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Filtration-equivalent $\aleph_1$-separable abelian groups of cardinality $\aleph_1$

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ABSTRACT

We show that it is consistent with ordinary set theory *ZFC* and the generalized continuum hypothesis that there exist two $\aleph_1$-separable abelian groups of cardinality $\aleph_1$ which are filtration-equivalent and one is a Whitehead group but the other is not. This solves one of the open problems from Eklof and Mekler (2002) [2].

0. Introduction

An $\aleph_1$-separable abelian group is an abelian group G such that every countable subgroup is contained in a free direct summand of G. This property is apparently stronger than the property of being strongly $\aleph_1$-free; however, the two properties coincide for groups of cardinality at most $\aleph_1$ in models of Martin's Axiom (*MA*) and the negation of the continuum hypothesis ($\neg CH$). Over the years the variety and abundance of $\aleph_1$-separable groups obtained by various constructions has demonstrated the failure of certain attempts to classify $\aleph_1$-separable groups of cardinality $\aleph_1$. In brief, one can say that positive results towards classification can be given assuming $MA + \neg CH$ and negative results are obtained assuming *CH* or even the axiom of constructibility $V = L$. A good survey is for instance [2, Chapter VIII].

There are four principal methods of constructing $\aleph_1$-free groups: as the union of an ascending chain of countable free groups; in terms of generators and relations; as a subgroup of a divisible group; and as a pure subgroup of $\mathbb{Z}^{\omega_1}$. In the study of $\aleph_1$-separable groups it turned out to be helpful to consider the concept of *filtration-equivalence*, a relation between two $\aleph_1$-separable groups. Recall that two groups A and B of cardinality $\aleph_1$ are called *filtration-equivalent* if they have filtrations $\{A_\nu : \nu \in \omega_1\}$ and $\{B_\nu : \nu \in \omega_1\}$ respectively such that for all $\nu \in \omega_1$, there is an isomorphism $\Phi_\nu : A_\nu \to B_\nu$ satisfying $\Phi_\nu[A_\mu] = B_\mu$ for all $\mu \le \nu$. Such an isomorphism is called *level-preserving*. Note that it is not required that Φ_τ extends Φ_ν when $\tau \ge \nu$ and note also that filtration-equivalent groups A and B are also quotient-equivalent, i.e. for all $\nu \in \omega_1$ we have $A_{\nu+1}/A_\nu \cong B_{\nu+1}/B_\nu$.

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Under the hypothesis of Martin's Axiom the notion of filtration-equivalence represents the end of the search; more precisely, assuming $MA+\neg CH$, filtration-equivalent $\aleph_1$-separable groups are isomorphic. Assuming even the proper forcing axiom (*PFA*) every $\aleph_1$-separable group (of cardinality $\aleph_1$) is of a special *standard form*. However, in L there exist non-isomorphic $\aleph_1$-separable groups of cardinality $\aleph_1$ which are filtration-equivalent.

In [2, Open problems on the structure of Ext Nr.6] (see also [1]) Eklof and Mekler asked whether or not it is consistent with *ZFC* that there exist two filtration-equivalent $\aleph_1$-separable groups of cardinality $\aleph_1$ such that one is a Whitehead group and the other is not. Recall that a Whitehead group is an abelian group G satisfying $\mathrm{Ext}(G,\mathbb{Z})=0$. The class of Whitehead groups is closed under direct sums and subgroups and contains the class of free abelian groups. However, the question whether all Whitehead groups are free is undecidable in ZFC as was shown by the first author in [7,8]. Similarly, we shall show in this paper that the answer to the question by Eklof and Mekler is affirmative even assuming *GCH*.

All groups are abelian and notation is in accordance with [4] and [2]. For further details on $\aleph_1$-separable groups and set theory we refer to [2], [5], and [6].

1. The construction

Using special ladder systems we construct $\aleph_1$-separable abelian groups of cardinality $\aleph_1$ with a prescribed Γ-invariant. The construction is similar to the one given in [2, Chapter XIII, Section 0].

Throughout this paper let $S\subseteq\omega_1$ be a stationary and co-stationary subset of ω_1. Since $\lim(\omega_1)$ is a closed and unbounded subset of ω_1 we may assume without loss of generality that S consists of limit ordinals of cofinality ω only. We shall further require that ω^2 divides δ for every $\delta\in S$. We recall the definition of a ladder and a ladder system.

Definition 1.1. We use the following notions:

(i) A ladder on $\delta\in S$ is a strictly increasing sequence $\eta_\delta=\{\eta_\delta(n):n\in\omega\}$ of non-limit ordinals less than δ which is cofinal in δ, i.e. $\sup\{\eta_\delta(n):n\in\omega\}=\delta$.

(ii) The ladder η_δ is a special ladder if there exists a sequence $0<k_0^{\eta_\delta}<k_1^{\eta_\delta}<\cdots<k_n^{\eta_\delta}<\cdots$ of natural numbers such that
 (a) $\eta_\delta(k_n^{\eta_\delta}+i)+\omega=\eta_\delta(k_n^{\eta_\delta}+j)+\omega$ for all $i,j<k_{n+1}^{\eta_\delta}-k_n^{\eta_\delta}$;
 (b) $\eta_\delta(k_n^{\eta_\delta})+\omega<\eta_\delta(k_{n+1}^{\eta_\delta})$.
 for all $n\in\omega$.

Note that the existence of S and certain ladders on S is well known. However, any limit ordinal δ of the form $\delta=\alpha+\omega$ obviously does not allow the existence of a special ladder. This is the reason why we have required that ω^2 divides δ for every $\delta\in S$, and hence no $\delta\in S$ can be of the form $\delta=\alpha+\omega$. For $\delta\in S$ all ladders are special but we will continue to use the word special because this concept makes sense also if δ is not a multiple of ω^2.

Example 1.2. The following are natural examples of the ladders η on δ:

(i) Let $k_n^\eta=2n$ for all $n\in\omega$. Then η is special if and only if

$$\eta(2n)+\omega=\eta(2n+1)+\omega<\eta(2n+2);$$

(ii) Let $k_n^\eta=n$ for all $n\in\omega$. Then η is special if and only if

$$\eta(n)+\omega<\eta(n+1).$$

For $\delta\in S$ we let Δ_δ be the set of all special ladders on δ.

We now collect ladder systems containing special ladders.

Definition 1.3. A system $\bar\eta=\langle\eta_\delta:\delta\in S\rangle$ of (special) ladders is called a (special) ladder system on S.

We put

$$E=\{\bar\eta:\bar\eta \text{ is a special ladder system}\}.$$

For later use we also define

$$E_\alpha=\{\bar\eta_\alpha:\bar\eta\in E,\ \bar\eta_\alpha=\langle\eta_\delta:\delta\in S\cap\alpha\rangle\}$$

for $\alpha<\omega_1$. On the set of special ladders we define the *ω-range* function as follows:

$$\mathrm{rd}(\eta)=\left\langle\eta(k_n^\eta)+\omega:n\in\omega\right\rangle.$$

Note that $\mathrm{rd}(\eta)$ determines all values of $\eta(n)+\omega$ ($n\in\omega$) since the ladder η is special. Moreover, if $\bar\eta\in E$, then put $\mathrm{rd}(\bar\eta)=\langle\mathrm{rd}(\eta_\delta):\delta\in S\rangle$ and similarly $\mathrm{rd}_\alpha(\bar\eta)=\mathrm{rd}_\alpha(\bar\eta_\alpha)=\langle\mathrm{rd}(\eta_\delta):\delta\in S\cap\alpha\rangle$ for $\alpha<\omega_1$.

Using the special ladder systems we can now define our desired groups. Let $\bar\eta\in E$ be a special ladder system and put $k_n^\delta=k_n^{\eta_\delta}$ for all $\delta\in S$ and $n\in\omega$. Moreover, let $t_n^\delta=k_{n+1}^\delta-k_n^\delta$ for all $n\in\omega$. We define a $\mathbb{Q}$-module

$$F=\bigoplus_{\beta<\omega_1}x_\beta\mathbb{Q}\oplus\bigoplus_{\delta\in S,n\in\omega}y_{\delta,n}\mathbb{Q}$$

freely generated (as a vectorspace) by the independent elements x_β ($\beta<\omega_1$) and $y_{\delta,n}$ ($\delta\in S$, $n\in\omega$). Our desired group will be constructed as a subgroup of F. Therefore, given a group $G\subseteq F$, we define a *canonical* $\aleph_1$*-filtration* of G by letting

$$G^\alpha=\left\langle G\cap(\{x_\beta:\beta<\alpha+\omega\}\cup\{y_{\delta,n}:\delta\in S\cap\alpha,n\in\omega\})\right\rangle_*\subseteq G$$

for $\alpha < \omega_1$. Here $\langle\cdots\rangle_*$ denotes the purification of $\langle\cdots\rangle$ in G. Then $\{G^\alpha : \alpha < \omega_1\}$ is an increasing chain of pure subgroups of G such that $G = \bigcup_{\alpha<\omega_1} G^\alpha$. However, the chain is not continuous since for instance $G^\omega \neq \bigcup_{n\in\omega} G^n$ but this is not needed in what follows and we will still call it a filtration. For simplicity let $y_\delta = y_{\delta,0}$ for $\delta \in S$. Let $\psi : \omega \to \omega$ be a fixed function with $\psi(n) \neq 0$ for all $n \in \omega$ and choose integers $a_l^{\delta,n}$ for $l < t_n^\delta$ such that $\gcd(a_l^{\delta,n} : l < t_n^\delta) = 1$ for all $n \in \omega$ and $\delta \in S$. We define elements $z_{\delta,n} \in F$ via

$$z_{\delta,n} = \prod_{i=0}^{n-1} \frac{1}{\psi(i)} y_\delta + \sum_{i=0}^{n-1} \prod_{j=i}^{n-1} \frac{1}{\psi(j)} \left(\sum_{l<t_i^\delta} a_l^{\delta,i} x_{\eta_\delta(k_i^\delta+l)} \right) \tag{1.1}$$

for $\delta \in S$ and $n \in \omega$. Furthermore we put $z_{\delta,0} = y_\delta$ and let $\bar{a} = \left\langle a_l^{\delta,n} : l < t_n^\delta, n \in \omega, \delta \in S \right\rangle$.

Let $G_{\bar{\eta}}^{\psi,\bar{a}} = \left\langle x_\beta, z_{\delta,n} : \beta < \omega_1, \delta \in S, n \in \omega \right\rangle \subseteq F$. Then easy calculations show that the generating relations satisfied by the generators of $G_{\bar{\eta}}^{\psi,\bar{a}}$ are

$$\psi(n) z_{\delta,n+1} = z_{\delta,n} + \sum_{l<t_n^\delta} a_l^{\delta,n} x_{\eta_\delta(k_n^\delta+l)} \tag{1.2}$$

for $\delta \in S$ and $n \in \omega$.

Lemma 1.4. *Let $G_{\bar{\eta}}^{\psi,\bar{a}} = \left\langle x_\beta, z_{\delta,n} : \beta < \omega_1, \delta \in S, n \in \omega \right\rangle \subseteq F$ be as above. Then $G_{\bar{\eta}}^{\psi,\bar{a}}$ admits a free presentation of the form*

$$0 \to Y \to X \to G_{\bar{\eta}}^{\psi,\bar{a}} \to 0$$

where X is the free group $X = \bigoplus_{\beta<\omega_1} \mathbb{Z}x_\beta \oplus \bigoplus_{\delta\in S, n\in\omega} \mathbb{Z}z_{\delta,n}$ and Y is the subgroup of X generated by the elements $\psi(n)z_{\delta,n+1} - z_{\delta,n} - \sum_{l<t_n^\delta} a_l^{\delta,n} x_{\eta_\delta(k_n^\delta+l)}$ for $\delta \in S$ and $n \in \omega$.

Proof. That the elements $\psi(n)z_{\delta,n+1} - z_{\delta,n} - \sum_{l<t_n^\delta} a_l^{\delta,n} x_{\eta_\delta(k_n^\delta+l)}$ for $\delta \in S$ and $n \in \omega$ are in the kernel Y is clear and that they generate Y is easily established and therefore left to the reader. □

To simplify notation we shall omit in what follows the superscript $(\psi, \bar{a})$ since the function ψ and the vector $\bar{a}$ of integers will always be clear from the context. However, the reader should keep in mind that for every ladder system $\bar{\eta}$ the group $G_{\bar{\eta}} = G_{\bar{\eta}}^{\psi,\bar{a}}$ always depends on the additional parameters ψ and $\bar{a}$. We consider Example 1.2 again.

Example 1.5. The following hold:

(i) Let $\bar{\eta}$ be a special ladder system consisting of special ladders as defined in Example 1.2(i) and choose $a_0^{\delta,n} = 1, a_1^{\delta,n} = -1$ for all $\delta \in S$ and $n \in \omega$. Then $G_{\bar{\eta}}$ satisfies the following relations

$$\psi(n) z_{\delta,n+1} = z_{\delta,n} + x_{\eta_\delta(2n)} - x_{\eta_\delta(2n+1)}.$$

(ii) Let $\bar{\eta}$ be a special ladder system consisting of special ladders as defined in Example 1.2(ii) and choose $a_0^{\delta,n} = 1$ for all $\delta \in S$ and $n \in \omega$. Then $G_{\bar{\eta}}$ satisfies the following relations

$$\psi(n) z_{\delta,n+1} = z_{\delta,n} + x_{\eta_\delta(n)}.$$

We now prove some properties of the constructed groups $G_{\bar{\eta}}$.

Lemma 1.6. *Let $\bar{\eta} \in E$. Then the group $G_{\bar{\eta}}$ is a torsion-free $\aleph_1$-separable abelian group of size $\aleph_1$ with $\Gamma(G_{\bar{\eta}}) = \tilde{S}$.*

Proof. Let $\bar{\eta} \in E$ be a special ladder system. Clearly the group $G_{\bar{\eta}}$ is a torsion-free group of cardinality $\aleph_1$. We first prove that $G_{\bar{\eta}}$ is $\aleph_1$-free. Therefore, let H be a finite rank subgroup of $G_{\bar{\eta}}$. Then there exists a finite subset

$$T \subseteq \{x_\beta : \beta < \omega_1\} \cup \{z_{\delta,n} : \delta \in S, n \in \omega\}$$

such that

$$H \subseteq \langle t : t \in T\rangle_* \subseteq G_{\bar{\eta}}.$$

Let $T_S = \{\delta \in S : z_{\delta,n} \in T \text{ for some } n \in \omega\}$. We can enlarge T (not changing T_S) so that there exists an integer m such that

- for $\delta \in T_S$ we have $z_{\delta,n} \in T$ if and only if $n \leq m$;
- for $y_\delta = z_{\delta,0} \in T$ we have $x_{\eta_\delta(n)} \in T$ if and only if $n < k_{m+1}^\delta$.

Then using Eq. (1.2) it is not hard to see that $\langle t : t \in T\rangle_*$ is freely generated by the elements $\{z_{\delta,m} : y_\delta \in T\} \cup \{x_{\eta_\delta(n)} : n < k^\delta_{m+1}, y_\delta \in T\}$. Thus H is free and therefore $G_{\bar\eta}$ is $\aleph_1$-free.

It remains to prove that $G_{\bar\eta}$ is $\aleph_1$-separable. Therefore let $\{G^\alpha_{\bar\eta} : \alpha < \omega_1\}$ be the canonical $\aleph_1$-filtration of $G_{\bar\eta}$. We shall now define for all $\nu \notin S$ a projection $\pi_\nu : G_{\bar\eta} \to G^\nu_{\bar\eta}$ such that $\pi_\nu \restriction_{G^\nu_{\bar\eta}} = id \restriction_{G^\nu_{\bar\eta}}$. Let $\nu \notin S$ be given. For every $\mu \geq \nu + \omega$ let $\pi_\nu(x_\mu) = 0$; for $\delta \in S$ with $\delta > \nu$ let n_δ be maximal with $\eta_\delta(k^\delta_{n_\delta}) < \nu$. Hence $\eta_\delta(k^\delta_{n_\delta} + i) < \nu + \omega$ for all $i < t^\delta_{n_\delta}$ and $\eta_\delta(k^\delta_{n_\delta+1}) > \nu + \omega$. Let $\pi_\nu(z_{\delta,n}) = 0$ for all $n \geq n_\delta$. Moreover, put

$$\pi_\nu(y_\delta) = -\sum_{i=0}^{n_\delta-1}\prod_{j=0}^{i-1}\psi(j+1)\sum_{l<t^\delta_i} a^{\delta,i}_l x_{\eta_\delta(k^\delta_i+l)}$$

and finally

$$\pi_\nu(z_{\delta,n}) = -\sum_{i=n}^{n_\delta-1}\prod_{j=n}^{i-1}\psi(j+1)\sum_{l<t^\delta_i} a^{\delta,i}_l x_{\eta_\delta(k^\delta_i+l)}$$

for all $n < n_\delta$. Letting $\pi_\nu \restriction_{G^\nu_{\bar\eta}} = id \restriction_{G^\nu_{\bar\eta}}$ it is now straightforward to check that π_ν is a well-defined homomorphism as claimed using Eq. (1.2). Finally, $\Gamma(G_{\bar\eta}) = \tilde{S}$ follows immediately by an easy checking that $G^{\nu+1}_{\bar\eta}/G^\nu_{\bar\eta}$ is not free for $\nu \in S$. □

We now prove that a special ladder system is sufficiently *separated*.

Lemma 1.7. *Let $\bar\eta \in E$ and $\alpha < \omega_1$. Then there exists a sequence of integers $\langle m_\delta : \delta \in S \cap \alpha\rangle$ such that the sets $\{\eta_\delta(k^\delta_n) + \omega : n \geq m_\delta\}$ $(\delta \in S \cap \alpha)$ are pairwise disjoint. In particular, the sets $\{\eta_\delta(k^\delta_n + i) : n \geq m_\delta, i < t^\delta_n\}$ $(\delta \in S \cap \alpha)$ are pairwise disjoint.*

Proof. Let $\bar\eta \in E$ and $\alpha < \omega_1$ be given. Since α is countable we may enumerate $S \cap \alpha$ by ω, say $S \cap \alpha = \{\delta_k : k \in \omega\}$. We shall now define inductively the sequence $\langle m_{\delta_k} : k \in \omega\rangle$ such that for every $k \in \omega$ the sets

$$\{\eta_{\delta_j}(k^{\delta_j}_n) + \omega : n \geq m_{\delta_j}\} \quad (j \leq k) \text{ are pairwise disjoint.} \tag{1.3}$$

We start with $k = 1$, hence η_{δ_0} and η_{δ_1} are given. In order to carry on the induction we shall prove a stronger result. Let m_{δ_0} be fixed but arbitrary. We claim that there is m_{δ_1} such that (1.3) holds for $k = 1$. Assume first that $\delta_0 < \delta_1$. Since $S \subseteq \lim(\omega_1)$ and $\omega^2|\delta$ for all $\delta \in S$ we obtain $\delta_1 > \delta_0 + \omega$. Hence it is easy to see that m_{δ_1} exists such that (1.3) is satisfied for $k = 1$ because η_{δ_1} is a ladder with $\sup(\mathrm{Im}(\eta_{\delta_1})) = \delta_1$.

Assume $\delta_1 < \delta_0$, then $\delta_0 > \delta_1 + \omega$. Thus there is m'_{δ_1} such that $\{\eta_{\delta_0}(k^{\delta_0}_n) + \omega : n \geq m'_{\delta_1}\}$ and $\{\eta_{\delta_1}(k^{\delta_1}_n) + \omega : n \geq m'_{\delta_1}\}$ are disjoint. Increasing m'_{δ_1} sufficiently we obtain $m_{\delta_1} \geq m'_{\delta_1}$ such that (1.3) holds.

The inductive step is now immediate. Given k such that $m_{\delta_0}, m_{\delta_1}, \ldots, m_{\delta_{k-1}}$ satisfy (1.3) we obtain integers s_j for $j < k$ such that $\{\eta_{\delta_j}(k^{\delta_j}_n) + \omega : n \geq m_{\delta_j}\}$ and $\{\eta_{\delta_k}(k^{\delta_k}_n) + \omega : n \geq s_j\}$ are pairwise disjoint for every $j < k$. Choosing $m_{\delta_k} = \max\{s_j : j < k\}$ we satisfy (1.3). □

Note that Lemma 1.7 can give the same sequence of integers for different $\bar\eta, \bar\nu \in E$ if $\mathrm{rd}(\bar\eta) = \mathrm{rd}(\bar\nu)$. Nevertheless, the next lemma shows that special ladder systems $\bar\eta, \bar\nu \in E$ with $\mathrm{rd}(\bar\eta) = \mathrm{rd}(\bar\nu)$ do not overlap very much.

Lemma 1.8. *Let $\bar\eta, \bar\nu \in E$ and $\alpha < \omega_1$ such that $\mathrm{rd}(\bar\eta) = \mathrm{rd}(\bar\nu)$. Moreover, let $\langle m_\delta : \delta \in S \cap \alpha\rangle$ be the sequence from Lemma 1.7. If $\eta_\delta(k^{\eta_\delta}_n + j) = \nu_{\delta'}(k^{\nu_{\delta'}}_m + i)$ for some $n \geq m_\delta$, $m \geq m_{\delta'}$, and $i < t^{\nu_{\delta'}}_m, j < t^{\nu_\delta}_n$. Then $\delta = \delta'$ and $m = n$.*

Proof. Assume that $\eta_\delta(k^{\eta_\delta}_n + j) = \nu_{\delta'}(k^{\nu_{\delta'}}_m + i)$ for some $n \geq m_\delta$, $m \geq m_{\delta'}$, and $i < t^{\nu_{\delta'}}_m, j < t^{\nu_\delta}_n$. Then

$$\eta_\delta(k^{\eta_\delta}_n + j) + \omega = \eta_\delta(k^{\eta_\delta}_n) + \omega = \nu_{\delta'}(k^{\nu_{\delta'}}_m + i) + \omega = \nu_{\delta'}(k^{\nu_{\delta'}}_m) + \omega = \eta_{\delta'}(k^{\eta_{\delta'}}_m) + \omega$$

since $\mathrm{rd}(\bar\eta) = \mathrm{rd}(\bar\nu)$. Thus $\delta = \delta'$ by Lemma 1.7. Moreover, $m = n$ follows since η_δ is a special ladder. □

Recall that two groups A and B of cardinality $\aleph_1$ are called *filtration-equivalent* if they have filtrations $\{A_\nu : \nu \in \omega_1\}$ and $\{B_\nu : \nu \in \omega_1\}$ respectively such that for all $\nu \in \omega_1$, there is an isomorphism $\Phi_\nu : A_\nu \to B_\nu$ satisfying $\Phi_\nu[A_\mu] = B_\mu$ for all $\mu \leq \nu$. Such an isomorphism is called *level-preserving*. Note that we do not require that Φ_τ extends Φ_ν when $\tau \geq \nu$ and note that filtration-equivalent groups A and B are also quotient-equivalent, i.e. for all $\nu \in \omega_1$ we have $A_{\nu+1}/A_\nu \cong B_{\nu+1}/B_\nu$.

Proposition 1.9. *Let $\bar\eta, \bar\nu \in E$ such that $\mathrm{rd}(\bar\eta) = \mathrm{rd}(\bar\nu)$. Then the groups $G_{\bar\eta}$ and $G_{\bar\nu}$ are filtration-equivalent.*

Proof. Let $\bar\eta$ and $\bar\nu$ be given. By construction we have

$$G_{\bar\eta} = \langle x_\beta, z_{\delta,n} : \beta < \omega_1, \delta \in S, n \in \omega\rangle$$

and

$$G_{\bar\nu} = \langle x_\beta, w_{\delta,n} : \beta < \omega_1, \delta \in S, n \in \omega\rangle$$

such that the elements $z_{\delta,n}$ and $w_{\delta,n}$ ($\delta \in S, n \in \omega$) are defined as in (1.1) for $\bar{\eta}$ and $\bar{\nu}$ respectively. Hence, the generating relations satisfied in $G_{\bar{\eta}}$ and $G_{\bar{\nu}}$ are the relations in Eq. (1.2) (see Lemma 1.4). Since filtration-equivalence is a transitive property it suffices to prove the result when $G_{\bar{\eta}}$ is of the simplest form ($k_n^{\eta_\delta} = n, t_n^{\eta_\delta} = 1, a_0^{\eta_\delta,n} = 1$), and hence

$$\psi(n) z_{\delta,n+1} = z_{\delta,n} + x_{\eta_\delta(n)}$$

and

$$\psi(n) w_{\delta,n+1} = w_{\delta,n} + \sum_{l<t_n^\delta} a_l^{\delta,n} x_{\nu_\delta(k_n^\delta+l)}.$$

Note that the parameters k_n^δ, t_n^δ and $a_l^{\delta,n}$ depend on ν_δ. Moreover, we shall assume for simplicity and without loss of generality that $a_0^{\delta,n} = 1$ for every $\delta \in S, n \in \omega$ since $\gcd(a_l^{\delta,n} : l < t_n^\delta) = 1$. Hence we may replace the basis element $x_{\nu_\delta(k_n^\delta)}$ by the new basis element $\sum_{l<t_n^\delta} a_l^{\delta,n} x_{\nu_\delta(k_n^\delta+l)}$.

Let $G_{\bar{\eta}} = \bigcup_{\alpha<\omega_1} G_{\bar{\eta}}^\alpha$ and $G_{\bar{\nu}} = \bigcup_{\alpha<\omega_1} G_{\bar{\nu}}^\alpha$ be the canonical $\aleph_1$-filtrations of $G_{\bar{\eta}}$ and $G_{\bar{\nu}}$ respectively. For each $\alpha < \omega_1$ we now define a level-preserving isomorphism from $G_{\bar{\eta}}^\alpha$ onto $G_{\bar{\nu}}^\alpha$. Let $\alpha < \omega_1$ be fixed. Since by assumption $\mathrm{rd}(\bar{\eta}) = \mathrm{rd}(\bar{\nu})$ we may choose a sequence $\langle m_\delta : \delta \in S \cap \alpha \rangle$ as in Lemma 1.7 for $\bar{\eta}$ and $\bar{\nu}$ simultaneously. Let $\hat{\pi} : G_{\bar{\eta}}^\alpha \to G_{\bar{\nu}}^\alpha$ be defined via

- $\hat{\pi}(x_{\eta_\delta(n)}) = \sum_{l<t_n^\delta} a_l^{\delta,n} x_{\nu_\delta(k_n^\delta+l)}$ *for all* $n \geq m_\delta, \delta \in S \cap \alpha$
- $\hat{\pi}(x_{\nu_\delta(k_n^\delta)}) = x_{\eta_\delta(n)}$ *for all* $n \geq m_\delta, \delta \in S \cap \alpha$ if $\eta_\delta(n) \neq \nu_\delta(k_n^\delta)$
- $\hat{\pi}(x_\beta) = x_\beta$ *for every* $\beta < \alpha + \omega$ *otherwise and*
- $\hat{\pi}(z_{\delta,n}) = w_{\delta,n}$ *for all* $n \geq m_\delta, \delta \in S \cap \alpha$.

Recursively we may define $\hat{\pi}(z_{\delta,n})$ for $n < m_\delta$ ($\delta \in S \cap \alpha$) using the definition of $\hat{\pi}$ on x_β ($\beta < \alpha + \omega$) and on z_{δ,m_δ}. By the choice of the sequence $\langle m_\delta : \delta \in S \cap \alpha \rangle$ it is now easy to see that $\hat{\pi}$ is a level-preserving isomorphism from $G_{\bar{\eta}}^\alpha$ onto $G_{\bar{\nu}}^\alpha$ and hence the groups $G_{\bar{\eta}}$ and $G_{\bar{\nu}}$ are filtration-equivalent. □

2. The consistency result

From now on we let $\psi : \omega \to \omega$ be given by $\psi(n) = n!$ with the convention that $0! = 1$. In order to force that the group $G_{\bar{\eta}}$ is a Whitehead group we recall the definition of the uniformization property.

Definition 2.1. If λ is a cardinal and $\bar{\eta}$ is a ladder system on S we say that $\bar{\eta}$ has λ–*uniformization* if for every family $\{c_\delta : \delta \in S\}$ of colorings $c_\delta : \omega \to \lambda$, there exist $\Psi : \omega_1 \to \lambda$ and $\Psi^* : S \to \omega$ such that $\Psi(\eta_\delta(n)) = c_\delta(n)$ for all $n \geq \Psi^*(\delta)$ and $\delta \in S$.

The following lemma is by now standard (compare [2, Chapter XIII, Proposition 0.2]). However, the construction in [2, Chapter XIII, Section 0] is slightly different from our construction since $x_{\eta_\delta(k_n^{\eta_\delta}+i)}$ ($i < t_n^{\eta_\delta}$) appear in Eq. (1.2) at the same time. Therefore, we give the adjusted proof of the next lemma in a particular case for the convenience of the reader. However, we shall only apply it for $\bar{\eta}$ of the simplest form as in Example 1.5(ii).

Lemma 2.2. *If $\bar{\eta}$ is a ladder system which has $\aleph_0$-uniformization, then the group $G_{\bar{\eta}}$ satisfies* $\mathrm{Ext}(G_{\bar{\eta}}, N) = 0$ *for every countable abelian group N. If $\bar{\eta}$ has 2-uniformization then $G_{\bar{\eta}}$ is a Whitehead group.*

Proof. Let N be a countable abelian group. For simplicity we shall assume the setting of Example 1.5(i). The general proof is similar. By construction we may regard $G_{\bar{\eta}}$ as the quotient P/Q of the free group $P = \bigoplus_{\beta<\aleph_1} x_\beta \mathbb{Z} \oplus \bigoplus_{\delta\in S, n\in\omega} z_{\delta,n}\mathbb{Z}$ and its subgroup Q generated by the elements

$$g_{\delta,n} = n! z_{\delta,n+1} - z_{\delta,n} - x_{\eta_\delta(2n+1)} + x_{\eta_\delta(2n)}$$

for $\delta \in S$ and $n \in \omega$. In order to show that $\mathrm{Ext}(G_{\bar{\eta}}, N) = 0$ it therefore suffices to prove that every homomorphism $\varphi : Q \to N$ has an extension $\tilde{\varphi} : P \to N$. Thus let $\varphi \in \mathrm{Hom}(Q, N)$ be given. We fix a bijection $b : N \to \aleph_0$ and define $c_\delta : \omega \to \omega$ for $\delta \in S$ as follows: Let $n \in \omega$ and put

$$c_\delta(2n) = b(\varphi(g_{\delta,n})) \quad \text{and} \quad c_\delta(2n+1) = b(2\varphi(g_{\delta,n})).$$

By the uniformization property there exists $f : \omega_1 \to \omega$ such that for all $\delta \in S$ there exists $k_\delta \in \omega$ such that

$$c_\delta(n) = f(\eta_\delta(n)) \quad \text{for all } n > k_\delta.$$

We define $\tilde{\varphi} : P \to N$ as follows: Let $\alpha < \omega_1$

- If $\alpha = \eta_\delta(n)$ for some $\delta \in S$ and $n > k_\delta$ then put $\tilde{\varphi}(x_\alpha) = b^{-1}(f(\alpha))$; note that $\alpha \notin S$;
- If $\alpha \notin S$ and $\alpha \neq \eta_\delta(n)$ for any $\delta \in S$ and $n > k_\delta$ then put $\tilde{\varphi}(x_\alpha) = 0$;
- if $\alpha \in S$ and $2n > k_\alpha$ then put $\tilde{\varphi}(z_{\alpha,n}) = 0$;
- if $\alpha \in S$ and $2n \leq k_\alpha$ then we define $\tilde{\varphi}(z_{\alpha,n})$ inductively and distinguish the following four cases:

– if $\eta_\alpha(2n) = \eta_\delta(k)$ for some $k > k_\delta$ and $\eta_\alpha(2n+1) = \eta_{\delta'}(k')$ for some $k' > k_{\delta'}$ then put

$$\tilde{\varphi}(z_{\alpha,n}) = b^{-1}(f(\eta_\alpha(2n+1))) - b^{-1}(f(\eta_\alpha(2n))) - \varphi(g_{\alpha,n}) + n!\tilde{\varphi}(z_{\alpha,n+1});$$

– if $\eta_\alpha(2n) = \eta_\delta(k)$ for some $k > k_\delta$ but $\eta_\alpha(2n+1) \neq \eta_{\delta'}(k')$ for all $k' > k_{\delta'}$ and $\delta' \in S$ then put

$$\tilde{\varphi}(z_{\alpha,n}) = -b^{-1}(f(\eta_\alpha(2n))) - \varphi(g_{\alpha,n}) + n!\tilde{\varphi}(z_{\alpha,n+1});$$

– if $\eta_\alpha(2n) \neq \eta_\delta(k)$ for all $k > k_\delta$ and $\delta \in S$ but $\eta_\alpha(2n+1) = \eta_{\delta'}(k')$ for some $k' > k_{\delta'}$ then put

$$\tilde{\varphi}(z_{\alpha,n}) = b^{-1}(f(\eta_\alpha(2n+1))) - \varphi(g_{\alpha,n}) + n!\tilde{\varphi}(z_{\alpha,n+1});$$

– if $\eta_\alpha(2n) \neq \eta_\delta(k)$ for all $k > k_\delta$ and $\delta \in S$ and also $\eta_\alpha(2n+1) \neq \eta_{\delta'}(k')$ for all $k' > k_{\delta'}$ and $\delta' \in S$ then put

$$\tilde{\varphi}(z_{\alpha,n}) = -\varphi(g_{\alpha,n}) + n!\tilde{\varphi}(z_{\alpha,n+1}).$$

It now remains to show that $\tilde{\varphi}$ is an extension of φ, and hence satisfies $\tilde{\varphi}(g_{\alpha,n}) = \varphi(g_{\alpha,n})$ for all $\alpha \in S$ and $n \in \omega$. Clearly we have

$$\tilde{\varphi}(g_{\alpha,n}) = n!\tilde{\varphi}(z_{\alpha,n+1}) - \tilde{\varphi}(z_{\alpha,n}) - \tilde{\varphi}(x_{\eta_\alpha(2n)}) + \tilde{\varphi}(x_{\eta_\alpha(2n+1)}).$$

If $\alpha \in S$ and $2n > k_\alpha$ then

$$\tilde{\varphi}(x_{\eta_\alpha(2n)}) = b^{-1}(f(\eta_\alpha(2n))) = b^{-1}(c_\alpha(2n)) = \varphi(g_{\alpha,n})$$

and similarly $\tilde{\varphi}(x_{\eta_\alpha(2n+1)}) = 2\varphi(g_{\alpha,n})$. Furthermore, $\tilde{\varphi}(z_{\alpha,n}) = \tilde{\varphi}(z_{\alpha,n+1}) = 0$ and hence

$$\tilde{\varphi}(g_{\alpha,n}) = -\varphi(g_{\alpha,n}) + 2\varphi(g_{\alpha,n}) = \varphi(g_{\alpha,n}).$$

All other cases can be checked similarly by easy calculations and are therefore left to the reader.

The second statement follows similarly using [2, Chapter XIII, Lemma 0.3] □

Similarly, we can prove the next result which is essentially [2, Chapter XIII, Proposition 0.6]. Recall that a ladder system $\bar{\eta}$ is called *tree-like* if $\eta_\delta(n) = \eta_{\delta'}(m)$ for some $\delta, \delta' \in S$ and $n, m \in \omega$ implies $m = n$ and $\eta_\delta(k) = \eta_{\delta'}(k)$ for all $k \le n$.

Lemma 2.3. *Let $\bar{\eta}$ be a special tree-like ladder system. If $G_{\bar{\eta}}$ satisfies* $\mathrm{Ext}(G_{\bar{\eta}}, \mathbb{Z}^{(\omega)}) = 0$*, then $\bar{\eta}$ has $\aleph_0$-uniformization. Similarly, if $G_{\bar{\eta}}$ is a Whitehead group, then $\bar{\eta}$ has 2-uniformization.*

Proof. As in the proof of Lemma 2.2 we shall assume for simplicity the setting of Example 1.5 and let $G_{\bar{\eta}} = P/Q$. Let $\{a_{nmj} : n, m, j \in \omega\}$ be a basis of $\mathbb{Z}^{(\omega)}$. Given an ω-coloring $\{c_\delta : \delta \in S\}$ define $\varphi : Q \to \mathbb{Z}^{(\omega)}$ by

$$\varphi(g_{\delta,n}) = a_{nc_\delta(2n+1)c_\delta(2n+2)}.$$

By hypothesis there exists an extension of φ to $\tilde{\varphi} : P \to \mathbb{Z}^{(\omega)}$. Define $\Psi^*(\delta)$ to be the least integer $n' > 4$ such that

$$\varphi(z_{\delta,0}) \in \langle a_{lmj} : l < n', m, j \in \omega \rangle.$$

It suffices to show that if $\eta_\delta(k) = \eta_\gamma(k)$ where $k \ge \Psi^*(\delta), \Psi^*(\gamma)$ then $c_\delta(k) = c_\gamma(k)$. In this case $\Psi(\eta_\delta(k))) = c_\delta(k)$ when $k \ge \Psi^*(\delta)$ is as required. Thus let $k \ge \Psi^*(\delta), \Psi^*(\gamma)$ and $\eta_\delta(k) = \eta_\gamma(k)$. Then $k = 2n+1$ or $2n+2$ for some n with $2 \le n \in \omega$. Let $\tilde{\varphi}'$ be the composition of $\tilde{\varphi}$ with the projection of $\mathbb{Z}^{(\omega)}$ onto $\langle a_{kmj} : m, j \in \omega \rangle$. Then

$$\tilde{\varphi}'(z_{\delta,0}) = \tilde{\varphi}'(z_{\gamma,0}) = 0.$$

Since $\bar{\eta}$ is tree-like we have $x_{\eta_\delta(s)} = x_{\eta_\gamma(s)}$ for all $s \le k$. Using this and the fact that $\tilde{\varphi}'(g_{\delta,n'}) = \tilde{\varphi}'(g_{\gamma,n'}) = 0$ for all $n' < k$ we can show by induction that

$$\tilde{\varphi}'(z_{\delta,n}) = \tilde{\varphi}'(z_{\gamma,n}).$$

Hence

$$a_{nc_\delta(2n+1)c_\delta(2n+2)} - a_{nc_\gamma(2n+1)c_\gamma(2n+2)} = \varphi(g_{\delta,n}) - \varphi(g_{\gamma,n}) = n!(\tilde{\varphi}'(z_{\gamma,n+1}) - \tilde{\varphi}'(z_{\delta,n+1})).$$

Therefore $n!$ divides $a_{nc_\delta(2n+1)c_\delta(2n+2)} - a_{nc_\gamma(2n+1)c_\gamma(2n+2)}$; so $a_{nc_\delta(2n+1)c_\delta(2n+2)}$ must equal $a_{nc_\gamma(2n+1)c_\gamma(2n+2)}$ since they are basis elements and hence $c_\delta(k) = c_\gamma(k)$ since either $k = 2n+1$ or $k = 2n+2$.

The second statement follows similarly with the appropriate adjustments and [2, Chapter XIII, Proposition 0.6]. □

We are now ready to prove the main theorem. Therefore let $\bar{\nu}$ be a special ladder system and $\bar{c} = \langle c_\delta : \omega \to \{0,1\} | \delta \in S \rangle$ a sequence of colorings. We define a group $H_{\bar{\nu},\bar{c}}$ as follows. Similar to the group $G_{\bar{\nu}}$ constructed in the previous section we let F' be the $\mathbb{Q}$-module

$$F' = \hat{w}\mathbb{Q} \oplus \bigoplus_{\alpha<\omega_1} \hat{x}_\alpha \mathbb{Q} \oplus \bigoplus_{\delta \in S, n \in \omega} y'_{\delta,n}\mathbb{Q}$$

and $H_{\bar{\nu},\bar{c}}$ be the subgroup of F' generated by

$$H_{\bar{\nu},\bar{c}} = \left\langle \hat{w}, \hat{x}_\beta, \hat{z}_{\delta,n} : \beta < \omega_1, \delta \in S, n \in \omega \right\rangle \subseteq_* F',$$

where the $\hat{z}_{\delta,n}$ are chosen subject to the relations

$$n!\hat{z}_{\delta,n+1} = \hat{z}_{\delta,n} + \sum_{i<t_n^\delta} a_l^{\delta,n} \hat{x}_{\nu_\delta(k_n^\delta+i)} + c_\delta(n)\hat{w}$$

for $\delta \in S$ and $n \in \omega$. We define a natural mapping $h_{\bar{\nu},\bar{c}} : H_{\bar{\nu},\bar{c}} \to G_{\bar{\nu}}$ via

- $h_{\bar{\nu},\bar{c}}(\hat{x}_\beta) = x_\beta$ for all $\beta < \omega_1$;
- $h_{\bar{\nu},\bar{c}}(\hat{z}_{\delta,n}) = z_{\delta,n}$ for all $\delta \in S$ and $n \in \omega$;
- $h_{\bar{\nu},\bar{c}}(\hat{w}) = 0$.

Obviously, the kernel of $h_{\bar{\nu},\bar{c}}$ is isomorphic to $\mathbb{Z}$, in fact $\ker(h_{\bar{\nu},\bar{c}}) = \hat{w}\mathbb{Z}$. Thus $h_{\bar{\nu},\bar{c}}$ induces a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_{\bar{\nu},\bar{c}} \longrightarrow G_{\bar{\nu}} \longrightarrow 0. \qquad \text{(E)}$$

As for $G_{\bar{\nu}}$ we also define a filtration for $H_{\bar{\nu},\bar{c}}$ by letting

$$H^\alpha_{\bar{\nu},\bar{c}} = \left\langle H_{\bar{\nu},\bar{c}} \cap (\{\hat{\omega}, \hat{x}_\beta : \beta < \alpha + \omega\} \cup \{y'_{\delta,n} : \delta \in S \cap \alpha, n \in \omega\}) \right\rangle_* \subseteq H_{\bar{\nu},\bar{c}}$$

for $\alpha < \omega_1$.

The idea for proving the main theorem is to build an extension model of ZFC in which GCH holds and in which we can force two special ladder systems $\bar{\eta}$ and $\bar{\nu}$ with $\mathrm{rd}(\bar{\eta}) = \mathrm{rd}(\bar{\nu})$ such that $\bar{\eta}$ has the 2-uniformization property, and hence $G_{\bar{\eta}}$ is a Whitehead group but at the same time we force a coloring $\bar{c}$ such that the sequence (E) does not split, and hence $G_{\bar{\nu}}$ is not a Whitehead group. For notational reasons we call a special ladder system of the simplest form as in Example 1.5(ii) a *simple special ladder system*.

Theorem 2.4. *There exists a model of ZFC in which GCH holds and for some special ladder systems $\bar{\eta}, \bar{\nu} \in E$ with $\mathrm{rd}(\bar{\eta}) = \mathrm{rd}(\bar{\nu})$, the group $G_{\bar{\eta}}$ is a Whitehead group, but $G_{\bar{\nu}}$ is not.*

Proof. Essentially the proof is given in [8] (see also [7] and [9]). Therefore we only recall the basic steps of the proof. Suppose we start with a ground model V in which *GCH* holds. Let $\bar{\eta}$ be a (special) ladder system. It was shown in [7, Theorem 1.1] that there exists a forcing notion $(P, \leq)$ such that:

- $|P| = \aleph_2$, P satisfies the $\aleph_2$-chain condition and adds no new sequences of length ω; hence, if V satisfies *GCH*, then also the extension model V^P satisfies *GCH*;
- every stationary set remains stationary in V^P;
- $\bar{\eta}$ has the 2-uniformization property (even the $\aleph_0$-uniformization property (see [7, Theorem 2.1])).

The forcing notion $(P, \leq)$ was obtained by a countable support iteration (of length $\aleph_2$); at each step using a basic forcing extension and taking inverse limits at stages of cofinality $\aleph_0$. We briefly recall the basic step to be iterated. Let $\bar{c} = \langle c_\delta : \delta \in S\rangle$ be a system of colors which has to be uniformized. Here each $c_\delta : \omega \to 2$. Define $P_{\bar{c}}$ as the set of all functions f such that

(i) $f : \alpha \to 2$ for some $\alpha < \omega_1$;
(ii) for all $\delta \leq \alpha$, $\delta \in S$ there is n_δ such that $f(\eta_\delta(m)) = c_\delta(m)$ for all $m \geq n_\delta$.

$P_{\bar{c}}$ is ordered naturally and it is easy to see that the set $D_\alpha = \{f \in P_{\bar{c}} : \alpha \subseteq \mathrm{Dom}(f)\}$ is dense for every $\alpha < \omega_1$ and hence a generic filter will give the desired uniformizing Ψ.

Now, assume that $V \vDash GCH$ is given. We shall define a countable support iteration $\bar{Q} = \left\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \right\rangle$ as follows: We start with an initial forcing (compare also [3]).

Definition 2.5. Let P_0 consist of all triples $\langle \bar{\eta}, \bar{\nu}, \bar{c}\rangle$ such that for some $\alpha < \omega_1$ we have

- $\bar{\eta}, \bar{\nu} \in E_\alpha$ are special ladder systems on $S \cap \alpha$
- $\bar{\eta}$ is simple
- $\bar{c} = \langle c_\delta : \omega \to 2 \mid \delta \in S \cap \alpha\rangle$
- $\mathrm{rd}_\alpha(\bar{\eta}) = \mathrm{rd}_\alpha(\bar{\nu})$.

We may think of the conditions in P_0 as partial special ladder systems on $S \cap \alpha$ for some $\alpha < \omega_1$ and a corresponding partial coloring. It is easy to check that a P_0-generic filter G gives a pair of special ladder systems $\bar{\eta}, \bar{\nu}$ on S (in the extension model $V[G]$) with the same ω-range and a global coloring $\bar{c}$. Moreover, $\bar{\eta}$ will be simple. Let $\tilde{\bar{\eta}}, \tilde{\bar{\nu}}$ and $\tilde{\bar{c}}, \tilde{c}_\delta$ be the corresponding P_0-names which are defined naturally. Note that P_0 is ω-closed and satisfies the $\aleph_2$-chain condition, so GCH holds in $V[G]$ since it holds in V. Applying the forcing described above to $V' = V[G]$ we can force that in $(V')^P$ the ladder system $\tilde{\bar{\eta}}$ has the 2-uniformization property and hence the group $G_{\tilde{\bar{\eta}}}$ is a Whitehead group. Here, we let $P = \left\langle P_\alpha, \dot{Q}_\alpha : 1 \leq \alpha < \omega_2 \right\rangle$, hence in

$V^{\bar{Q}} = (V')^P$ the generalized continuum hypothesis holds. We have to show that the group $G_{\tilde{\bar{\nu}}}$ is not a Whitehead group. As indicated this shall be done by showing that the sequence (E) cannot be forced to split.

For the sake of contradiction assume that (E) splits. Hence for some $p \in \bar{Q}$ we have

$$p \Vdash \text{“}f_{\tilde{\bar{\nu}},\tilde{\bar{c}}} \in \mathrm{Hom}(G_{\tilde{\bar{\nu}}}, H_{\tilde{\bar{\nu}},\tilde{\bar{c}}}) \text{ is a right inverse of } h_{\tilde{\bar{\nu}},\tilde{\bar{c}}}.\text{"}$$

Since $\bar{Q}$ satisfies the $\aleph_2$ chain condition we can replace $\bar{Q}$ by P_α for some $\alpha < \omega_2$.

For an infinite cardinal κ let $H(\kappa)$ be the class of sets hereditarily of cardinality $< \kappa$, i.e. $H(\kappa) = \{X : |TC(X)| < \kappa\}$ where $TC(X)$ is the transitive closure of the set X. As in [8] there is an elementary submodel $N \prec (H(\aleph_2), \varepsilon)$ such that

- $|N| = \aleph_0$;
- $p, f_{\tilde{\bar{\nu}},\tilde{\bar{c}}} \in N_0$;
- $N = \bigcup_{n\in\omega} N_n$ with $N_n \prec (H(\aleph_2), \varepsilon)$ elementary submodels such that $N_n \in N_{n+1}$.

We let $\delta = N \cap \omega_1 \in S$ and $\delta_n = N_n \cap \omega_1$ for $n \in \omega$. Note that δ can be chosen from S because the set of δ's that can be obtained from N's is a club and therefore meets the stationary set S. Choose η'_δ such that $\eta'_\delta(n) \in [\delta_n, \delta_{n+1}]$ for all $n \in \omega$ and η'_δ is simple and special.

As in [7, Lemma 1.8] and [8, Theorem 2.1] (see also [3]) we define inductively a sequence of finite sets of conditions in the following way:

In stage n let $\eta'_\delta(n) = \gamma$. We have a finite tree $\langle p^n_t : t \in T_n\rangle \in N_{n+1}$ of conditions and let $k^\delta_{n+1} = k^\delta_n + |\max(T_n)| + 1$. Moreover, if $t \in \max(T_n)$, then

$$p^n_t \text{ forces a value to } f_{\tilde{\bar{\nu}},\tilde{\bar{c}}} \restriction G_{\tilde{\bar{\nu}}\restriction_{\gamma+\omega}}.$$

We now choose $\nu_\delta(k^\delta_n + i)$ in $[\gamma, \gamma + \omega]$ for $i < t^\delta_n$ so that ν_δ becomes special and $\mathrm{rd}(\eta'_\delta) = \mathrm{rd}(\nu_\delta)$. We have that, if $t \in \max(T_n)$, then

$$p^n_t \Vdash \text{“}f_{\tilde{\bar{\nu}},\tilde{\bar{c}}}(x_{\nu_\delta(k^\delta_n+i)}) - \hat{x}_{\nu_\delta(k^\delta_n+i)} = \hat{w}\tilde{b}_{t,n,i}\text{"}$$

for every $i < t^\delta_n$. By linear algebra we may choose $a^{\delta,n}_l$ for $l < t^\delta_n$ such that $\gcd(a^{\delta,n}_l : l < t^\delta_n) = 1$ and if $t \in \max(T_n)$, then

$$p^n_t \Vdash \text{“}\sum_{i<t^\delta_n} a^{\delta,n}_i \tilde{b}_{t,n,i} = 0\text{"}.$$

Finally, we choose $c_\delta(n)$ arbitrarily. In the inverse limit we hence obtain a triple $\langle \bar{\eta}', \bar{\nu}', \bar{c}'\rangle$ which we may increase to $\langle \bar{\eta}'\langle\eta'_\delta\rangle, \bar{\nu}'\langle\nu_\delta\rangle, \bar{c}'\langle c_\delta\rangle\rangle$. Now we can find $p^* \in P_\alpha$ above p^n_t for some $t \in \max(T_n)$ and all $n \in \omega$. Note that the c_δ was chosen arbitrarily, so there are $2^{\aleph_0}$ possible choices for the same ν_δ.

Now assume that G is a generic filter containing the condition p^*. Then $\hat{z}_{\delta,0} - f_{\tilde{\bar{\nu}},\tilde{\bar{c}}}(z_{\delta,0}) \in \hat{w}\mathbb{Z}$. Moreover,

$$n! f_{\bar{\nu},\bar{c}}(z_{\delta,n+1}) = f_{\bar{\nu},\bar{c}}(z_{\delta,n}) + \sum_{i<t^\delta_n} a^{\delta,n}_i f_{\bar{\nu},\bar{c}}(x_{\nu_\delta(k^\delta_n+i)})$$

for every $n \in \omega$. Similarly, we have

$$n!\hat{z}_{\delta,n+1} = \hat{z}_{\delta,n} + \sum_{i<t^\delta_n} a^{\delta,n}_i \hat{x}_{\nu_\delta(k^\delta_n+i)} + c_\delta(n)\hat{w}.$$

Subtracting the two equations yields

$$n!(f_{\bar{\nu},\bar{c}}(z_{\delta,n+1}) - \hat{z}_{\delta,n+1}) = (f_{\bar{\nu},\bar{c}}(z_{\delta,n}) - \hat{z}_{\delta,n}) + \sum_{i<t^\delta_n} a^{\delta,n}_i (f_{\bar{\nu},\bar{c}}(x_{\nu_\delta(k^\delta_n+i)}) - \hat{x}_{\nu_\delta(k^\delta_n+i)}) - c_\delta(n)\hat{w}.$$

But by our choice we have

$$\sum_{i<t^\delta_n} a^{\delta,n}_i (f_{\bar{\nu},\bar{c}}(x_{\nu_\delta(k^\delta_n+i)}) - \hat{x}_{\nu_\delta(k^\delta_n+i)}) = \sum_{i<t^\delta_n} a^{\delta,n}_i b_{t,n,i} = 0.$$

Therefore, we get

$$n!(f_{\bar{\nu},\bar{c}}(z_{\delta,n+1}) - \hat{z}_{\delta,n+1}) = (f_{\bar{\nu},\bar{c}}(z_{\delta,n}) - \hat{z}_{\delta,n}) - c_\delta(n)\hat{w} \in \mathbb{Z}\hat{w}. \tag{2.1}$$

Since $\mathbb{Z}$ is countable there exist generic filters G_1 and G_2 (and corresponding triples $(\bar{\eta}^1, \bar{\nu}^1, \bar{c}^1)$ and $(\bar{\eta}^2, \bar{\nu}^2, \bar{c}^2)$) such that $c^1_\delta \neq c^2_\delta$, $\nu^1_\delta = \nu^2_\delta = \nu_\delta$ but

$$\hat{z}_{\delta,0} - f_{\bar{\nu}^1,\bar{c}^1}(z_{\delta,0}) = \hat{z}_{\delta,0} - f_{\bar{\nu}^2,\bar{c}^2}(z_{\delta,0}) \in \mathbb{Z}\hat{w}.$$

Let n be minimal such that $c^1_\delta(n) \neq c^2_\delta(n)$. Then an easy induction using Eq. (2.1) shows that

$$f_{\bar{\nu}^1,\bar{c}^1}(z^1_{\delta,l}) - \hat{z}^1_{\delta,l} = f_{\bar{\nu}^2,\bar{c}^2}(z^2_{\delta,l}) - \hat{z}^2_{\delta,l} \in \mathbb{Z}\hat{w}$$

for every $l \le n$. Note that $\hat{z}_{\delta,i}$ depends on G_i $(i = 1, 2)$. We finally calculate

$$\begin{aligned} & n!(f_{\bar{\nu}^1,\bar{c}^1}(z^1_{\delta,n+1}) - \hat{z}^1_{\delta,n+1}) - n!(f_{\bar{\nu}^2,\bar{c}^2}(z^2_{\delta,n+1}) - \hat{z}^2_{\delta,n+1}) \\ & \quad = (f_{\bar{\nu}^1,\bar{c}^1}(z^1_{\delta,n}) - \hat{z}^1_{\delta,n}) - (f_{\bar{\nu}^2,\bar{c}^2}(z^2_{\delta,n}) - \hat{z}^2_{\delta,n}) + (c^1_\delta(n) - c^2_\delta(n))\hat{w}. \end{aligned}$$

By Eq. (2.1) we conclude

$$n!(f_{\bar{\nu}^1,\bar{c}^1}(z^1_{\delta,n+1}) - \hat{z}^1_{\delta,n+1}) - n!(f_{\bar{\nu}^2,\bar{c}^2}(z^2_{\delta,n+1}) - \hat{z}^2_{\delta,n+1}) = (c^1_\delta(n) - c^2_\delta(n))\hat{w}.$$

However, the left side is divisible by 2 but the right side is $\hat{w}$ or $-\hat{w}$, hence not divisible by 2 — a contradiction. Note that all the differences are elements of the pure subgroup $\hat{w}\mathbb{Z}$ by Eq. (2.1). Hence the above calculations take place in $\hat{w}\mathbb{Z}$ which is in the ground model, although the elements we are talking about come from different (incompatible) extension models. Thus the sequence (E) cannot be forced to split and this finishes the proof. □

At this point we would like to remark that it is not clear if the special ladder system $\hat{\bar{\eta}}$ in the above proof is still simple?

Corollary 2.6. *It is consistent with ZFC and GCH that there exist two filtration-equivalent $\aleph_1$-separable abelian groups of cardinality $\aleph_1$ such that one is Whitehead and the other is not.*

Proof. Applying Theorem 2.4 to a model V of GCH we get an extension model of V in which there exist abelian groups $G_{\bar{\eta}}$ and $G_{\bar{\nu}}$ for $\bar{\eta}, \bar{\nu} \in E$ such that $G_{\bar{\eta}}$ is a Whitehead group but $G_{\bar{\nu}}$ is not. Since $\mathrm{rd}(\bar{\eta}) = \mathrm{rd}(\bar{\nu})$ we deduce that $G_{\bar{\eta}}$ and $G_{\bar{\nu}}$ are filtration-equivalent by Proposition 1.9. □

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