

## PERFECTLY MEAGER SETS AND UNIVERSALLY NULL SETS

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ABSTRACT. We will show that there is no ZFC example of a set distinguishing between universally null and perfectly meager sets.

### 1. INTRODUCTION

Consider the following three families of sets of reals:

**Definition 1.** Let  $X \subseteq \mathbb{R}$ . Then:

- (1)  $X$  is perfectly meager if for every perfect set  $P \subseteq \mathbb{R}$ ,  $P \cap X$  is meager in  $P$ .
- (2)  $X$  is universally meager if every Borel isomorphic image of  $X$  is meager.
- (3)  $X$  is universal null if every Borel isomorphic image of  $X$  has Lebesgue measure zero.

Let **PM**, **UM** and **UN** denote these families respectively.

The family **UM** was studied recently by Zakrzewski [13], and identified as an analog of **UN**.

One gets an equivalent definition of **UN** by replacing “Borel isomorphic” with “homeomorphic”, but this is not the case with **UM**.

Let  $\mathcal{M}$  and  $\mathcal{N}$  denote the  $\sigma$ -ideals of meager and of measure zero subsets of the reals, respectively.

For a  $\sigma$ -ideal  $\mathcal{J} \subseteq P(\mathbb{R})$  let

$$\text{non}(\mathcal{J}) = \min\{|X| : X \subseteq \mathbb{R} \text{ \& } X \notin \mathcal{J}\}.$$

There are many ZFC examples of uncountable sets that are in  $\mathbf{UM} \cap \mathbf{UN}$ . These include  $\omega_1 \omega_1^*$ -gaps, a selector from the constituents of a non-Borel  $\mathbf{\Pi}_1^1$  set, etc. (see [9]). All these sets have size  $\aleph_1$ , since Miller [8] showed that, consistently, no set of size  $2^{\aleph_0}$  is in  $\mathbf{UM} \cup \mathbf{UN}$ .

Grzegorek found other constructions in ZFC that produce sets of (consistently) different sizes.

**Theorem 2** (Grzegorek, [6]). (1) *There exists a set  $X \in \mathbf{UN}$  such that  $|X| = \text{non}(\mathcal{N})$ ,*

(2) *There exists a set  $X \in \mathbf{UM}$  such that  $|X| = \text{non}(\mathcal{M})$ .*

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The problem of whether the equality  $\mathbf{UM} = \mathbf{UN}$  is consistent is open. However, both inclusions are consistent with ZFC;  $\mathbf{UM} \subsetneq \mathbf{UN}$  holds in a model obtained by adding  $\aleph_2$  Cohen reals, and  $\mathbf{UN} \subsetneq \mathbf{UM}$  holds in a model obtained by adding  $\aleph_2$  random reals (side-by-side) (see [4], [9], [8]).

In this paper we investigate the connection between families  $\mathbf{UN}$  and  $\mathbf{PM}$ , and show that both inclusions  $\mathbf{PM} \subseteq \mathbf{UN}$  and  $\mathbf{UN} \subseteq \mathbf{PM}$  are consistent with ZFC as well. Observe that trivially  $\mathbf{UM} \subseteq \mathbf{PM}$ , thus we only need to check that  $\mathbf{PM} \subseteq \mathbf{UN}$  is consistent. Recall that  $\mathbf{PM} \neq \mathbf{UM}$  is consistent ([12]) as well as  $\mathbf{PM} = \mathbf{UM}$  ([2]). We will show that:

**Theorem 3.** *It is consistent with ZFC that  $\mathbf{PM} \subseteq [\mathbb{R}]^{\leq \aleph_1} \subseteq \mathbf{UN}$ .*

## 2. FORCING

Suppose that  $X \subseteq 2^\omega$  is a perfectly meager set in  $\mathbf{V}$ . Let  $\tilde{P}$  be a fixed closed subset of  $2^\omega \times 2^\omega$  which is universal for perfect sets in  $2^\omega$ . In other words, for every perfect set  $P \subseteq 2^\omega$  there exists an  $x$  such that  $P = (\tilde{P})_x = \{y : (x, y) \in \tilde{P}\}$ . Note that this property is absolute. Since  $X$  is perfectly meager, we can find sets  $\tilde{Q}^n \subseteq 2^\omega \times 2^\omega$  such that for every  $x \in 2^\omega$  and  $n \in \omega$ ,

- (1)  $(\tilde{Q}^n)_x$  is a closed nowhere dense subset of  $(\tilde{P})_x$ ,
- (2)  $X \cap (\tilde{P})_x \subseteq \left(\bigcup_{n \in \omega} \tilde{Q}^n\right)_x$ .

Clearly, the set  $\bigcup_{n \in \omega} \tilde{Q}^n$  witnesses that  $X \in \mathbf{PM}$  since

$$X \subseteq 2^\omega \setminus \bigcup_{x \in 2^\omega} (\tilde{P} \setminus \bigcup_{n \in \omega} \tilde{Q}^n)_x.$$

Note that the last inclusion makes sense even if  $X$  is not a subset of  $\mathbf{V}$ . Suppose that  $\mathbf{V}' \subseteq \mathbf{V}$  and  $X \subseteq \mathbf{V}$  is a set of reals. We will say  $\mathbf{V}' \models X \in \mathbf{PM}$  if there exists a family  $\{\tilde{Q}^n : n \in \omega\} \in \mathbf{V}'$  such that  $X \cap (\tilde{P})_x \subseteq \left(\bigcup_{n \in \omega} \tilde{Q}^n\right)_x$  for every real  $x \in \mathbf{V}'$ .

The property of being perfectly meager is not absolute, so whether  $X$  is perfectly meager in  $\mathbf{V}'$  has no bearing onto whether  $X$  is perfectly meager in  $\mathbf{V}$ . For example, if  $x \in \mathbf{V}$  is a Cohen real over  $\mathbf{V}'$ , then the set  $\{x\}$  is perfectly meager in  $\mathbf{V}$  but not in  $\mathbf{V}'$ .

**Lemma 4.** *Let  $\{\mathcal{P}_\alpha, \dot{Q}_\alpha : \alpha < \omega_2\}$  be a countable support iteration of proper forcing notions over  $\mathbf{V} \models \text{CH}$ . Suppose that  $X \subseteq \mathbf{V}^{\mathcal{P}_{\omega_2}} \cap \mathbb{R}$  is a perfectly meager set. Then there exists an  $\omega_1$ -club  $C \subseteq \omega_2$  such that for every  $\alpha \in C$ ,*

$$\mathbf{V}^{\mathcal{P}_\alpha} \models X \in \mathbf{PM}.$$

*Proof.* Let  $\{\tilde{Q}^n : n \in \omega\} \in \mathbf{V}^{\mathcal{P}_{\omega_2}}$  be a family witnessing that  $X$  is perfectly meager. Let  $C$  consist of those ordinals of cofinality  $\omega_1$  such that  $\tilde{Q}^n \cap ((2^\omega \cap \mathbf{V}^{\mathcal{P}_\alpha}) \times 2^\omega) \in \mathbf{V}^{\mathcal{P}_\alpha}$  for every  $n$ . The usual argument involving Skolem-Löwenheim theorem shows that  $C$  has the required property.  $\square$

Our objective is to find a set of general conditions on a forcing notion  $\mathbb{P}$  such that the countable support iteration of  $\mathbb{P}$  of length  $\omega_2$  produces a model where  $\mathbf{PM} \subseteq [\mathbb{R}]^{\leq \aleph_1} \subseteq \mathbf{UN}$ . These conditions are sufficient for the class of forcing notions defined using norms [10].

These conditions are the following:

- (1)  $\mathbf{V}^{\mathbb{P}} \models \mathbf{V} \cap 2^\omega \in \mathcal{N}$ ,
- (2)  $\mathbf{V}^{\mathbb{P}} \models \mathbf{V} \cap 2^\omega \notin \mathcal{M}$ ,
- (3)  $\mathbb{P}$  is  $\omega^\omega$ -bounding, that is,  $\omega^\omega \cap \mathbf{V}$  is a dominating family in  $\omega^\omega \cap \mathbf{V}^{\mathbb{P}}$ ,
- (4)  $\mathbb{P}$  adds a real  $x_{\mathbb{P}} \in 2^\omega$  such that  $\mathbf{V} \models \{x_{\mathbb{P}}\} \notin \mathbf{PM}$ ,
- (5)  $\mathbb{P}$  generic real is minimal, that is, if  $g$  is  $\mathbb{P}$ -generic over  $\mathbf{V}$  and  $x \in \mathbf{V}[g] \cap 2^\omega$ , then  $x \in \mathbf{V}$  or  $g \in \mathbf{V}[x]$ .

Condition (1) is necessary to make all sets of size  $\aleph_1$  universally null, and condition (2) is necessary to avoid making all  $\aleph_1$  sets perfectly meager. Recall that (2) and (3) together are essentially equivalent to  $\mathbf{V}^{\mathbb{P}} \models \mathbf{V} \cap \mathcal{M}$  is cofinal in  $\mathcal{M}$ .

For the forcing notions  $\mathbb{P}$  that we have in mind the following property holds: for every real  $x \in \mathbf{V}^{\mathbb{P}}$  there exists a continuous function  $f \in \mathbf{V}$  such that  $x = f(x_G)$ , where  $x_G$  is a generic real.

Condition (5) guarantees that in the above context  $f$  can be chosen to be a homeomorphism. In particular, if  $X$  is a set of reals of size  $\aleph_2$ , then  $X$  will contain a homeomorphic image of a sequence of generic reals.

The following forcing notion appeared in [5]; it is similar (but not identical) to the infinitely equal real forcing from [7].

For a tree  $p$  and  $t \in p$ , let  $\text{succ}_p(t)$  be the set of all immediate successors of  $t$  in  $p$ ,  $p_t = \{v \in p : t \subseteq v \text{ or } v \subseteq t\}$  the subtree of  $p$  determined by  $t$ ,  $p|n$  the  $n$ -th level of  $p$ , and  $[p]$  the set of branches of  $p$ . By identifying  $s \in \omega^{<\omega}$  with the full-branching tree having root  $s$ , we can also denote  $[s] = \{f \in \omega^\omega : s \subseteq f\}$ .

Fix a strictly increasing function  $f \in \omega^\omega$  and let  $\mathbf{X} = \prod_{n \in \omega} f(n)$ . Note that  $\mathbf{X}$  is a Polish space homeomorphic to  $2^\omega$ . For technical reasons we require that  $f(n) = 2^{\bar{f}(n)}$  for  $n \in \omega$ .

Let  $\mathbb{E}\mathbb{E}$  be the following forcing notion:  $p \in \mathbb{E}\mathbb{E}$  if

- (1)  $p$  is a nonempty subtree of  $\omega^{<\omega}$ ,
- (2)  $s(n) < f(n)$  for all  $s \in p$  and  $n \in \text{dom}(s)$ ,
- (3) for all  $s \in p$  there exists an extension  $t$  of  $s$  such that  $t \cap n \in p$  for all  $n < f(|t|)$ .

For  $p, q \in \mathbb{E}\mathbb{E}$ ,  $p \geq q$  if  $p \subseteq q$ . Without loss of generality we can assume that  $|\text{succ}_p(s)| = 1$  or  $\text{succ}_p(s) = f(|p|)$  for all  $p \in \mathbb{E}\mathbb{E}$  and  $s \in p$ . Conditions of this type form a dense subset of  $\mathbb{E}\mathbb{E}$ . Let

$$\text{split}(p) = \{s \in p : |\text{succ}_p(s)| > 1\} = \bigcup_{n \in \omega} \text{split}_n(p),$$

where  $\text{split}_n(p) = \{s \in \text{split}(p) : |\{t \subsetneq s : t \in \text{split}(p)\}| = n\}$ .

For  $p, q \in \mathbb{E}\mathbb{E}$ ,  $n \in \omega$ , we let

$$p \geq_n q \iff p \geq q \ \& \ \text{split}_n(q) = \text{split}_n(p).$$

**Lemma 5** ([5]). (1)  $\mathbb{E}\mathbb{E}$  satisfies Axiom A, so it is proper,

- (2)  $\mathbf{V}^{\mathbb{E}\mathbb{E}} \models \mathbf{V} \cap 2^\omega \in \mathcal{N}$ ,
- (3)  $\mathbf{V}^{\mathbb{E}\mathbb{E}} \models \mathbf{V} \cap 2^\omega \notin \mathcal{M}$ ,
- (4) for every maximal antichain  $\mathcal{A} \subseteq \mathbb{E}\mathbb{E}$ ,  $p \in \mathbb{E}\mathbb{E}$ , and  $n \in \omega$  there exists  $q \geq_n p$  such that  $\{r \in \mathcal{A} : r \text{ is compatible with } q\}$  is finite,
- (5) for every family of maximal antichains  $\{\mathcal{A}_n : n \in \omega\}$  and  $p \in \mathbb{E}\mathbb{E}$  there exists  $q \geq p$  such that for every  $n$ ,  $\{r \in \mathcal{A}_n : r \text{ is compatible with } q\}$  is finite,
- (6)  $\mathbb{E}\mathbb{E}$  is  $\omega^\omega$  bounding,
- (7)  $\mathbf{V}^{\mathbb{E}\mathbb{E}} \models \mathbf{V} \cap \mathcal{M}$  is cofinal in  $\mathcal{M}$ . □

Note that for  $p \in \mathbb{E}\mathbb{E}$  the set  $[p]$  is a compact subset of  $\mathbf{X} = \prod_n f(n)$ . Moreover, there is a canonical isomorphism between  $[p]$  and  $2^\omega$  defined as follows:

For every  $n$  let  $\{s_0^n, \dots, s_{f(n)}^n\}$  be a fixed enumeration of 0-1 sequences of length  $\tilde{f}(n)$  (recall that  $f(n) = 2^{\tilde{f}(n)}$ ). Define  $F : [p] \rightarrow 2^\omega$  as

$$F(x) = s_{x(n_0+1)}^{n_0} \widehat{\ } s_{x(n_1+1)}^{n_1} \widehat{\ } \dots,$$

where  $n_0, n_1, \dots$  is the increasing enumeration of the set  $\{n : x \upharpoonright n \in \text{split}(p)\}$ .

**Lemma 6.** *Let  $p \in \mathbb{E}\mathbb{E}$  and suppose that  $H \subseteq [p]$  is a meager set in  $[p]$ . For every  $n \in \omega$  there exists  $q \geq_n p$  such that  $[q] \cap H = \emptyset$ . In particular,  $\Vdash_{\mathbb{E}\mathbb{E}} \mathbf{V} \models \{\dot{g}\} \notin \text{PM}$ .*

*Proof.* Let  $H \subseteq [p]$  be a meager set, and let  $n \in \omega$ . Fix a descending sequence of open sets  $\langle U_k : k \in \omega \rangle$  such that each  $U_k$  is dense in  $[p]$  and  $H \cap \bigcap_k U_k = \emptyset$ . By induction build a sequence  $\langle p_k : k \in \omega \rangle$  such that  $p_0 = p$ , and for every  $k$ ,

- (1)  $p_{k+1} \geq_{n+k+1} p_k \in \mathbb{E}\mathbb{E}$ ,
- (2)  $[p_{k+1}] \subseteq U_k$ .

Suppose that  $p_k$  is given. For every  $v \in \text{split}_{n+k+1}(p_k)$  find  $q_v \geq (p_k)_v$  such that  $[q_v] \subseteq U_k$ . Let  $p_{k+1} = \bigcup \{q_v : v \in \text{split}_{n+k+1}(p_k)\}$ . Condition  $q = \lim_k p_k$  has the required property.

Suppose that  $\{\widetilde{Q}^n : n \in \omega\} \in \mathbf{V}$  is a possible witness that  $\{\dot{g}\}$  is perfectly meager, and let  $p \in \mathbb{E}\mathbb{E}$ . Find  $x \in \mathbf{V}$  such that  $[p] = (P)_x$  and let  $q \geq p$  be such that  $[q] \cap \left(\bigcup_n \widetilde{Q}^n\right)_x = \emptyset$ . Clearly,

$$q \Vdash_{\mathbb{E}\mathbb{E}} \{\dot{g}\} \in \bigcup_{x \in \mathbf{V}} \left( P \setminus \bigcup_n \widetilde{Q}^n \right)_x.$$

In particular,  $q \Vdash_{\mathbb{E}\mathbb{E}} \mathbf{V} \models \{\dot{g}\} \notin \text{PM}$ .  $\square$

**Lemma 7.** *Suppose that  $p \in \mathbb{E}\mathbb{E}$  and  $p \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \in 2^\omega$ . For every  $n \in \omega$  there exist  $q \geq_n p$  and a continuous function  $F : [q] \rightarrow 2^\omega$  such that  $q \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} = F(\dot{g})$ , where  $\dot{g}$  is the canonical name for the generic real.*

*Moreover, we can require that for every  $v \in \text{split}_n(q)$  and any  $x_1, x_2 \in [q_v]$ ,  $F(x_1) \upharpoonright n = F(x_2) \upharpoonright n$ .*

*Proof.* The first part is a special case of a more general fact. For  $n \in \omega$  let  $\mathcal{A}_n \subseteq \mathbb{E}\mathbb{E}$  be a maximal antichain below  $p$  such that  $\forall r \in \mathcal{A}_n \exists s \in 2^n r \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \upharpoonright n = s$ . Use Lemma 5(5) to find  $q \geq p$  such that for every  $n \in \omega$ ,  $\{r \in \mathcal{A}_n : r \text{ is compatible with } q\}$  is finite. Let  $\mathcal{A}'_n = \{r \in \mathcal{A}_n : r \text{ is compatible with } q\}$ . Without loss of generality we can assume that  $[q] \subseteq \bigcup_{r \in \mathcal{A}'_n} [r]$ . It follows that  $[r] \cap [q]$  is clopen in  $[q]$  for every  $r \in \mathcal{A}'_n$ . Define  $F : [q] \rightarrow 2^\omega$  as  $F(x) = y$  if for every  $n \in \omega$  there exists  $r \in \mathcal{A}'_n$  such that  $x \in [r]$  and  $r \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \upharpoonright n = y \upharpoonright n$ . It is easy to see that  $F$  is a continuous function that has the required properties.

To show the second part we need to build  $q$  in such a way that for every  $v \in \text{split}_n(q)$ , there is  $r \in \mathcal{A}'_n$  such that  $q_v \geq r$ .  $\square$

**Lemma 8.** *Suppose that  $p \in \mathbb{E}\mathbb{E}$ ,  $n \in \omega$  and  $p \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \in 2^\omega$ . Let  $F : [q] \rightarrow 2^\omega$  be a continuous function such that  $p \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} = F(\dot{g})$ .*

*There exists  $q \geq p$  such that  $F \upharpoonright [q]$  is constant, or there exists  $q \geq_n p$  such that  $F \upharpoonright [q]$  is one-to-one. In particular, the generic real is minimal.*

*Proof.* Consider the following two cases:

CASE 1.  $p \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \notin \mathbf{V}$ . Let  $x \in \mathbf{V}$  and  $q \geq p$  be such that  $q \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} = x$ . Clearly  $F \upharpoonright [q]$  is constant with value  $x$ .

CASE 2.  $p \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \notin \mathbf{V}$ .

Build by induction a sequence of conditions  $\langle p_k : k \in \omega \rangle$  such that  $p_0 = p$  and for every  $k$ ,

- (1)  $p_{k+1} \geq_{n+k+1} p_k$ ,
- (2) sets  $\left\{ F^{\omega} \left( [(p_{k+1})_s] \right) : s \in \text{split}_{n+k+1}(p_{k+1}) \right\}$  are pairwise disjoint and have diameter  $< 2^{-k}$ .

Suppose that  $p_k$  is given. Note that  $F^{\omega} \left( [(p_k)_s] \right)$  is uncountable for every  $s \in p_k$ . For  $v \in \text{split}_{n+k+1}(p_k)$  choose pairwise different reals  $x_v \in F^{\omega} \left( [(p_k)_v] \right)$ . It is not important now but will be relevant in the sequel, that we can choose these reals “effectively” from a fixed countable subset of  $[p_k]$ . Let  $\ell > k$  be such that sequences  $x_v \upharpoonright \ell$  are also pairwise different. For every  $v \in \text{split}_{n+k+1}(p_k)$  let  $s_v \in \text{split}(p_k)$  be such that for every  $z \in [(p_k)_{s_v}]$ ,  $F(z) \upharpoonright \ell = x_v \upharpoonright \ell$ . If  $F$  is as in the second part of Lemma 7, then we can find  $s_v$  in  $\text{split}_{\ell}(p_k)$ . Define  $p_{k+1} = \bigcup \{ (p_k)_{s_v} : v \in \text{split}_{n+k+1}(p_k) \}$ . Observe that  $q = \lim_k p_k$  has the required property.  $\square$

Note that the above lemma shows that the reals added by  $\mathbb{E}\mathbb{E}$  are minimal. Infinitely equal forcing from [7] or [4] does not have this property.

### 3. ITERATION OF $\mathbb{E}\mathbb{E}$

Let  $\alpha \leq \omega_2$  be an ordinal and suppose that  $\mathbb{E}\mathbb{E}_{\alpha}$  is a countable support iteration of  $\mathbb{E}\mathbb{E}$  of length  $\alpha$ . In other words,  $p \in \mathbb{E}\mathbb{E}_{\alpha}$  is

- (1)  $p$  is a function and  $\text{dom}(p) = \alpha$ ,
- (2)  $\text{supp}(p) = \{ \beta : p(\beta) \neq \emptyset \}$  is countable,
- (3)  $\forall \beta < \alpha$   $p \upharpoonright \beta \Vdash_{\mathbb{E}\mathbb{E}_{\beta}} p(\beta) \in \mathbb{E}\mathbb{E}$ .

For  $F \in [\alpha]^{<\omega}$ ,  $n \in \omega$ , and  $p, q \in \mathbb{E}\mathbb{E}_{\alpha}$  define

$$q \geq_{F,n} p \iff q \geq p \ \& \ \forall \beta \in F \ q \upharpoonright \beta \Vdash_{\mathbb{E}\mathbb{E}_{\beta}} q(\beta) \geq_n p(\beta).$$

The following fact is well-known.

**Theorem 9** ([5], [7], [3]). *Suppose that  $p \in \mathbb{E}\mathbb{E}_{\alpha}$ ,  $F \in [\alpha]^{<\omega}$ , and  $n \in \omega$ . Then:*

- (1) *for every maximal antichain  $\mathcal{A} \subseteq \mathbb{E}\mathbb{E}_{\alpha}$ , there exists  $q \geq_{F,n} p$  such that  $\{ r \in \mathcal{A} : r \text{ is compatible with } q \}$  is finite,*
- (2) *for every family of maximal antichains  $\{ \mathcal{A}_n : n \in \omega \}$  there exists  $q \geq p$  such that for every  $n$ ,  $\{ r \in \mathcal{A}_n : r \text{ is compatible with } q \}$  is finite,*
- (3)  $\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}} \models [\mathbb{R}]^{<2^{\aleph_0}} \subseteq \mathcal{N}$ ,
- (4)  $\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}} \models \mathcal{M} \cap \mathbf{V}$  is cofinal in  $\mathcal{M}$ .  $\square$

For  $p \in \mathbb{E}\mathbb{E}_{\alpha}$  let  $\text{cl}(p)$  be the smallest set  $w \subseteq \alpha$  such that  $p$  can be evaluated using generic reals  $\langle \dot{g}_{\beta} : \beta \in w \rangle$ . In other words,  $\text{cl}(p)$  consists of those  $\beta < \alpha$  such that the transitive closure of  $p$  contains an  $\mathbb{E}\mathbb{E}_{\beta}$ -name for an element of  $\mathbb{E}\mathbb{E}$ . It is well-known [11] that  $\{ p \in \mathbb{E}\mathbb{E}_{\alpha} : \text{cl}(p) \in [\alpha]^{<\omega} \}$  is dense in  $\mathbb{E}\mathbb{E}_{\alpha}$ .

Suppose that  $p \in \mathbb{E}\mathbb{E}_\alpha$ ,  $w = \text{cl}(p)$  is countable and  $\alpha_p = \text{ot}(\text{cl}(p))$ . Let  $\mathbb{E}\mathbb{E}_w$  be the countable support iteration of  $\mathbb{E}\mathbb{E}$  with the domain  $w$ . In other words, consider the countable support iteration  $\langle \mathcal{P}_\beta, \dot{Q}_\beta : \beta < \text{sup}(w) \rangle$  such that

$$\forall \beta < \text{sup}(w) \Vdash_{\mathcal{P}_\beta} \dot{Q}_\beta \simeq \begin{cases} \mathbb{E}\mathbb{E} & \text{if } \beta \in w, \\ \emptyset & \text{if } \beta \notin w. \end{cases}$$

It is clear that  $\mathbb{E}\mathbb{E}_w \simeq \mathbb{E}\mathbb{E}_{\alpha_p}$ . Moreover, we can view condition  $p$  as a member of  $\mathbb{E}\mathbb{E}_w$ .

For the rest of the section we will consider only the iteration of  $\mathbb{E}\mathbb{E}$  of countable length  $\alpha$  and show that  $\mathbb{E}\mathbb{E}_\alpha$  has the same properties as  $\mathbb{E}\mathbb{E}$ .

Let  $\alpha$  be a countable ordinal and  $p \in \mathbb{E}\mathbb{E}_\alpha$ . Define  $\bar{p} \subseteq \mathbf{X}^\alpha$  as follows:

$\langle x_\beta : \beta < \alpha \rangle \in \bar{p}$  if for every  $\beta < \alpha$ ,

$$x_\beta \in \left[ p(\beta) [\langle x_\gamma : \gamma < \beta \rangle] \right].$$

Note that  $p(\beta) [\langle x_\gamma : \gamma < \beta \rangle]$  is the interpretation of  $p(\beta)$  using reals  $\langle x_\gamma : \gamma < \beta \rangle$  so it may be undefined if these reals are not sufficiently generic.

For a set  $G \subseteq \mathbf{X}^\alpha$ ,  $u \subseteq \alpha$ , and  $x \in \mathbf{X}^u$  let

$$(G)_x = \{y \in \mathbf{X}^{\alpha \setminus u} : \exists z \in G \ z \upharpoonright u = x \ \& \ z \upharpoonright (\alpha \setminus u) = y\},$$

and for  $\beta \in \alpha$  let  $(G)_\beta = \{x(\beta) : x \in G\}$ .

We say that  $p \in \mathbb{E}\mathbb{E}_\alpha$  is good if

- (1)  $\bar{p}$  is compact,
- (2) for every  $\beta < \alpha$  and  $x \in \overline{p \upharpoonright \beta}$ ,  $\overline{p[x]} = (p)_x$  and  $\overline{p(\beta)[x]} = ((p)_x)_\beta$ ,
- (3)  $\bar{p}$  is homeomorphic to  $\mathbf{X}^\alpha$  via a homeomorphism  $h$  such that for every  $\beta < \alpha$  and  $x \in \overline{p \upharpoonright \beta}$ ,  $h \upharpoonright ((p)_x)_\beta$  is a homeomorphism between  $((p)_x)_\beta$  and  $\mathbf{X}$ .

**Lemma 10.**  $\{p \in \mathbb{E}\mathbb{E}_\alpha : \bar{p} \text{ is good}\}$  is dense in  $\mathbb{E}\mathbb{E}_\alpha$ .

*Proof.* CASE 1.  $\alpha = \beta + 1$ .

Fix  $p \in \mathbb{E}\mathbb{E}_\alpha$  and for  $n \in \omega$  let  $\mathcal{A}_n$  be a maximal antichain below  $p \upharpoonright \beta$  such that

- (1)  $\forall r \in \mathcal{A}_n \ \bar{r}$  is compact,
- (2)  $\forall r \in \mathcal{A}_n \ \exists t \subseteq \prod_{j < n} f(j) \ r \Vdash_{\mathbb{E}\mathbb{E}_\beta} p(\beta) \upharpoonright n = t$ .

Fix a sequence  $\langle F_n : n \in \omega \rangle$  such that for  $n \in \omega$ ,

- (1)  $F_n \in [\beta]^{<\omega}$ ,
- (2)  $F_n \subseteq F_{n+1}$ ,
- (3)  $\bigcup_n F_n = \beta$ .

By induction build a sequence  $\langle q_n : n \in \omega \rangle$  such that for  $n \in \omega$ ,

- (1)  $\bar{q}_n$  is compact,
- (2)  $q_{n+1} \geq_{F_n, n} q_n$ ,
- (3)  $\exists \mathcal{A}'_n \in [\mathcal{A}_n]^{<\omega} \ \bar{q}_n \subseteq \bigcup_{r \in \mathcal{A}'_n} \bar{r}$ .

Let  $q_\omega = \lim_n q_n$ . As in the proof of Lemma 7 we show that there exists a continuous function  $F : \bar{q}_\omega \rightarrow \mathbb{E}\mathbb{E}$  (encode elements of  $\mathbb{E}\mathbb{E}$  as reals) such that

$$q_\omega \Vdash_{\mathbb{E}\mathbb{E}_\beta} p(\beta) = F(\langle \dot{g}_\gamma : \gamma < \beta \rangle).$$

Consider  $q = q_\omega \hat{\ } p(\beta) \geq p$ . Clearly,  $\bar{q} = \{\langle x, y \rangle : x \in \bar{q}_\omega, y \in [F(x)]\}$  is compact in  $\mathbf{X}^\alpha$ . Remaining requirements are met as well.

CASE 2.  $\alpha$  is limit.

Given  $p \in \mathbb{E}\mathbb{E}_\alpha$  fix sequences  $\langle F_n : n \in \omega \rangle$  and  $\langle \alpha_n : n \in \omega \rangle$  such that

- (1)  $F_n \in [\alpha_n]^{<\omega}$ ,
- (2)  $F_n \subseteq F_{n+1}$ ,
- (3)  $\bigcup_n F_n = \alpha$ ,
- (4)  $\sup_n \alpha_n = \alpha$ .

By induction build a sequence  $\langle q_n : n \in \omega \rangle$  such that for  $n \in \omega$ ,

- (1)  $q_n \in \mathbb{E}\mathbb{E}_\alpha$ ,
- (2)  $\text{supp}(q_n) \subseteq \alpha_n$ ,
- (3)  $q_{n+1} \geq_{F_n, n} q_n$ ,
- (4)  $q_n \upharpoonright \alpha_n \geq p \upharpoonright \alpha_n$ ,
- (5)  $q_n \upharpoonright \alpha_n$  is compact in  $\mathbf{X}^{\alpha_n}$ .

Let  $q = \lim_n q_n$ . Note that  $\bar{q} = \bigcap_n \overline{q_n \upharpoonright \alpha_n} \times \mathbf{X}^{\alpha \setminus \alpha_n}$  is as required.  $\square$

From now on we will always work with conditions  $p$  such that  $\bar{p}$  is good. We noticed earlier that for every condition  $p \in \mathbb{E}\mathbb{E}$ ,  $[p]$  is canonically isomorphic to  $2^\omega$ . In exactly the same way we can verify that if  $p \in \mathbb{E}\mathbb{E}_\alpha$  and  $\bar{p}$  is good, then  $\bar{p}$  is isomorphic to  $(2^\omega)^\alpha$ .

As in Lemma 7 we show that:

**Lemma 11.** *Suppose that  $p \in \mathbb{E}\mathbb{E}_\alpha$  and  $p \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} \in 2^\omega$ . Then there exist  $q \geq p$  and a continuous function  $F : \bar{p} \rightarrow 2^\omega$  such that  $q \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} = F(\dot{\mathbf{g}})$ , where  $\dot{\mathbf{g}} = \langle \dot{g}_\beta : \beta < \alpha \rangle$  is the sequence of generic reals.*

**Lemma 12.** *Let  $p \in \mathbb{E}\mathbb{E}_\alpha$  and suppose that  $H \subseteq \bar{p}$  is a meager set in  $\bar{p}$ . For every  $F \in [\alpha]^{<\omega}$  and  $n \in \omega$  there exists  $q \geq_{F, n} p$  such that  $\bar{q} \cap H = \emptyset$ .*

*Proof.* As before, without loss of generality we can assume that  $\alpha$  is countable.

Induction on  $\alpha$ .

CASE 1.  $\alpha = \beta + 1$ .

Suppose that  $p \in \mathbb{E}\mathbb{E}_\alpha$  and  $H \subseteq \bar{p} \subseteq \mathbf{X}^\beta \times \mathbf{X}$  is meager, and let  $F \in [\alpha]^{<\omega}$  and  $n \in \omega$  be given.

Let

$$H' = \{x \in \bar{p} \upharpoonright \beta : (H)_x \text{ is not meager in } [p(\beta)[x]] = ((\bar{p})_x)_\beta\}.$$

Using the fact that  $\bar{p}$  is homeomorphic to  $(2^\omega)^\alpha$  via homeomorphism respecting vertical sections, and by Kuratowski-Ulam theorem, we conclude that  $H'$  is a meager set in  $\bar{p} \upharpoonright \beta$ .

Recall the following classical lemma:

**Lemma 13** ([1]). *Suppose that  $H \subseteq 2^\omega \times 2^\omega$  is a Borel set.*

- (1) *Assume  $(H)_x$  is meager for all  $x$ . Then there exists a sequence of Borel sets  $\{G_n : n \in \omega\} \subseteq 2^\omega \times 2^\omega$  such that*
  - (a)  $(G_n)_x$  is a closed nowhere dense set for all  $x \in 2^\omega$ ,
  - (b)  $H \subseteq \bigcup_{n \in \omega} G_n$ .

By the inductive hypothesis we can find  $q^* \geq_{F \upharpoonright \beta, n} p \upharpoonright \beta$  such that  $\bar{q}^* \cap H' = \emptyset$ . By Lemma 6 for every  $x \in \bar{q}^*$  there exists  $q_x \geq_n p(\beta)[x]$  such that  $[q_x] \cap (H)_x = \emptyset$ . Moreover, by Lemma 13, the mapping  $x \mapsto q_x$  can be chosen to be Borel, and subsequently, by shrinking  $q^*$ , continuous. Let  $q \in \mathbb{E}\mathbb{E}_\alpha$  be defined such that  $q \upharpoonright \beta = q^*$  and  $q^* \Vdash_{\mathbb{E}\mathbb{E}_\beta} q(\beta) = q_{\dot{g}_\beta}$ . It is clear that  $q$  has the required properties.

CASE 2.  $\alpha$  is limit.

Fix sequences  $\langle F_n : n \in \omega \rangle$  and  $\langle \alpha_n : n \in \omega \rangle$  such that

- (1)  $F_n \in [\alpha_n]^{<\omega}$ ,
- (2)  $F_n \subseteq F_{n+1}$ ,
- (3)  $\bigcup_n F_n = \alpha$ ,
- (4)  $\sup_n \alpha_n = \alpha$ .

By induction build a sequence  $\langle q_n : n \in \omega \rangle$  such that for  $n \in \omega$ ,

- (1)  $q_n \in \mathbb{E}\mathbb{E}_\alpha$ ,
- (2)  $\text{supp}(q_n) \subseteq \alpha_n$ ,
- (3)  $q_{n+1} \geq_{F_n, n} q_n$ ,
- (4)  $q_n \upharpoonright \alpha_n \geq p \upharpoonright \alpha_n$ ,
- (5)  $\overline{q_n \upharpoonright \alpha_n} \cap H_n = \emptyset$ , where  $H_n = \left\{ x \in \overline{q_n \upharpoonright \alpha_n} : (H)_x \text{ is not meager in } \overline{p \upharpoonright x} \right\}$ .

As before (5) is possible by Kuratowski-Ulam theorem. Let  $q = \lim_n q_n$ . It is clear that  $\overline{q} \cap H = \emptyset$ .  $\square$

The following lemma is an analog of Lemma 8.

**Lemma 14.** *Suppose that  $p \in \mathbb{E}\mathbb{E}_\alpha$ ,  $n \in \omega$  and  $p \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} \in 2^\omega$ . Let  $F : \overline{p} \rightarrow 2^\omega$  be a continuous function such that  $p \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} = F(\dot{\mathbf{g}})$ , where  $\dot{\mathbf{g}} = \langle \dot{g}_\beta : \beta < \alpha \rangle$  is the sequence of generic reals. There exists  $q \geq p$  such that exactly one of the following conditions holds:*

- (1)  $F \upharpoonright \overline{q}$  is constant,
- (2) there exists  $\beta < \alpha$  such that  $F \upharpoonright \overline{q \upharpoonright \beta}$  is one-to-one and for every  $x \in \overline{q \upharpoonright \beta}$ ,  $F \upharpoonright (\overline{q \upharpoonright \beta})_x$  is constant,
- (3)  $F \upharpoonright \overline{q}$  is one-to-one.

*Proof.* We have three cases:

CASE 1. There exists  $q \geq p$  such that  $q \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} \in \mathbf{V}$ . Without loss of generality we can assume that for some  $x \in \mathbf{V} \cap 2^\omega$ ,  $q \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} = x$ . It follows that  $F \upharpoonright \overline{q}$  is constant.

CASE 2. There exists  $q \geq p$  such that  $q \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \exists \beta < \alpha \dot{x} \in \mathbf{V}^{\mathbb{E}\mathbb{E}_\beta}$ . By shrinking  $q$  we can assume that there exists a continuous function  $G : \overline{q \upharpoonright \beta} \rightarrow 2^\omega$  such that  $q \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} = G(\dot{\mathbf{g}} \upharpoonright \beta)$ . In particular, for  $x \in [q]$ ,  $F(x) = G(x \upharpoonright \beta)$ . If  $\beta$  was minimal, then, using the argument below, we can also assume that  $G$  is one-to-one.

Suppose that  $q \in \mathbb{E}\mathbb{E}_\alpha$ ,  $F \in [\alpha]^{<\omega}$ , and  $n \in \omega$ . Without loss of generality we can assume that for every  $\beta \in F$ ,  $q \upharpoonright \beta$  determines the value of  $\text{split}_n(q(\beta))$  (up to finitely many values). Suppose that  $\sigma : F \rightarrow \omega^{<\omega}$  is a function such that  $\sigma(\beta) \in \text{split}_n(q(\beta))$  for  $\beta \in F$ . Let  $(q)_\sigma$  be the condition defined as

$$\forall \beta < \alpha (q)_\sigma \upharpoonright \beta \Vdash_{\mathbb{E}\mathbb{E}_\beta} (q)_\sigma(\beta) = \begin{cases} q(\beta) & \text{if } \beta \notin F, \\ (q(\beta))_{\sigma(\beta)} & \text{if } \beta \in F. \end{cases}$$

Let  $\Sigma_{F,n}$  be the finite set of all mappings  $\sigma$  satisfying the requirements.

**Lemma 15.** *Suppose that  $F \in [\alpha]^{<\omega}$ ,  $n \in \omega$  and*

$$p \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} = F(\dot{\mathbf{g}}) \ \& \ \forall \beta < \alpha \dot{x} \notin \mathbf{V}^{\mathbb{E}\mathbb{E}_\beta}.$$

*There exists  $q \geq_{F,n} p$  such that the sets  $\left\{ F^m(\overline{(q)_\sigma}) : \sigma \in \Sigma_{F,n} \right\}$  are pairwise disjoint.*



*Proof.* Induction on  $|F|$  and  $\alpha$ . If  $F = \{\beta\}$  this is essentially Lemma 8.

Let  $\{v_j : j < k^*\}$  be an enumeration of  $\text{split}_n(p(\beta))$ . For  $v \in \text{split}_n(p)$  choose pairwise different reals  $x_v \in F''(\overline{(p)_v})$ . Note that this choice can be made canonically from, for example, the countable dense set of leftmost branches of subtrees of  $p$ . Let  $\ell > k$  be such that sequences  $x_v \upharpoonright \ell$  are also pairwise different. Define conditions  $\langle r_j : j \leq k^* \rangle, \langle q_j : j \leq k^* \rangle$  such that for every  $j \leq k^*$ ,

- (1)  $r_j \in \mathbb{E}\mathbb{E}_\beta$ ,
- (2)  $r_{j+1} \geq r_j$ ,
- (3)  $r_j \Vdash_{\mathbb{E}\mathbb{E}_\beta} q_j \geq (p)_{v_j} \upharpoonright [\beta, \alpha]$ ,
- (4)  $\forall z \in \overline{r_j \frown q_j}, F(z) \upharpoonright \ell = F(x_{v_j}) \upharpoonright \ell$ .

Let  $q \upharpoonright \beta = q_{k^*}$  and  $q \upharpoonright [\beta, \alpha] = \bigcup_{j < k^*} q_j$ .

Suppose that  $|F| = k + 1$  and let  $\beta = \max(F)$ .

By the part already proved, for each  $\mathbf{x} = \langle x_\gamma : \gamma < \beta \rangle \in \overline{p \upharpoonright \beta}$  find a condition  $q_{\mathbf{x}} \geq_n p \upharpoonright [\beta, \alpha][\mathbf{x}]$  such that the sets  $\left\{ F''(\overline{(q_{\mathbf{x}})_s}) : s \in \text{split}_n(q_{\mathbf{x}}) \right\}$  are pairwise disjoint. Note that we can do it in such a way that the mapping  $\mathbf{x} \mapsto q_{\mathbf{x}}$  is continuous. (As before we first choose  $q_{\mathbf{x}}$  in a Borel way, and then shrink  $p \upharpoonright \beta$  to make this mapping continuous.) That defines an  $\mathbb{E}\mathbb{E}_\beta$ -name for an element of  $\mathbb{E}\mathbb{E}_{\beta, \alpha}$ , which we call  $q^*$ .

Next, let  $F' = F \setminus \{\beta\}$  and apply the inductive hypothesis to find  $q' \geq_{F', n} p \upharpoonright \beta$  such that  $\left\{ F''(\overline{(q')_\sigma}) : \sigma \in \Sigma_{F', n} \right\}$  are pairwise disjoint. Let  $q \in \mathbb{E}\mathbb{E}_\alpha$  be defined as  $q \upharpoonright \beta = q'$  and  $q \upharpoonright \beta \Vdash_{\mathbb{E}\mathbb{E}_\beta} q \upharpoonright [\beta, \alpha] = q^*$ .

It is clear that  $q$  is as required.  $\square$

CASE 3.  $p \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \forall \beta < \alpha \dot{x} \notin \mathbf{V}^{\mathbb{E}\mathbb{E}_\beta}$ .

Let  $\langle F_n : k \in \omega \rangle$  be an increasing sequence of finite sets such that  $\bigcup_n F_n = \alpha$ .

By induction build a sequence of conditions  $\langle p_n : n \in \omega \rangle$  such that  $p_0 = p$  and for every  $n$ ,

- (1)  $p_{n+1} \geq_{F_n, n} p_n$ ,
- (2) sets  $\left\{ F''(\overline{(p_n)_\sigma}) : \sigma \in \Sigma_{F_n, n} \right\}$  are pairwise disjoint.

Let  $q = \lim_n p_n$ .

Suppose that  $\mathbf{x} = \langle x_\beta : \beta < \alpha \rangle$  and  $\mathbf{x}' = \langle x'_\beta : \beta < \alpha \rangle$  are two distinct points in  $\overline{q}$ . Let  $\beta$  be the first ordinal such that  $x_\beta \neq x'_\beta$ . Let  $n$  be so large that  $\beta \in F_n$  and there are two distinct  $\sigma, \sigma' \in \Sigma_{F_n, n}$  such that  $\mathbf{x} \in \overline{(p_n)_\sigma}$  and  $\mathbf{x}' \in \overline{(p_n)_{\sigma'}}$ . Since  $F''(\overline{(p_n)_\sigma}) \cap F''(\overline{(p_n)_{\sigma'}}) = \emptyset$ , it follows that  $F(\mathbf{x}) \neq F(\mathbf{x}')$ .  $\square$

#### 4. A MODEL WHERE $\mathbf{PM} \subseteq \mathbf{UN}$

Let  $\mathbb{E}\mathbb{E}_{\omega_2}$  be the countable support iteration of  $\mathbb{E}\mathbb{E}$  of length  $\aleph_2$ . We will show that in  $\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}}$ ,  $\mathbf{PM} \subseteq \mathbf{UN}$ .

By Theorem 9(2),  $\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}} \models [\mathbb{R}]^{<2^{\aleph_0}} \subseteq \mathbf{UN}$ , thus we have to show that

$$\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}} \models \mathbf{PM} \subseteq [\mathbb{R}]^{<2^{\aleph_0}}.$$

Suppose that  $X \in \mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}}$  is a set of reals of size  $\aleph_2$ . Let  $\{\dot{x}_\alpha : \alpha < \omega_2\}$  be the set of names for elements of  $X$  such that  $\Vdash_{\mathbb{E}\mathbb{E}_{\omega_2}} \forall \alpha \neq \beta \dot{x}_\alpha \neq \dot{x}_\beta$ . Apply Lemma 11 and find for each  $\alpha < \omega_2$  a set  $w_\alpha \in [\omega_2]^{<\omega}$ , a condition  $p_\alpha \in \mathbb{E}\mathbb{E}_{w_\alpha}$ , and a continuous

function  $F_\alpha : \overline{p_\alpha} \rightarrow 2^\omega$  such that  $p_\alpha \Vdash_{\mathbb{E}\mathbb{E}_{\omega_2}} \dot{x}_\alpha = F_\alpha(\langle \dot{q}_\beta : \beta \in w_\alpha \rangle)$ . We can assume that  $w_\alpha$  is minimal. In other words,  $p_\alpha \Vdash_{\mathbb{E}\mathbb{E}_{\omega_2}} \forall \beta < \sup(w_\alpha) \dot{x}_\alpha \notin \mathbf{V}^{\mathbb{E}\mathbb{E}_\beta}$ . In particular, without loss of generality we can assume  $F_\alpha$  is one-to-one, so it is a homeomorphism.

By thinning out we can assume that  $\text{ot}(w_\alpha) = \gamma$ ,  $F_\alpha = F$  and  $\overline{p_\alpha} = \overline{p}$ . Moreover, since  $\mathbf{V} \models \text{CH}$ , we can assume that  $w_\alpha \cap w_\beta = w^*$  for  $\alpha \neq \beta$ . Finally, without loss of generality we can assume that  $w^* = \emptyset$ .

Let  $P = F''(\overline{p})$ . Since  $F$  is a homeomorphism,  $P$  is perfect. We will show that  $X \cap P$  is not meager in  $\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}}$  (relative to  $P$ ).

Assume otherwise and let  $H \subseteq P$  be a meager set such that for some  $p^* \in \mathbb{E}\mathbb{E}_{\omega_2}$ ,  $p^* \Vdash_{\mathbb{E}\mathbb{E}_{\omega_2}} X \cap P \subseteq H$ . By Theorem 9(4) we can assume that  $H \in \mathbf{V}$ . Set  $G = (F)^{-1}(H)$  and notice that  $G$  is a meager subset of  $\overline{p}$ .

Find  $\alpha < \omega_2$  such that  $w_\alpha \cap \text{cl}(p^*) = \emptyset$ . By Lemma 12 there exists  $q \geq p$ ,  $q \in \mathbb{E}\mathbb{E}_{w_\alpha} \simeq \mathbb{E}\mathbb{E}_\gamma$  such that  $\overline{q} \cap G = \emptyset$ .

Since  $p^*$  and  $q$  are compatible let  $r \geq p^*, q$ . It follows that

$$r \Vdash_{\mathbb{E}\mathbb{E}_{\omega_2}} \dot{x}_\alpha = F_\alpha(\langle \dot{q}_\beta : \beta \in w_\alpha \rangle) \notin H,$$

which finishes the proof.

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