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ON MODELS WITH POWER-LIKE ORDERINGS

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Abstract. We prove here theorems of the form: if T has a model M in which $P_1(M)$ is κ_1-like ordered, $P_2(M)$ is κ_2-like ordered ..., and $Q_1(M)$ is of power $\lambda_1, \cdots$, then T has a model N in which $P_1(M)$ is κ'_1-like ordered ..., $Q_1(N)$ is of power $\lambda'_1, \cdots$. (In this article κ is a strong-limit singular cardinal, and κ' is a singular cardinal.)

We also sometimes add the condition that M, N omits some types. The results are seemingly the best possible, i.e. according to our knowledge about n-cardinal problems (or, more precisely, a certain variant of them).

§0. Introduction.

DEFINITION 0.1. In a model M, $P(M)$ is κ-like (ordered) if $<$ (or more exactly $<^M$) orders $P(M)$, $P(M)$ is of power κ, and every head of $P(M)$ is of power $<\kappa$, i.e. if $l \in P(M)$ then $|\{a \in P(M): a < l\}| < \kappa$. [$P(M) = \{a: a$ an element of M, $M \models P[a]\}$.]

DEFINITION 0.2. A model M is a $\langle \kappa_1, \kappa_2, \cdots | \lambda_1, \lambda_2, \cdots \rangle$-model if $P_1(M)$ is κ_1-ordered, $P_2(M)$ is κ_2-ordered, $\cdots$; $|Q_1(M)| = \lambda_1$, $|Q_2(M)| = \lambda_2, \cdots$ (this notation includes also $\langle \kappa_1, \kappa_2 | \rangle$).

DEFINITION 0.3. (1) $\langle \chi, \zeta \rangle: \langle \kappa_1, \cdots | \lambda_1, \cdots \rangle \to \langle \kappa'_1, \cdots | \lambda'_1, \cdots \rangle$ if : *if* L is a first-order language of power $\leq \chi$, T a theory in L, Γ a set of $\leq \zeta$ types in L, and T has a $\langle \kappa_1, \cdots | \lambda_1, \cdots \rangle$-model omitting every $p \in \Gamma$, *then* T has a $\langle \kappa'_1, \cdots | \lambda'_1, \cdots \rangle$-model omitting every $p \in \Gamma$.

(2) If $\zeta = 0$ we shall write χ instead of $\langle \chi, 0 \rangle$.

REMARK. This generalizes a notation of Vaught in [17]. *Here κ will always be a strong-limit singular cardinal (κ is strong-limit if and only if $\lambda < \kappa$ implies $2^\lambda < \kappa$) and κ' will always be a singular cardinal.*

Our purpose will be to prove relations of the form

$$\langle \chi, \zeta \rangle: \langle \kappa_1, \cdots | \lambda_1, \cdots \rangle \to \langle \kappa'_1, \cdots | \lambda'_1, \cdots \rangle$$

and call them "transfer theorems". By Fuhrken [7] these relations are equivalent to corresponding transfer theorems for languages with generalized quantifiers [$(Q_\alpha x)\psi(x)$—there exists at least $\aleph_\alpha$ x's satisfying ψ].

Our paper continues the work of Keisler in [10] and [11]. Later I learned he had proven some more unpublished results (covered by this paper). In [10], Keisler proves that if $\chi < \kappa'$ then $\chi: \langle \kappa | \rangle \to \langle \kappa' | \rangle$. His proof consists of two parts: In the first part he builds a set of sentences Σ, depending recursively in L, and shows that if T is a theory in the language L, $T \cup \Sigma$ is consistent and $|L| < \kappa'$, then T has a $\langle \kappa' | \rangle$-model. In the second part he proves that if T has a $\langle \kappa | \rangle$-model, then $T \cup \Sigma$ is consistent (this is proved by a polar partition theorem). The point is finding Σ, and proving the second part. The proof of the first part is easy.

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In this paper we shall use similar methods, with similar Σ's, but we shall use a refined partition theorem, which we prove by a well-known partition theorem which appears in Erdös, Hajnal and Rado [4] (Theorem 1.1 in §1 of this paper). It is a reformulation of [4, Theorem 3] but we include it for completeness.

REMARK. Our method has one fault relative to that of Keisler: it does not apply to κ an inaccessible cardinal. But we can always use the following known theorem which appears in Fuhrken [7].

If P^M is κ-like, κ a strongly inaccessible cardinal, $A \subset M$, $|A| < \kappa$, μ a regular cardinal $<\kappa$, then M has an elementary submodel N, such that $A \subset |N|$, P^N is κ_1-like, κ_1 is a strong limit singular cardinal of cofinality μ.

Our method will enable us to reduce transfer problems to problems which do not mention κ-like orderings. In many cases we can translate problems of the form $\chi: \langle\kappa_1, \cdots \mid \lambda_1, \cdots\rangle \to \langle\kappa_1', \cdots \mid \lambda_1', \cdots\rangle$ to problems of the form

$$\chi: \langle|\mathrm{cf}\,\kappa_1, \cdots, \lambda_1, \cdots\rangle \to \langle|\mathrm{cf}\,\kappa_1', \cdots, \lambda_1', \cdots\rangle.$$

In many cases we can deduce compactness and completeness theorems for languages with generalized quantifiers, and also for the theory of the class of $\langle\kappa_1, \cdots \mid \lambda_1, \cdots\rangle$-models, for certain $\langle\kappa_1, \cdots \mid \lambda_1, \cdots\rangle$. But as it goes exactly like Keisler [10] we do not mention it.

We do not investigate transfer theorems in which it seems there is no new point.

Now we shall mention our results. In §1, we define our notations. In §2, we prove theorems about skeletons which are, in fact, partition theorems, using a theorem from [4]. In §3, we investigate relations of the form $\chi: \langle\kappa \mid \lambda\rangle \to \langle\kappa' \mid \lambda'\rangle$. Our results are:

If $\chi \leq \lambda' < \kappa'$, $\lambda < \kappa$, then $\chi: \langle\kappa \mid \lambda\rangle \to \langle\kappa' \mid \lambda'\rangle$ when at least one of the following conditions is satisfied.

(A) $\mathrm{cf}\,\kappa' \leq \lambda'$,

(B) $\mathrm{cf}\,\kappa \geq \beth_{(\lambda, \omega)}$,

(C) $\mathrm{cf}\,\kappa > \lambda$, $\mathrm{cf}\,\kappa' = \lambda'^+$, $\lambda' = \sum_{\mu<\lambda}(\lambda')^\mu$.

In §4, we investigate outer-cofinality problems following Keisler and Morley [12]. We use this for proving the following result (in fact a more general result than this):

If $\chi < \kappa_2' < \kappa_1'$, $\kappa_2 < \kappa_1$ and $\mathrm{cf}\,\kappa_1 = \mathrm{cf}\,\kappa_2 \Rightarrow \mathrm{cf}\,\kappa_1' = \mathrm{cf}\,\kappa_2'$ then $\chi: \langle\kappa_1, \kappa_2|\rangle \to \langle\kappa_1', \kappa_2'|\rangle$.

In §5, we investigate relations of the form $\langle\chi, \zeta\rangle: \langle\kappa \mid \rangle \to \langle\kappa' \mid\rangle$. We again reduce those problems to problems with no κ-like ordering (Theorem 5.7).

On the other hand, Theorem 5.7 is a generalization of the following theorem, of M. and V. Morley, which was later and independently discovered by Kunen, and later by J. Schmerl and myself, but was not as yet published. (See Morley [23] and Barwise and Kunen [19].)

THEOREM 0.1. *$\mu_\lambda = \beth_{\delta(\lambda)}$ where*

(1) *μ_λ is the first cardinal such that if T, $|T| \leq \lambda$, is a theory which has a model omitting p of power $\geq \mu_\lambda$, then T has a model omitting p in arbitrarily large cardinals.*

(2) *$\delta(\lambda)$ is the first ordinal such that if T, $|T| \leq \lambda$, is a theory which has a model omitting p, of order type $\geq \delta$, then T has a not-well-ordered model omitting p. (There is a straightforward generalization for omitting a set of types of power $\leq \zeta$.)*

Our results are

If $\kappa = \beth_\delta$, cf $\kappa' = \mu$, then $\langle \chi, \zeta \rangle : \langle \kappa \mid \rangle \rightarrow \langle \kappa' \mid \rangle$ if at least one of the following conditions is satisfied:

(A) cf $\delta = \lambda$; $\lambda = \mu$ or $\lambda > \mu$, $\lambda > \chi$; and δ is divided by $(2^{\chi+\mu})^+$,

(B) a particular case of (A) is cf $\delta \geq (2^{(\chi+\mu)})^+$,

(C) $\zeta = 1$; and cf $\delta = \mu$ or cf $\delta > \chi + \mu$; and $\delta \geq \delta(\chi + \text{cf}\, \delta)$ (by Theorem 5.7 and Lemma 5.5(1)),

(D) $\zeta = 1$, $\chi = \aleph_0$, and cf $\delta > \aleph_0$, $\mu = \aleph_1$ (by Theorem 5.7 and Lemma 5.9),

(E) $\zeta = 1$, $\chi = \aleph_0$, $\mu = \aleph_0$ and δ is divided by $\aleph_1$ (by Theorems 5.7, 5.5 and 5.6.2).

There are also some negative results—see Lemmas 5.10, 5.11, 5.12. For example,

By (A) $\langle \aleph_0, 2^{\aleph_0} \rangle : \langle \beth[(2^{\aleph_0})^+ \times \omega] \mid \rangle \rightarrow \langle \aleph_{i+\omega} \mid \rangle$,

By (D) $\langle \aleph_0, 1 \rangle : \langle \beth_{\aleph_1} \mid \rangle \rightarrow \langle \aleph_{i+\aleph_1} \mid \rangle$,

By (E) $\langle \aleph_0, 1 \rangle : \langle \beth(\aleph_1^\omega) \mid \rangle \rightarrow \langle \aleph_{i+\aleph_0} \mid \rangle$ ($\aleph_1^\omega$-ordinal exponentiation).

One of the negative results is

$$\text{not } \langle \chi, 1 \rangle : \langle \beth_\delta \mid \rangle \rightarrow \langle \aleph_{\aleph_1} \mid \rangle, \text{ cf}\, \chi > \omega, \text{ cf}\, \delta = \chi^+ < \delta < \chi^{++}.$$

Many of the results of this paper appeared in the abstract [15].

ADDED JANUARY 1972.

(1) *The value of* $\delta(\lambda)$. Kunen and Barwise [19] proved that the bounds of $\delta(\lambda)$ $(\lambda^+ < \delta(\lambda) < (2^\lambda)^+$ when cf $\lambda > \aleph_0)$ cannot be improved. That is, it is consistent with ZFC that $(\exists\lambda)[\text{cf}\, \lambda > \aleph_0 \wedge \delta(\lambda) < \lambda^{++} \wedge 2^\lambda > \lambda^+]$ and also that $(\exists\lambda)[\text{cf}\, \lambda > \aleph_0 \wedge \delta(\lambda) > 2^\lambda \wedge 2^\lambda > \lambda^+]$. They also compute δ for admissible sublanguages of $L_{\omega_1\omega}$.

(2) *Completeness and compactness.* For a very nice and natural axiomatization of the language with the generalized quantifier $(\exists^{>\aleph_0} x)$ (and many other important results) see Keisler [22].

The added axioms say that sets with two elements are countable, the union of countably many countable sets is countable, and a subset of a countable set is countable. On $L_{\omega_1\omega}$ in general see Keisler [21]. For a compactness result see Shelah [32], [33]. A class of models is χ-compact, if whenever every finite subset of a theory of power $\leq \chi$ has a model, the theory has a model. The result is that *if* the class of $\langle \lambda_1, \cdots \mid \mu_1, \cdots \rangle$-models is $\aleph_0$-compact *then* it is χ-compact, for $\chi \leq \mu_i$, $\chi < \lambda_i$ (provided that the number of λ's and μ's is countable).

Also the results of Ehrenfeucht and Mostowski [35] on the existence of models with a group of automorphisms, and of Ehrenfeucht [36] on the existence of a model which realizes few types are generalized. By using ultraproducts, Fuhrken [7] proved that if $\lambda_i^{\aleph_0} = \mu_i$, cf $\lambda_i > \aleph_0$, $\lambda < \lambda_i \Rightarrow \lambda^{\aleph_0} < \lambda_i$ *then* the class of $\langle \lambda_1, \cdots, \lambda_i, \cdots \mid \mu_i, \cdots, \mu_i, \cdots \rangle$-models is $\aleph_0$-compact. See also Ebbinghaus [38].

However, there is no nice axiomatization for the language with the added quantifier $(\exists^{\geq \beth_\omega} x)$. Also, it is not known whether for every λ, μ the class of $\langle \mid \lambda, \mu \rangle$-models is $\aleph_0$-compact.

(3) *Inaccessible-like models.* In Schmerl and Shelah [27], [28] the following is proved: If λ is strongly inaccessible, and is in M_ω in the Mahlo hierarchy (see Levy [37]) and $\mu > \chi$ *then* $\chi : \langle \lambda \mid \rangle \rightarrow \langle \mu \mid \rangle$. If λ is in $M_{\delta(\chi,\zeta)}$ in the Mahlo hierarchy then $\langle \chi, \zeta \rangle : \langle \lambda \mid \rangle \rightarrow \langle \mu \mid \rangle$ for $\mu > \chi$. By Schmerl [25], [26] those results cannot be

improved. In the second case we can use the methods of §5 here. By MacDowell and Specker [24], $\aleph_0: \langle \aleph_0 | \rangle \to \langle \lambda | \rangle$ for every λ.

(4) *Singular nonstrong-limit cardinals.* It is known (see Fuhrken [7]) that there is a sentence which has a $\langle \kappa' | \rangle$-model iff κ' is singular nonstrong-limit cardinal. It is easy to see that there is a sentence which has a $\langle \kappa' | \rangle$-model iff κ' is singular and $(\exists \lambda < \kappa')(\kappa' < \text{Ded } \lambda)[(\exists \lambda < \kappa')(\kappa' < \text{Ded* } \lambda)]$, where

DEFINITION 0.4. Ded λ [Ded* λ] is the first cardinal μ such that there is no tree with λ nodes and μ branches [each branch of height χ for some χ].

This suggests

Conjecture 0.A. If κ', κ'' are limit cardinals, and $(\exists \lambda < \kappa'')$ (Ded* $\lambda \geq \kappa''$), $\chi < \kappa''$ *then* $\chi: \langle \kappa' | \rangle \to \langle \kappa'' | \rangle$.

Other related natural conjectures arise (e.g. the parallel of the theorems of Remark 3 or κ'' which is neither strong-limit nor satisfies the condition in the conjecture). However even the related two-cardinals conjecture has not been proved.

Conjecture 0.B. If $\mu \leq \lambda <$ Ded* μ, $\chi \leq \mu$, *then* $\chi: \langle | \aleph_{\alpha+\omega}, \aleph_\alpha \rangle \to \langle | \lambda, \mu \rangle$.

Again we do not bother to formulate related conjectures. This Conjecture 0.B is implied by the following conjecture:

Conjecture 0.C. If T has an $\langle | \aleph_{\alpha+\omega}, \aleph_\alpha \rangle$-model, *then* $T \cup \Sigma$ is consistent where

$$\Sigma = \{z_1 = \tau(x_{\eta_1}, \cdots, x_{\eta_n}) \wedge z_2 = \tau(x_{\nu_1}, \cdots, x_{\nu_n}) \to [Q_2(z_1) \to z_1 = z_2]: \eta_1, \cdots, \eta_n, \nu_1, \cdots, \nu_n \text{ are sequences of ones and zeros of length } \omega; \text{ and for some } m, \eta_i \mid m = \nu_i \mid m, i \neq j \Rightarrow \eta_i \mid m \neq \eta_j \mid m\} \cup \{Q_1(x_\eta) \mid \text{every } \eta\}.$$

For this it suffices to prove the following:

Conjecture 0.D. Suppose f is an n-place function from $\aleph_{\alpha+\omega}$ to subsets of $\aleph_{\alpha+\omega}$ of cardinality $\aleph_\alpha$. Then there is $A \subset \aleph_\alpha$, $|A| = \aleph_\alpha$ such that $a_1, \cdots, a_n \in A \Rightarrow [f(a_1, \cdots, a_n) \cap A] \subset \{a_1, \cdots, a_n\}$.

(5) *n-cardinal theorems from finite cardinals.*

DEFINITION 0.5. $\chi: \{\langle \lambda_1^i, \cdots \mid \mu_1^i, \cdots \rangle \mid i \in I\} \to \langle \lambda_1', \cdots \mid \mu_1', \cdots \rangle$ if any theory T, $|T| \leq \chi$, which for every $i \in I$ has a $\langle \lambda_1^i, \cdots \mid \mu_1^i, \cdots \rangle$-model *has also a* $\langle \lambda_1', \cdots \mid \mu_1', \cdots \rangle$-model.

Question. When does $\aleph_0: \langle | n_i, m_i \rangle | i < \omega\} \to \langle | \lambda, \mu \rangle$?

It is easy to see that the Vaught and Chang theorems for gap one (see [2]) and Vaught theorem for cardinals for a part [18] can be generalized easily to this case (e.g. if $(\forall h < \omega)(\exists i)(n_i \geq m_i^h)$ then $\aleph_0: \{\langle n_i, m_i \rangle: i < \omega\} \to \langle | \mu^+, \mu \rangle$ when $\mu = \sum_{\chi < \mu} \mu^\chi$).

Conjecture 0.E. If $(\forall h < \omega)(\exists i)(n_i \geq m_i^h)$ and $\mu \leq \lambda <$ Ded* μ *then* μ: $\{\langle | n_i, m_i \rangle: i < \omega\} \to \langle | \lambda, \mu \rangle$.

Or even

Conjecture 0.F. For every set $\{(n_i, m_i) \mid i < \omega\}$, $n_i \geq m_i$, there is n^* such that $\aleph_0: \{\langle | n_i, m_i \rangle: i \in I\} \to \langle | \lambda, \mu \rangle$ iff $\mu \leq \lambda \leq \beth(\mu, n^*)$; $n^* < \omega$ or $n^* = \infty$.

(6) *From n-cardinal transfers to power-like transfers.*

THEOREM 0.2. (A) *If* κ', κ'' *are limit cardinals* κ'' *singular*, cf $\kappa'' = \lambda$, $\kappa'' = \sum_{i<\lambda} \lambda_i$, *the cardinal* κ' *is weakly inaccessible, and* (*) *for every function* $f: \kappa' \to \kappa'$; $\lambda: \{\langle | \cdots, \lambda^i, \cdots \rangle_{i<\lambda}: \langle \lambda^i: i < \lambda \rangle$ *is an increasing sequence of cardinals* $< \kappa'$, $\lambda^{i+1} > f(\lambda^i)\} \to \langle | \cdots, \lambda_i, \cdots \rangle$ *then* $\lambda: \langle \kappa' | \rangle \to \langle \kappa'' | \rangle$.

(B) *If the class of* $\langle | \cdots, \lambda_i, \cdots \rangle_{i<\lambda}$*-models is* λ*-compact, we can replace* (*) *by* $\aleph_0 : \{\langle | \lambda', \cdots, \lambda^n \rangle : \lambda^1 < \cdots < \lambda^n < \kappa'\} \to \langle | \lambda_{i_1}, \cdots, \lambda_{i_n} \rangle \lambda^{i+1} > f(\lambda^i)$ *for every* $i_1 < \cdots < i_n < \lambda, n < \omega$.

(C) *If* $(\forall \mu)\ (\mu < \kappa' \to \mu^\lambda < \kappa')$ *we can choose the* λ_i*'s so that the class of* $\langle | \cdots, \lambda_i, \cdots \rangle_{i<\lambda}$*-models is* λ*-compact (i.e. such that* $\lambda_i^\lambda = \lambda_i$*; the compactness is proved by ultraproducts, by Fuhrken* [7]).

This is proved by interpreting each Q_i as an elementary submodel for $L(T)$ which is an end extension of Q_j for $j < i$. We can find such Q_i's for a $\langle \kappa' | \rangle$-model of T by [32]. I think this was first proved by Silver. He proved that (G.C.H.) $\aleph_0 : \langle \kappa' | \rangle \to \langle \kappa'' | \rangle$, κ'' singular, κ' inaccessible.

ADDED IN PROOF MAY 18, 1972. Seemingly Conjecture 0.E is proved.

(7) *Why limit cardinals*? By Fuhrken [7] there is no point in dealing with transfer theorems for $\langle \lambda^+ | \rangle$-models. Because for limit μ, $\chi : \langle \lambda^+ | \rangle \not\to \langle \mu | \rangle$ and

$$\langle \chi, \zeta \rangle : \langle \lambda_1^+, \lambda_2, \cdots | \mu_1, \cdots \rangle \to \langle \lambda_1'^+, \lambda_2', \cdots | \mu_1', \cdots \rangle$$

is equivalent to

$$\langle \chi, \zeta \rangle : \langle \lambda_2, \cdots | \lambda_1^+, \lambda_1, \mu_1, \cdots \rangle \to \langle \lambda_2', \cdots | \lambda_1'^+, \lambda_1', \mu_1', \cdots \rangle.$$

(8) *Stable theories.* On stable theories see [29]. There are very strong transfers for stable theories and they will appear. For example, let $L(Q^1, \cdots, Q^n)$ be a language with n added quantifiers, T a complete stable theory in it, and suppose T has a model if Q^i is interpreted by $(\exists^{\geq \lambda^i} x)$ where $\lambda^1 < \cdots < \lambda^n$ are regular. Then for every μ^i's, $|T| < \mu^1 < \cdots < \mu^n$, T has a model when $(Q^i x)$ is interpreted by $(\exists^{\geq \mu^i} x)$.

For categoricity see Shelah [30], [31] and Viner [34].

(9) *Ehrenfeucht games.* H. Friedman [20] and S. Viner [34] and L. D. Lipner [39] independently generalized Ehrenfeucht games.

(10) *Craig and Beth theorems.* Friedman [20] proved that the language $L(\exists^{>\aleph_0} x)$ fails to satisfy the Beth theorem, improving a previous result of Keisler and Silver for the Craig theorem.

§1. Notations. Natural numbers will be denoted by m, n, q, p.

Ordinals will be denoted by $i, j, k, l, \alpha, \beta, \gamma$; and limit ordinals by δ. We assume $i = \{j : j < i\}$, and that a cardinal is the first ordinal of its power. Cardinals will be denoted by $\lambda, \mu, \chi, \kappa, \zeta$. Let cf δ be the first cardinal μ such that there exists an increasing sequence $i_j; j < \mu$, $\bigcup_{j<\mu} i_j = \delta$. Let $\aleph_i$ be the ith infinite cardinal. Let us define by induction: $\beth(\lambda, 0) = \lambda$, $\beth(\lambda, i+1) = 2^{\beth(\lambda, i)}$, $\beth(\lambda, \delta) = \bigcup_{i<\delta} \beth(\lambda, i)$; $\beth_i = \beth(i) = \beth(\aleph_0, i)$. If A is a set, $|A|$ will be its cardinality. λ^+ is the first cardinal $> \lambda$.

A sequence $\bar{s}$ will be a function from an ordinal $l(\bar{s})$, and its ith element will be $s_i = s(i)$. The domain and range of a sequence will be denoted respectively by Dom $\bar{s}$, Rang $\bar{s}$. A pseudosequence $\bar{s}$ is a function from an ordered set, the nth element of which is n. A head H of an ordered set A is a subset of A such that $a \in A$, $b \in H$, $a < b$ implies $a \in H$. $B \subset A$ is cofinal with $a \in A$, if $(\forall b \in A)\ [b < a \Rightarrow (\exists c \in B)(b < c < a)]$, and $B \subset \{c \in A : c < a\}$. $\bar{s}$ is cofinal with A if Rang $\bar{s}$ is. L will denote a (first-order) language, and without loss of generality we shall assume $|L| \geq \aleph_0$. F, G will denote function symbols, R a predicate, and P, Q one place predicates. Formulas will be denoted by φ, ψ, θ. Variables will be x, y, z, and finite sequences of variables $\bar{x}, \bar{y}, \bar{z}$. We shall not differentiate strictly between individual constants and the corresponding elements in an L-model. M, N will denote models, $|M|$ will be the set of elements of M, and hence $\|M\|$—its cardinality. If M is an L-model, R a predicate in L, then R^M will be the corresponding relation; and if F is a function symbol in L, F^M will be the corresponding function. (Sometimes we write F, R instead of F^M, R^M. If R is one-place, we write $R(M)$ instead of R^M.) Sometimes we write $a \in M$ instead of $a \in |M|$. If M is an L-model then $L(M) = L$. T shall denote a theory, i.e. set of sentences. $L(T) = L$ if T is a set of sentences in L, and L is the minimal such language. Th(M)—the theory of M is the set of sentences in $L(M)$ which M satisfies. T is a theory in L if $L(T) \subset L$. By a, b, c we shall denote elements of models, by $\bar{a}, \bar{b}, \bar{c}$ finite sequences of such elements. We write $\langle a_0, \cdots, a_n\rangle = \bar{a} \in A$ instead of $a_0, \cdots \in A$. We say that p is a Δ-type (on A, $A \subset |M|$) if p consists of formulas $\varphi(x_0, \cdots, x_{n-1}, \bar{a})$, $\varphi \in \Delta$, $\bar{a} \in A$. Usually we assume $n = 1$. p is a type in L, if p is a Δ-type, where Δ is the set of formulas of L. The Δ-type $\bar{a} \in |M|$ realizes over A is $\{\varphi(\bar{x}, \bar{c})\colon \bar{c} \in A, \varphi \in \Delta, M \vDash \varphi[\bar{a}, \bar{c}]\}$ where $M \vDash \psi[\bar{b}]$ means $\psi[\bar{b}]$ is satisfied in M. If Δ is not mentioned it is the set of all suitable formulas in the language. M realizes p if there exists $\bar{a} \in |M|$ such that $\varphi(x, \bar{c}) \in p$ implies $M \vDash \varphi[\bar{a}, \bar{c}]$. M omits p if it does not realize it. Types will be denoted by p, and sets of types by Γ. (It is clear when p is type and when it is a natural number.) A sequence $\langle a_i\colon i < k\rangle$, $a_i \in M$, is n-Δ-indiscernible (in M) over $A\,(\subset |M|)$ if for every $\bar{c} \in A, \varphi \in \Delta$, $i_1 < i_2 < \cdots < i_n < k$, $j_1 < j_2 < \cdots < j_n < k$, $M \vDash \varphi[a_{i_1}, \cdots, a_{i_n}, \bar{c}]$ if and only if $M \vDash \varphi[a_{j_1}, \cdots, a_{j_n}, \bar{c}]$. (Similarly for a pseudosequence.)

A theorem from Erdös, Hajnal and Rado [4] implies that

THEOREM 1.1. *If A, $B \subset |M|$, $|B| \geq \beth(|A| + |\Delta|, n+1)$, then there is a sequence of elements of B, which is n-Δ-indiscernible over A, and of length $\geq |A| + |\Delta|$.*

For simplicity only we shall assume that $<$ (a two place predicate) belongs to every language, and for every model $<$ (or more precisely $<^M$) orders the model.

If φ is a formula, $Q(x)$ a one place predicate, then φ^Q – φ, *relativized* to Q, is defined by induction: If φ is atomic $\varphi^Q = \varphi$, $(\varphi_1 \wedge \varphi_2)^Q = \varphi_1^Q \wedge \varphi_2^Q$, $(\neg\varphi_1)^Q = \neg\varphi_1^Q$, $[(\exists x)\varphi]^Q = (\exists x)(Q(x) \wedge \varphi^Q)$ (the other connectives and quantifiers are defined by those).

§2. Skeletons.

DEFINITION 2.1. (1) The (pseudo-) sequence $\bar{U} = \langle U_i\colon i < \mu\rangle$ is a (pseudo-) skeleton of $P(M)$ if

(A) For every $i < \mu$, U_i is an increasing infinite (pseudo-) sequence of elements of $P(M)$. (*As U_i is an increasing sequence, it is completely determined by its set of elements, hence we shall not differentiate between U_i and* Rang U_i.)

(B) If $i < j$ then every element in U_i is smaller than every element in U_j.

(C) For every $a \in P(M)$, there exists $i < \mu$ such that $a < U_i(0)$.

For skeletons only we demand also

(D) $P(M)$ is κ-like ordered, cf $\kappa = \mu$.

(E) $\sum_{i<\mu}|U_i| = \kappa$.

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(2) Every U_i will be called a column of the skeleton.

REMARK. In the following definition only skeletons are mentioned but we shall use them freely also for pseudoskeletons.

DEFINITION 2.2. Let $\langle U_i: i\rangle$ be a skeleton of $P(M)$, and c an element of $P(M)$ (and H a head of $P(M)$). Let $\bar{H}$ be the minimal head (of $P(M)$) $c \in \bar{H}$ ($H \subset \bar{H}$) such that if $a \in \bar{H}$, $b \notin \bar{H}$, then there exists i, such that $a \leq U_i(t) < b$, for every $t \in \operatorname{Dom} U_i$ which is big enough.

The sequences $\langle a_m: m < n\rangle$ and $\langle b_m: m < n\rangle$ (of elements of the skeleton) *are similar* over c (over H) if and only if

(1) $a_k \in U_i$ if and only if $b_k \in U_i$ (for every i);

(2) $a_k < a_m$ if and only if $b_k < b_m$ (and so $a_k = a_m$ if and only if $b_k = b_m$);

(3) if $a_k \in \bar{H}$ or $b_k \in \bar{H}$ then $a_k = b_k$.

REMARK. If $c(H)$ are not mentioned, we mean that the similarity is over the empty head.

DEFINITION 2.3. (1) A skeleton (of $P(M)$) is Δ-n-good over A if and only if every two sequences, of length n, from the skeleton, which are similar over any head H (of $P(M)$) realize the same Δ-type over $A \cup H$. (Hence if $[F(\bar{x}, \bar{z}) = y] \in \Delta$, $\bar{a}$, $\bar{a}'$ are similar sequences over H from the skeleton, $\bar{b} \in A \cup H$, then $F(\bar{a}', \bar{b}) \in H$ implies $F(\bar{a}', b) = F(\bar{a}, \bar{b})$.)

(2) The skeleton is Δ-n-excellent over A if it is Δ-n-good over A, and for every function $F(\bar{x}, \bar{y})$ such that $[P(F(\bar{x}, \bar{y})) \to (z > F(\bar{x}, \bar{y}))] \in \Delta$ (we shall say in short that F appears in Δ) and for every $\bar{c} \in A$ and $\bar{a} \in \bigcup\{\operatorname{Rang} U_i: i < i_0 < \mu\}$,

$$M \models P[F[\bar{a}, \bar{c}]] \to U_{i_0}(0) > F(\bar{a}, \bar{c}).$$

(3) A skeleton is Δ-good if it is Δ-n-good for every natural number n. Similarly we define Δ-excellence and we omit Δ if it is the set of all formulas in $L(M)$.

DEFINITION 2.4. The skeleton $\langle V_i: i < \mu\rangle$ is a subskeleton of the skeleton $\langle U_i: i < \mu\rangle$ if and only if there exists an increasing sequence $\langle j_i: i < \mu\rangle$ such that V_i is a subsequence of U_{j_i} for every $i < \mu$.

PROPOSITION 2.1. (1) *Two sequences in a (pseudo-) subskeleton are similar over c in the skeleton if they are similar over c in the subskeleton. (If we replace c by H a parallel proposition holds.)*

(2) *A subskeleton of a Δ-n-good (Δ-n-excellent) (pseudo-) skeleton is a Δ-n-good (Δ-n-excellent) (pseudo-) skeleton.*

(3) *A subskeleton of a subskeleton of $\bar{U}$ is a subskeleton of U.*

PROOF. The proof is immediate. We shall use this proposition without mention also of pseudoskeletons.

THEOREM 2.2. *If $|\Delta| + |Q(M)| < \kappa$, then every skeleton of $P(M)$ has a subskeleton which is Δ-n-good over Q.*

PROOF. We shall prove by induction on m the following statement:

(*) If $\langle U_i: i < \operatorname{cf} \kappa\rangle$ is a skeleton, then for every m it has a subskeleton $\langle U_i^m: i < \operatorname{cf} \kappa\rangle$ such that if $\bar{a} = \langle a_0, \cdots, a_{n-1}\rangle$ is a sequence in which appear elements of at most m columns, $\bar{b}$ is similar to $\bar{a}$ over H, then they realize the same Δ-type over $Q(M) \cup H$ (clearly $\bar{a}$ and $\bar{b}$ are sequences from the skeleton).

If we succeed in proving (*) for $m = n$, clearly Theorem 2.2 will be proven.

For $m = 0$, (*) is trivial (the skeleton itself can serve as the required subskeleton).

Suppose we have proven (*) for m by finding $\langle U_i^m : i < \operatorname{cf}\kappa\rangle$, we shall prove it for $m + 1$. Let $\kappa = \sum_{i<\operatorname{cf}\kappa}\mu_i$, where $\mu_i < \kappa$, and μ_i is an increasing sequence.

Now we shall define, by induction on i, U_i^{m+1} and j_i such that U_i^{m+1} will be a subsequence of $U_{j_i}^m$. Suppose we have defined U_i^{m+1}, j_i for every $i < k < \operatorname{cf}\kappa$. As $\{j_i : i < k < \operatorname{cf}\kappa\}$ is a set of $<\operatorname{cf}\kappa$ ordinals smaller than $\operatorname{cf}\kappa$, and $\operatorname{cf}\kappa$ is a regular cardinal, there exists $j < \operatorname{cf}\kappa$ such that $j_i < j$ for every $i < k$. Let $H_k = \{c : c \in P(M),\ i < k,\ t \in \operatorname{Dom} U_{j_i}^m,\ c < U_{j_i}^m(t)\}$, and clearly H_k is a head of $P(M)$. As $P(M)$ is κ-like ordered $|H_k| < \kappa$. As $\sum_{i<\operatorname{cf}\kappa}|U_i^m| = \kappa$, and $|U_i^m| < \kappa$, there exists $j_k < \operatorname{cf}\kappa$, $i < k \rightarrow j_i < j_k$, such that $|U_{j_k}^m| \geq \beth(|H_k| + |Q(M)| + \mu_k + \operatorname{cf}\kappa + |\Delta|, n + 1)$. By Lemma 1.1, $U_{j_k}^m$ has a subsequence U_k^{m+1}, $|U_k^{m+1}| \geq \mu_k$, which is Δ-n-indiscernible over $H_k \cup Q(M) \cup \{U_i^m(l) : l < \omega, i < \operatorname{cf}\kappa\}$. It is clear that $\langle U_k^{m+1} : k < \operatorname{cf}\kappa\rangle$ is a subskeleton of $\langle U_i : i < \operatorname{cf}\kappa\rangle$. So it remains to prove that it satisfies (*) for $m + 1$.

Let $\bar{a} = \langle a_0, \cdots, a_{n-1}\rangle$, $\bar{b} = \langle b_0, \cdots, b_{n-1}\rangle$ be two similar sequences in $\langle U_k^{m+1} : k < \operatorname{cf}\kappa\rangle$ over a head H; and suppose that in $\bar{a}$ appear elements from $m + 1$ column, the first of which is U_α^{m+1}, and without loss of generality the elements from U_α^{m+1} will be $a_0, \cdots, a_p$. Let $H^1 = \{c \in P(M) : c < U_\alpha^{m+1}(t)\}$ and it is not hard to see that without loss of generality we can assume $H \subset H^1$.

It is not hard to find a sequence $\bar{a}^1 = \langle a_0^1, \cdots, a_{n-1}^1\rangle$ in $\langle U_i^m : i < \operatorname{cf}\kappa\rangle$ such that

(1) $\bar{a}$, $\bar{a}^1$ are similar over H,

(2) $a_0^1 = a_0, \cdots, a_p^1 = a_p$,

(3) if $p < k < n$ then $a_k^1 = U_{j_\beta}^m(l)$ for some $l < \omega$, $\beta < \operatorname{cf}\kappa$.

Similarly we can define $\bar{b}^1$ such that in addition $\bar{b}^1 = \langle b_0, \cdots, b_p, a_{p+1}^1, \cdots, a_{n-1}^1\rangle$.

Now it is easily seen that $\bar{a}$ and $\bar{a}^1$ are similar sequences on H^1 in $\langle U_i^m : i < \operatorname{cf}\kappa\rangle$. Also $\langle a_{p+1}, \cdots, a_{n-1}\rangle$, $\langle a_{p+1}^1, \cdots, a_{n-1}^1\rangle$ are similar sequences over H^1 in $\langle U_i^m : i < \operatorname{cf}\kappa\rangle$, and so they satisfy the same Δ-type over $Q(M) \cup H^1$. As $H \subset H^1$, and $a_0, \cdots, a_p \in H^1$, $a_0 = a_0^1, \cdots, a_p = a_p^1$, clearly $\bar{a} = \langle a_0, \cdots, a_p, a_{p+1}, \cdots, a_{n-1}\rangle$ and $\bar{a}^1 = \langle a_0^1, \cdots, a_p^1, a_{p+1}^1, \cdots, a_{n-1}^1\rangle$ satisfy the same Δ-type over $Q(M) \cup H$.

Similarly $\bar{b}$ and $\bar{b}^1$ satisfy the same Δ-type over $Q(M) \cup H$. So it remains to prove only that $\bar{a}^1$ and $\bar{b}^1$ satisfy the same Δ-type over $Q(M) \cup H$. As $a_{p+1}^1 = b_{p+1}^1, \cdots, a_{n-1}^1 = b_{n-1}^1$, and $a_{p+1}^1, \cdots, a_{n-1}^1 \in \{U_i^m(l) : i < \operatorname{cf}\kappa, l < \omega\}$, it suffices to prove that $\langle a_0^1, \cdots, a_p^1\rangle$, $\langle b_0^1, \cdots, b_p^1\rangle$ satisfy the same Δ-type over $Q(M) \cup H \cup \{U_i^m(l) : i < \operatorname{cf}\kappa, l < \omega\}$. As $H \subset H_\alpha$, and U_α^{m+1} is Δ-n-indiscernible over $Q(M) \cup H_\alpha \cup \{U_i^m(l) : i < \operatorname{cf}\kappa, l < \omega\}$, and $p + 1 \leq n$ this is clearly true.

So we have proven statement (*) by induction. Taking $m = n$, we prove Theorem 2.2.

THEOREM 2.3. *If $|\Delta| + |Q(M)| < \operatorname{cf}\kappa$ then every skeleton of $P(M)$ has a subskeleton which is Δ-n-excellent over Q.*

PROOF. Let $\bar{U} = \langle U_i : i < \operatorname{cf}\kappa\rangle$ be the skeleton. By Theorem 2.2, it has a Δ-$2n$-good subskeleton $\bar{W} = \langle W_i : i < \operatorname{cf}\kappa\rangle$. Let $\kappa = \sum_{i<\operatorname{cf}\kappa}\mu_i$, $\operatorname{cf}\kappa < \mu_i < \kappa$, and without loss of generality assume $|W_i| > \mu_i$. Let $V_i = \{W_i(k \times \omega + n + 1) : k \times \omega + n + 1 < \mu_i^+\}$ and $\bar{V} = \langle V_i : i < \operatorname{cf}\kappa\rangle$.

Let, for $i < \operatorname{cf}\kappa$, $A_i = \{V_j(t) : j < i, t \in \operatorname{Dom} V_j\}$, and let $B_i = \{F[\bar{a}, \bar{c}]$: F

appears in Δ (i.e. $[P(F(\bar{x}, \bar{y})) \to z > F(\bar{x}, \bar{y})] \in \Delta$), $\bar{c} \in Q(M)$, $\bar{a} \in A_i$, $l(\bar{a}) \leq n\} \cap P(M)$.

It is sufficient to prove that every B_i is a bound set in $P(M)$. For suppose it is true, and we shall define an increasing sequence j_i, $j_i < \mathrm{cf}\,\kappa$ for $i < \mathrm{cf}\,\kappa$, such that $\langle V_{j_i}: i < \mathrm{cf}\,\kappa\rangle$ is the required subskeleton.

Suppose j_i has been defined for $i < k < \mathrm{cf}\,\kappa$, and let $j^0 < \mathrm{cf}\,\kappa$ be an upper bound for $\{j_i: i < k\}$. As $\sum_{i<\mathrm{cf}\kappa} |V_i| = \kappa$, $|V_i| < \kappa$, there exists $j_k > j^0$, $j_k < \mathrm{cf}\,\kappa$, such that $|V_{j_k}| \geq \mu_k$, and for every $b \in B_{j^0}$, $b < V_{j_k}(0)$. Clearly $\langle V_{j_i}: i < \mathrm{cf}\,\kappa\rangle$ is the required subskeleton.

So it suffices to prove that each B_i is a bounded set in $P(M)$. Suppose B_i is not bound. Now we shall define an equivalence relation on B_i: $F_1(\bar{a}_1, \bar{c}_1) \sim F_2(\bar{a}_2, \bar{c}_2)$ if $F_1 = F_2$, $\bar{c}_1 = \bar{c}_2$, and $\bar{a}_1$, $\bar{a}_2$ are similar sequences. As the number of function symbols appearing in Δ is $\leq |\Delta| < \mathrm{cf}\,\kappa$ (or they both are finite and so $< \mathrm{cf}\,\kappa$); and the number of $\bar{c} \in Q(M)$ is also $< |Q(M)|^+ + \aleph_0 \leq \mathrm{cf}\,\kappa$; and the number of similarity types of sequences from B_i is $< |i|^+ + \aleph_0 \leq \mathrm{cf}\,\kappa$; it is clear that the number of equivalence classes is $< \mathrm{cf}\,\kappa$. We can conclude from this that there exists an equivalence class B^0 which is unbounded in $P(M)$. It is easy to find a subset B^1 of B^0 which is unbound in $P(M)$, $B^1 \subset P(M)$, and $B^1 = \{F(\bar{a}_i, \bar{c}): i < \mathrm{cf}\,\kappa\}$, such that $i < j < \mathrm{cf}\,\kappa$ implies $F(\bar{a}_i, \bar{c}) < F(\bar{a}_j, \bar{c})$. $\langle \bar{a}_i(0): i < \mathrm{cf}\,\kappa\rangle$ has no infinite decreasing subsequence (by the definition of skeleton). From this it is easily seen that either it has an increasing subsequence of length $\mathrm{cf}\,\kappa$, or for some a, $|\{i: \bar{a}_i(0) = a\}| = \mathrm{cf}\,\kappa$. So B^1 has a subset B^1_0, such that $B^1_0 = \{F(\bar{a}_{j_i}, \bar{c}): i < \mathrm{cf}\,\kappa\}$ and is unbound in $P(M)$ and $i < k$ implies $\bar{a}_{j_i}(0) < \bar{a}_{j_k}(0)$, or $i < k$ implies $\bar{a}_{j_i}(0) = \bar{a}_{j_k}(0)$. Doing the same for $1, \cdots, n-1$, we get $A^1 = B^1_{n-1}$ which satisfies

(1) $A^1 \subset B^1$, $|A^1| = \mathrm{cf}\,\kappa$, and so A^1 is unbound in $P(M)$,

(2) $A^1 = \{F(\bar{a}^i, \bar{c}): i < \mathrm{cf}\,\kappa\}$,

(3) $i < j$ implies $F(\bar{a}^i, \bar{c}) < F(\bar{a}^j, \bar{c})$,

(4) for $p < n$, either $i < j$ implies $\bar{a}^i(p) = \bar{a}^j(p)$ or $i < j$ implies $\bar{a}^i(p) < \bar{a}^j(p)$.

Now it is not hard to find in the skeleton $\overline{W}$ (not $\overline{V}$) a sequence $\bar{a}$ such that

(1) $\bar{a}$ is similar to $\bar{a}^0$.

(2) If $p < n$, and $i < j$ implies $\bar{a}^i(p) = \bar{a}^j(p)$ then $\bar{a}(p) = \bar{a}^0(p)$.

(3) If $p < n$ and $i < j$ implies $\bar{a}^i(p) < \bar{a}^j(p)$ then for each i, $\bar{a}^i(p) < \bar{a}(p)$; and if $b \in \bigcup\{\mathrm{Rang}\, V_i: i < \mathrm{cf}\,\kappa\}$ and for each i, $\bar{a}^i(p) < b$, then $\bar{a}(p) < b$.

This is possible as $V_i = \{W_i(k \times \omega + n + 1): k \times \omega + n + 1 < \mu_i^+\}$.

It is not hard to see that for every i there exists $j > i$ such that $\bar{a}^i a^j$ and $\bar{a}^i \bar{a}$ are similar sequences in $\overline{W}$. As $\overline{W}$ is Δ-$2n$-good we can conclude that these sequences realize the same Δ-type over $Q(M)$. So from $P[F(\bar{a}^j, \bar{c})]$ we can conclude $P[F(\bar{a}, \bar{c})]$, and from $F(\bar{a}^i, \bar{c}) < F(\bar{a}^j, \bar{c})$ we can conclude $F(\bar{a}^i, \bar{c}) < F(\bar{a}, \bar{c})$. So $F(\bar{a}, \bar{c}) \in P(M)$, and for every i ($< \mathrm{cf}\,\kappa$), $F(\bar{a}^i, \bar{c}) < F(\bar{a}, \bar{c})$. This contradicts the unboundedness of A^1, and hence we finish the proof.

§3. Applications of skeletons for transfer theorems. In this section we shall use the theorems about the existence of good and excellent skeletons for proving relations of the form $\chi: \langle \kappa \mid \lambda\rangle \to \langle \kappa' \mid \lambda'\rangle$.

DEFINITION 3.1. A theory T (in L) is closed if (1) for any formula $\varphi(y, \bar{x})$, there exists a function symbol $F(\bar{x})$ such that $(\forall \bar{x})[(\exists y)\varphi(y, \bar{x}) \to \varphi(F(\bar{x}), \bar{x})] \in T$, (2) for every function symbol $F, F_1, F_2, \cdots$, there exists a function symbol G such that $(\forall \bar{y}_1, \bar{y}_2, \cdots)[F(F_1(\bar{y}_1), F_2(\bar{y}_2), \cdots) = G(\bar{y}_1, \bar{y}_2, \cdots)] \in T$.

THEOREM 3.1. *For every language L there exists a language L_{SK} and a theory T_{SK} (in L_{SK}) such that*

(1) *every L-model is the reduct of a model of T_{SK};*

(2) *every theory T in L_{SK}, $T \supset T_{SK}$, is closed;*

(3) *if M is a model of T_{SK}, $|N| \subset |M|$, and $|N|$ is closed under the functions of M, then N is an elementary submodel of M.*

PROOF. This is a well-known theorem. We shall use it without mentioning it explicitly.

THEOREM 3.2. (1) *We can define $\psi = \psi_{SH}(\varphi, R, R_1, n)$ as a sentence depending only on the formula φ and the predicates R, R_1 such that*

(A) *If M is a model of ψ, then R_1^M is an equivalence relation between the elements of R^M, and $R^M \subset P^M$, and for every $a \in R^M$, there is $b \in R^M$, $M \vDash \neg R_1[a, b]$, $a < b$.*

Let us define for $a \in R^M$, U_a as a pseudosequence whose set of elements is $\{b: M \vDash R_1[b, a]\}$.

(B) *If $M \vDash \psi$, then $\langle U_a: a \in R(M)\rangle$ is a $\{\varphi\}$-n-*good *pseudoskeleton of $P(M)$ over $Q(M)$.*

(C) *Let $T_{SH}^n = T \cup \{\psi_{SH}(\varphi, R, R_1, n): \varphi \in L(T)\}$. (We assume $R, R_1 \notin L(T)$.) Then if M is a $\langle\kappa \mid \lambda\rangle$-model of T, $\lambda + |T| < \kappa$; then M is a reduct of a model of T_{SH}^n.*

We write $T_{SH} = \bigcup_{n<\omega} T_{SH}^n$.

(2) *The same as* (1) *except that we replace SH by ST, goodness by excellence, and in* (C) *we replace $|T| + \lambda < \kappa$ by $|T| + \lambda < \operatorname{cf} \kappa$.*

PROOF. The proof follows immediately from Theorem 2.2, 2.3; and in fact, T_{SH}^n, T_{ST}^n are a formalization of what was said there, i.e. $\psi_{SH}(\varphi, R, R_1, n)$ will be the conjunction of the following sentences:

(1) $(\forall x)(R(x) \to P(x))$,

(2) $(\forall x)(R(x) \leftrightarrow R_1(x, x))$,

(3) $(\forall xy)(R_1(x, y) \to R_1(y, x))$,

(4) $(\forall xyz)[R_1(x, y) \wedge R_1(y, z) \to R_1(x, z)]$,

(5) a sentence telling that: if $\bar{x}_1$, $\bar{x}_2$ are "similar" sequences (of length n) over z, $R(z)$ and $\bigwedge_m z > \bar{y}(m)$ then $\varphi(\bar{x}_1, \bar{y}) \leftrightarrow \varphi(\bar{x}_2, \bar{y})$,

(6) $(\forall x)(\exists y)(R(x) \to R(y) \wedge x < y \wedge \neg R_1(x, y))$,

(7) $(\forall xyz)(R(x) \wedge R(y) \wedge R(z) \wedge x < y \wedge y < z \wedge \neg R_1(x, y) \to \neg R_1(x, z))$.

THEOREM 3.3. *If M is a model of $T \cup T_{SH}$ where T is a closed theory and there exists a sequence $\langle a_i: i < \lambda\rangle$, $a_i \in R(M)$, such that $i < j < \lambda$ implies $M \vDash \neg R_1[a_i, a_j]$, the sequence is unbounded in $P(M)$, and $\kappa > \|M\|$, $\kappa > \operatorname{cf} \kappa = \lambda$, then there exists models N, M_1 of T such that M_1 is an elementary $L(T)$-submodel of M and of N, $Q(M) = Q(M_1) = Q(N)$ and $P(N)$ is κ-like ordered. If M is a model of $T \cup T_{ST}$, the unboundedness of the sequence is not necessary.*

PROOF. Suppose $\kappa = \sum\{\mu_i: i < \operatorname{cf} \kappa\}$, $\mu_i < \kappa$, M is a model of $T_{SH} \cup T$ and $\langle a_i: i < \lambda\rangle$ is unbound in $R(M)$. Let L_1 be L_{SH} with the new constants $\{c: c \in Q(M)\} \cup \{b_i^j: i < \operatorname{cf} \kappa, j < \mu_i\}$. Let

$$T^1 = T \cup \{\varphi(b_{i_1}^{j_1}, b_{i_2}^{j_2}, \cdots, b_{i_m}^{j_m}, c_1, \cdots, c_p) \leftrightarrow \varphi(b_{i_1}^{k_1}, \cdots, b_{i_m}^{k_m}, c_1, \cdots, c_p): c_1, \cdots, c_p \in Q(M), \text{ and } i_\alpha = i_\beta \text{ implies } j_\alpha > j_\beta \leftrightarrow k_\alpha > k_\beta\}$$
$$\cup \{\varphi(b_{i_1}^{j_1}, \cdots, b_{i_m}^{j_m}, c_1, \cdots, c_p): j_1, \cdots, j_m < \omega, M \models \varphi[U_{a_{i_1}}(j_1), \cdots, U_{a_{i_m}}(j_m), c_1, \cdots]\}.$$

Clearly every finite subset of T^1 has a model, and so by the compactness theorem, T^1 has a model N_1. Let N be the elementary submodel of N^1 whose set of elements is the closure by the functions in N^1 of $A = \{c: c \in Q(M)\} \cup \{b_i^j: j < \mu_i, i < \operatorname{cf} \kappa\}$.

It is not hard to see that all the conditions are satisfied. The only nontotally trivial one is that $P(N)$ is κ-like ordered. We shall prove it.

Firstly, we shall prove that $\{b_i^0: i < \operatorname{cf} \kappa\}$ is unbound in $P(N)$. For suppose $P[F(d_1, d_2, \cdots)]$ and $F(d_1, d_2, \cdots) > b_i^0$ for every i, where $d_1, \cdots \in A$. We an easily define $d_1^1, d_2^1, \cdots$ such that

(0) $d_1^1, d_2^1, \cdots \in A$,

(1) $d_m \in \{b_i^j: j < \mu_i\}$ iff $d_m^1 \in \{b_i^j: j < \mu_i\}$ for any m and i,

(2) if $d_m \in Q(M)$ or $d_m^1 \in Q(M)$ then $d_m = d_m^1$,

(3) $d_m < d_n$ $(d_m = d_n)$ iff $d_m^1 < d_n^1$ $(d_m^1 = d_n^1)$,

(4) if $d_m = b_i^0$ or $d_m^1 = b_i^0$ then $d_m = d_m^1$,

(5) if $d_m \notin Q(M)$, then for some $i < \operatorname{cf} \kappa$ and $0 < j < \omega$, $d_m^1 = b_i^j$.

From the definition of T_1 it is easily seen that for any $i < \operatorname{cf} \kappa$, $F(d_1^1, d_2^1, \cdots) > b_i^0$, and $P[F[d_1^1, d_2^1, \cdots]]$ (all this holds in N). Now define $d_1^2, d_2^2, \cdots$: if $d_m^1 \in Q(M)$ then $d_m^2 = d_m^1$; and if $d_m^1 = b_i^j$ then $d_m^2 = U_{a_i}(j)$. From the definition of T_1 it is clear that (in M) $P[F(d_1^2, d_2^2, \cdots)]$ and for each $i < \operatorname{cf} \kappa$, $F(d_1^2, d_2^2, \cdots) > a_i$. So $\{a_i: i < \operatorname{cf} \kappa\}$ is a bound set in $P(M)$, a contradiction.

We have proven that $\{b_i^0: i < \operatorname{cf} \kappa\}$ is unbound in $P(N)$. In order to prove that $P(N)$ is κ-like ordered it is sufficient to prove that for every $i < \operatorname{cf} \kappa$, $|\{c \in P(N): c < b_i^0\}| < \kappa$. Let $i_0 < \operatorname{cf} \kappa$ be fixed, and let $B = Q(M) \cup \{b_i^j: i < i_0 \text{ or } j < \omega \text{ and } i < \operatorname{cf} \kappa\}$, and let B_1 be the closure of B under the function of N. Clearly $|B_1| < \kappa$. Now as in the previous paragraph, it is easy to see that if $d_1, d_2, \cdots \in A$, $P[F(d_1, \cdots)]$ holds, and $F(d_1, \cdots) < b_{i_0}^0$ then $F(d_1, \cdots) = F(d_1^1, \cdots)$ where $d_1^1, \cdots \in B$, and so $F(d_1, \cdots) \in B_1$. Hence $|\{c \in P(N): c < b_{i_0}^0\}| \leq |B_1| < \kappa$ and so $P(N)$ is κ-like ordered.

Now there remains the case where M is a model of $T \cup T_{ST}$, and $\langle a_i: i < \lambda\rangle$ is bound in $P(M)$. Then the closure of $Q(M) \cup \{U_a(n): a \in \{a_i: i < \lambda\}, n < \omega\}$ by the functions of $L(T)$ is an elementary $L(T)$-submodel of M, M^0, and a model of $T \cup T_{ST}$. As $\langle U_a: a \in R(M)\rangle$ is an excellent pseudoskeleton, $\{a_i: i < \lambda\}$ is unbound in $P(M^0)$. Hence the remainder of the proof is as in the previous case.

Now we shall deduce from Theorems 3.2, 3.3 some conclusions, which are the aim of this section.

Conclusion 3.4. If $\kappa > \lambda$, $\kappa' > \lambda' \geq \operatorname{cf} \kappa'$, $\chi \leq \lambda'$ then $\chi: \langle \kappa \mid \lambda\rangle \to \langle \kappa' \mid \lambda'\rangle$.

Proof. Let T, $|T| \leq \chi$ have a $\langle \kappa \mid \lambda\rangle$-model. By Theorem 3.2, $T_1 = (T \cup T_{SK}) \cup (T \cup T_{SK})_{SH}$ is consistent. By Theorem 3.3, it is sufficient to find a model M of T_1 in which there exists in $P(M)$ an unbound increasing sequence $\langle a_i: i < \operatorname{cf} \kappa'\rangle$, $a_i \in R(M)$, $i \neq j \Rightarrow \neg R_1[a_i, a_j]$, and such that $|Q(M)| = \lambda'$.

It is not hard to find a model M_0 of T_1 such that $\|M_0\| = |Q(M_0)| = \lambda'$. Let us define by transfinite induction M_i for $i \leq \operatorname{cf} \kappa'$: M_0 has been defined. M_{i+1} will be an elementary extension such that $\|M_{i+1}\| = \|M_i\| = \lambda'$ and there exists $a_i \in |M_{i+1}|$, $a_i \in R(M_{i+1})$, and $b \in R(M_i) \Rightarrow b < a_i, \neg R_1[b, a_i]$. Clearly M_μ ($\mu = \mathrm{cf}\,\kappa'$) is the required model.

CONCLUSION 3.5. If $\mathrm{cf}\,\kappa' = \lambda'^+$, $\lambda' = \sum\{(\lambda')^\mu : \mu < \lambda'\}$, $\chi \le \lambda'$, and $\mathrm{cf}\,\kappa > \lambda$ then $\chi: \langle \kappa \mid \lambda \rangle \to \langle \kappa' \mid \lambda' \rangle$.

PROOF. Suppose $|T| \le \chi$ and T has a $\langle \kappa \mid \lambda \rangle$-model. We should prove it has a $\langle \kappa' \mid \lambda' \rangle$-model. By Theorem 3.2, every finite subset of $T_1 = (T \cup T_{SK}) \cup (T \cup T_{SK})_{ST}$ has a model M, in which $R(M)$ is (cf κ)-like ordered, and $|Q(M)| < \mathrm{cf}\,\kappa$. This implies by Chang [2] that T_1 has a model N, $\|N\| = \lambda'^+$, such that $R(M)$ is (λ'^+)-like ordered, and $|Q(N)| = \lambda'$. As $R(M)$ is λ'^+-like ordered, we can find in it an increasing sequence of non-R_1-equivalent elements of length λ'^+, and hence Theorem 3.3 implies our conclusion.

CONCLUSION 3.6. If $\mathrm{cf}\,\kappa \ge \beth(\lambda, \omega)$, $\chi \le \lambda' < \kappa'$, then $\chi: \langle \kappa \mid \lambda \rangle \to \langle \kappa' \mid \lambda' \rangle$.

PROOF. The proof is as in the previous conclusions, using the two-cardinal theorem appearing in Vaught [18] and also in Morley [13]. (In fact, we should take care for the existence of the increasing sequence we want.)

LEMMA 3.7. (1) *For every theory T there exists a theory T_1, $|T_1| \le |T| + \aleph_0$, such that*

T has a $\langle \mid \lambda_1, \cdots, \lambda_m, \mathrm{cf}\,\kappa_1, \cdots, \mathrm{cf}\,\kappa_n \rangle$-model iff T_1 has a $\langle \kappa_1, \cdots, \kappa_n \mid \lambda_1, \cdots, \lambda_m \rangle$-model.

(2) *For every theory T there exists a theory T_1, $|T_1| \le |T| + \aleph_0$, such that*

T_1 has a $\langle \kappa \mid \lambda \rangle$-model, iff T has a model M, $|Q(M)| = \lambda$, and in $P(M)$ there exists an increasing unbounded sequence of length cf κ.

PROOF. The proof is immediate.

COROLLARY 3.8. *There exists a sentence φ which has a $\langle \kappa' \mid \lambda' \rangle$-model iff* $\mathrm{cf}\,\kappa' \le \lambda'$ ($\mathrm{cf}\,\kappa' \le \lambda'^+$) ($\mathrm{cf}\,\kappa' \le \aleph_{\beta+n}$ *where* $\lambda' = \aleph_\beta$) ($\mathrm{cf}\,\kappa' \le \beth(\lambda', n)$).

PROOF. The proof is an immediate consequence of Vaught [18].

§4. Outer cofinality and transfer theorems.

Such problems were discussed in Keisler and Morley [12] for models of ZF. We shall generalize their results, and use them for transfer theorems. We used our results about outer cofinality to prove that if $\kappa_1 > \kappa_2$, $\kappa_1' > \kappa_2' > \mathrm{cf}\,\kappa_1'$, $\chi < \kappa_2'$, and $\mathrm{cf}\,\kappa_1 \ne \mathrm{cf}\,\kappa_2$ or $\mathrm{cf}\,\kappa_1' = \mathrm{cf}\,\kappa_2'$ then $\chi: \langle \kappa_1, \kappa_2 \mid \rangle \to \langle \kappa_1', \kappa_2' \mid \rangle$ (in fact, a more general theorem, Theorem 4.5).

DEFINITION 4.1. If $a, b \in P(M)$, then cf $a \le b$ in P, i.e. the cofinality of a is not greater than b in P if for some function symbol $F(x, \bar{z})$ in $L(M)$, and sequence $\bar{c}$ in M,

$$M \models (\forall x < a)[P(x) \to (\exists y < b)(P(y) \wedge x < F(y, \bar{c}) \wedge F(y, \bar{c}) < a)],$$

otherwise the cofinality of a is greater than b. If cf $a > b$ for every $b < a$, $b \in P(M)$, then a is called regular (in P). If $P(M) = M$, we omit the words "in P".

DEFINITION 4.2. In the notations of Definition 4.1, $\mathrm{ocf}\,P_1(M) = \lambda$ if there exists an increasing sequence $\langle a_i : i < \lambda \rangle$, $a_i \in P_1(M)$, and there is no $a^1 \in P_1(M)$, $a_i < a^1$, for every i; and there does not exist such a sequence for $k < \lambda$. Such a sequence (not necessarily of length λ) will be called a sequence cofinal to $P_1(M)$. $\mathrm{ocf}\,P_1(M)$ is

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called the *outer* cofinality of $P_1(M)$. We define $\operatorname{ocf} a = \operatorname{ocf}(\{b \in M: b < a\})$. $\operatorname{ocf}_P a = \operatorname{ocf}(\{b \in P(M): b < a\})$.

THEOREM 4.1. (1) *There is a type, i.e. a set of formulas, such that a is regular iff it realizes that type (i.e. satisfies each of the formulas (we can replace in* (1), (3) *regular by regular in P)).*

(2) *If a is regular in M, and M_1 an elementary extension of M, then a is regular in M_1.*

(3) *If T is a complete theory, and a an individual constant, and in one model of T, a is regular, then it is regular in every model of T.*

(4) *If $a \in P(M)$ and $\{d \in P(M): d < a\}$ is λ-like ordered, and λ is regular, then, a is regular in P.*

(5) *If $\{a_i: i \in I\}$ are individual constants, and for every finite subset J of I and T_1 of T, there is a model of T_1 in which $\{a_i: i \in J\}$ are regular, then T has a model in which $\{a_i: i \in I\}$ are regular.*

PROOF. The proof is immediate.

THEOREM 4.2. *If a is a limit element of M, $\operatorname{Th}(M)$ is closed, then M has an elementary extension M_1 such that*

(1) *There is $a_1 \in M_1$, $a_1 < a$, such that $b \in M$, $b < a$ implies $b < a$.*

(2) *If $b \in M$ and $\operatorname{cf} b > a$ then there is no $b_1 \in M_1$, $b_1 < b$ such that $b_2 \in M$, $b_2 < b$ implies $b_2 < b_1$.*

PROOF. Suppose $\operatorname{ocf} a = \lambda$, and let D be a nonprincipal ultrafilter on λ, such that $j < \lambda$ implies $\{i: j < i < \lambda\} \in D$.

Let $\langle a_i: i < \lambda\rangle$ be a sequence in M cofinal to a. Let $a^0 = \langle a_i: i < \lambda\rangle/D$, and for $c \in M$, we identify c and $\langle c: i < \lambda\rangle/D$. (On ultrapowers, see [5].) Let M_1 be an elementary submodel of M^λ/D whose set of elements is the closure of $|M| \cup \{a^0\}$. Clearly $a^0 < a$, and $a^1 \in M$, $a^1 < a$ implies $a^1 < a^0$. Suppose $b \in M$, $\operatorname{cf} b > a$, and there is $b_1 < b$, $b_1 \in M_1$, such that $b_2 \in M$, $b_2 < b$ implies $b_2 < b_1$. We should show that this leads to a contradiction. So let $b_1 = F(a^0, \bar{c})$, where $\bar{c}$ is a sequence from $|M|$. If $b_2 \in M$, $b_2 < b$, then $M^I/D \vDash b_2 < F(a^0, \bar{c}) \wedge F(a^0, \bar{c}) < b$. By the fundamental theorem about ultraproducts (see [5]) this implies, if $\bar{c} = \langle c_0, \cdots, c_n\rangle$, that $\{i < \lambda: M \vDash b_2 < F(a_i, c_0, \cdots, c_n) \wedge F(a_i, c_0, \cdots, c_n) < b\} \in D$, and hence, for some $i < \lambda$, $M \vDash b_2 < F(a_i, \bar{c}) \wedge F(a_i, \bar{c}) < b$. As $a_i < a$, this contradicts $\operatorname{cf} b > a$, which we assumed. So M_1 is the required extension.

THEOREM 4.3. *If $T = \operatorname{Th}(M)$ is closed, and $a_0 > a_1 > \cdots > a_n$ are limit elements of $P(M)$, and for each $k < l \le n$, $\operatorname{cf} a_k > a_l$, and $\delta_0, \cdots, \delta_n$ are limit ordinals, then M has an elementary extension N, in which, for $k = 0, \cdots, n$, there is a sequence in N, of length δ_k, which is cofinal to a_k. Hence $\operatorname{ocf} a_k = \operatorname{cf} \delta_k$.*

PROOF. We define by induction M_i^m for $i \le \delta_m$, $m \le n$: $M_0^0 = M$. If M_i^m is defined, then M_{i+1}^m will be an elementary extension of M_i^m, in which there is $a_i^m < a_m$, such that $a \in M_i^m$, $a < a_m$ implies $a < a_i^m$; and if $\operatorname{cf} b > a_m$, $b \in M_i^m$, then there is no b_1, $b_1 < b$, such that $b_2 \in M_i^m$, $b_2 < b$ implies $b_2 < b_1$. (Such an extension exists by Theorem 4.2.) For limit ordinals δ, $M_\delta^m = \bigcup_{i<\delta} M_i^m$, and $M_0^{m+1} = M_{\delta_m}^m$.

Let $N = M_{\delta_n}^n$. Clearly for each $m \le n$, $\langle a_i^m: i < \delta_m\rangle$ is a cofinal sequence.

REMARK. This theorem is true also for an infinite number of a_i's.

THEOREM 4.4. *Suppose each finite subset of T has a model M in which $P_m(M)$ is*

λ_m-like ordered for $m \le n$ and λ_m is regular. Suppose μ_m, $m \le n$, are regular cardinals such that $\lambda_m = \lambda_p$ implies $\mu_m = \mu_p$.

Then T has a model N in which, for $m = 0, \cdots, n$, $\operatorname{ocf} P_m(N) = \mu_m$. Moreover $\|N\| \le |T| + \aleph_0 + \mu_0 + \cdots + \mu_m$.

REMARK. This theorem is true also for an infinite number of P's.

PROOF. We can easily find T_1, $T \subset T_1$, $|T_1| \le |T| + \aleph_0$, such that

(1) T_1 is closed.

(2) If M_1 is a model of T_1, then $\operatorname{ocf} P_m(M_1) = \operatorname{ocf} a_m$, for $m = 0, \cdots, n$, and $a_m \in P(M_1)$.

(3) Every finite subset of T_1 has a model M_1 in which, for $m = 0, \cdots, n$, $\{a : a \in P(M_1), a < a_m\}$ is λ_m-like ordered.

(4) If $\lambda_m = \lambda_p$ then $a_m = a_p$.

We adjoin to T sentences saying that $\{x : P_n(x)\}$ and $\{x : P(x) \wedge x < a_m\}$ have equal cofinality, and one of them is λ-like ordered iff the second is λ-like ordered (this is done by saying for each head of the one there is a one-one mapping into a head of the other). Then we used Theorem 3.1.

By Theorem 4.1 (4) and (5), T_1 has a model in which each a_m is regular. By Theorem 4.3, T_1 has a model N_1 in which $\operatorname{cf} a_m = \mu_m$. It is easy to find an elementary submodel N of N_1, $\|N\| \le |T_1| + \sum_{m \le n} \mu_m = |T| + \aleph_0 + \mu_0 + \cdots + \mu_m$, and for each m, N_1 contains a_m and a sequence $\langle a_i^m : i < \mu_m \rangle$ which is cofinal to a_m in N_1. Clearly also in N, $\operatorname{ocf} a_m = \mu_m$, and so N is the required model.

THEOREM 4.5. *Suppose that, for each $m, p < n$,*

(1) $\kappa_m > \kappa_p$ *iff* $\kappa'_m > \kappa'_p$,

(2) $\operatorname{cf} \kappa'_m < \kappa'_p$,

(3) $\kappa'_m > \chi$,

(4) $\operatorname{cf} \kappa_m = \operatorname{cf} \kappa_p$ *implies* $\operatorname{cf} \kappa'_m = \operatorname{cf} \kappa'_p$.

Then $\chi : \langle \kappa_0, \cdots, \kappa_{n-1} | \rangle \to \langle \kappa'_0, \cdots, \kappa'_{n-1} | \rangle$.

PROOF. Without loss of generality $\kappa_1 > \kappa_2 > \cdots > \kappa_{n-1}$. Then the proof is by iterating Theorem 3.2, and then using Theorems 4.4 and 3.3.

THEOREM 4.6. *If M is a model of ZF, $a, b \in |M|$, $a \ne b$ are regular cardinals, then M has an elementary extension in which $\operatorname{ocf}_{\text{ord}} a = \lambda$, $\operatorname{ocf}_{\text{ord}} b = \mu$ (i.e. the outer cofinality of $\{c \in M : c$ is an ordinal $< a\}$ is λ, and similarly for b).*

REMARK. This improves Theorem 3.2 from Keisler and Morley [12], and solves a question they asked on p. 58.

PROOF. By [12], M has an elementary extension N, $\|N\| = \chi^+$ (χ a regular cardinal $> \|M\|$), and $\operatorname{ocf}_{\text{ord}} a = \chi$, $\operatorname{ocf}_{\text{ord}} b = \chi^+$. We add to N two one-place relations—R_1^N, R_2^N—such that R_1^N is χ-like ordered and R_1^N is an unbound subset of $\{c \in N : c$ an ordinal $< a\}$. Similarly for R_2 and b. Then the proof follows easily by Theorem 4.4.

Alternatively, this can be proved directly by Theorem 4.3.

§5. On transfer theorems with omitting types.

DEFINITION 5.1. If T is a theory, Γ a set of types, then $Ec(T, \Gamma)$ will be the class of models of T which omits every type $p \in \Gamma$.

DEFINITION 5.2. $\langle \chi, \zeta \rangle : \langle \kappa_1, \cdots | \lambda_1, \cdots \rangle \to \langle \kappa'_1, \cdots | \lambda'_1, \cdots \rangle$ if for every language L, $|L| \leq \chi$ and theory T (in L) and set Γ of $\leq \zeta$ types in L if in $Ec(T, \Gamma)$ there is a $\langle \kappa_1, \cdots \mid \lambda_1, \cdots \rangle$-model, then in $Ec(T, \Gamma)$ there is a $\langle \kappa'_1, \cdots \mid \lambda'_1, \cdots \rangle$-model. (We always assume ζ is a cardinal greater than zero, although most of the results remain true for $\zeta = 0$.)

DEFINITION 5.3. (1) $\mathrm{Tp}(M)$ is the type of order by which $<$ (more exactly $<^M$) orders M.

(2) $\mathrm{Tp}(P(M))$ is the type of order by which $<$ orders $P(M)$.

(3) $\mathrm{Tp}(T, \Gamma) = \{\mathrm{Tp}(M): M \in Ec(T, \Gamma)\}$.

DEFINITION 5.4. $\langle \chi, \zeta \rangle: [\delta] \to [\mu]$ if: *if* L is a language of power $\leq \chi$, T a theory in L, Γ a set of types in L, $|\Gamma| \leq \zeta$, and $\delta \in \mathrm{Tp}(T, \Gamma)$, *then* there exists $M \in Ec(T, \Gamma)$ of outer cofinality μ which is not well-ordered.

DEFINITION 5.5. $\delta(\chi, \zeta)$ is the first ordinal such that: if T and Γ are in the language L, $|L| \leq \chi$, $|L| \leq \zeta$, and $\delta(\chi, \zeta) \leq k \in \mathrm{Tp}(T, \Gamma)$, then in $Ec(T, \Gamma)$ there is a non-well-ordered model.

THEOREM 5.1. (1) *If* $\langle \chi, \zeta \rangle: [\delta] \to [\mu]$, *and* T *and* Γ *are in* L, $|L| \leq \chi$, $|\Gamma| \leq \zeta$, $M \in Ec(T, \Gamma)$, *and* $\mathrm{Tp}(P(M)) = \delta$, *then there is* $M \in Ec(T, \Gamma)$ *such that* $P(M)$ *is not well-ordered and its outer cofinality is* μ.

(2) *If* T, Γ *are in* L, $|L| \leq \chi$, $|\Gamma| \leq \zeta$, *and* $M \in Ec(T, \Gamma)$, $\mathrm{Tp}(P(M)) \geq \delta(\chi, \zeta)$, *then in* $Ec(T, \Gamma)$ *there is an* M *for which* $P(M)$ *is not well-ordered.*

PROOF. The proof is immediate.

THEOREM 5.2. *If* $\langle \chi, \zeta \rangle: [\delta] \to [\mu]$, *and for every ordinal* $i, j < \delta$, $i + j < \delta$, *then for every singular cardinal* κ, κ', $\kappa = \beth_{k+\delta}$, $\mathrm{cf}\, \kappa' = \mu$, $\langle \chi, \zeta \rangle: \langle \kappa \mid \rangle \to \langle \kappa' \mid \rangle$.

PROOF. Let L be a language, $|L| \leq \chi$, T a theory in L, Γ a set of types in L, $|\Gamma| \leq \zeta$. Let, also, $M \in Ec(T, \Gamma)$ be a $\langle \kappa \mid \rangle$-model, i.e. P_1^M is κ-like ordered. We should prove that in $Ec(T, \Gamma)$ there is a $\langle \kappa' \mid \rangle$-model. Without loss of generality assume T is closed (see Theorem 3.1). We shall also assume, without loss of generality that $\mathrm{Tp}(P_1(M)) = \kappa$. (See Fuhrken [6]. He proved that *if* $<'$ is an ordering of $P_1(M)$ of type κ, M_1, M are elementarily equivalent *then* $P_1(M_1)$ is λ-like ordered by $<$ iff $P_1(M_1)$ is λ-like ordered by $<'$.) So let $P_1(M) = \{c_i: i < \kappa\}$, such that $i < j$ iff $c_i < c_j$. Let also $\langle \tau_i: i < \mathrm{cf}\, \kappa \rangle$ be an increasing sequence of ordinals $< \delta$, $\bigcup_i \tau_i = \delta$, and $\tau_i + \tau_i < \tau_{i+1}$. (It is easy to construct such a sequence as $i, j < \delta \Rightarrow i + j < \delta$. Clearly $\mathrm{cf}\, \delta = \mathrm{cf}\, \kappa$.)

For every ordinal α, $\bigcup_{j<i_0} \tau_j \leq \alpha < \alpha_0 = \tau_{i_0} < \delta$, let $S_\alpha^n = \{\langle X_{a,i}: i_0 \leq i < \mathrm{cf}\, \kappa \rangle: a \in A_\alpha^n\}$ be the set of n-good skeletons of $P_1(M)$, such that $|X_{a,i}| = \beth_{k+\tau_i+\alpha}$ and $\bigcup_{i<j} X_{a,i} \subset \{c_e: e < \beth_{k+\tau_j}\}$. Clearly $S_\alpha^0 \neq 0$.

By the proof of Theorem 2.2, it is easily seen that

(*) *If* $\beta + (n+2)^2 < \alpha$, *and* $a \in A_\alpha^n$, $\bigcup_{j<i_0} \tau_i \leq \alpha < \tau_{i_0}$ *then there exists* $b \in A_\beta^{n+1}$ *such that for every* i, $i_0 \leq i < \delta$, $X_{b,i} \subset X_{a,i}$.

Now we shall define a model N. The set of elements of N will be

$$|M| \cup \{i: i < \delta\} \cup \bigcup \{A_\alpha^n: n < \omega, \alpha < \delta\}.$$

(Without loss of generality we assume those sets are disjoint.)

The set of relations and functions of N will consist of the following:

(1) Q^1, Q^2, Q^3, Q^4 which will be (resp.) $|M|$, $\{i: i < \delta\} = \delta$, $\bigcup\{A_\alpha^n: n < \omega, \alpha < \delta\}$, and $\{\tau_i: i < \delta\}$.

(2) The relations of M and functions of M (if $\{a_0, \cdots, a_n\} \not\subset |M|$ we can define $F^N[a_0, \cdots, a_n] = 0$ $(\in \delta)$).

(3) Ordinal addition among the ordinals $< \delta$; and $<^M$ is extended to a total ordering of $|N|$, such that on δ it will be the order between ordinals. Also 0, 1, 2, 3, $\cdots$ will be individual constants in N.

(4) Functions F_1, F_2, F_3 such that

$F_1(a) = n$ iff $a \in A^n_\alpha$ for some α,

$F_2(a) = \alpha$ iff $a \in A^n_\alpha$ for some n,

$F_3(a) = \tau_{i_0}$ iff $a \in A^n_\alpha$, $\bigcup_{i<i_0}\tau_i \leq \alpha < \tau_{i_0}$.

(5) A relation R such that $N \models R[a, i, b]$ iff $b \in X_{a,i}$.

(6) A relation SUB such that $N \models SUB[b, a]$ iff $a \in A^n_\alpha$, $\bigcup_{i<i_0}\tau_i \leq \alpha < \tau_{i_0}$, and for every i, $i_0 \leq i < \delta$, $X_{b,i} \subset X_{a,i}$.

(7) More function symbols, such that Th(N) will be closed, and of power $\leq \chi$. (It is clear by Theorem 3.1 that we can do it.)

(We assume implicitly that all the new symbols do not belong to L.)

By Theorem 5.1, it follows that there exists a model N_1, elementarily equivalent to N; such that $(Q^2)^{N_1}$ is not well-ordered and has outer cofinality μ; and N_1 omits every type p^{Q^1}, for $p \in \Gamma$, where $p^{Q^1} = \{\varphi^{Q^1} : \varphi \in p\}$ (φ^{Q^1} is φ relativized to Q^1). It is clear that also the outer cofinality of $(Q^4)^{N_1}$ is μ.

As $(Q^2)^{N_1}$ is not well-ordered, let $\{\alpha_n : n < \omega\}$ be a decreasing sequence in $(Q^2)^{N_1}$. Clearly $N_1 \models \alpha_{n+m} + m \leq \alpha_n$ for every n, m. Hence we can find a subsequence $\{\beta_n : n < \omega\}$ of $\{\alpha_n : n < \omega\}$ such that $\beta_{n+1} + (n+2)^2 \leq \beta_n$. By what was said in (*) (which appears just before the definition of N in this proof) we can find a_n, $n < \omega$, in $(Q^3)^{N_1}$ such that $F_1(a_n) = n$, $F_2(a_n) = \beta_n$, $F_3(a_n) \leq F_3(a_0) = \tau$, and for every n, $N_1 \models SUB[a_{n+1}, a_n]$.

Let $\{\gamma_l : l < \mu\}$ be an increasing cofinal sequence in $(Q^4)^{N_1}$, $\tau < \gamma_0$.

Let $X_{n,i} = \{c \in (Q^1)^{N_1} : N_1 \models R[a_n, \gamma_i, c]\}$.

By the elementary equivalence of N and N_1, clearly $\langle X_{n,i} : i < \mu\rangle$ is an n-good pseudoskeleton of $P_1^{N_1}$. As $N_1 \models SUB[a_{n+1}, a_n]$, clearly $n < m$ implies $X_{m,i} \subset X_{n,i}$.

The rest of the proof is obvious and similar to that of the proof of Theorem 3.3.

Let $\kappa' = \sum_{i<\mu}\lambda_i$, $\lambda_i < \kappa'$, let $\{d_{i,j} : j < \lambda_i, i < \mu\}$ be a set of new individual constants. Let $T_1 = T \cup \{d_{i,j} < d_{i_1,j_1} : i < i_1 \text{ or } i = i_1, j < j_1\} \cup \{\psi(d_{i_0,j_0}, d_{i_1,j_1}, \cdots, d_{i_n,j_n}) :$ there exists $c_{i_0,j_0}, \cdots, c_{i_n,j_n}$ such that

(1) $d_{i_l,j_l} < d_{i_\alpha,j_\alpha}$ iff $c_{i_l,j_l} < c_{i_k,j_k}$ (for $l, \alpha \in n+1$),

(2) $c_{i_l,j_l} \in X_{n+1,i_l}$ (for every $l \in n+1$),

(3) $N_1 \models \psi[c_{i_0,j_0}, \cdots, c_{i_n,j_n}]$,

(4) ψ is a formula in $L(T)\}$.

Clearly T_1 is consistent, and hence has a $L(T)$-model N_2. N_2 has an elementary submodel N_3 which is the closure of $\{d_{i,j} : j < \lambda_i, i < \mu\}$. Clearly N_3 is the required model, and hence this finishes the proof.

THEOREM 5.3. *For every T, Γ in L there exists T_1, Γ_1 in L_1 such that*

(1) $|L_1| + \aleph_0 = |L| + \aleph_0$, $|\Gamma_1| = |\Gamma|$.

(2) *There exists a $\langle\kappa'|\rangle$-model in $Ec(T_1, \Gamma_1)$ iff one of the following conditions is satisfied:*

(A) *There exists $M \in Ec(T, \Gamma)$ which is not well-ordered and whose outer cofinality is* cf κ'.

(B) $\kappa' = \aleph_{k+\delta}$, *and there exists* $\delta_1 \in \mathrm{Tp}(T, \Gamma)$ *such that there is an increasing sequence of ordinals cofinal in* δ_1 *of length* δ (*hence* $\delta \le \delta_1$, $\mathrm{cf}\,\kappa' = \mathrm{cf}\,\delta = \mathrm{cf}\,\delta_1$).

REMARK. (1) By small changes in T_1 we can change condition (B) a little. For example we can replace $\kappa' = \aleph_{k+\delta}$ by $\kappa' = \bigcup_{i<\delta}\lambda_i$ (where $\lambda_i \le \lambda_{i+1} < (2^{\lambda_i})^+$; for every limit ordinal $\delta_2 < \delta$, $\bigcup_{i<\delta_2}\lambda_i = \lambda_{\delta_2}$).

We can replace $(2^{\lambda_i})^+$ by any cardinal function F such that there is a sentence which has a $\langle|\lambda^1, \lambda^2\rangle$-model iff $\lambda^1 \le \lambda^2 < F(\lambda^1)$.

(2) By this theorem, from any example proving $\langle\chi, \zeta\rangle: [\delta] \nrightarrow [\mu]$, we can construct an example proving $\langle\chi, \zeta\rangle: \langle\kappa|\rangle \nrightarrow \langle\kappa'|\rangle$.

PROOF. T_1 will consist of

(A) a sentence saying $<$ is an order (of the whole model);

(B) the sentences of T relativized to R;

(C) a sentence "saying" R is cofinal in the model, i.e., $(\forall x)(\exists y)(x < y \wedge R(y))$. If M is a model of T_1, $a \in R(M)$, let $|a| = |\{b \in M: b < a\}|$;

(D) a sentence telling that if x is the predecessor of y in R then $|y| \le |x|^+$, i.e.

$$(\forall xy)[[R(x) \wedge \neg(\exists z)(R(z) \wedge x < z < y) \wedge x < y] \\ \rightarrow [(\forall z_1 z_2 z_3)(z_1 < z_2 < z_3 < y \\ \rightarrow F(z_1, z_3) < x \wedge F(z_1, z_3) \ne F(z_2, z_3))]];$$

(E) a sentence telling that if $a \in R(M)$ has no predecessor then

$$|a| \le \bigcup\{|b|: b < a, b \in R(M)\},$$

i.e.,

$$(\forall x)[R(x) \wedge (\forall y)(\exists z)[y < x \wedge R(y) \rightarrow R(z) \wedge y < z < x] \\ \rightarrow (\forall y)(\exists z)[y < x \rightarrow R(z) \wedge y < z < x]].$$

Γ_1 will be Γ relativized to R, i.e., $\Gamma_1 = \{p^Q: p \in \Gamma\}$ where $p^Q = \{\varphi^Q: \varphi \in p\}$. Clearly T_1, Γ_1 satisfy the theorem.

LEMMA 5.4. (1) *If there is an increasing sequence of ordinals* $<\delta$ *which is cofinal with* δ, *and has length* δ_1, *and* $\langle\chi, \zeta\rangle: [\delta_1] \rightarrow [\mu]$, *then* $\langle\chi, \zeta\rangle: [\delta] \rightarrow [\mu]$.

(2) *If* $\chi < \chi_1$, $\zeta \le \zeta_1$ *then* $\delta(\chi, \zeta) \le \delta(\chi_1, \zeta_1)$.

(3) *If* $\chi \le \chi_1$, $\zeta \le \zeta_1$, *and* $\langle\chi_1, \zeta_1\rangle: [\delta] \rightarrow [\mu]$ *then* $\langle\chi, \zeta\rangle: [\delta] \rightarrow [\mu]$.

(4) *If* $\zeta \le \chi$ *then* $\delta(\chi, \zeta) = \delta(\chi, 1)$.

(5) *If* $\zeta \le \chi$, *then* $\langle\chi, \zeta\rangle: [\delta] \rightarrow [\mu]$ *iff* $\langle\chi, 1\rangle: [\delta] \rightarrow [\mu]$.

(6) *If* $\zeta_1 \ge 2^\chi$ *then* $\delta(\chi, \zeta_1) = \delta(\chi, 2^\chi)$. (*Similarly for* $\langle\chi, \zeta\rangle: [\delta] \rightarrow [\mu]$.)

(7) *If* $\mathrm{cf}\,\delta > \kappa$, *then* $\langle\chi, \zeta\rangle: [\kappa] \rightarrow [\mu] \Rightarrow \langle\chi, \zeta\rangle: [\delta] \rightarrow [\mu]$.

PROOF. For (1), (2) and (3) the proof is immediate.

(4) This is a particular case of $h_\chi = n_\chi$, in fact, which appears in Chang [3, pp. 47–48] [n_χ is the Hanf number for omitting a type in a language $\le\chi$; h_χ is the Hanf number of sentences of $L_{\chi^+,\omega}$]. As $1 \le \zeta$, by (1) $\delta(\chi, 1) \le \delta(\chi, \zeta)$. Suppose T, Γ are in the language L, $|L| \le \chi$, $|\Gamma| \le \zeta$. Let $\Gamma = \{p_i: i < i_0 < \zeta\}$, $p_i = \{\varphi_{i,j}(x): j < \chi\}$ (this is possible as $|p_i| \le \chi$, and we allow many appearances of one formula).

Let

$$T_1 = T \cup \{(\forall x)Q(F_i(x)): i < \zeta\} \cup \{(\forall x)[F_i(x) = c_j \rightarrow \neg\varphi_{i,j}(x)]: i < \zeta, j < \chi\}.$$

(Clearly we assume $Q \notin L$, $F_i \notin L$.)

Let $\Gamma_1 = \{p\}$, $p = \{Q(x)\} \cup \{\neg x = c_i : i < \chi\}$. Clearly T_1, Γ_1 are in a language L_1, $|L_1| \leq \chi$, and $|\Gamma_1| = 1$. Also clearly $\mathrm{Tp}(T_1, \Gamma_1) = \mathrm{Tp}(T, \Gamma)$. If $\delta(\chi, 1) < \delta(\chi, \zeta)$, we shall get easily a contradiction by this construction.

(5) The same proof as of (4).

(6) There are no more 2^χ types in a language of power $\leq \chi$.

(7) is proven by 5.4 (1) and the downward Löwenheim-Skolem-Tarski theorem.

LEMMA 5.5. (1) *If* $\operatorname{cf} \delta = \mu$, $\delta \geq \delta(\chi + \mu, \zeta)$ *then* $\langle \chi, \zeta \rangle : [\delta] \to [\mu]$.

(2) *If* $\mu > \chi$, $\mu \geq \mu_1$, μ, μ_1 *regular and* $\langle \chi, \zeta \rangle : [\delta] \to [\mu]$ *then* $\langle \chi, \zeta \rangle : [\delta] \to [\mu_1]$.

PROOF. (1) Suppose T, Γ are in L, $|L| \leq \chi$, $|\Gamma| \leq \zeta$, and $M \in Ec(T, \Gamma)$, $\delta = \mathrm{Tp}(M)$. As $\operatorname{cf} \delta = \mu$, we can find in M a cofinal sequence $\{c_i : i < \mu\}$. Let $\Gamma_1 = \Gamma \cup \{p\}, p = \{x > c_i : i < \mu\}$, and $T_1 = T \cup \{c_i < c_j : i < j < \mu\}$. Clearly T_1, Γ_1 are in a language L_1 of power $\leq \chi + \mu$, and $|\Gamma_1| = |\Gamma| + 1$. Using Lemma 5.4 (4), it is clear that $\delta \geq \delta(\chi + \mu, \zeta + 1) = \delta(\chi + \mu, \zeta)$. Hence there is a model $M \in Ec(T_1, \Gamma_1)$, which is not well-ordered. As it omits p_1 its outer cofinality is clearly μ. Hence, by the definition $\langle \chi, \zeta \rangle : [\delta] \to [\mu]$.

(2) is immediate.

THEOREM 5.6. (1) $\chi^+ \leq \delta(\chi, \zeta) \leq (2^\chi)^+$, *and if* $\operatorname{cf} \chi > \omega$, *then* $\chi^+ < \delta(\chi, 1)$.

(2) *If* $\chi = \beth_\delta$, $\operatorname{cf} \delta = \omega$, *then* $\delta(\chi, 1) = \chi^+$ (*in particular*, $\delta(\aleph_0, 1) = \aleph_1$). *Moreover, if* $\operatorname{cf} \chi = \omega$, *then* $\delta(\chi, 1) \leq (\sum_{\lambda < \chi} 2^\lambda)^+$.

(3) $\delta(\chi, 2^\chi) = (2^\chi)^+$.

REMARK. Clearly, these results with Lemma 5.5 give as immediate corollaries relations of the form $\langle \chi, \zeta \rangle : [\delta] \to [\mu]$, which by Theorem 5.2 proves relations of the form $\langle \chi, \zeta \rangle : \langle \kappa | \rangle \to \langle \kappa' | \rangle$.

PROOF. By Theorem 0.1 quoted in the Introduction ($\mu_\chi = \beth_{\delta(x)}$), we can prove these results by previous results on μ_χ. (1) follows by Morley [13], M. and V. Morley [14] and Chang [1] and [3]. (2) follows by Morley [13] (for $\chi = \aleph_0$), and Helling [8]; and (3) is proved in Shelah [16]. (The results in (1) and (2) appear in Chang [3, pp. 47–48].)

In fact we use only the easy direction in Theorem 0.1: $\beth_{\delta(x)} \leq \mu_\chi$; for in the cases we use the other direction, the proofs we depend upon prove our result.

THEOREM 5.7. *For every infinite cardinal* χ, ζ, μ, *and ordinal* δ, $j, i < \delta \Rightarrow i + j < \delta$ (*or equivalently* $i < \delta \Rightarrow i + i < \delta$) *the following conditions are equivalent.*

(A) $\langle \chi, \zeta \rangle : [\delta] \to [\mu]$;

(B) *for every* κ, κ', $\kappa = \beth_{k+\delta}$, $\operatorname{cf} \kappa' = \mu$; $\langle \chi, \zeta \rangle : \langle \kappa | \rangle \to \langle \kappa' | \rangle$.

PROOF. In Theorem 5.2 we prove that (A) implies (B). Suppose not (A), and we shall prove not (B). By definition there are T, Γ in L, $|L| \leq \chi$, $|\Gamma| \leq \zeta$, such that $\delta \in \mathrm{Tp}(T, \Gamma)$, but there is no $M \in Ec(T, \Gamma)$ with outer cofinality μ, which is not well-ordered.

Let T_1, Γ_1 be those constructed from T, Γ in Theorem 5.3, i.e. $|T_1| \leq \chi$, $|\Gamma_1| \leq |\Gamma|$ and there is a $\langle \kappa' | \rangle$-model in $Ec(T_1, \Gamma_1)$ iff

(α) there is $M \in Ec(T, \Gamma)$ with outer cofinality $\operatorname{cf} \kappa'$ which is not well-ordered; or

(β) $\kappa' = \beth_{k+\delta_0}$, and there is $\delta_1 \in \mathrm{Tp}(T, \Gamma)$ such that there is an increasing sequence of ordinals cofinal in δ_1 of length δ_0.

Now let $\delta^0 = (2^{\chi+\mu})^+ \times \mu$. Clearly $\operatorname{cf} \delta^0 = \mu$, and (by Lemma 5.5) $\langle \chi, \zeta \rangle : [\delta^0] \to [\mu]$. Let $\kappa = \beth_\delta$, $\kappa' = \aleph_{\delta^0}$. We know that not $\langle \chi, \zeta \rangle : [\delta] \to [\mu]$, and will prove not $\langle \chi, \zeta \rangle : \langle \kappa | \rangle \to \langle \kappa' | \rangle$, hence not (B). We know that T, Γ are in L, $|L| \leq \chi$,

$|\Gamma| \leq \zeta$, and (by (β)), that there is a $\langle \kappa | \rangle$-model in $Ec(T_1, \Gamma_1)$. If there is a $\langle \kappa' | \rangle$-model in $Ec(T_1, \Gamma_1)$ then (α) or (β) is satisfied. By the choice of T, Γ there is no $M \in Ec(T, \Gamma)$ with outer cofinality $\mu = \operatorname{cf} \delta' = \operatorname{cf} \kappa'$ which is not well-ordered, hence (α) is not satisfied. Suppose (β) is satisfied, then, as clearly $k < \delta^0$ implies $k + k < \delta^0$, there is $\delta_1 \in \mathrm{Tp}(T, \Gamma)$ such that there is an increasing sequence of ordinals cofinal in δ_1 of length δ^0. But by Lemma 5.4, $\langle \chi, \zeta \rangle: [\delta^0] \to [\mu]$ implies $\langle \chi, \zeta \rangle: [\delta_1] \to [\mu]$, this implies that in $Ec(T, \Gamma)$ there is a not well-ordered model of outer cofinality μ, a contradiction. Hence condition (β) is not satisfied, so $\langle \chi, \zeta \rangle$: $\langle \kappa | \rangle \not\to \langle \kappa' | \rangle$, and by this we end the proof.

REMARK. (1) Because of this theorem, we shall deal only with relations of the form $\langle \chi, \zeta \rangle: [\delta] \to [\mu]$, and shall not mention the obvious conclusions of the form $\langle \chi, \zeta \rangle: \langle \kappa | \rangle \to \langle \kappa' | \rangle$. (See, for example, 5.5, 5.6 and 5.4, which have such obvious conclusions.)

(2) We cannot expect to prove more transfer theorems, before we prove weaker two-cardinals transfer theorems with omitting types (i.e. relations of the form $\langle \chi, \zeta \rangle: \langle | \lambda_1, \lambda_2 \rangle \to \langle | \lambda_1', \lambda_2' \rangle$).

LEMMA 5.8. *If* $\operatorname{cf} \delta \geq \beth[(2^\chi)^+]$ (*or even if* $\operatorname{cf} \delta \geq \beth_{\delta(\chi,\zeta)}$) *then* $\langle \chi, \zeta \rangle: [\delta] \to [\mu]$ *for every* μ.

PROOF. This is done by a slight change in the proofs of Morley [13] and Chang [1].

LEMMA 5.9. *If* $\operatorname{cf} \delta > \omega$ *then* $\langle \chi, 1 \rangle: [\delta] \to [\aleph_1]$.

PROOF. By Lemma 5.4 (7), it is sufficient to prove $\langle \chi, 1 \rangle: [\aleph_1] \to [\aleph_1]$. Hence let T, Γ be in a countable language L, $\Gamma = \{p\}$, and $\aleph_1 \in \mathrm{Tp}(T, \Gamma)$, or $\aleph_1 = \mathrm{Tp}(M)$, $M \in Ec(T, \Gamma)$. From Keisler [9, Theorem 2.1], it follows that we can find a model N of a countable language such that M is a reduct of N; and if $N_1 \equiv N$, $\|N_1\| = \aleph_0$, and N_1 omits p, then N_1 has an elementary extension N_2 of outer cofinality $\aleph_1$ which omits p. By Lemma 5.5 and Theorem 5.6(2), there is N_1, $N_1 \equiv N$, $\|N_1\| = \aleph_0$, N_1 omits p and N_1 is not well-ordered. By the definition of N we get our conclusion.

We shall try to get a few negative results.

LEMMA 5.10. *If there are* T, Γ *in* L, $|L| \leq \chi$, $|\Gamma| \leq \zeta$, *such that there is* $M \in Ec(T, \Gamma)$, $\|M\| = \operatorname{cf} \delta$, *but there is no* $M \in Ec(T, \Gamma)$, $\|M\| = \mu$, *then* $\langle \chi, \zeta \rangle: [\delta] \not\to [\mu]$.

PROOF. The proof is immediate.

LEMMA 5.11. *If* $\operatorname{cf} \delta \leq \chi$, *then* $\langle \chi, \zeta \rangle: [\delta] \to [\mu]$ *iff* $\delta(\chi, \zeta) \leq \delta$ *and* $\operatorname{cf} \delta = \operatorname{cf} \mu$.

PROOF. By Lemma 5.5, it is immediate.

LEMMA 5.12. (1) *If* $\operatorname{cf} \aleph_\alpha > \omega$, $n < \omega$, *then* $\langle \aleph_\alpha, 1 \rangle: [\aleph_{\alpha+n}] \not\to [\aleph_{\alpha+n}]$.

(2) *If there is* T, Γ *in* L, $|L| \leq \aleph_\alpha$, $|\Gamma| = 1$, *such that in* $Ec(T, \Gamma)$ *there is a* $\langle | \aleph_{\alpha+n+2}, \aleph_{\alpha+n+1} \rangle$-*model, but not a* $\langle | \aleph_{\alpha+n+1}, \aleph_{\alpha+n} \rangle$-*model, and* $\operatorname{cf} \aleph_\alpha > \omega$, *then* $\langle \aleph_\alpha, 1 \rangle: [\aleph_{\alpha+n+2}] \not\to [\aleph_{\alpha+n+1}]$.

(3) *If* $\operatorname{cf} \chi > \omega$, *then there are* T, Γ *in* L, $|L| = \chi$, $|\Gamma| = 1$, *such that*

(A) *Every* δ, $\chi^+ < \delta < \chi^{++}$, $\operatorname{cf} \delta = \chi^+$, *belongs to* $\mathrm{Tp}(T, \Gamma)$.

(B) *Every model in* $Ec(T, \Gamma)$ *which is not well-ordered has outer cofinality* ω.

PROOF. All the parts of this theorem are easily proved by the following theorem of Chang from [3]:

For every predicate P, Q there is a sentence ψ such that if in a model M, $i = \mathrm{Tp}(P(M))$, $\mathrm{cf}\, i > \omega$, then $Q(M)$ is well-ordered, and $\mathrm{Tp}(Q(M)) \leq |i|^+$. Moreover for every λ is a model in which $\mathrm{Tp}(P(M)) = \lambda$, $\mathrm{Tp}(Q(M)) = \lambda^+$.

$$T = \{(\forall x)Q(x)\} \cup \{\text{the axioms of order}\} \cup \{c_i < c_j : i < j < \chi\} \cup \{\psi\},$$
$$\Gamma = \{p\}, \qquad p = \{P(x)\} \cup \{x \neq c_i : i < \chi\}$$

clearly proves (1) for $n = 1$ (this was Chang's theorem). By iterating the construction we can prove (1), and from it (2), (3) follow easily.

(4) can be proved by adding to the above defined T a sentence saying P, Q are with equicofinalities.

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