_{\xi_\ell,\varepsilon_\ell} \ne \emptyset \bmod D_{\bar{\eta}}$

(E) if $\delta_1 \in S \cap \delta, \bar{\eta}$ is a δ-candidate and $\bar{\eta} \restriction \delta_1 = \langle \eta_i \restriction \delta_1 : i \le \lambda \rangle$ is a δ_1-candidate *then* $D_{\bar{\eta}\restriction\delta_1} \subseteq D_{\bar{\eta}}$

$(F)_1$ $\eta \in w_{\delta,\zeta}$ *iff* $(\exists \delta')(\delta' \in S \cap (\delta+1)\ \&\ \eta \restriction \delta \in I_{\delta',\zeta})$

$(F)_2$ if $\bar{\eta} = \langle \eta_i : i \le \lambda \rangle$ is a δ candidate and $\eta_\lambda \in w_{\delta,\zeta}$ *then* $\{i < \lambda : \eta_i \in w_{\delta,\zeta}\} \in D_{\bar{\eta}}$ and $\langle (\exists \delta' \in S \cap (\delta+1))(\eta_\lambda \restriction \delta' \in J_{\delta',\zeta}) \rangle = \mathrm{LIM}_{D_{\bar{\eta}}} \langle (\exists \delta' \in S \cap (\delta+1))(\eta_i \restriction \delta' \in J_{\delta',\zeta}) : i < \lambda \rangle$

$(F)_3$ $w_{\delta,\zeta}$ satisfies the following

(a) it is empty if $\zeta < \zeta_{<\delta}$

(b) is a pair if $\zeta \in [\zeta_{<\delta}, \zeta_\delta)$

(c) otherwise $w_{\delta,\zeta}$ is the disjoint union $w^0_{\delta,\zeta} \cup w^1_{\delta,\zeta} \cup w^2_{\delta,\zeta}$ where $w^0_{\delta,\zeta} = \{\eta \in T_\delta : (\exists \delta' \in S \cap (\delta+1))(\eta \restriction \delta' \in w_{\delta',\zeta})\}$, $w^1_{\delta,\zeta} = \{\eta \in T_\delta : \eta \notin w^0_{\delta,\zeta}\}$ and for no κ-candidate $\bar{\eta}$ is $\eta \triangleleft \eta_\lambda\}$, $w^2_{\delta,\zeta} = \{\eta \in T_\delta : \eta \notin w^0_{\delta,\zeta} \cup w^1_{\delta,\zeta}$ and for some δ-candidate

$$\bar{\eta}, \eta_\lambda = \eta$$

and

$$(\forall i < \lambda)(\exists \delta' \in S \cap \delta)(\eta_i \restriction \delta' \in w_{\delta',\zeta})$$

and the set

$$\{i < \lambda : (\exists \delta' \in S \cap \delta)(\eta_i \restriction \delta' \in J_{\delta,\zeta})\}$$

or its complement belongs to $D_{\bar{\eta}\restriction\delta^*}$ for some $\delta^* < \delta\}$

$(F)_4$ $I_{\delta,\zeta} = w^2_{\delta,\zeta} \cup w^1_{\delta,\zeta}$

(G) if $\bar{\eta}$ is a δ-candidate and $B \subseteq \lambda, f^*(B) \in \mathbf{B}^*_{\mu\times(\delta+1)}$, *then* $B \in D_{\bar{\eta}} \vee (\lambda \backslash B) \in D_{\bar{\eta}}$.

We can ask more explicitly: there is an ultrafilter $D'_{\bar{\eta}}$ on the Boolean Algebra $\mathbf{B}^*_{\mu\times(\delta+1)}$ such that $D_{\bar{\eta}} = \{B \subseteq \lambda : f^*(B) \in D'_{\bar{\eta}}\}$.

The rest of the proof is split into carrying the construction and proving it is enough.

Stage F: This is Enough: First for every κ-candidate $\bar\eta$ lets $D_{\bar\eta} = \cup\{D_{\bar\nu,\delta} : \delta \in S, \bar\nu$ is a δ-candidate and $i \le \lambda \Rightarrow \nu_i \triangleleft \eta_i\}$. Easily $D_{\bar\eta}$ is a uniform ultrafilter on λ. Let us define the space. The set of points of the space is $T_\kappa = {}^\kappa\mu$ and a subbase of clopen sets will be u_ζ: for $\zeta < \mu \times \kappa$ where u_ζ is defined as $u_\zeta =: \cup\{(T_\kappa)^{[\nu]} : \nu \in J_\zeta\}$ and $J_\zeta =: \bigcup_{\delta\in S} J_{\delta,\zeta}$. Now note that

(α) $I_\zeta = \cup\{I_{\delta,\zeta} : \delta \in S\}$ is an antichain and $\forall\rho \in T_\kappa \exists!\delta(\rho \restriction \delta \in I_{\delta,\zeta})$
[Why? We prove this by induction on $\rho(0)$ and it is straightforward. In details, it is an antichain by the choice $I_{\delta,\zeta} = w^2_{\delta,\zeta}, w^2_{\delta,\zeta} \subseteq T_\delta\backslash w^0_{\delta,\zeta}$. As for the second phrase by the first there is at most one such δ; let $\rho \in T_\kappa$ and assume we have proved it for every $\rho' \in T_\kappa$ such that $\rho'(0) < \rho(0)$. By the definition of κ-candidate, if there is no κ-candidate $\bar\eta$ with $\eta_\lambda = \rho$, *then* for every large enough $\delta \in S$, there is no δ-candidate $\bar\eta$ with $\eta_\lambda = \rho \restriction \delta$, hence for any such $\delta, \rho \restriction \delta$ belongs to $w^0_{\delta,\zeta}$ or to $w^1_{\delta,\zeta}$, in the first case for some $\delta' \in \delta \cap S$ we have $(\rho \restriction \delta) \restriction \delta' \in I_{\delta',\zeta}$ so $\rho \restriction \delta' \in I_{\delta',\zeta}$ and we are done, in the second case $\rho \restriction \delta \in w^1_{\delta,\zeta} \subseteq I_{\delta,\zeta}$ and we are done. So assume that there is a κ-candidate $\bar\eta$ with $\eta_\lambda = \rho$, by the definition of a candidate it is unique and $i < \lambda \Rightarrow \eta_i(0) < \rho(0)$, so for each $i < \lambda$ there is $\delta_i \in S$ such that $\eta_i \restriction \delta_i \in I_{\delta_i,\zeta}$ and let $\gamma = \mathrm{Min}\{\gamma < \mu : \langle \eta_i \restriction \gamma : i < \lambda\rangle$ is with no repetition$\}$. Let $A = \{i < \lambda : \eta_i \restriction \delta_i \in J_{\delta,\zeta}\}$ so for some $\beta < \mu$ we have $f_2^*(A) \in \mathbf{B}^*_\beta$. For $\delta \in S$, which is $> \sup[\{\gamma, \delta_i : i < \lambda\}]$ we get $\rho \restriction \delta \in w_{\delta,\zeta}$ and we can finish as before.]

(β) X is a $\underline{T_3}$ space
[why? as we use a clopen basis we really need just to separate points which holds by clause (C), i.e. if $\nu_1 \neq \nu_2 \in X$ then for some $\delta \in S$ we have $\nu_1 \restriction \delta \neq \nu_2 \restriction \delta$ and apply clause (C) to $\nu_1 \restriction \delta, \nu_2 \restriction \delta$]

(γ) $|X| = \mu^\kappa = 2^\kappa$
[why? as T_κ is the set of points of X]

(δ) suppose $Y = \{\eta_i : i < \lambda\} \subseteq X = T_\kappa$ and $\bigwedge_{i<j} \eta_i \neq \eta_j$. We need to show that $|c\ell(Y)|$ large, i.e. has cardinality 2^κ.

Choose γ such that $\langle \eta_i \restriction \gamma : i < \lambda\rangle$ is with no repetitions.

Let

$$W_{\bar\eta} = \{<>\} \cup \{\rho : \text{for some } \alpha \le \kappa, \rho \in T_\alpha, \rho(0) \text{ codes } \langle \eta_i \restriction \gamma : i < \lambda\rangle, \\ \rho(0) > \sup\{\eta_i(0) : i < \lambda\} \text{ and} \\ (\forall\beta < \ell g(\rho))(\beta \text{ odd } \Rightarrow \rho(\beta) \text{ codes } \langle \eta_i \restriction \beta : i < \lambda\rangle^\frown\langle \rho \restriction \beta\rangle\}.$$

So clearly:

(i) $W_{\bar\eta} \cap T_1 \neq \emptyset$

(ii) $W_{\bar\eta}$ is a subtree of $(\bigcup_{\alpha\leq\kappa} T_\alpha, \triangleleft)$ (i.e. closed under initial segments, closed under limits),

(iii) every $\rho \in W_{\bar\eta} \cap T_\alpha$ where $\alpha < \kappa$ has a successor and if α is even has μ successors.

So $|W_{\bar\eta} \cap T_\kappa| = \mu^\kappa$.

So enough to prove

(*) if $\rho \in W_{\bar\eta} \cap T_\kappa$ then $\rho \in c\ell\{\eta_i : i < \lambda\}$.

Let $\bar\eta = \langle \eta_i : i < \lambda\rangle, \eta_\lambda = \rho, \bar\eta' = \bar\eta\char`^\langle\rho\rangle$ and the filter $D_{\bar\eta'} = \cup\{D_{\langle \bar\eta'_i \restriction \delta : i \leq \lambda\rangle} : \delta \in S$ and $\delta \geq \gamma\}$ is a filter by clause (E) and even ultrafilter by clause (G).

Now for every ζ, by clause $(\mathrm{F})_2$ for δ large enough

$$\text{Truth Value}(\rho \in u_\zeta) = \lim\nolimits_{D_{\langle \eta'_i \restriction \delta : i \leq \delta\rangle}} \langle \text{Truth Value}(\eta_i \in u_\zeta) : i < \lambda\rangle.$$

As $\{u_\zeta : \zeta < \mu \times \kappa\}$ is a clopen basis of the topology, we are done.

Stage G: The construction:
We arrive to stage $\delta \in S$. So for every δ-candidate $\bar\eta = \langle \eta_i : i \leq \lambda\rangle$, let

$$D'_{\bar\eta} = \cup\{D_{\langle \eta_i \restriction \delta_1 : i \leq \lambda\rangle} : \delta_1 \in \delta \cap S \text{ and } \langle \eta_i \restriction \delta_1 : i \leq \lambda\rangle \text{ a } \delta_1\text{-candidate}\}.$$

NOTE. $|T_\delta| = \mu$ by the choice of κ.
Let $<^*_\delta$ be a well ordering of T_δ such that: $\nu_1(0) < \nu_2(0) \Rightarrow \nu_1 <^*_\delta \nu_2$.
Hence

(*) $\langle \eta_i : i \leq \lambda\rangle$ is a δ-candidate $\Rightarrow \bigwedge_{i<\lambda} \eta_i <^*_\delta \eta_\lambda$.

So let $\{\langle \nu_{1,\zeta}, \nu_{2,\zeta}\rangle : \zeta_{<\delta} \leq \zeta < \zeta_\delta\}$ list $\{(\nu_1, \nu_2) : \nu_1 <^*_\delta \nu_2\}$; such a list exists as $\zeta_\delta \geq \zeta_{<\delta} + \mu$ and $|T_\delta| = \mu$. Now we choose by induction on $\zeta < \zeta_\delta$ the following

(α) $D^\zeta_{\bar\eta}$ for $\bar\eta$ a δ-candidate when $\zeta \geq \zeta_{<\delta}$

(β) $w^*_{\delta,\zeta}, I_{\delta,\zeta}, J_{\delta,\zeta}$

(γ) $D^{\zeta_{<\delta}}_{\bar\eta}$ is $D'_{\bar\eta}$ which was defined above

(δ) $D^\zeta_{\bar\eta}$ for ζ in $[\zeta_{<\delta}, \zeta_\delta]$ is increasing continuous

(ϵ) if $s^n < \omega, \zeta_{<\delta} \leq \zeta \leq \xi_1 < \xi_2 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda^+$ then $\bigcap_{\ell=1}^{n} A_{\xi_\ell,\varepsilon_\ell} \neq \emptyset \bmod D^\zeta_{\bar\eta}$

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(ζ) $D^{\zeta+1}_{\bar\eta}, I_{\delta,\zeta}, J_{\delta,\zeta}$ satisfies the requirement (F)$_2$

(η) $\nu_{1,\zeta} \in J_{\delta,\zeta} \Leftrightarrow \nu_{2,\zeta} \notin J_{\delta,\zeta}$ *or* $\nu_{1,\zeta}, \nu_{2,\zeta} \in w^0_{\delta,\zeta}$

(θ) $D^{\zeta}_{\bar\eta}$ is $D^{\delta'}_{\bar\eta} + \{A_{\zeta_1,\varepsilon_\eta(\zeta_0)} : \zeta_1 < \zeta\}$ for some function $\varepsilon_{\bar\eta} : [\zeta_{<\delta}, \zeta) \to \lambda$.

NOTE. For $\zeta = 0$, condition (ε) holds by the induction hypothesis (i.e. clause (D)) and choice of D'_η (and choice of $A_{\xi,\varepsilon}$'s if for no $\delta_1, \bar\eta \restriction \delta_1$ is a δ_1-candidate.

(ι) if $\zeta < \zeta_{<\delta}$ *then:*

$$w_{\delta,\zeta} = w^0_{\delta,\zeta} \cup w^1_{\delta,\zeta} \cup w^2_{\delta,\zeta} \text{ are defined as in } (F)_2$$

$$I^\zeta_{\delta,\zeta} = w^1_{\delta,\zeta} \cup w^2_{\delta,\zeta}$$

$$\begin{aligned} J^\zeta_{\delta,\zeta} = \{\eta \in T_\delta : & \delta \in w^2_{\delta,\zeta} \text{ and for some } \delta\text{-candidate } \bar\eta \text{ we have } \eta_\lambda = \eta \\ & \text{hence } (\forall i < \lambda)(\exists \delta' \in S \cap \delta)[\eta_i \restriction \delta' \in w_{\delta',\zeta}] \\ & \text{and } \{i < \lambda : (\exists \delta' \in S \cap \delta)[\eta_i \restriction \delta' \in J_{\delta',\zeta}]\} \text{ belongs to } D'_{\bar\eta}\}. \end{aligned}$$

[Note in the context above, by the induction hypothesis $(\exists \delta' \in S \cap \delta)[\eta_i \restriction \delta' \in w_{\delta',\zeta}]$ is equivalent to $(\exists \delta' \in S \cap \delta)[\eta_i \restriction \delta' \in I_{\delta',\zeta}]$ and thus δ' is unique. Of course, they have to satisfy the relevant requirements from (A)-(G)].

The cases $\zeta \leq \zeta_{<\delta}, \zeta$ limit are easy.

The crucial point is: we have $\langle D^\zeta_{\bar\eta} : \bar\eta \text{ a } \delta\text{-candidate}\rangle$ and $\zeta \in [\zeta_{<\delta}, \zeta_\delta)$ and we should define $w_{\delta,\zeta}, I_{\delta,\zeta}$ and $D^{\zeta+1}_{\bar\eta}$ to which the last stage is dedicated.

Stage H: Define by induction on $n < \omega$,

$$w^\zeta_0 = \{\nu_{1,\zeta}, \nu_{2,\zeta}\}$$

$$w^\zeta_{n+1} = \{\eta^\rho_i : i < \lambda, \rho \in w_n \text{ and } \bar\eta^\rho \text{ is a } \delta\text{-candidate with } \eta^\rho_\lambda = \rho\}.$$

Note that $\eta^\rho_i <^*_\delta \rho$.

Let $w = w_{\delta,\zeta} = I_{\delta,\zeta} = \bigcup_{n<\omega} w^\zeta_n$, so $|w_{\delta,\zeta}| \leq \lambda$.

We need: to choose $J_{\alpha,\zeta} \cap w_{\delta,\zeta}$ so that the cases of (ζ) (i.e. (F)$_2$) for $\bar\eta^\rho, \rho \in w$ hold and condition (η) (i.e. (C) for $\nu_{1,\zeta}, \nu_{2,\zeta}$) holds.

Let $w'_{\delta,\zeta} = \{\rho \in w_{\delta,\zeta} : \bar\eta^\rho \text{ is well defined}\}$, (so $w'_{\delta,\zeta} \subseteq w_{\delta,\zeta}$). Let $w'_{\delta,\zeta} = \{\rho[\zeta,\varepsilon] : \varepsilon < \varepsilon^* \leq \lambda\}$. Now we define $D^{\zeta+1}_{\bar\eta^{\rho[\zeta,\varepsilon]}}$ as $D^\zeta_{\eta^{\rho[\zeta,\varepsilon]}} + A_{\zeta,\varepsilon}$, clearly "legal".

Let $A'_{\zeta,\varepsilon} = \{i < \lambda : i \in A_{\zeta,\varepsilon}$ and $i > \varepsilon$ and $\eta_i^{\rho[\zeta,\varepsilon]} \notin \{\eta_{i_1}^{\rho[\zeta,\varepsilon_1]} : \varepsilon_1 < i$ and $i_1 < i\}$ and $\eta_i^{\rho[\zeta,\varepsilon]} \neq \nu_{1,\zeta}, \nu_{2,\zeta}\}$.

Observe

$(*)_1$ $A_{\zeta,\varepsilon} \backslash A'_\varepsilon$ is not stationary by Fodor's lemma as $\langle \eta_i^{\rho[\varepsilon]} : i < \lambda \rangle$ is with no repetition.

Now we shall prove that

$(*)_2$ the sets $\{\eta_i^{\rho[\varepsilon]} : i \in A'_\varepsilon\}$ for $\varepsilon > \varepsilon^*$ are pairwise disjoint.

So toward contradiction suppose $i_1 \in A'_{\varepsilon_1}, i_2 \in A'_{\varepsilon_2}, \varepsilon_1 < \varepsilon_2 < \varepsilon^*$ and $\eta_{i_1}^{\rho[\zeta,\varepsilon_1]} = \eta_{i_2}^{\rho[\zeta,\varepsilon_2]}$ and try to get a contradiction.

Case 1: $i_2 > i_1$.

As $i_1 \in A'_{\varepsilon_1}$ we have $i_1 > \varepsilon_1$ similarly $i_2 > \varepsilon_2$ but $\varepsilon_1 < \varepsilon_2$ so $i_2 > \varepsilon_2 > \varepsilon_1$, and by the assumption $i_2 > i_1$. So $\eta_{i_1}^{\rho[\zeta,\varepsilon_1]}$ belongs to the set $\{\eta_i^{\rho[\zeta,\varepsilon]} : \varepsilon < i_2 \ \& \ i < i_2\}$ so $\eta_{i_2}^{\rho[\zeta,\varepsilon_2]} \neq \eta_{i_1}^{\rho[\zeta,\varepsilon_1]}$ as $\eta_{i_2}^{\rho[\zeta,\varepsilon_2]}$ does not belong to this set as $i_2 \in A'_{\varepsilon_2}$.

Case 2: $i_2 < i_1$.

As $i_2 \in A'_{\zeta,\varepsilon_2}$ necessarily $\varepsilon_2 < i_2$. So $\varepsilon_2 < i_2 < i_1$ so $\eta_{i_2}^{\rho[\zeta,\varepsilon_2]} \in \{\eta_i^{\rho[\varepsilon]} : \varepsilon < i_1 \ \& \ \ell^i < i_1\}$ but $\eta_{i_2}^{\rho[\zeta,\varepsilon_1]}$ does not belong to this set as $i_1 \in A'_{\varepsilon_1}$ hence $\eta_{i_1}^{[\zeta,\varepsilon_1]}, \eta_{i_2}^{[\zeta,\varepsilon_2]}$ cannot be equal.

Case 3: $i_1 = i_2$.

As $i_1 \in A'_{\varepsilon_1}$ we have $i_1 \in A_{\zeta,\varepsilon_1}$ similarly $i_2 \in A_{\zeta,\varepsilon_2}$ but those sets are disjoint; a contradiction. So $(*)_2$ holds.

Now define $w_n^{\zeta,\ell}$ for $\ell = 1, 2, n < \omega$ by induction on

$$n : w_0^{\zeta,\ell} = \{\nu_{\ell,\zeta}\}$$

$$w_{n+1}^{\zeta,\ell} = \{\eta_i^{\rho[\zeta,\varepsilon]} : \rho[\zeta,\varepsilon] \in w_n^{\zeta,\ell} \text{ and } i \in A'_\varepsilon \text{ and } \varepsilon < \varepsilon^*\}.$$

Let $w^{\zeta,\ell} = \bigcup\limits_{n<\omega} w_n^{\zeta,\ell}$, now by $(*)_2$, $w^{\zeta,1} \cap w^{\zeta,2} = \emptyset$ (note the clause $\eta_i^{\rho[\zeta,\varepsilon]} \neq \nu_{1,\zeta}$ in the definition of A'_ε). So we define

$$J_{\delta,\zeta} = w^{\zeta,2}.$$

Now it is easy to check clause (F), i.e. (ζ) and we are done. $\square_{1.1}$

* * *

§2 The singular case and the full result

THEOREM 2.1. *Assume $\lambda > \aleph_0$. Let $\mu = 2^\lambda, \kappa = \mathrm{Min}\{\kappa : 2^\kappa > \mu\}$. There is a Hausdorff space X with a clopen basis with $|X| = 2^\kappa$ such that for $Y \subseteq \lambda$ closed $|Y| < |X| \Rightarrow |Y| < \lambda$.*

PROOF. For λ singular we should replace the filter D_0 on λ. So let $\lambda = \sum_{j<\mathrm{cf}(\lambda)} \lambda_j, \lambda_j$ strictly increasing $\bar\lambda = \langle \lambda_j : j < \mathrm{cf}(\lambda)\rangle$. Let $D^*_{\bar\lambda} = \{A \subseteq \lambda :$ for every $j < \mathrm{cf}(\lambda)$ large enough $A \cap \lambda_j^+$ contains a club of $\lambda_j^+\}$.

We can find a partition $\langle A^j_\alpha : \alpha < \lambda_j^+\rangle$ of $\lambda_j^+ \backslash \lambda_j$ to stationary sets; let us stipulate $A^j_\alpha = \emptyset$ when $\lambda_j^+ \le \alpha < \lambda$ and let $\bar A^* = \langle A_\alpha = \bigcup_{j<\mathrm{cf}(\lambda)} A^j_\alpha : \alpha < \lambda\rangle$ (so $A_\alpha \neq \emptyset \bmod D^*_{\bar\lambda}$ and $\alpha < \beta < \lambda \Rightarrow A_\alpha \cap A_\lambda = \emptyset$). Let $\{f_\xi : \xi < \mu \times \kappa\}$ be a family of functions from λ to λ such that if $n < \omega, \xi_1 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda$ then $\{\alpha < \lambda : f_{\varepsilon_\ell}(\alpha) = \varepsilon_\ell$ for $\ell = 1, \ldots, n\}$ is not empty (exists by [1]). Now for $\xi < \mu \times \kappa$ and $\varepsilon < \lambda$ we let $A_{\xi,\varepsilon} = \cup\{A_\alpha : f_\xi(\alpha) = \varepsilon\}$. Clearly $\xi < \mu \times \kappa$ & $\varepsilon_1 < \varepsilon_2 < \lambda \Rightarrow A_{\xi,\varepsilon_1} \cap A_{\xi,\varepsilon_2} = \emptyset$, and also: if $n < \omega, \xi_1 < \ldots < \xi_n < \mu \times \kappa$ and $\varepsilon_1, \ldots, \varepsilon_n < \lambda$ then $\bigcap_{\ell=1}^n A_{\zeta_\ell,\varepsilon_\ell} \neq \emptyset \bmod D^*_{\bar\lambda}$. Let D_0 be a maximal filter on λ extending $D^*_{\bar\lambda}$ and still satisfying $\bigcap_{\ell=1}^n A_{\xi_\ell,\varepsilon_\ell} \neq \emptyset \bmod D_0$ for $n, \xi_\ell, \varepsilon_\ell (\ell < n)$ as above.

Now the proof proceeds as before. All is the same except in stage II where we use λ regular, D_0 contains all clubs of λ.

The point is that we define A'_ε as before, the main question is: why $A'_\varepsilon = A_\varepsilon \bmod D^*_{\bar\lambda}$.

Choose $j^* < \mathrm{cf}(\lambda)$ such that:

$$\varepsilon < \lambda_{j^*}.$$

So it is enough to show

$$(*) \quad \text{if } j^* \le j < \mathrm{cf}(\lambda) \text{ then } A'_\varepsilon \cap [\lambda_j, \lambda_j^+) = A_\varepsilon \cap [\lambda_j, \lambda_j^+) \bmod D_{\lambda_j^+}$$

(where $D_{\lambda_j^+}$-the club filter on λ_j^+).

Looking at the definition of $A'_{\zeta,\varepsilon}$,

$$A'_{\zeta,\varepsilon} \cap [\lambda_j, \lambda_j^+) = \{i \in [\lambda_j, \lambda_j^+) : i \in A_{\zeta,\varepsilon} \cap [\lambda_j, \lambda_j^+) \text{ and } \eta_{i_1}^{\rho[\zeta,\varepsilon]} \notin \{\eta_{i_1}^{\rho[\zeta,\varepsilon_1]} : \varepsilon_1 < i \text{ and } i_1 < i\} \text{ and } \eta_i^{\rho[\varepsilon]} \neq \nu_{1,\zeta}\}$$

as $\langle \eta_i^{\rho^{[\zeta,\epsilon]}} : \lambda_j \leq i < \lambda_j^+ \rangle$ is with no repetition and Fodor's theorem holds (can formulate the demand on D). Just check that the use of $A'_{\zeta,\varepsilon}$ in §1 still works.

CONCLUSION 2.2. *If $\lambda \geq \aleph_0, \kappa = Min\{\kappa : 2^\kappa > 2^\lambda\}$, then there is a T_3-space $\lambda, |X| = 2^\kappa$ with no closed subspace of cardinality $\in [\lambda, 2^\kappa)$.* $\square_{2.1}$

* * *

We still would like to replace 2^κ by 2^{2^λ}.

THEOREM 2.3. *For $\lambda \geq \aleph_0$ there is a T_3 space X with clopen basis such that: no closed subspace has cardinality in $[\lambda, 2^{2^\lambda})$.*

PROOF. Like the proof of Theorem 1.1 with $\kappa = 2^\mu$.

The only problem is that $T_\delta - {}^\delta\mu$ may have cardinality $> 2^\mu$ so we have to redefine a δ-candidate (as there are too many $\eta_i \restriction \gamma$ to code) and in the crucial Stages G and H we have the list $\{(\nu_{1,\varepsilon}^\delta, \nu_{2,\varepsilon}^\delta) : \varepsilon < |T_\delta|\}$ but possibly $|T_\delta| > 2^\mu$. Still $|T_\delta| \leq \mu^{|\delta|} \leq 2^\mu$.

Stage B':

Let $Cd : \mu \to \mathcal{H}_{<\lambda^+}(\mu)$ be such that for every $x \in \mathcal{H}_{<\lambda^+}(\mu)$ for μ ordinals $\alpha < \mu$ we have $Cd(\alpha) = \lambda$.

Stage C':

For limit $\delta \leq \kappa$ we call $\bar\eta$ a δ-candidate if:

- (a) $\bar\eta = \langle \eta_i : i \leq \lambda \rangle$
- (b) $\eta_i \in T_\delta$
- (c) for some $\gamma, \langle \eta_i \restriction \gamma : i < \lambda \rangle$ is with no repetition
- (d) for odd $\beta < \delta$ we have $Cd(\eta_\lambda(\beta)) = \langle (\eta_i(\beta - 1), \eta_i(\beta)) : i < \lambda \rangle$
- (e) $Cd(\eta_\lambda(0)) = \{(i, j, \gamma, \eta_i(\gamma), \eta_j(\gamma)) : i < j < \lambda$ and γ minimal such that $(\forall i < j < \lambda)\eta_i(\gamma) \neq \eta_j(\gamma)\}$
- (f) $\eta_\lambda(0) > \sup\{\eta_i(0) : i < \lambda\}$.

So

- $(*)_1$ if $\langle \eta_i : i \leq \lambda \rangle$ is a δ_1-candidate, $\delta_0 < \delta_1$ limit and $(\exists \gamma < \delta_0)(\langle \eta_i \restriction \gamma : i \leq \lambda \rangle$ with no repetitions *then* $\langle \eta_i \restriction \delta_0 : i \leq \lambda \rangle$ is a δ_0-candidate
- $(*)_2$ if $\eta_i \in T_\kappa$ for $i < \kappa$ are pairwise distinct *then* for 2^μ sequence $\eta_\lambda \in T_\kappa, \langle \eta_i : i \leq \lambda \rangle$ is a κ-candidate.

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Stage H:

For each $\varepsilon < |T_\delta|$ we can choose $w_n^\varepsilon, w_{\delta,\varepsilon}, w_n^{\varepsilon,\ell}, w^{\varepsilon,\ell}$ as in Stage H there. We choose $u_\varepsilon \in [\delta]^{\leq\lambda}$ such that: if $\bar\eta$ is a δ-candidate $\eta_\lambda \in w_{\delta,\varepsilon}$ (so $\eta_i \in w_{\delta,\varepsilon}$ for $i < \lambda$) then $0 \in u_\varepsilon$ & $i < j < \lambda \Rightarrow$ $\mathrm{Min}\{\gamma : \eta_i(\gamma) \neq \eta_j\gamma)\} \in u_\varepsilon$.

By Engelking–Karlowicz [1] there are functions $H^\delta_\Upsilon : T_\delta \to \mathcal{H}_{<\lambda^+}(\mu)$ for $\Upsilon \in [\zeta_{<\delta}, \zeta_\delta)$ such that for every $w \subset [T_\delta]^\lambda$ and $h : w \to \lambda^+$ there is $\Upsilon \in [\zeta_{<\delta}, \zeta_\delta)$ such that $h \subseteq H^\delta$.

As $\lambda^+ \leq \mu$ without loss of generality $|\mathrm{Rang}(H^\delta_\Upsilon)| \leq \lambda$ (divide H^δ_Υ to $\leq 2^\lambda = \mu$ functions).

For each $\varepsilon < |T_\delta|$ let $h^\varepsilon_\delta : w^\varepsilon_\delta \to \mathcal{H}_{<\lambda^+}(\mu)$ be $h^\varepsilon_\delta(\eta) = (h^{\varepsilon,0}_\delta(\eta), h^{\varepsilon,1}_\delta(\eta), h^{\varepsilon,2}_\delta(\eta))$ where

$$h^{\varepsilon,0}_\delta(\eta) = \mathrm{otp}(\{\nu \in w^\varepsilon_\delta : \nu <^*_\delta \eta\}, <^*_\delta)$$

$$h^{\varepsilon,1}_\delta(\eta) = \{\langle\gamma, \eta(\gamma)\rangle : \gamma \in u\}$$

$$h^{\varepsilon,2}_\delta(\eta) = \text{ truth value of } \eta \in w^{\varepsilon,0}$$

(is in $\mathcal{H}_{<\lambda^+}$ as $|w^\varepsilon_\delta| \leq \lambda$); let

$$\Upsilon_\varepsilon = \mathrm{Min}\{\Upsilon \in [\zeta_{<\delta}, \zeta_\delta) : h^\varepsilon_\delta \subseteq H^\delta_\Upsilon\}$$

(well defined). Let $\gamma^\delta_\Upsilon =: \sup\{\gamma < \lambda^+ : \gamma$ is the first cardinal in some sequence $\bar\lambda$ from $(\mathrm{Rang}(H^\delta_\Upsilon)\}$, let g^δ_Υ be a one-to-one function from γ^δ_Υ into λ.

Next we can define the $D^\Upsilon_{\bar\eta}$ for $\bar\eta$ a δ-candidate; for $\Upsilon < \mu$:

$$D^{\Upsilon+1}_{\bar\eta} = D^\Upsilon_{\bar\eta} + A_{\Upsilon,\gamma^\delta_\Upsilon}.$$

In Stage $\Upsilon \in [\zeta_{<\delta}, \zeta_\delta)$ we deal with all $\varepsilon < |T_\delta|$ such that $\Upsilon_\varepsilon = \Upsilon$. Now we treat the choice of $I_{\delta,\zeta}, J_{\delta,\zeta}, w_{\delta,\zeta}$. We can finish as before (but dealing with many cases at once). $\square_{2.3}$

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