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Martin's axiom and Δ^2_1 well-ordering of the reals

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Abstract. Assuming an inaccessible cardinal κ, there is a generic extension in which $MA + 2^{\aleph_0} = \kappa$ holds and the reals have a Δ^2_1 well-ordering.

1 Introduction

The aim of this paper is to describe a technique that allows the encoding of an arbitrary set of ordinals by a Δ^2_1 formula in a generic extension which is cofinality preserving. This encoding is robust enough to coexist with MA (Martin's Axiom). Specifically, we will show, for any model of ZFC set theory with an inaccessible cardinal κ, the existence of a cardinal preserving generic extension in which $2^{\aleph_0} = \kappa + MA+$ *there is a* Δ^2_1 *well-ordering of* $\mathbb{R}$.

Let us explain what is meant by a Δ^2_1 well-order. We refer here to the structure $\langle H, \in \rangle$ where $H = H(\aleph_1)$ is the collection of all hereditarily countable sets. A Σ^2_k formula is a second-order formula of the form $\exists X_1 \subseteq H \ \forall X_2 \subseteq H \ldots \varphi(X_1, \ldots, X_k, \ a_1 \ldots, a_n)$ with k alternations of set quantifiers (unary predicates, X_i), and where φ is a first-order formula (in which quantification is over H) with predicate names $X_1, \ldots, X_k$, and variables $a_1, \ldots, a_n$ (which vary over H). A Δ^2_k formula is one that is equivalent to a Σ^2_k and to a Π^2_k formula. A Δ^2_1 well-ordering is one that is given by a Δ^2_1 formula $\psi(x, y)$ that defines a well-ordering of $\mathbb{R}$. Obviously, a Σ^2_1 linear ordering of $\mathbb{R}$ is also a Π^2_1 ordering.

An alternative definition of Σ^2_k formulas, which connects to the usual definition of Σ^1_n (projective) sets, is to look at third-order formulas over $\langle \mathbb{N}, +, \ldots \rangle$, that is, second order formulas over $\mathbb{R}$.

Our result cannot be improved to give a projective well-ordering of $\mathbb{R}$ because of a theorem of Shelah and Woodin [4] which proves that there is no well-ordering of $\mathbb{R}$ in $L(\mathbb{R})$, assuming some large cardinal. Since any projective order is in $L[\mathbb{R}]$,

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and as a small extension, such as the one described here, will not destroy any large cardinal above κ, the Δ^2_1 well-order cannot be improved to a projective well-order.

Though this paper can be read independently, it is a continuation of our [1] work where another coding technique is described which does not add any new reals. Both that work and the present are motivated by a theorem of Woodin [5] which shows that if CH holds and there is a measurable cardinal which is Woodin, then there is no Σ^2_1 well-order of the reals. In view of this result, a natural question is what happens if the CH is removed? Woodin has obtained the following result: Assuming an inaccessible cardinal κ, there is a c.c.c. forcing extension in which $\kappa = 2^{\aleph_0}$ and

1. there is a Δ^2_1 well-ordering of $\mathbb{R}$.
2. Martin's axiom holds for σ-centered posets.

Since the poset used to get this extension has cardinality κ, it does not destroy whatever large cardinal properties the ground model has above κ, and hence the assumption of CH is necessary for Woodin's theorem.

The theorem proved in this paper is a slight improvement of this theorem in that MA replaces the restricted version for σ-centered posets, but our main point is to describe a different encoding technique.

We were also motivated by the following related result of Solovay:

There is a forcing poset of size $2^{2^{\aleph_0}}$ such that the following holds in the extension

1. $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.
2. *MA* for σ-centered posets,
3. there is a Δ^2_1 well-ordering of the reals.

Let us emphasize that no inaccessible cardinal is needed for Solovay's result. Let us also mention here the main result of Abraham and Shelah [1]

> There is a generic extension *that adds no new countable sets* in which there exists a Σ^2_2 well-order of $\mathbb{R}$.

The theorem proved in this paper will now be formally stated.

Theorem. *Let κ be an inaccessible cardinal, and assume GCH holds below κ. Then there is a forcing extension that changes no cofinalities and in which*

1. $2^{\aleph_0} = \kappa$*+ Martin's Axiom, and*
2. there is a Δ^2_1 well-ordering of $\mathbb{R}$.

In a forthcoming work [2] we will show that the inaccessible is dispensable (but the continuum is $\aleph_2$ in this work).

2 Overview

The idea of the proof is quite simple, and we first give a general description. The generic extension is a length κ mixed–support iteration consisting of two components: The first component iterates c.c.c. posets with the aim of finally obtaining Martin's Axiom. The second component is doing the coding. Quite arbitrarily, we have chosen the set (called lim) of limit ordinals below κ to be the support of the c.c.c. component, and the set of successor ordinals (succ) to support the coding component. The iteration is a finite/Easton iteration. This means that the domain of each condition is finite on the limit ordinals, and has cardinality $< \rho$ below every inaccessible cardinal $\rho \leq \kappa$.

For a regular cardinal λ, F_λ denotes the club filter on λ. We say that a family $H \subseteq F_\lambda$ *generates* F_λ iff $\forall C \in F_\lambda\ \exists D \in H\ (D \subseteq C)$. The least cardinality of a generating family for F_λ is called here "the generating number for λ". A crucial question (in this paper) to ask about a regular cardinal λ is whether its generating number is λ^+ or higher: it is through answers to these questions that the encoding works.

If $2^\lambda = \lambda^+$, then the generating number for λ is λ^+ of course, but it is easy to increase it by forcing, say, λ^{++} new subsets of λ with conditions of size $< \lambda$. We denote with $C(\lambda, \mu)$ the poset that introduces μ subsets to λ with conditions of size $< \lambda$.

$$C(\lambda, \mu) = \{f \mid \text{dom}(f) \subseteq \lambda \times \mu,\ \ \text{range}(f) = 2, \mid f \mid < \lambda\}$$

where $\mid f \mid$ is the cardinality of the function f. Equivalently, one can demand $\text{dom}(f) \subseteq \mu$ in the definition. Clearly $C(\lambda, \mu)$ is λ-closed, and if $\lambda^{<\lambda} = \lambda$, then it satisfies the λ^+-c.c.

The closure in λ of each generic subset of λ is a closed unbounded set that contains no old club set. We will iterate such posets, varying λ, and taking care of MA as well.

In the final generic extension, $2^{\aleph_0} = \kappa$, Martin's Axiom holds, and the sequence of answers to the questions about the generating numbers for $\lambda < \kappa$ encodes a well-ordering of $\mathbb{R}$ which is Δ^2_1. As will be explained below, these questions are asked only for even (infinite) successors below κ, that is, cardinals of the form $\aleph_{\delta+2n}$ where $\delta > 0$ is a limit ordinal and $1 \leq n < \omega$ (call this set of even successor cardinals **es**). It is convenient to use an enumeration of **es** that uses all the successor ordinals as indices: **es**$= \{\lambda_j \mid j < \kappa \text{ is a successor ordinal}\}$. So $\lambda_1 = \aleph_2$ is the first infinite even successor, $\lambda_2 = \aleph_4, \ldots, \lambda_{\omega+1} = \aleph_{\omega+2}, \lambda_{\omega+2} = \aleph_{\omega+4}$ etc. In general,

$$\text{if } \alpha = \delta + n + 1 \text{ where } \delta \in \text{lim and } n < \omega, \text{ then } \lambda_\alpha = \aleph_{\delta+2(n+1)}.$$

In the final model, the well-ordering of $\mathbb{R}$ is the sequence of reals $\langle r_\xi \mid \xi < \kappa\rangle$ where $r_\xi \subseteq \omega$ is encoded by setting $\alpha = \omega\xi$ and

$$n \in r_\xi \text{ iff the generating number for } \lambda = \lambda_{\alpha+n+1} \text{ is } \lambda^{++}.$$

Why is it necessary to skip cardinals and to space the λ_α's two cardinals apart? Suppose that $r \subseteq \omega$ is the first real we want to encode. If $0 \in r$, then the first coding poset is $C(\aleph_2, \aleph_4)$. Recall that *GCH* is assumed, and hence cardinals are not collapsed, and $2^{\aleph_2} = \aleph_4$ after this forcing. Now if $1 \in r$, we may want to continue forcing with $c(\aleph_3, \aleph_5)$, but this will collapse $\aleph_4$ since $2^{\aleph_2} = \aleph_4$. Thus we must start the next iteration at least two cardinals apart, and forcing with $C(\aleph_4, \aleph_6)$ is fine. In general, $\lambda_{\alpha+1} = \lambda_\alpha^{++}$, enables the proof that cardinals are not collapsed in the extension.

The coding component of our forcing will be an iteration of posets of type $C(\lambda_\alpha, \lambda_\alpha^{++})$ for well chosen α's. This choice will be made to obtain the desired coding by determining the generating number for $\lambda \in$ **es**.

Let us take a closer, but still informal, view of the forcing poset. If we denote with P_α the αth stage of the iteration, then our final poset is P_κ. For limit δ's, P_δ is the mixed support limit of $\langle P_i | i < \delta\rangle$ with finite/Easton support. This means that $f \in P_\delta$ iff f is a partial function defined on δ such that $f \restriction i \in P_i$ for every $i < \delta$, and dom(f) contains only finitely many limit ordinals (this is the c.c.c. component), and $|\text{dom}(f) \cap \mu| < \mu$ for any inaccessible cardinal μ (this is the Easton support requirement of the coding component). At successor stages $P_{j+1} \cong P_j * Q_j$ is a two-step iteration, where Q_j is a poset in V^{P_j} characterized by the following. For limit $j < \kappa$, Q_j is in V^{P_j} a c.c.c. forcing. And for successor $j < \kappa$ of the form $\delta + i$, where $i \in \omega$ and $\delta \in \lim$, Q_j is either the trivial poset, or $C(\lambda_j, \lambda_j^{++})$ which is the poset for adding λ_j^{++} many subsets to $\lambda_j = \aleph_{\delta+2i}$. The decision as to the character of Q_j will be described later; the role of Q_j is to encode one bit of information about some real. This decision is made generically, in V^{P_j}, and it depends on the real in V^{P_j} that is being encoded.

So P_1 is some c.c.c. poset, and P_2 is P_1 followed by either the trivial poset or by $C(\aleph_2, \aleph_2^{++})$. In the latter case, forcing with P_2 makes $2^{\aleph_2} = \aleph_4$.

The iteration continues in a similar fashion. To illustrate one of the main points, let us see (only intuitively now) why $\aleph_1$ is not collapsed. We will show that every $f : \omega_1 \to On$ in V^{P_κ} (where *On* is the class of ordinals) has a countable approximation in V, that is, a function f' such that, for every $\alpha \in \omega_1$, $f(\alpha) \in f'(\alpha)$ where $f'(\alpha)$ is a countable set of ordinals.

Observe first that the Easton component of P_κ is $< \aleph_2$ closed. This means that if an increasing sequence $\langle p_i | i < \omega_1\rangle$ of conditions in P_κ have the same c.c.c. component ($p_i \restriction \lim = p_j \restriction \lim$), then there is an upper bound in P_κ to the sequence. We say that p is a *pure* extension of q if p extends q and both have the same restriction to lim (same c.c.c. component). Now, if $f : \omega_1 \to On$ is a function in V^{P_κ}, we define an increasing sequence $\langle p_i | i < \omega_1\rangle$ of conditions in P_κ such that $i < j \Rightarrow p_j$ is a pure extension of p_i: To obtain p_{i+1} extend p_i in countably many steps; at each step find first an extension q' of the previous step q that forces a new value for $f(i)$ (if there is one) and then take only the pure extension of q imposed by q'. It turns out that this process will never take more than countably many steps, or else we get a contradiction to the assumption that at limit stages c.c.c. posets are iterated. The upper bound $p \in P_\kappa$ of this pure increasing sequence "knows", for each $i < \omega_1$, all the countable many possible values for $f(i)$.

We arrange the iteration in such a way that for every real $r \in V^{P_\kappa}$ there is a unique limit ordinal $\delta = \delta(r)$ so that, for every $k \in \omega$, $k \in r$ iff the generating number for $\lambda = \lambda_{\delta+(k+1)}$ is λ^{++}, where $\lambda = \aleph_{\delta+2(k+1)}$).

Now the well-ordering on $\mathbb{R}$ is defined by

$$r_1 \prec r_2 \text{ iff } \delta(r_1) < \delta(r_2).$$

This formula is certainly first-order expressible in $H(\kappa)$ (the collection of sets of cardinality hereditarily $< \kappa$ in the extension), but why is it Σ^2_1? Why can we reduce it to second–order quantification over $H(\aleph_1)$? The point is that $2^{\aleph_0} = \kappa + MA$, and we can speak correctly within $H(\aleph_1)$ about $H(\kappa)$, and it takes a single second-order quantification to do that (this trick was used by Solovay in his theorem cited above; we will outline it now, and it will be explained in more detail later.) To express $r_1 \prec r_2$, just say:

> There is a relation R over $H = H(\aleph_1)$, such that (H, R) satisfies enough of set theory (when R interprets the membership relation $\in$), such that R is well-founded and such that every real is "found" in (H, R); moreover, (H, R) satisfies the following statement: "*every limit ordinal has the form $\delta(r)$ for some real r, and $\delta(\hat{r}_1) < \delta(\hat{r}_2)$*", where $\hat{r}$ is the construction of $r \subseteq \omega$ in the model (H, R).

Since R is well-founded, (H, R) is collapsed to some $\in$ structure, M, which turns out to be $H(\kappa)$ as we want. The main points to notice in order to prove this are that (1) M cannot contain less than κ ordinals because it contains all the reals, and a definable well-ordering of $\mathbb{R}$. (2) What M considers to be a cardinal is really a cardinal, because any possible collapsing function in $H(\kappa)$ can be encoded by a real (with the almost disjoint set technique which is applicable because of Martin's Axiom). Since this encoding real is in M, $H(\kappa)$ is included in M. (3) M does not contain more ordinals than κ. This is so since every limit ordinal δ is connected to a single real which is encoded along the segment $[\aleph_{\delta+2}, \aleph_{\delta+\omega})$ by the characteristic of the club filters. Thus M is $H(\kappa)$.

The details of this proof are written in the sequel.

3 Mixed support iteration

In this section we describe how to iterate, with mixed support (Mitchell's type support), c.c.c. posets and λ-complete posets, where the support of the c.c.c. component is finite, and the support of the complete component is of Easton type—bounded below inaccessibles.

Let κ be an inaccessible cardinal, and $\lambda < \kappa$ a regular cardinal $> \aleph_1$. The non c.c.c posets in the iteration are all assumed to be λ closed. For definiteness we have chosen the support of the c.c.c. posets to be the limit ordinals below κ, denoted here lim (0 is in lim), and the λ-complete forcings are supported by the successors below κ, denoted "succ".

For an ordinal $\mu \leq \kappa$, a *mixed support* iteration of length μ is defined here to be a sequence of posets $\langle P_i | i \leq \mu \rangle$ such that

1. The members of each P_i are partial functions defined on i.
2. For limit $\delta \leq \mu$, P_δ is the mixed support *limit* of $\langle P_i | i < \delta \rangle$. This means the following. P_δ consists of all the partial functions f defined on δ such that
   a) $f \restriction i \in P_i$ for every $i < \delta$.
   b) $\mathrm{Dom}(f) \cap \lim$ is finite.
   c) In case δ is inaccessible, $|\mathrm{Dom}(f) \cap \mathrm{succ}| < \delta$.
   The partial order on P_δ is defined by $f \leq g$ iff for all $i < \delta$ $f \restriction i \leq g \restriction i$ in P_i.
3. For successors $\eta + 1 \leq \mu$, $P_{\eta+1} \simeq P_\eta * Q_\eta$ where Q_η is a name of a poset in the universe of terms V^{P_η}. So $f \in P_{\eta+1}$ iff $f \restriction \eta \in P = P_\eta$ and $f \restriction \eta \Vdash_P f(\eta) \in Q_\eta$. The partial order on $P_{\eta+1}$ is defined as usual.
4. For any limit ordinal $\delta < \mu$, Q_δ is in V^{P_δ} a c.c.c. forcing (i.e., the empty condition in P_δ forces that). For successors $\alpha < \mu$, Q_α is λ-closed in V^{P_α} (closed under sequences of length $< \lambda$).

The notation $\Vdash_\eta$ can be used for $\Vdash_{P_\eta}$. It is convenient to define two conditions p and q in P to be *equivalent* iff they are compatible with the same conditions in P. However, it is customary not to deal with equivalence classes, and to write $p = q$ instead of $[p] = [q]$, and we shall accept this convention.

For $i < \mu$ (μ is the length of the iteration) the restriction map $f \mapsto f \restriction i$ is a projection of P_μ onto P_i. But for an arbitrary set $A \subseteq i$, $f \restriction A$ is not necessarily a condition, and, even when it is a condition, it is possible that $[f] = [g]$ and $f \restriction A \neq g \restriction A$. Therefore, the notation $f \restriction A$ refers to the function f itself and not to its equivalence class.

The set of functions $f \restriction \lim$, for $f \in P_\mu$, is called the "c.c.c. component" of P_μ. And the functions of the form $f \restriction \mathrm{succ}$ form the "complete component" of P_μ. Let us say that f_2 is a *pure* extension of f_1 in P_μ iff $f_1 \leq f_2$ and $f_1 \restriction \lim = f_2 \restriction \lim$. Thus, a pure extension of f_1 does not touch the c.c.c. component. (This definition refers to the functions f_1 and f_2 and not to their equivalence classes in P_μ.)

The following lemma is an obvious consequence of the assumed λ-completeness of the posets in the complete component.

Lemma 3.1. *P_μ is $< \lambda$ pure closed. That is, any purely increasing sequence $\langle q_i | i < \tau \rangle$ of length $\tau < \lambda$ (q_j is a pure extension of q_i for $i < j$) has a least upper bound in P_μ, which is a pure extension of each q_i.*

Suppose now that $q \in P_\mu$, and r is in the c.c.c. component of P_μ. Then the sum $h = q + r$ is the function defined by

$$h(i) = \begin{cases} r(i) & \text{if} \quad i \in \mathrm{dom}(r) \\ q(i) & \text{if} \quad i \in \mathrm{dom}(q) \setminus \mathrm{dom}(r) \end{cases} .$$

Whenever the notation $h = q + r$ is used, it is tacitly assumed that for every i, $h \restriction i \Vdash_i$ $h(i) \in Q_i$ *and* $r(i)$ *extends* $q(i)$. Hence $q + r \in P_\mu$ extends q. We have the following two easy lemmas on pure extensions given with no proof.

Lemma 3.2. *If $p_1 \leq p_2$ in P_μ, then there is a pure extension q of p_1 such that, setting $r = p_2 \restriction \lim$, we have*

$$[p_2] = [q + r] .$$

Thus any extension is a combination of a pure extension with a finitely supported c.c.c. component.

Lemma 3.3. *If $p_0 + r$ is a condition and p_1 is a pure extension of p_0, then $p_1 + r$ is a condition that extends $p_0 + r$.*

The c.c.c. component of P_μ is certainly not a c.c.c. iteration, but the following quasi c.c.c. property still carries over from the usual argument that iteration with finite support of c.c.c. posets is again c.c.c.

Lemma 3.4. *Assume that ω_1 is preserved by $P_{\mu'}$ for every $\mu' < \mu$. Let $\{r_\xi | \xi < \omega_1\}$ be an uncountable subset of the c.c.c. component of P_μ. If $q \in P_\mu$ is such that $q + r_\xi \in P_\mu$ can be formed for every $\xi < \omega_1$, then*

1. *For some $\xi_1 \neq \xi_2$, $q + r_{\xi_1}$ and $q + r_{\xi_2}$ are compatible in P_μ.*
2. *There is some r in the c.c.c. component of P_μ such that $q + r \in P_\mu$ and*

$$q + r \Vdash_\mu \quad \text{there are unboundedly many } \xi < \omega_1 \text{ with } q + r_\xi \in G \text{ (the generic filter).}$$

Proof. Obviously, *(2)* implies *(1)* (because the posets are separative, and $p \Vdash$ "$q + r_\xi \in G$" implies $q + r_\xi \leq p$). So we will only prove (2), by induction on μ.

Recall first that for any c.c.c. poset Q and uncountable subset $A \subseteq Q$ there is a condition $a \in A$ such that $a \Vdash_Q$ $A \cap G$ *is uncountable*. (Obvious warning: This does not mean there are uncountably many $a' \in A$ with $a' \leq a$.)

If μ is limit, there is no problem in using the familiar Δ-argument in case $cf(\mu) = \omega_1$, and the obvious application of the inductive assumption when $cf(\mu) \neq \omega_1$. For example, in case $cf(\mu) = \omega_1$, form a Δ-system out of $\mathrm{dom}(r_\xi), \xi < \omega_1$, and let $d \subseteq i_0 < \mu$ be the fixed finite core of the system. Then apply the inductive assumption to $q \restriction i_0$ and to $r'_\xi = r_\xi \restriction d$, for ξ in the Δ-system. This gives some r_0 in the c.c.c. component of P_{i_0} which satisfies *2* above for $q \restriction i_0$ and the conditions r'_ξ. It is not too difficult to see that $q + r_0$ is as required (use the fact that the c.c.c component of every condition has a finite support).

In case $\mu = j + 1$ and j is a limit ordinal (for this is the interesting case), then $P_\mu \simeq P_j * Q(j)$, where $Q(j)$ is a c.c.c. poset in V^{P_j}. Set $q' = q \restriction j$, and $r'_\xi = r_\xi \restriction j$. Apply induction to find r' such that

$$q' + r' \Vdash_j \quad \textit{for unboundedly many } \xi < \omega_1, \quad q' + r'_\xi \in G_j \textit{ (the generic filter over } P_j\textit{)}.$$

Then define a name σ in V^{P_j} of a subset of ω_1 such that

$$[q' \Vdash_j \xi \in \sigma] \text{ iff } q' + r'_\xi \in G_j .$$

Since
(1) $q' + r'$ forces that σ is unbounded in ω_1,
(2) ω_1 is not collapsed in V^{P_j} by our assumption,
(3) $Q(j)$ is c.c.c.,
there is, by the remark made at the beginning of the proof, a name $a \in V^{P_j}$ such that $q' + r' \Vdash_j$ "*a is some $r_\xi(j)$ for r'_ξ in G_j such that $a \Vdash_{Q(j)}$* (for unboundedly many $\zeta \in \sigma$, $r_\zeta(j) \in H$) ". (H is the $Q(j)$ generic filter.

Now it is immediate to combine r' and a to a function r which is as required.

□

The main property of the mixed support iteration is the following.

Lemma 3.5. *Assume P_μ is a mixed support iteration as described above of c.c.c. and λ-complete posets. For every cardinal $\lambda' < \lambda$, every $f : \lambda' \to On$ in V^{P_μ} has a countable approximation in V (that is, a function g defined on λ' such that for every $\alpha < \lambda'$, $g(\alpha)$ is countable and $f(\alpha) \in g(\alpha)$.)*

Proof. By induction on μ. Observe first that the lemma implies that any set of infinite cardinality $\lambda' < \lambda$ in the extension is covered by a ground model set of the same cardinality. Hence cardinals $\leq \lambda$ are not collapsed in V^{P_μ}. The lemma also implies that, for regular uncountable $\lambda' < \lambda$, any club subset of λ' in V^{P_μ} contains an old club set in V.

It is obvious that any c.c.c. extension or λ-complete extension has the property described in the theorem, namely that functions on λ' have countable approximations. Hence, in case $\mu = \mu_0 + 1$, the theorem is obvious: First get the approximation in $V^{P_{\mu_0}}$ (assume without loss of generality that the first approximation has the form $g : \lambda' \times \aleph_0 \to On$), and then use induction to get a second approximation in V.

So assume that μ is a limit ordinal, and $f \in V^{P_\mu}$ is a function defined on $\lambda' < \lambda$. We are going to define a pure increasing sequence $\langle q_\xi | \xi < \lambda' \rangle$ in P_μ such that for every $\alpha < \lambda'$ there is a countable set $g(\alpha)$ and

$$q_{\alpha+1} \Vdash f(\alpha) \in g(\alpha).$$

If this construction can be carried on, then use the $< \lambda$ pure completeness of P_μ to find an upper bound q to this sequence. Then $q \Vdash g$ *is a countable approximation to f*.

The definition of $q_{\xi+1}$ is done by defining (1) a pure increasing sequence $\langle q(\alpha) | \alpha < \alpha_0 \rangle$ where $q(0) = q_\xi$, and (2) for each α, a finite function r_α in the c.c.c. component of P_μ so that, for $\alpha \neq \alpha'$, $q(\alpha) + r_\alpha$ and $q(\alpha') + r_{\alpha'}$ force different values for $f(\xi)$. The definition of this sequence is continued as long as possible, and the following argument shows that it must stop for some $\alpha_0 < \omega_1$, and then $q_{\xi+1}$ is the pure supremum of this countable sequence, and $g(\xi)$ is the set of all values forced there to be $f(\xi)$. Indeed, otherwise, $q(\alpha)$ can be defined for every $\alpha < \omega_1$ and we let q be the upper bound of this pure increasing sequence (recall that $\aleph_1 < \lambda$). Then $q + r_\alpha$ is in P_μ for every $\alpha < \omega_1$ and it forces different values for $f(\xi)$. This contradicts the quasi c.c.c. Lemma 3.4. □

4 Definition of the forcing extension

The description of the poset P_κ, used for the coding proof, is given in this section by defining a mixed-support iteration $\langle P_\mu | \mu \leq \kappa \rangle$ as outlined in Sect. 2.

At successor stages: $P_{\mu+1} \cong P_\mu * Q_\mu$ where Q_μ is a poset in V^{P_μ} defined thus. If $\mu = \delta \in \lim$, then Q_δ is in V^{P_δ} a c.c.c. poset of cardinality, say, $\leq \aleph_\delta$. (P_1 is a c.c.c. poset, say the countable Cohen poset.) The choice of Q_δ is determined by some bookkeeping function, aimed to ensure that Martin's Axiom holds in V^{P_κ}. (The cardinality limitation is to ensure the right cardinalities to show that cardinals are not collapsed.)

For successor ordinals of the form $j = \delta + i$ where δ is limit and $0 < i < \omega$, Q_j is defined to be in V^{P_j} either the trivial poset (containing a single condition) or the poset $C(\lambda_j, \lambda_j^{++})$, where $\lambda_j = \aleph_{\delta+2i}$. To determine which alternative to take, define a function g that gives, for every limit $\delta < \kappa$, a name $g(\delta) \in V^{P_\delta}$ such that, for every $\alpha < \kappa$, every real in V^{P_α} is some $g(\delta)$ for $\delta \geq \alpha$. Suppose that $g(\delta)$ is interpreted as $r \subseteq \omega$ in $V[G_\delta]$ (the generic extension via P_δ); then this determines Q_j, for every j in the interval $(\delta, \delta + \omega)$, which has the form $j = \delta + i_0 + 1$, by

$$Q_j \text{ is non-trivial iff } i_0 \in r.$$

In order to prove that P_κ possesses the required properties (such as not collapsing cardinals), we decompose P_κ at any stage $\alpha < \kappa$, and write $P_\kappa \cong P_\alpha * P^\alpha_\kappa$, where P_α is the iteration up to α, and P^α_κ is the remainder of the iteration. It is not hard to realize that P^α_κ is just like P_κ except that $\lambda_1 = \aleph_2$ is replaced with $\lambda_{\alpha+1} = \aleph_{\alpha+2}$. For this reason, we must first describe P^α_κ and analyze its properties.

For each ordinal $\alpha < \kappa$ a mixed support iteration $\langle P^\alpha_\mu | \alpha \leq \mu \leq \kappa \rangle$ will be defined by induction on μ. The poset used to obtain the theorem is P^0_κ, but the P^α_κ are necessary as well since the decomposition $P^0_\kappa \simeq P^0_\alpha * (P^\alpha_\kappa)^{V^{P^0_\alpha}}$ is used to show the desirable properties of the iteration. This may also explain why we choose the index μ of P^α_μ to start from α and not from 0. The conditions in P^α_μ are functions defined on the ordinal interval $[\alpha, \mu)$.

To begin with, P^α_α is the trivial poset $\{\emptyset\}$ containing only one condition (the empty function). The definition of $P^\alpha_{j+1} \simeq P^\alpha_j * Q^\alpha(j)$ depends on whether $j \in \lim$ or $j \in$ succ. If $j \in \lim$ then $Q^\alpha(j)$ is in $V^{P^\alpha_j}$ a c.c.c. poset of cardinality $\leq \aleph_j$ (for definiteness). The choice of $Q^\alpha(j)$ for $j \in \lim$ is determined by some bookkeeping function which we do not specify now, the aim of which is to obtain Martin's Axiom in $V^{P^0_\kappa}$.

If j is a successor ordinal of the form $j = \delta + i$ where δ is limit and $0 < i < \omega$, we require that $Q^\alpha(j)$ is in $V^{P_j^\alpha}$ either the trivial poset, or $C(\lambda_j, \lambda_j^{++})$ where $\lambda_j = \aleph_{\delta+2i}$ (all in the sense of $V^{P_j^\alpha}$). The exact description of $Q^\alpha(j)$ (i.e., the decision as to whether it is the trivial poset or the one that introduces λ_j^{++} club subsets to λ_j) is not needed to prove that cardinals are not collapsed.

Lemma 4.1. *For every successor α, P_μ^α is λ_α pure closed.*

Proof. The complete component of P_μ^α consists of posets of the form $C(\lambda_j, \lambda_j^{++})$ which are λ_j closed. Since $\lambda_\alpha \leq \lambda_j$ for all these j's, the lemma follows. □

Lemma 4.2. *For every μ such that $\alpha \leq \mu \leq \kappa$, P_μ^α changes no cofinalities and hence preserves cardinals. In fact, this is deduced from the following properties of the mixed support iteration P_μ^α.*

1. *For limit $\mu \leq \kappa$, the cardinality of P_μ^α is $\leq \aleph_\mu^+$, and if $\mu > \alpha$ is inaccessible, then $|P_\mu^\alpha| = \aleph_\mu$.*
2. *If μ is successor, $\mu = j + 1$, then P_μ^α satisfies the λ_j^+-c.c. and its cardinality is $\leq \lambda_j^{++}$ (where $\lambda_j = \aleph_{\delta+2i}$ if $j = \delta + i$ for δ limit and $0 \leq i < \omega$). Thus the GCH continues to hold in $V^{P_\mu^\alpha}$ for λ_j^+ and higher cardinals.*
3. *For each i such that $\alpha < i < \mu$ $P_\mu^\alpha \cong P_i^\alpha * (P_\mu^i)^{V'}$ where V' is $V^{P_i^\alpha}$.*

Proof. Let us see first how 1,2,3 are used to show by induction that P_μ^α preserves cofinalities. So let $g : \eta \to \sigma$ be a cofinal function in $V^{P_\mu^\alpha}$ where η is a regular cardinal. We have to show that $cf(\sigma) \leq \eta$ in V as well. Assume first $\mu = j + 1$ is a successor ordinal, and then $P_\mu^\alpha \cong P_j^\alpha * Q^\alpha(j)$. The case $j \in \lim$ is obvious since $Q^\alpha(j)$ is then a c.c.c. poset. So assume that j is a successor ordinal now, and λ_j is thus defined. The case $\lambda_j \leq \eta$ follows from the λ_j^+-c.c of P_μ^α. In case $\lambda_j > \eta$ use the λ_j completeness of $Q^\alpha(j)$ and induction.

Now assume that μ is limit. The proof divides into two cases. Suppose, for some successor j with $\alpha \leq j < \mu$, $\eta < \lambda_j$. Then $P_\mu^\alpha \cong P_j^\alpha * (P_\mu^j)^{V'}$ where V' is $V^{P_j^\alpha}$. Lemma 3.5 was formulated for quite a general mixed support iteration, and it can be applied in V' to P_μ^j to yield that the function g has a countable approximation in V'. We may then apply the inductive hypothesis.

In case $\eta \geq \lambda_j$ for all such j's, $\eta \geq \aleph_\mu$. Apply cardinality or chain condition arguments: It follows in this case that P_μ^α satisfies the η^+-c.c. and hence the cofinality of σ in V is $\leq \eta$.

So now we prove the three properties by induction on μ. The proof of *1* and *2* are fairly standard, and uses, besides the definition of the Easton support, the inductive assumptions and the restrictions on the cardinalities of the posets.

To prove *3*, we shall define a map $f \mapsto \langle f \restriction i, f/i \rangle$ of P_μ^α into $P_i^\alpha * (P_\mu^i)^{V'}$ as follows. Clearly, $f \restriction i \in P_i^\alpha$. To define the name f/i in $V^{P_i^\alpha}$, we assume a V generic filter, G, over P_i^α, place ourselves in $V[G]$, and define the function $(f/i)[G]$ which interprets f/i (for every $\xi \in \mathrm{dom}(f)$, $f/i[G](\xi)$ is a name in $(P_\xi^i)^{V[G]}$ naturally defined). Let us check that this map is onto a dense subset of the two-step iteration. So let $\langle h, \tau \rangle \in P_i^\alpha * (P_\mu^i)^{V'}$. By extending h we may assume that h 'knows' the finite domain of the c.c.c. component of τ. That is, for some finite set $E_0 \subseteq \mu$, $h \Vdash_i$ dom $(\tau) \cap$ lim $= E_0$. Let $E_1 = \{\eta \in \text{succ} \mid$ some extension of h in P^α_i forces $\eta \in \text{dom}(\tau)\}$. Because the cardinality of P^α_i is $< \aleph_{i+\omega}$, E_1 is bounded below inaccessible cardinals, and can serve as Easton support of a condition. □

5 The proof of the theorem

All the technical machinery is assembled, and we only have to apply it. The iteration has the form P^0_κ and the definition of the function h that decides the value of $Q(j)$ is made so that Martin's Axiom holds in $V^{P^0_\kappa}$, and for every real r in $V^{P^0_\kappa}$ there is a unique limit ordinal $\delta(r)$ such that

$$i \in r \text{ iff for } j = \delta(r) + i + 1,\ Q(j) \text{ is } C(\lambda_j, \lambda_j^{++}).$$

Lemma 5.1. *For every successor $j < \kappa$, $Q(j)$ is $C(\lambda_j, \lambda^{++}_j)$ iff the club filter on λ_j in $V^{P^0_\kappa}$ has generating number λ^{++}_j .*

To prove the lemma, observe that any function $f : \lambda_j \to On$ has a countable approximation in P^0_{j+1}. This is so by Lemmas 4.1 and 3.5, because $P^0_\kappa = P^0_{j+1} * (P^{j+1}_\kappa)^{V'}$, and P^{j+1}_κ is in V' a mixed support iteration of c.c.c. and λ_{j+1}-closed posets. So every club subset of λ_j in $V^{P^0_\kappa}$ contains a club in $P^0_j * Q(j)$, and then the generating number of λ_j in $V^{P^0_\kappa}$ and $V^{P^0_{j+1}}$ are the same. But in $V^{P^0_j}$, $2^{\lambda_j} = \lambda^+_j$ (by Lemma 4.2(2)), and hence the generating number in $V^{P^0_\kappa}$ is determined in $P^0_j * Q(j)$ as follows. If $Q(j)$ is trivial, then the generating number remains λ^+_j, and if $Q(j)$ is $C(\lambda_j, \lambda^{++}_j)$, then the generating number is λ^{++}_j of course.

The definition of the well-ordering of $\mathbb{R}$ in $V^{P^0_\kappa}$ is now clear: $r_1 \prec r_2$ iff $\delta(r_1) < \delta(r_2)$. Why is $\prec$ a Σ^2_1 relation? The answer was outlined in Sect. 2, and now more details are given.

The "almost disjoint sets encoding technique" was introduced in Jensen and Solovay [3], and the reader can find there a detailed exposition; we only give an outline. Assume μ is a cardinal, and $s = \langle s_\xi | \xi < \mu \rangle$ a collection of pairwise almost disjoint subsets of ω. Let $X \subseteq \mu$ be any subset. Then the following c.c.c. poset P introduces a real $a \subseteq \omega$ such that, together with s, a encodes X. In fact, $\xi \in X$ iff $s_\xi \cap a$ is finite.

A condition $(e, c) \in P$ is a pair such that e is a finite partial function from ω to 2, and $c \subseteq X$ is finite. The order relation expresses the intuition that e gives finite information on a, and c is a promise that for $\xi \in c$ the generic subset will not add any more members of $a \cap s_\xi$. So (e_1, c_1) extends (e_2, c_2) iff $e_2 \subseteq e_1$, $c_2 \subseteq c_1$, and for $\xi \in c_2$, $s_\xi \cap E_1 \subseteq E_2$ (where $E_i = \{k | e_i(k) = 1\}$).

The intuitive meaning of this order relation becomes clear by the following definition. Let $G \subseteq P$ be generic; then set

$$a = \{k | e(k) = 1 \text{ for some } (e, c) \in G\}.$$

It can be seen that, $a \cap s_\xi$ is finite for $\xi \in X$, and is infinite for $\xi \notin X$.

This almost disjoint set encoding is used to prove that the Σ^2_1 definition given in Sect. 2 is really equivalent to the well ordering $\prec$. The main point is this. Suppose Martin's Axiom $+ 2^{\aleph_0} = \kappa$, and M is a transitive model of some part of ZFC containing all the reals and a well-order of them (which is a class in M). Then M contains all the bounded subsets of κ as well. Why? Well, let $X \subseteq \mu < \kappa$ be any bounded set. Since M contains a set of μ reals, it also contains a sequence of μ pairwise almost disjoint subsets of ω (taken, for example, as branches of $2^{\overset{\omega}{\smile}}$). By Martin's Axiom, there is a set $a \subseteq \omega$ that encodes X. As $a \in M$, $X \in M$ as well.

6 A weakening of the GCH assumption

The theorem required GCH (below κ) to ensure that cardinal are not collapsed. In this section this assumption is weakened somewhat in demanding that $2^\mu = \mu^+$ only on some closed unbounded set of cardinals $\mu < \kappa$.

To see this, let $\langle \mu_i \mid i < \kappa \rangle$ be an enumeration of a club set of limit cardinals, such that $2^{\mu_i} = \mu_i^+$, and $\mathrm{cf}(\mu_{i+1}) > \mu_i^+$, and $(\mu_{i+1})^{\leq \mu_i} = \mu_{i+1}$.

The construction is basically the same as before, but μ_i replaces λ_i and the main point is this: For a successor $j = \delta + i$, where $\delta < \kappa$ is limit and $0 < i < \omega$, $P_{i+1} = P_i * Q(j)$ where $Q(j)$ is now a poset that adds either μ_j or μ_{j+1} subsets to μ_j^+. Now if M is as before a transitive model that contains all the reals, then the club sequence can be reconstructed by asking the questions about the generating numbers. If one starts with μ_0, then the original sequence is reconstructed; starting with another cardinal may result in another club. However, this club intersects the original sequence of the μ_i's, and hence both sequences have an equal end-section. Hence we must demand that the well-ordering of $\mathbb{R}$ is determined by any end section of the club.

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