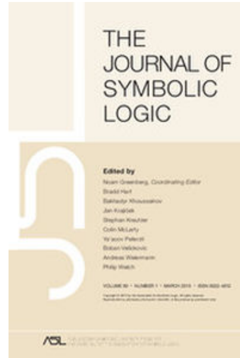
Sh:76



Independence of strong partition relation for small cardinals, and the free-subset problem

Saharon Shelah

The Journal of Symbolic Logic / Volume 45 / Issue 03 / September 1980, pp 505 - 509
DOI: 10.2307/2273418, Published online: 12 March 2014

THE JOURNAL OF SYMBOLIC LOGIC
Volume 45, Number 3, Sept. 1980

INDEPENDENCE OF STRONG PARTITION RELATION FOR SMALL CARDINALS, AND THE FREE-SUBSET PROBLEM

SAHARON SHELAH[1]

Abstract. We prove the independence of a strong partition relation on $\aleph_\omega$, answering a question of Erdös and Hajnal. We then give an almost complete answer to the free subset problem.

We were interested in

1. *Problem.* Is the following consistent e.g. with G.C.H.?

For any function f from $S_{\aleph_0}(\aleph_\omega)$ (= the finite subsets of $\aleph_\omega$) to $\{0, 1\}$, there are disjoint $S_n \subseteq \aleph_\omega$, $|S_n| = \aleph_n$, such that if U, V are finite subsets of $\aleph_\omega$ and $(\forall n)$ $(|U \cap S_n| = |V \cap S_n| \leq r_n)$ then $f(u) = f(v)$ where

Version 1. $r_n = n$,

or

Version 2. $r_n = r^*$ for some finite r^*.

Considering the well-known results on measurable and real valued measurable, this seems essentially the best we can hope for $\aleph_\omega$. Note that if we require also $|U|$, $|V| < r^{**}$ for some fixed r^{**}, this was answered positively by Erdös, Hajnal and Rado (see e.g. [EHMR]).

Erdös and Hajnal [EH 1] asked this question for $r^* = 1$ (corrected version see [EH 2, p. 27, problem 29]). Our main result is a consistency result for this case, (Theorem 3) and a generalization to all cardinals (Theorem 4).[2]

Let us turn to the free subset problem.

2. DEFINITION. $\mathrm{Fr}_\mu(\kappa, \lambda)$ means that for every algebra M, with μ (finitary) operations, and κ elements, there is a set A of λ elements which is free, i.e., no $a \in A$ belong to the closure of $A - \{a\}$ by the operations of M. We omit μ if $\mu = \aleph_0$.

On this problem see Erdös and Hajnal [EH 1], Devlin [D], and Devlin and Paris [DP].

Baumgartner (essentially) proved: if $V = L$ then $\mathrm{Fr}(\kappa, \aleph_0)$ iff $\kappa \to (\omega)_2^{<\omega}$ (see [DP, p. 334] for this and more). Hence if $V = L$ then $\mathrm{Fr}(\aleph_\omega, \aleph_0)$ fails, but if Problem 1 has a positive answer even for $r^* = 1$, then $\mathrm{Fr}(\aleph_\omega, \aleph_0)$ holds. So if $V = L$, then Problem 1 has a negative answer. So together we get an essentially complete answer to [EH 1, Problem 24].

Received November 21, 1977; revised January 2, 1979.

[1]The author would like to thank the United States-Israel Binational Science Foundation for partially supporting this research by grant 1110.

[2]Meanwhile the author has proved the consistency of a positive answer to Version 1 (assuming the existence of infinitely many compact cardinals).

0022-4812/80/4503-0007/$02.25

Sh:76

Now [EH 3] states and [D, p. 315] proves that $\mathrm{Fr}(\aleph_\alpha, n)$ iff $\alpha \geq n$, and not $\mathrm{Fr}(\aleph_\omega, \aleph_1)$. Let H_α (the αth Hajnal cardinal) be the least κ such that $\mathrm{Fr}(\kappa, \aleph_\alpha)$. Devlin [D, p. 318, Theorem 14] proves H_α is increasing.

We prove in this paper that not $\mathrm{Fr}(\aleph_\alpha, |\alpha|^+)$ (so $H_\alpha \geq \aleph_{\aleph_\alpha}$) but that it is consistent with ZFC + G.C.H. that $\mathrm{Fr}(\aleph_\kappa, \kappa)$ when $\aleph_\kappa > \kappa$. In fact $\mathrm{Fr}_{\aleph_\alpha}(\aleph_{\alpha+\kappa}, \kappa)$ when $\aleph_{\alpha+\kappa} > \kappa$ (our consistency results are assuming the consistency of "ZFC + enough measurable cardinals exist"). In particular we answer Problem 2 of Devlin [D, p. 319].

Notice that a positive answer to 1 for $r^* = 2$, implies every Banach space of cardinality $\geq \aleph_\omega$ has an infinite sequence which is a commutative base.

We now finish the introduction and start the mathematical development.

3. THEOREM. *Suppose the consistency of "ZFC + there are infinitely many measurables."*

Then it is consistent that: ZFC + G.C.H. + "*if for each* n, $f_n\colon S_{\aleph_0}(\aleph_\omega) \to \aleph_{2n+1}$ *then there are* $S_n \subseteq \aleph_{2n+2}$, $|S_n| = \aleph_{2n+2}$ $(n < \omega)$ *such that for any* $k < \omega$, $a_l, b_l \in S_{n(l)}$ $(l < k)$, $n \leq n(0) \leq n(1) < \cdots < n(k-1)$, $f_n(\{a_0, \ldots, a_{k-1}\}) = f_n(\{b_0, \ldots, b_{k-1}\})$".

REMARK. It might help the reader to know that the basic ideas of the proof are

(3a) If κ_n is measurable for $n < \omega$, $\kappa_0 = \aleph_0$, $\kappa_n < \kappa_{n+1}$, D_n a κ_n-complete normal ultrafilter on κ_n for $n > 0$, $\kappa = \sum_{n<\omega} \kappa_n$, then the sequence $\langle \kappa_n : n < \omega\rangle$ satisfies a combinatorial property ((3c) defined below) parallel to the one we want the sequence $\langle \aleph_n : n < \omega\rangle$ to satisfy.

(3b) This property is preserved when we collapse each κ_n to $\aleph_n$. The (known) property is

(3c) If $f_n\colon S_{\aleph_0}(\kappa_{n+1}) \to \kappa_n^+$, then there are $A_n \in D_{n+1}$ such that for $v \in S_{\aleph_0}(\kappa_{n+1})$, $v \subseteq \bigcup_n A_n$, $n < \omega$, $f_n(v)$ depends only on $\langle |v \cap A_n| : n < \omega\rangle$.

Proof. Note that by simple coding, we can restrict ourselves in the conclusion of the theorem to the case $n(l) = n + l$. Let $\aleph_0 = \kappa_0 < \kappa_1 < \kappa_2 < \cdots < \kappa_n$ be measurable and assume G.C.H. Let D_n be a complete normal ultrafilter over κ_n for $n > 0$.

Let $P_n = (P_n, \leq)$ be the Levi forcing conditions for collapsing all cardinals μ, $\kappa_n^+ < \mu < \kappa_{n+1}$ to κ_n^+ (so P_n is closed under ascending chains of length $< \kappa_n^+$), and ϕ_n its empty condition. The proof that P_n satisfies the κ_{n+1}-chain condition shows in fact:

(*) If $p_\alpha \in P_n$ for $\alpha < \kappa_{n+1}$ then for some $A \in D_{n+1}$ and p, $\alpha \in A \Rightarrow p \subseteq p_\alpha$, and $\alpha \neq \beta \in A$ implies $\mathrm{Dom}\, p_\alpha \cap \mathrm{Dom}\, p_\alpha = \mathrm{Dom}\, p$. Hence $\{p_\alpha : \alpha \in A\}$ are pairwise compatible; moreover the union of any $\leq \kappa_n$ of them is a condition in P_n, and any $p^* \geq p$ is compatible with every p_α $(\alpha \in A)$ except $\leq \kappa_n$ of them.

Let $P = \prod_{n<\omega} P_n$ be ordered by $\langle p_n : n < \omega\rangle \leq \langle q_n : n < \omega\rangle$ iff $\bigwedge_{n<\omega}[p_n \leq q_n]$; we assume the P_n's are pairwise disjoint, elements of P are denoted by $\bar{p}$. Clearly if G is generic for P, $\aleph_{2n}^{V[G]} = \kappa_n$, $\aleph_{2n+1}^{V[G]} = \kappa_n^+$. Suppose $\tau = \langle f_n : n < \omega\rangle$ is a term, $\bar{p} \in P$, $\bar{p} \Vdash_P$ "$\langle f_n : n < \omega\rangle$ as in the theorem".

We define by induction on $n < \omega$ conditions $p_l^n \in P_l$ $(l < \omega)$, sets $A_l^n \in D_{l+1}$, and conditions $q_\alpha^n \in P_l$ $(l < \omega, \kappa_l < \alpha < \kappa_{l+1})$ such that

(1) If $p_l^n \leq p \in P_l$, then $\{\alpha < \kappa_{l+1} : p, q_\alpha^n \text{ compatible (in } P_l)\} \in D_{l+1}$.

(2) $p_l^n \leq p_l^{n+1}$, $p_l^n \leq q_\alpha^n$, when $\kappa_l < \alpha < \kappa_{l+1}$, $\alpha \in A_l^{n+1}$.

(3) $q_\alpha^n \leq q_\alpha^{n+1}$.

(4) $A_l^{n+1} \subseteq A_l^n$, $A_l^n \in D_{l+1}$, $\kappa_l \cap A_l^0 = \emptyset$, and for $\alpha \neq \beta \in A_l^n$, $\mathrm{Dom}\ q_\alpha^n \cap \mathrm{Dom}\ q_\beta^n = \mathrm{Dom}\ p_l^n$.

(5) For each $k, r < \omega$, $n \leq r + 1$, and any $\alpha(k + r) \in A_{k+r}^n, \ldots, \alpha(k) \in A_k^n$, let $\bar{p}^* = \langle \phi_0, \ldots, \phi_{k+r-n}, q^n_{\alpha(k+r-n+1)}, \ldots, q^n_{\alpha(k+r)}, p^n_{k+r+1}, \ldots \rangle$; *if* $\bar{p}^* \leq \bar{p}' = \langle p_0', p_1', \ldots \rangle$, $\bar{p}' \Vdash_P f_k(\{\alpha(k), \ldots, \alpha(k + r)\}) = c$ *then* $\langle p_0', \ldots, p'_{k+r-n}, q^n_{\alpha(k+r-n+1)}), \ldots, q^n_{\alpha(k+r)}, \ldots, p^n_{k+r+1}, \ldots \rangle \Vdash_P f_k(\{\alpha(k), \ldots, \alpha(k + r)\}) = c$.

(6) For each $k, r < \omega$, $n = r + 2$ and $\beta(k + i)$, $\alpha(k + i) \in A^n_{k+i}$, for $i \leq r$, c and $p_0' \in P_0, \ldots, p'_{k-1} \in P_{k-1}$, the following are equivalent:

(i) $\langle p_0', \ldots, p'_{k-1}, q^n_{\alpha(k)}, \ldots, q^n_{\alpha(k+r)}, p^n_{k+r+1}, p^n_{k+r+2}, \ldots \rangle \Vdash_P f_k(\{\alpha(k), \ldots, \alpha(k+r)\}) = c$,

(ii) $\langle p_0', \ldots, p'_{k-1}, q^n_{\beta(k)}, \ldots, q^n_{\beta(k+r)}, p^n_{k+r+1}, p^n_{k+r+2}, \ldots \rangle \Vdash_P f_k(\{\beta(k), \ldots, \beta(k + r)\}) = c$.

Case I. $n = 0$. We define by induction on $j < \omega$ conditions $t_{l,j} \in P_l$ for $l \geq j$, increasing with j. For $j = 0$, $t_{l,j} = p_l$. For $j + 1$, we define $t_{j+1,j+1}\ t_{j+2,j+1}, \ldots$ such that the parallel of (5) holds, i.e.

(8) *If* $k, r < \omega$, $k + r \leq j$, $c < \aleph_{2k+1}$, $\alpha(k + r) < \kappa_{(k+r)+1}, \ldots, \alpha(k) < \kappa_{k+1}$, $\bar{p}^* = \langle \phi_0, \ldots, \phi_j, t_{j+1,j+1}, t_{j+2,j+1}, \ldots \rangle$ and $\bar{p}^* \leq \bar{p}' \in P$ and $\bar{p}' \Vdash_P f_k(\{\alpha(\bar{p}^*/k), \ldots, \alpha(k + r)\}) = c$ *then* $\langle p_0', \ldots, p_j', t_{j+1,j+1}, t_{j+2,j+1}, \ldots \rangle \Vdash_P f_k(\{\alpha(k), \ldots, \alpha(k + r)\}) = c$.

This is possible as the number of possible $\langle k, r, c, p_0', \ldots, p_j', \alpha(k), \ldots, \alpha(k + r)\rangle$ is $\leq \kappa_{j+1}$, and $\prod_{l \geq j+1} P_l$ is κ^+_{j+1}-complete.

Case II. $n = m + 1$ and we have defined for m. First define the q_α^n's so that (5) becomes true and then choose p_l^n, A_l^n such that (6) and (1)—(4) become true (using the fact (*) mentioned in the beginning of the proof, and property (3c) mentioned in the remark to the theorem).

Now let $p_l^\omega = \bigcup_{n<\omega} p_l^n$, $q_\alpha^\omega = \bigcup_{n<\omega} q_\alpha^n$, $A_l^\omega = \bigcap_{n<\omega} A_l^n \in D_{l+1}$; by the definition of P_n the parallel of (1), (2) holds.

Let $\bar{p}^* = \langle p_0^\omega, p_1^\omega, \ldots \rangle$ and let $G \subseteq P$, $\bar{p} \in G$ be generic, G_l its projection on P_l then for each $l < \omega$, by (1) $S_{l+1} = \{\alpha \in A_l^\omega : q_\alpha^\omega \in G_l\}$ is unbounded in κ_{l+1}, so clearly $V[G] \models$ " $|S_l| = \aleph_{2l+2}$". By (5) and (6) above we get that the S_l's are as required.

REMARK. I thank M. Magidor for telling me that S is unbounded, thus in Theorem 3 we get $|S_n| = \aleph_{2n}$ (before I used a condition and got $|S_n| = \aleph_{2n-2}$).

Similarly

4. THEOREM. *Assuming e.g. Consists(ZFC + $\exists$ supercompact) we get the consistency of ZFC + G.C.H + the following*:

Let $\alpha(i)$ $(i < i^*)$ *be strictly increasing*, $i^* \leq \aleph_{\alpha(0)}$, $\alpha(i) + 1 < \alpha(i + 1)$, $\alpha(*) = \sup_{i<i^*} \alpha(i)$, $\alpha(i)$ *even*, $f_i = S_{\aleph_0}(\aleph_{\alpha(*)}) \rightarrow \aleph_{\alpha(i)+1}$, *then there are* $S_i \subseteq \aleph_{\alpha(0)}$, $|S_i| = \aleph_{\alpha(i)+2}$, *such that for* $i \leq i(0) < i(1) < \cdots < i(m - 1) < i^*$, $a_l, b_l \in S_{i(l)}$, $f_i(\{a_0, \ldots, a_{m-1}\}) = f_i(\{b_0, \ldots, b_{m-1}\})$.

Now we can answer a question of Devlin [D].

5. *Conclusion*. If $\chi \geq \aleph_{\beta+\kappa} > \kappa$, $\alpha \leq \beta$ then $\mathrm{Fr}_{\aleph_\alpha}(\chi, \kappa)$, in the model of set theory constructed in 4.

PROOF. Let M be an algebra with $\aleph_\alpha$ operations, we can assume $\aleph_{\beta+\kappa} \subseteq |M|$, $\aleph_{\beta+\kappa}$ as a set of ordinals in 4, $i^* = \kappa$, $\alpha(i) = \beta + 2i$ and define

$$f_i(U) = \begin{cases} 0 & U \text{ is included in the closure of} \\ & U - \{\gamma : \aleph_{\alpha(i)} < \gamma < \aleph_{\alpha(i)+2}\} \text{ in } M, \\ 1 & \text{otherwise.} \end{cases}$$

Now apply 4.

6. THEOREM. $\mathrm{Fr}(\aleph_\alpha, |\alpha|^+)$ *does not hold.*

REMARK. This matched almost exactly with Conclusion 5.

Theorem 6 follows by 7, by induction on α ([D, p. 315, Theorem 6] contains a theorem of Erdös and Hajnal, which is 7(1) for $\alpha \leq \omega$).

7. LEMMA. (1) *If* $\mathrm{Fr}_\chi(\lambda, \mu)$ *fail then* $\mathrm{Fr}_\chi(\lambda^+, \mu)$ *fail*

(2) *If* $(\forall\lambda^* < \lambda)\neg\mathrm{Fr}_\chi(\lambda^*, \mu)$ *then*

(a) $\mathrm{Fr}_\chi(\lambda, \mu^+)$ *fail,*

(b) $\mathrm{Fr}_\chi(\lambda, \mu)$ *fail if* (α) $\lambda < \aleph_{\mathrm{cf}\,\mu}$ *or* (β) $\mathrm{cf}\,\lambda \leq \chi$, $\mathrm{cf}\,\lambda \neq \mathrm{cf}\,\mu$.

PROOF. (1) We define an algebra M on λ^+: we first have $\leq \chi$ operations on λ exemplifying $\neg\mathrm{Fr}_\chi(\lambda, \mu)$, functions f, g such that

$$(*) \qquad \lambda \leq \alpha < \lambda^+ \wedge \beta < \alpha \Rightarrow f(\alpha, \beta) < \lambda \wedge g(\alpha, f(\alpha, \beta)) = \beta.$$

Suppose $A = \{a_i : i < \mu\}$ is free in M w.l.o.g. a_i is increasing; if μ a_i's are $< \lambda$ we get contradiction trivially. So suppose $\lambda \leq a_i$.

Case (i). For some $i < j < \mu$, $f(a_{2i+1}, a_{2i}) = f(a_{2j+1}, a_{2j})$ then $a_{2j} = g(a_{2j+1}, f(a_{2i+1}, a_{2i}))$ contradicting the freeness of A.

Case (ii). Not (i). Then $b_i = f(a_{2i+1}, a_{2i}) < \lambda$. Now by the definition of $M \restriction \lambda$, $\{b_i : i < \mu\}$ cannot be free, so for some $j \notin \{i_1, \ldots, i_n\}$, and term τ, $b_j = \tau(b_{i_1}, \ldots, b_{i_n})$ *then*

$$a_{2j} = g(a_{2j+1}, \tau[f(a_{2i_1+1}, a_{2i_n+1}), \ldots, f(a_{2i_n+1}, a_{2i_n})]).$$

Contradiction again.

(2) For every $\alpha < \lambda$ there is an algebra M_α (with no free subset of power μ) with universe α and functions f_ξ^α $(\xi < \chi)$, and w.l.o.g. f_ξ^α is an $n(\xi)$-place function. Define M: its universe in λ. It has function f, g, such that $(\forall\beta < \alpha)[f(\alpha, \beta) < |\alpha| \wedge g(\alpha, f(\alpha, \beta)) = \beta]$ and it has functions f_ξ $(\xi < \chi)$, where f_ξ is an $(n(\xi) + 1)$-place function, and $f_\xi(\alpha, \beta_1, \ldots, \beta_{n(\xi)}) = f_\xi^\alpha(\beta_1, \ldots, \beta_{n(\xi)})$ if defined and zero otherwise. In (b) case (β) put an unbounded set of cardinals $< \lambda$ as individual constants. Now part (a) is trivially exemplified by M. Let us prove (b), so suppose $A \subseteq \lambda$ is free, $|A| = \mu$, and choose A with minimal sup A. By hypothesis, $\mathrm{cf}\,\lambda \neq \mathrm{cf}\,\mu$, hence for some $\alpha < \lambda$, $|A \cap \alpha| = \mu$, hence $\alpha^* = \sup A < \lambda$. If some $\beta > \alpha^*$ is in the closure of the empty set in M, we get a contradiction by M_β, so case (β) is proved.

Now $\{|\alpha| : \alpha \in A\}$ has cardinality $\geq \mathrm{cf}\,\mu$, otherwise for some $\kappa |\{\alpha \in A : |\alpha| = \kappa|\}$ $= \mu$ so $\kappa \leq \alpha^* \leq \kappa^+$, and as in 7(1) we get a contradiction to α^*'s minimality; but this contradicts (α), thus we are finished.

REFERENCES

[EH 1] P. ERDÖS and A. HAJNAL, *Unsolved problems in set theory*, ***U.C.L.A. Symposium*, 1967**; (Scott, Editor), ***Proceedings of Symposia in Pure Mathematics***, Vol. XII, Part I, American Mathematical Society, Providence, R.I., 1971, pp. 17–48.

[EH 2] ———, *Unsolved and solved problems in set theory*, ***Tarski 70th Birthday Symposium, Berkeley*, 1971** (Henkin, Editor), ***Proceedings of Symposia in Pure Mathematics***, Vol. XXV, American Mathematical Society, Providence, R.I., 1974, pp. 269–288.

[EH 3] ———, *On a problem of B. Jonsson*, ***Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques***, vol. 14(1966), pp. 19–23.

[EHMR] P. Erdös, A. Hajnal, A. Mate and R. Rado, *Partition calculus* (to appear).

[D] K. Devlin, *Some weak versions of large cardinal axioms*, ***Annals of Mathematical Logic***, vol. 5(1973), pp. 291–325.

[DP] K. Devlin and J. Paris, *More on the free subset problem*, ***Annals of Mathematical Logic***, vol. 5(1973), pp. 327–336.

HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL