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BOREL ORDERINGS

LEO HARRINGTON, DAVID MARKER AND SAHARON SHELAH

ABSTRACT. We show that any Borel linear order can be embedded in an order preserving way into 2^α for some countable ordinal α and that any thin Borel partial order can be written as a union of countably many Borel chains.

0. Introduction. We say that $\langle \mathbf{X}, \leq\rangle$ is a Borel order if $\mathbf{X}$ and $\leq$ are Borel subsets of $\mathbf{R}$ and $\mathbf{R}^2$ respectively and $\leq$ is reflexive and transitive. Friedman [**F**] initiated the study of Borel orders, suggesting that this was an interesting class of uncountable orderings which avoids many of the pathologies of the uncountable. A number of results in this direction were proved in [**F**, **H-S** and **S**]. In this paper we will give several structure theorems for Borel orderings which, in some sense, explain why pathologies are avoided.

$\langle \mathbf{X}, \leq\rangle$ is said to be *thin* if there is no perfect set of pairwise incomparable elements. Orderings that are thin admit reasonable structure theorems. There is little to say about nonthin orderings. The main results are

THEOREM 3.1. *If $\langle X, \leq\rangle$ is a thin Borel order, then for some $\alpha < \omega_1$ there is a Borel $f\colon \mathbf{X} \to 2^\alpha$ order preserving (where 2^α is ordered lexicographically).*

THEOREM 5.1. *If $\langle X, \leq\rangle$ is a thin Borel order, then $\mathbf{X}$ can be written as a countable union of Borel chains.*

These results were proved by the first and third authors. The second author added the results of §4.

To prove these results it is useful to look instead at the lightface (i.e. Δ^1_1) refinements. The proofs rely on the interplay of effective descriptive set theory and forcing. The tools we need are developed in §1 and §2. Other descriptive set-theoretic facts can be found in Moschovakis [**M**]. As usual in descriptive set theory we will interchange $\mathbf{R}$ and ω^ω whenever convenient.

Notationally, if $\leq$ is an order, $x \approx y$ if $x \leq y$ and $y \leq x$ and $x|y$ if x and y are incomparable. Lower case latin letters usually denote reals (with e, m, n, i, j reserved for integers). Upper case latin letters denote sets of reals and script letters denote sets of sets of reals.

1. The reflection lemmas. Let $W_0, W_1, W_2, \ldots$ be the usual enumeration of r.e. sets. Let $U \subseteq \omega \times \omega^\omega = \{(e,x)\colon \forall y\ \exists n\ (x|n, y|n, n) \in W_e\}$. Then U is ω-universal Π^1_1. Let $U_e = \{x\colon (e,x) \in U\}$.

DEFINITION 1.1. If $\mathscr{A} \subset \mathscr{P}(\mathbf{R})$ we say $\mathscr{A}$ is Π^1_1 on Π^1_1 if $\{e\colon U_e \in \mathscr{A}\}$ is Π^1_1.

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LEMMA 1.2 (REFLECTION). *If $\mathscr{A}$ is Π^1_1 on Π^1_1 and $Y \in \mathscr{A}$ is Π^1_1, then there is a Δ^1_1 $X \subseteq Y$ such that $X \in \mathscr{A}$.*

PROOF. Let $\Psi\colon U \to \omega_1$ be a Π^1_1-norm. Let n, $m \in \omega$ such that $\{e\colon U_e \in \mathscr{A}\} = U_n$ and $Y = U_m$. For $e \in \omega$ let $V_e = \{y \in Y\colon \Psi(m,y) < \Psi(n,e)\}$ (i.e.: $y \in V_e \Leftrightarrow y$ gets into Y, before U_e gets into $\mathscr{A}$). Each V_e is Π^1_1 and we can easily find a recursive function f such that $V_e = U_{f(e)}$. By the recursion theorem there is an $\hat{e} \in \omega$ s.t. $W_{\hat{e}} = W_{f(\hat{e})}$. Then by the definition of U and f, $V_{\hat{e}} = U_{f(\hat{e})} = U_{\hat{e}}$.

If $U_{\hat{e}} \notin \mathscr{A}$, then $\Psi(n,\hat{e}) = \infty$ and $V_{\hat{e}} = Y$. But $Y \in \mathscr{A}$ and $V_{\hat{e}} = U_{\hat{e}}$, so this is a contradiction and $U_{\hat{e}} \in \mathscr{A}$. If $\alpha = \Psi(n,\hat{e}) < \omega_1^{ck}$, then $V_{\hat{e}} = \{y\colon \Psi(m,y) < \alpha\}$ and this set is Δ^1_1. Thus $U_{\hat{e}}$ is the desired set. $\square$

DEFINITION 1.3. Let $\mathscr{A} \subset \mathscr{P}(\mathbf{R})^2$. We say $\mathscr{A}$ is Π^1_1 on Π^1_1 if $\{(e,e')\colon \mathscr{A}(U_e, U_{e'})\}$ is Π^1_1. $\mathscr{A}$ is *monotonic upward* if whenever $\mathscr{A}(Y,Z)$, $Y \subset Y'$ and $Z \subset Z'$, then $\mathscr{A}(Y'Z')$. We say $\mathscr{A}$ is *continuous downward* if whenever we have $Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$ and $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$ such that for all n $\mathscr{A}(Y_n, Z_n)$, then $\mathscr{A}(\bigcap Y_n, \bigcap Z_n)$.

LEMMA 1.4 (STRONG REFLECTION). *If $\mathscr{A} \subseteq \mathscr{P}(\mathbf{R})^2$ is Π^1_1 on Π^1_1, monotonic upward and continous downward, then if $Y \in \Pi^1_1$ and $\mathscr{A}(Y, \neg Y)$, there is a Δ^1_1 $X \subseteq Y$ such that $\mathscr{A}(X, \neg X)$.*

PROOF.

Claim. If $X \subseteq Y$ is Δ^1_1, there is Δ^1_1 $\hat{X} \supseteq X$ such that $\mathscr{A}(\hat{X}, \neg X)$ and $\hat{X} \subseteq Y$.

Consider $B(Z) = \{Z\colon A(Z, \neg X)$ and $X \subseteq Z\}$. Then B is Π^1_1 on Π^1_1. Since $\mathscr{A}$ is monotonic upward and $\neg Y \subseteq \neg X, B(Y)$. Thus by reflection there is a Δ^1_1 $\hat{X} \supseteq X$ such that $\mathscr{A}(\hat{X}, \neg X)$ and $\hat{X} \subseteq Y$.

Let X_0 be any Δ^1_1 subset of Y. Given X_n let $X_{n+1} \supseteq X_n$ be Δ^1_1 such that $\mathscr{A}(X_{n+1}, \neg X_n)$ and $X_{n+1} \subseteq Y$. The procedure for going from X_n to X_{n+1} is uniform so $\langle X_n\colon n \in \omega\rangle$ is Δ^1_1. Let $X = \bigcup X_n$. Then X is Δ^1_1 and $X \subseteq Y$. Since $\mathscr{A}(X_{n+1}, \neg X_n)$. By monotonicity for all n $\mathscr{A}(X, \neg X_n)$. Thus by continuity $\mathscr{A}(X, \bigcap \neg X_n)$. But $\neg X = \bigcap \neg X_n$, so $\mathscr{A}(X, \neg X)$. $\square$

There is a natural way for $\mathscr{A}$'s that satisfy the hypothesis of strong reflection to arise. Let $P(\overline{x}, \overline{y})$ be Π^1_1. Let $\mathscr{A}(X,Y) \Leftrightarrow \forall \overline{x} \notin X\ \forall \overline{y} \notin Y\ P(\overline{x}, \overline{y})$. Clearly $\mathscr{A}$ is Π^1_1 on Π^1_1, monotonic upward and continuous downward. The following corollary gives the flavor of applications of strong reflection.

COROLLARY 1.5. *If X is Σ^1_1 and linearly ordered by $\leq$ a Δ^1_1 ordering, then there is a Δ^1_1 $Y \supseteq X$ such that Y is linearly ordered by $\leq$.*

PROOF. Let $\mathscr{A}(Y,Z) \Leftrightarrow \forall x_0, x_1, \notin Y\ x_0 \leq x_1 \vee x_1 \leq x_0$.

Then $\mathscr{A}$ satisfies the hypothesis of strong reflection and $\mathscr{A}(\neg X, X)$. So by strong reflection there is a Δ^1_1 $Z \subseteq \neg X$ such that $\mathscr{A}(Z, \neg Z)$. Let $Y = \neg Z$. $\square$

The reflection lemmas provide a uniform treatment for a number of results which otherwise would be proved by an ad hoc mix of Σ^1_1-separation and Kreisel uniformization arguments.

2. Gandy forcing. Let $\mathbf{P} = \{A \in \Sigma^1_1\colon A \neq \varnothing\}$. We order $\mathbf{P}$ by inclusion (i.e. $A \leq B$ iff $A \subseteq B$). This notion of forcing was used by Harrington [**H**] to give a new proof of Silver's theorem on $\mathbf{\Pi}^1_1$ equivalence relations.

LEMMA 2.1. *If $G \subseteq \mathbf{P}$ is generic, then $\bigcap\{A\colon A \in G\} = \{a\}$ for some $a \in \mathbf{R}$.*

Let $A \in G$ say $x \in A \Leftrightarrow \exists f\ R(f,x)$ where R is Π^0_1. For $\sigma, \eta \in \omega^\omega$ let $A_{\sigma,\eta} = \{x \supset \sigma\colon \exists f \supseteq \eta\ R(f,x)\}$. For each n we can find $\sigma_n \eta_n \in \omega^n$ s.t. $A_{\sigma_n,\eta_n} \in G$ and $\sigma_n \subset \sigma_{n+1}$, $\tau_n \subset \tau_{n+1}$. Thus if $x = \lim_n \sigma_n$, and $f = \lim_n \eta_n$, then $R(f,x)$. For each n $\{y\colon y|n = x|n\} \in G$. Thus $\{x\} = \cap G$. □

If $b \in \mathbf{R}$ we can view b as coding a pair of reals $\langle b_0, b_1\rangle$.

LEMMA 2.2. *If b is $\mathbf{P}$-generic, then both b_0 and b_1 are $\mathbf{P}$-generic.*

PROOF. We show b_0 is $\mathbf{P}$-generic. Let $B \in \mathbf{P}$ and let $\mathscr{D} \subseteq \mathbf{P}$ be dense open. Let $B' = \{x\colon \exists y\ (x,y) \in B\}$, since $B' \in \mathbf{P}$ there is $C' \subseteq B'$ s.t. $C' \in \mathscr{D}$. Let $C = \{(x,y)\colon x \in C' \wedge (x,y) \in B\}$. Then $C \neq \emptyset$, $C \subseteq B$ and $c \Vdash \overset{\circ}{b} \in C'$. Thus $\mathscr{D}' = \{C \in \mathbf{P}\colon$ for some $C' \in \mathscr{D}$, $C \Vdash \overset{\circ}{b}_0 \in C'\}$ is dense, so b_0 is $\mathbf{P}$-generic. □

In our main results we will be dealing with modified products of Gandy forcing.

DEFINITION 2.3. $\mathbf{P}^n = \{A \subseteq \mathbf{R}^n\colon A$ is Σ^1_1 and $A \neq \emptyset\}$. Let E be a Σ^1_1 equivalence relation. Let $\mathbf{P}^n_E = \{A \subseteq \mathbf{R}^n\colon A$ is Σ^1_1 and if $(a_1,\dots,a_n) \in A$, then $\forall i,j \leq n\ a_i E a_j\}$. Let $\mathbf{P} \times_E \mathbf{P} = \{(Y,A) \in \mathbf{P} \times \mathbf{P}\colon (Y \times A) \cap E \neq \emptyset\}$ and $\mathbf{P}^2_E \times_E \mathbf{P}^2_E = \{(Y,A) \in \mathbf{P}^2_E \times \mathbf{P}^2_E\colon \exists (y_0,y_1) \in Y\ \exists (a_0,a_1) \in A\ y_0 E y_1 E a_0 E a_1\}$. Each of these sets is ordered by inclusion.

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LEMMA 2.4. *If a, b are $\mathbf{P} \times_E \mathbf{P}$ generic, then a and b are $\mathbf{P}$-generic.*

PROOF. Let $\mathscr{D} \subseteq \mathbf{P}$ be dense open, let $\langle A, B\rangle \in \mathbf{P} \times_E \mathbf{P}$. Let $A' = \{x \in A\colon \exists y \in B\ xEy\}$. Let $C \in \mathscr{D}$, $C \subseteq A'$. Then $\langle C, B\rangle \in \mathbf{P} \times_E \mathbf{P}$. Thus $\mathscr{D}' = \{\langle C, X\rangle\colon C \in \mathscr{D}\}$ is dense in $\mathbf{P} \times_E \mathbf{P}$. So a is $\mathbf{P}$-generic. □

LEMMA 2.5. *If (a_0,a_1), (b_0,b_1) are $\mathbf{P}^2_E \times_E \mathbf{P}^2_E$ generic, then each pair (a_i,b_j) is $\mathbf{P} \times_E \mathbf{P}$ generic.*

PROOF. We will prove this for a_0, b_0. Let $\mathscr{D} \subseteq \mathbf{P} \times_E \mathbf{P}$ be dense. Let $\langle A, B\rangle \in \mathbf{P}^2_E \times_E \mathbf{P}^2_E$. Let $\overline{A} = \{x\colon \exists y\ xEy \wedge (x,y) \in A\}$, $\overline{B} = \{x\colon \exists y\ xEy$ and $(x,y) \in B\}$. Then $\langle \overline{A}, \overline{B}\rangle \in \mathbf{P} \times_E \mathbf{P}$. Let $\langle C, D\rangle \leq \langle \overline{A}, \overline{B}\rangle$, $\langle C, D\rangle \in \mathscr{D}$. Let $\hat{C} = \{(x,y) \in A\colon xEy \wedge x \in C\}$, $\hat{D} = \{(x,y) \in B\colon xEy \wedge x \in D\}$. Since $\exists x_0 \in C\ \exists x_1 \in D\ x_0 E x_1$, $x_0 \in \overline{A}$ and $x_1 \in \overline{B}$, $\langle \hat{C}, \hat{D}\rangle \in \mathbf{P}^2_E \times_E \mathbf{P}^2_E$.

Thus $\mathscr{D}' = \{\langle X, Y\rangle \in \mathbf{P}^2 \times \mathbf{P}^2_E\colon \langle \{x\colon \exists y (x,y) \in X\}, \{x\colon \exists y\colon (x,y) \in Y\}\rangle \in \mathscr{D}\}$ is dense. □

We would like to have that if b_0, b_1 are $\mathbf{P} \times_E \mathbf{P}$ generic then $b_0 E b_1$. This seems unlikely in general but is true for an important class of equivalence relations E.

DEFINITION 2.6. We say E is *representable* if for every x, y if $x \not\!E y$ then there is Δ^1_1 set X such that $x \in X$, $y \notin X$ and $\forall x \in X\ \forall y \notin X\ x \not\!E y$.

LEMMA 2.7. *If E is representable and a, b are $\mathbf{P} \times_E \mathbf{P}$ generic then aEb.*

PROOF. Suppose not. Let $\langle A, B\rangle \Vdash \mathring{a} \not\!E \mathring{b}$. There is a Δ^1_1 set X such that $a \in X$, $b \notin X$ and for all $x \in X$, $y \notin X$ $x \not\!E y$. Let $\hat{A} = A \cap X$, $\hat{B} = B - X$. Since a, b are $\mathbf{P}$-generic, by Lemma 2.1 $\langle \hat{A}, \hat{B}\rangle \in \mathbf{P} \times_E \mathbf{P}$. But clearly $\langle \hat{A}, \hat{B}\rangle \notin \mathbf{P} \times_E \mathbf{P}$. □

In applications we will be interested in finding perfect sets of mutually generic reals. To obtain $\mathbf{P} \times \mathbf{P}$ generic reals this is done much as it would be in Cohen forcing. The only difficulty arises if we reach a condition A which is countable. By the effective perfect set theorem if $A \in \Sigma^1_1$ and A is countable, then A is a set of hyperarithmetic reals. In general we will only be doing forcing below conditions which do not contain any hyperarithmetic reals.

LEMMA 2.8. *If $A \in \mathbf{P}$ and $|A| > \aleph_0$, then we can add a perfect set of mutually $\mathbf{P}$-generic reals in A.*

PROOF. Let $\overline{A} = \{x \in A \colon x \notin \Delta^1_1\}$. Then $\overline{A} \in \mathbf{P}$ and we can do the usual arguments below $\overline{A}$. □

Obtaining perfect sets of mutually $\mathbf{P} \times_E \mathbf{P}$ generic reals is somewhat more delicate. We prove only the special case we need.

LEMMA 2.9. *Let $\leq$ be a Δ^1_1 ordering on $\mathbf{R}$. Suppose for each $A \in \mathbf{P}$ we have a Σ^1_1-equivalence relation E_A such that if $B \subseteq A$, then $xE_By \Rightarrow xE_Ay$. Suppose there is a Y s.t. for all $B \subseteq Y$, $B \in \mathbf{P}$ there are b_0, $b_1 \in B$ such that $\langle b_0, b_1\rangle$ are $\mathbf{P} \times_{E_B} \mathbf{P}$ generic and b_0 and b_1 are incomparable. Then there is a perfect subset of Y of pairwise incomparable elements.*

PROOF. Let $\hat{\mathbf{P}}$ be the set of finite functions $\varphi \colon t \to \{B \in \mathbf{P} \colon B \subseteq Y\}$ s.t.

(i) $t \subseteq 2^\omega$ is a finite tree,

(ii) $\varphi(\varnothing) = Y$,

(iii) if $\eta \subseteq \tau \in t$, then $\varphi(\eta) \supseteq \varphi(\tau)$,

(iv) for $\eta^\wedge 0$, $\eta^\wedge 1 \in t$ $\langle \varphi(\eta^\wedge 0), \varphi(\eta^\wedge 1)\rangle \Vdash_{E_{\varphi(\eta)}} \mathring{y}_0$ incomparable to $\mathring{y}_1$.

(v) Let $L_t = \{\eta \in t \colon \forall \tau \supset \eta\ \tau \notin t\}$, then there is $\langle a_\eta \colon \eta \in L_t\rangle$ s.t. $a_\eta \in \varphi(\eta)$ and if $\eta, \tau \in L_t$, then $a_\eta E_{\varphi(\eta \wedge \tau)} a_\tau$ ($\eta \wedge \tau = \sigma$ where $\sigma \subseteq \eta, \tau$ and $\eta(|\sigma|) \neq \tau(|\sigma|)$). Order $\hat{\mathbf{P}}$ by extension. Suppose $\Psi \colon T \to \mathbf{P}$ is generic. Let $\hat{\Psi} \colon [T] \to \mathbf{R}$ by $\hat{\Psi}(f) = \bigcap\{\Psi(f|n) \colon n \in \omega\}$ it is routine to see that if $f, g \in [T]$, then $\hat{\Psi}(f)$ and $\hat{\Psi}(g)$ are $\mathbf{P} \times_{E_{\psi(f \wedge g)}} \mathbf{P}$ generic and hence incomparable. We need only show that T is perfect.

Suppose $\varphi \colon t \to \mathbf{P}$ is in $\hat{\mathbf{P}}$ and $\eta \in L_t$. Let $A_\eta = \{x \in \varphi(\eta) \colon \exists \langle b_\sigma\rangle_{\sigma \in L_t}\ b_\eta = x$ and $b_\sigma E_{\varphi(\sigma \wedge \tau)} b_\tau$ for all $\sigma, \tau \in L_t\}$. Let B_0, $B_1 \subseteq A_\eta$, B_0, $B_1 \in \mathbf{P}$ s.t. $\langle B_0, B_1\rangle \Vdash \mathring{y}_0$ and $\mathring{y}_1$ are incomparable. Let $\hat{\varphi} \colon t \cup \{\eta^\wedge 0, \eta^\wedge 1\} \to \mathbf{P}$ s.t. $\hat{\varphi} \supseteq \varphi$ and $\hat{\varphi}(\eta^\wedge i) = B_i$. Then $\hat{\varphi} \in \hat{\mathbf{P}}$. Thus if $\Psi \colon T \to \mathbf{P}$ is generic, T is perfect.

3. Representing Δ^1_1 orderings.

THEOREM 3.1. *If* $\langle \mathbf{X} \leq\rangle$ *is a thin* Δ^1_1 *order, then there is* $\alpha < \omega_1^{ck}$ *and a* Δ^1_1 *order preserving* $f\colon \mathbf{X} \to 2^\alpha$.

PROOF. Without loss of generality $\mathbf{X} = \mathbf{R}$.

Let $\mathscr{F} = \{f\colon \mathbf{R} \to 2^\alpha\colon \alpha < \omega_1^{ck}, f \in \Delta^1_1 \text{ and } x \leq y \Rightarrow f(x) \leq f(y)\}$.

Claim 1. $\mathscr{F}$ is Π^1_1.

$n = \langle n_0, n_1\rangle$ codes an element of $\mathscr{F}$,

(1) n_0 codes a recursive ordinal α,

(2) n_1 is a Δ^1_1 code for a subset G of $\mathbf{R} \times 2^\alpha$,

(3) $\forall x \exists y \in \Delta^1_1(x)\ (G(x,y) \wedge \forall z (G(x,z) \to z = y))$ and

(4) $\forall x,y,z,w\ ((x \leq y \wedge G(x,z) \wedge G(y,w)) \to x \leq w$ [in 2^α]).

Since the quantifier $\exists y \in \Delta^1_1(x)$ is really universal, this is easily seen to be Π^1_1.

For $F \in \mathscr{F}$ we define an equivalence relation E_F by xE_Fy if and only if $F(x) = F(y)$. Let $E = \bigcap\{E_F\colon F \in \mathscr{F}\}$. Then $xEy \Leftrightarrow \forall F \in \Delta^1_1\ (F \in \mathscr{F} \Rightarrow F(x) = F(y))$. Since $\mathscr{F}$ is Π^1_1 and $\forall F \in \Delta^1_1$ is really existential, E is a Σ^1_1-equivalence relation.

Our main claim is that $E =\approx$. Before establishing the main claim we will show how to establish the theorem from it. For the moment assume $E =\approx$.

Consider the following $\mathscr{A} \subseteq \mathscr{P}(\mathbf{R})$, $\mathscr{A}(X) \Leftrightarrow \forall x,y\ (x \not\approx y \to \exists F \in \Delta^1_1 F \in X$, $F \in \mathscr{F}$ and $F(x) \neq F(y))$. Then $\mathscr{A}$ is Π^1_1 on Π^1_1 and $\mathscr{A}(\mathscr{F})$ so by reflection there is a Δ^1_1 $X \subseteq \mathscr{F}$ s.t. $\forall x,y\ x \not\approx y \to \exists F \in X\ F(x) \neq F(y))$.

Let F_0, F_1, $F_2, \ldots$ be a Δ^1_1 enumeration of X. Define F_0^*, $F_1^*, \ldots$ as follows.

(i) $F_0^* = F_0$.

(ii) Suppose $F_n^*\colon \mathbf{R} \to 2^\gamma$ and $F_{n+1}\colon \mathbf{R} \to 2^\alpha$. Define $F_{n+1}^*\colon \mathbf{R} \to 2^{\alpha+\gamma}$ by $F_{n+1}^*(x) = F_n^*(X)^\frown F_{n+1}(x)$.

It is easily seen that F_{n+1}^* is order preserving and if $F_i(x) \neq F_i(y)$ for any $i \leq n+1$, then $F_{n+1}^*(x) \neq F_{n+1}^*(y)$. Let $F^* = \lim_n F_n^*$. Then F^* is Δ^1_1 (note: By boundedness we can find $\delta < \omega_1^{ck}$ s.t. for all n $F_n\colon \mathbf{R} \to 2^\alpha$ for some $\alpha < \delta$. Thus $F^*\colon \mathbf{R} \to 2^\alpha$ for some $\alpha \leq \delta \cdot \omega$) and if $x \not\approx y$, $F^*(x) \neq F^*(y)$. Thus F is the desired function. Thus we need only establish that $E =\approx$.

Let $Z = \{x\colon \exists y\ xEy \wedge x \not\approx y\}$. Clearly Z is Σ^1_1. If $Z = \varnothing$, then $E =\approx$, so we assume for purposes of contradiction that $Z \neq \varnothing$. Let $\mathbf{P} = \{A \in \Sigma^1_1\colon A \neq \varnothing$ and $A \subseteq Z\}$.

Claim 2. If a and b are $\mathbf{P} \times_E \mathbf{P}$ generic, then it is not the case that $a < b$ (or symmetrically it is not the case that $b < a$.).

Let $\langle A, B\rangle \in \mathbf{P} \times_E \mathbf{P}$ s.t. $\langle A, B\rangle \Vdash \mathring{a} < \mathring{b}$.

Case 1. $\forall a \in A\ \forall b \in B\ (aEb \Rightarrow a \not\geq b)$.

Let $A_0 = \{x\colon \exists y \in A\ yEx \wedge y \geq x\}$. Let $B_0 = \{x\colon \exists y \in B\ yEx \wedge y \leq x\}$. Now A_0 and B_0 are Σ^1_1. A_0 is the downward closure of A in each E class and B_0 is the upward closure of B in each E-class. Clearly A_0 and B_0 are disjoint. Let $\mathscr{A}(X,Y) \Leftrightarrow \forall x,y\ (x \notin X \wedge y \notin Y \to x \not{E} y \vee y \not\leq x) \wedge \forall z\ (z \notin X \to z \notin B_0)$. Then $\mathscr{A}$ satisfies the requirements for strong reflection and $\mathscr{A}(\neg A_0, A_0)$. Thus there is a Δ^1_1 $C \supseteq A_0$ s.t. $\mathscr{A}(\neg C, C)$. Thus C is downward closed and disjoint from B_0.

Subclaim. There is $F^* \in \mathscr{F}$ s.t. $\forall x,y\ ((xE_F * y \wedge y \leq x \wedge x \in C) \to y \in C)$.

Let $\mathscr{A}(X) \Leftrightarrow \forall x,y\ ((\exists f \in \Delta^1_1 f \in X \wedge f(x) \neq f(y)) \vee y \not\leq x \vee x \notin C \vee y \in C)$. Then $\mathscr{A}$ is Π^1_1 on Π^1_1 and $\mathscr{A}(\mathscr{F})$. Thus by reflection there is a Δ^1_1 $X \subseteq \mathscr{F}$ s.t.

$\mathscr{A}(X)$. As above we can build $F^* \in \mathscr{F}$ such that $xE_{F^*}y$ iff for all $F \in X$ xE_Fy. So $\forall x, y\ ((xE_{F^*}y \wedge y \leq x \wedge x \in C) \rightarrow y \in C)$.

Suppose $F^*\mathbf{R} \rightarrow 2^\alpha$. Let $G\colon \mathbf{R} \rightarrow 2^{\alpha+1}$ by

$$G(X) = \begin{cases} F^*(x)^\wedge 0, & x \in C, \\ F^*(x)^\wedge 1, & x \notin C. \end{cases}$$

It is easy to see that $G \in \mathscr{F}$ and if $xE_{F^*}y$, $x \in C$ and $y \notin C$, then $x \not{E}_G y$. In particular if $a \in A$ and $b \in B$, then $a \not{E} b$, contradicting the assumption that $\langle A, B\rangle \in \mathbf{P} \times_E \mathbf{P}$.

Case 2. There are $a \in A$, $b \in B$ s.t. aEb and $b \leq a$.

Let $D = \{(a, b)\colon a \in A,\ b \in B,\ aEb \text{ and } b \leq a\} \in \mathbf{P}^2_E$. Let (a_1, b_1), (a_2, b_2) be $\mathbf{P}^2_E \times_E \mathbf{P}^2_E$ generic such that (a_1, b_1), $(a_2, b_2) \in D$. By 2.5 (a_1, b_2) and (a_2, b_1) are each $\mathbf{P} \times_E \mathbf{P}$ generic, so since $a_i \in A$, $b_i \in B$, $a_1 < b_2$ and $a_2 < b_1$. But $a_1 \geq b_1$ and $a_2 \geq b_2$ so $a_1 < a_1$ a contradiction.

This completes the proof of Claim 2.

Claim 3. If a and b are $\mathbf{P} \times_E \mathbf{P}$ generic, then it is not the case that $a \approx b$.

Suppose $\langle A, B\rangle \in \mathbf{P} \times_E \mathbf{P}$ and $\langle A, B\rangle \Vdash \mathring{a} \approx \mathring{b}$.

Case 1. $\exists a \in A$, $b \in B$ $aEb \wedge a < b$ (or $\exists a \in A$ $b \in B$ $aEb \wedge b < a$).

Let $D = \{(a, b)\colon a \in A,\ b \in B,\ aEb \text{ and } a < b\}$. Let (a_0, b_0), (a_1, b_1) be $\mathbf{P}^2_E \times_E \mathbf{P}^2_E$ generic with both generic filters containing D. Then by 2.5 (a_0, b_1) and (a_1, b_0) are $\mathbf{P} \times_E \mathbf{P}$ generic. Thus $a_0 < b_0 \approx a_1 < b_1 \approx a_0$ a contradiction.

Case 2. $\exists a \in A$, $b \in B$ $aEb \wedge a|b$.

Let $D = \{(a, b)\colon a \in A,\ b \in B,\ aEb \text{ and } a|b\}$. Let (a_0, b_0), (a_1, b_1) and (a_2, b_2) be mutually $\mathbf{P}^2_E \times_E \mathbf{P}^2_E$ generic with all three generic filters containing D. Then $a_0|b_0$ and by 2.5 (a_2, b_0), (a_2, b_1) and (a_1, b_1) are $\mathbf{P}\times_E\mathbf{P}$ generic. Thus $b_0 \approx a_2 \approx b_1 \approx a_0$. So we have a contradiction.

Case 3. $\forall a \in A$ $\forall b \in B$ $aEb \Rightarrow a \approx b$.

Let $C = \{x\colon x \in A \wedge \exists y \in B\ xEy\}$. Then C is Σ^1_1 and $\forall x, y \in C\ (xEy \Rightarrow x \approx y)$. Since $A \subseteq Z$, $C \subseteq Z$. Thus $\forall x \in C\ \exists y\ (xEy \wedge x \not\approx y)$. We claim this is impossible.

Subclaim. Let X be Σ^1_1. Suppose $\forall x, y \in X\ (xEy \Rightarrow x \approx y)$. Then $\forall x \in X\ \forall y\ (xEy \Rightarrow x \approx y)$.

Suppose not. Let $B^+ = \{x\colon \exists y \in X\ xEy \wedge (x > y \vee x|y)\}$ and $B^- = \{x\colon \exists y \in X\ xEy \wedge (x < y\ veex|y)\}$. One of these is nonempty assume it is B^+. Let $A_0 = \{x\colon \exists y \in X\ xEy \wedge x \leq y\}$. Since $E/X =\approx$, $A_0 \cap B^+ = \varnothing$. As in Case 1 of Claim 3 by strong reflection we can find a Δ^1_1 C s.t. $A_0 \subseteq C$, $B^+ \cap C = \varnothing$ and C is downward closed in each E class it intersects. By a second reflection argument we can find an $F \in \mathscr{F}$ s.t. if $x \in C \wedge xE_Fy \wedge x \geq y$, then $y \in C$. Letting

$$G(x) = \begin{cases} F(x)^\wedge 0, & x \in C, \\ F(x)^\wedge 1, & x \notin C. \end{cases}$$

We see that if xE_Fy and $x \in A_0$ and $y \in B^+$, then $x \not{E}_G y$, a contradiction. This establishes the subclaim and Claim 3.

Thus if a and b are $\mathbf{P} \times_E \mathbf{P}$ generic, they are incomparable in $\leq$. By 2.9 it is possible to add a perfect set of mutually $\mathbf{P} \times_E \mathbf{P}$ generic elements. But then in the generic extension there is a perfect set of pairwise incomparable elements. This is Σ^1_2 so by Shoenfield absoluteness, there is already a perfect set of pairwise incomparable elements. Thus $\leq$ is not thin, a contradiction.

COROLLARY 3.2 (HARRINGTON-SHELAH [HS]). *If $\leq$ is a thin Borel order, there are no ω_1-chains.*

Theorem 3.1 is similar in spirit to the following result in mathematical economics.

THEOREM 3.3 (DEBREU [D]). *If $(X, \leq)$ is a closed prelinear order then there is a continuous order preserving $f: X \to \mathbf{R}$.*

4. Separable Δ_1^1 orders.

DEFINITION 4.1. We say $(\mathbf{X}, \leq)$ is *separable* iff there is a countable $S \subseteq \mathbf{X}$ such that for any x_0, $x_1 \in \mathbf{X}$ if $x_0 < x_1$, there is $s \in S$ $x_0 < s < x_1$.

If $(\mathbf{X}, \leq)$ is a separable Borel prelinear order, then it is easy to see that there is a order preserving Borel $f: \mathbf{X} \to \mathbf{R}$. The main goal of this section is to prove the effective version of this result.

LEMMA 4.2. *If $(\mathbf{X}, \leq)$ is a Δ_1^1 prelinear order then either*

(i) *there is a perfect set of pairwise disjoint closed intervals $[a, b]$ with $a < b$ or*

(ii) *if $x_0 < x_1$, there is a Δ_1^1 downward closed Y such that $x_0 \in Y$ and $x_1 \notin Y$.*

PROOF. Without loss of generality $\mathbf{X} = \mathbf{R}$. We view each real x as coding a pair $\langle x_0, x_1 \rangle$. Let $A = \{x: x_0 < x_1 \wedge \forall Y \in \Delta_1^1 [\forall z, y ((z \leq y \wedge y \in Y) \to z \in Y)] \to (x_0 \notin Y \vee x_1 \in Y)\}$. The quantifier $\forall Y \in \Delta_1^1$ is essentially existential so A is Σ_1^1. Let $\mathbf{P} = \{B \in \Sigma_1^1: B \neq \emptyset\}$.

Claim 1. If $r \in \mathbf{R}$ and $\mathring{a}$ is a name for a generic real then $A \Vdash \neg(\mathring{a}_0 < r < \mathring{a}_1)$.

Suppose not. Let $B \subseteq A$ such that $B \Vdash \mathring{a}_0 < r < \mathring{a}_1$.

Case 1. There are $b, b' \in B$ s.t. $b_1 \leq b'_0$.

Let $C = \{c: c_0 \in B,\ c_1 \in B \wedge c_{10} \geq c_{01}\}$. Let c be $\mathbf{P}$ generic below C. Then by 2.2 c_0 and c_1 are also $\mathbf{P}$ generic so $c_{00} < r < c_{01}$ and $c_{10} < r < c_{11}$. But $c \in C$ so $c_{01} \leq c_{10}$. Thus $r < r$ a contradiction.

Case 2. For all $b, b' \in B$ $b_0 < b'_1$.

Let $B^- = \{x: \exists y \in B\ x \leq y_0\}$ and $B^+ = \{x: \exists y \in B\ x \geq y_1\}$. Then B^- and B^+ are disjoint Σ_1^1 sets which are downward and upward closed respectively. Let $\mathscr{A}(X, Y) \Leftrightarrow \forall z \notin X\ z \notin B^+ \wedge \forall x \notin X\ \forall y \notin Y\ y \nleq x$. Then $\mathscr{A}$ satisfies the hypothesis for strong reflection and $\mathscr{A}(\neg B^-, B^-)$. Thus by strong reflection there is a Δ_1^1 C such that $B^- \subseteq C$, C is downward closed and $C \cap B^+ = \emptyset$. But then if $b \in B$ $b_0 \in C$ and $b_1 \notin C$. Thus $B \nsubseteq A$ a contradiction.

Claim 2. If $r \in \mathbf{R}$, then $A \Vdash \mathring{a}_0 \neq r$ and $\mathring{a}_1 \neq r$. Suppose $B \in \mathbf{P}$, $B \subseteq A$ and $B \Vdash \mathring{a}_0 = r$. If $b \in B$ and $b_0 \neq r$, choose n such that $b_0(n) \neq r(n)$. Then $B' = \{c \in B: c(n) = b_0(n)\} \in \mathbf{P}$ and $B' \Vdash \mathring{a}_0 = r$. Thus for all $b \in B$ $b_0 = r$. But then $\{r\} = \{x: \exists b \in B\ b_0 = x\}$ is a Σ_1^1 singleton, so $r \in \Delta_1^1$. Then $Y = \{x: x \leq r\}$ is a downward closed Δ_1^1 set separating b_0 from b_1 for $b \in B$, a contradiction.

Claim 3. If a and b are $\mathbf{P} \times \mathbf{P}$ generic, below A then $[a_0, a_1] \cap [b_0, b_1] = \emptyset$.

This is clear from Claims 1 and 2 since b is generic over a.

To prove the lemma we find a perfect set of mutually $\mathbf{P}$ generic reals below A. By Claim 3 these give rise to a perfect set of pairwise disjoint intervals. □

THEOREM 4.3. *If $(\mathbf{X}, \leq)$ is a Δ_1^1 prelinear order, then either*

(i) *there is a perfect set of pairwise disjoint closed intervals $[a, b]$ with $a < b$, or*

(ii) *there is a Δ_1^1 $F: \mathbf{X} \to \mathbf{R}$ order preserving.*

PROOF. Assume (i) fails. Let $\mathscr{F} = \{f : \mathbf{X} \to \mathbf{R} : f$ is Δ^1_1 and $x \leq y \Rightarrow f(x) \leq f(y)\}$. As in 3.1 $\mathscr{F}$ is Π^1_1.

Claim 1. If $x < y$, then there is $f \in \mathscr{F}$ s.t. $f(x) < f(y)$. By Lemma 4.2 there is a downward closed $Y \in \Delta^1_1$ such that $x \in y$ and $y \notin Y$. Let

$$f(z) = \begin{cases} 0, & z \in Y, \\ 1, & z \notin Y. \end{cases}$$

Clearly f separates x and y.

Claim 2. There is a Δ^1_1 $C \subseteq \mathscr{F}$ such that for all $x < y$ there if $f \in C$ such that $f(x) < f(y)$.

Let $\mathscr{A}(Z) \Leftrightarrow \forall x, y \in \mathbf{X}$ $(x < y \to \exists f \in \Delta^1_1$ $f \in Z \wedge f(x) < f(y))$. Clearly $\mathscr{A}$ is Π^1_1 on Π^1_1 and $\mathscr{A}(\mathscr{F})$, so by reflection there is a Δ^1_1 $C \subseteq \mathscr{F}$ such that $\mathscr{A}(C)$.

Let f_0, f_1, f_2 be a Δ^1_1 enumeration of C. By composing f_n with an isomorphism $h_n : \mathbf{R} \to (1/2^{n+1}, 1/2^n)$ we may assume $F_n : \mathbf{X} \to (1/2^{n+1}, 1/2^n)$. Let $F(x) = \sum_{n=0}^{\infty} f_n(x)$. It is easy to see that F is well defined and Δ^1_1.

Suppose $x \leq y$, then for any $n \in \omega$ $f_n(x) \leq f_n(y)$. Thus $\sum_{n=0}^{\infty} f_n(x) \leq \sum_{n=0}^{\infty} f_n(y)$. If $x < y$, there is a least n s.t. $f_n(x) < f_n(y)$. Clearly $\sum_{n \neq i} f_i(x) \leq \sum_{n \neq i} f_i(y)$, so $F(x) = f_n(x) + \sum_{n \neq i} f_i(x) < f_n(y) + \sum_{n \neq i} f_i(y) = F(y)$. Thus F is order preserving. F is the desired function. $\square$

COROLLARY 4.4. *If* $(\mathbf{X}, \leq)$ *is a separable* Δ^1_1 *prelinear order there is a* Δ^1_1 $f : \mathbf{X} \to \mathbf{R}$ *order preserving.*

COROLLARY 4.5 (FRIEDMAN [**F**]). *If* $(\mathbf{X}, \leq)$ *is a Borel prelinear order then* $(\mathbf{X}, \leq)$ *is separable or there is a perfect set of totally isolated points.*

COROLLARY 4.6 (FRIEDMAN-SHELAH [**F**, **St**]). *There is no Borel Suslin line.*

5. Decomposing thin Δ^1_1 orders. The following result is an analog of Dilworth's result [**Di**] that every partial order of width n can be written as a union of n chains.

THEOREM 5.1. *If* $(\mathbf{X}, \leq)$ *is a thin* Δ^1_1 *prepartial order, then there are* Δ^1_1 *chains* $\langle \mathbf{X}_n : n \in \omega \rangle$ *such that* $\mathbf{X} = \bigcup_{n \in \omega} \mathbf{X}_n$.

PROOF. Without loss of generality assume $\mathbf{X} = \mathbf{R}$. Let $Z = \{Y \in \Sigma^1_1 :\ \leq$ is a prelinear ordering on $Y\}$. Let $W_0 = \bigcup\{Y : Y \in Z\}$. If $Y \in Z$, then by 1.5 there is a Δ^1_1 $X \supseteq Y$ such that $X \in Z$. Thus $z \in W_0 \Leftrightarrow \exists Y \in \Delta^1_1$ $(\forall x, y \in Y$ $(x \leq y \vee y \leq x) \wedge z \in Y)$. So by familiar arguments W_0 is Δ^1_1. Let $W = \neg W_0$. If $W = \varnothing$, then $\mathbf{X}$ is the union of countably many Δ^1_1 chains. So we assume $W \neq \varnothing$. We will reach a contradiction if we can show there is a nonempty linearly orderly Σ^1_1 $Y \subseteq W$.

For $Y \in \Sigma^1_1$, $Y \neq \varnothing$, $Y \subseteq W$, let $\mathscr{F}_Y = \{F \in \Delta^1_1 : \exists \alpha < \omega_1^{ck}$ $F : \mathbf{R} \to 2^\alpha$ s.t. $\forall x, y \in Y$ $F(x) < F(y) \to x < y\}$. For $F \in \mathscr{F}_Y$ let $x E_F y \Leftrightarrow F(x) = F(y)$ and $x E_Y y \Leftrightarrow \forall F \in \mathscr{F}_Y$ $F(x) = F(y)$. As in 3.1 each $\mathscr{F}_Y$ is Π^1_1 and each E_Y is Σ^1_1 and representable.

If $x, y \in Y$ and $x E_Y y$, then there is $F \in \mathscr{F}$ such that $F(x) \neq F(y)$. Thus x and y must be comparable. If for all $x, y \in Y$ $x E_Y y \Rightarrow x \approx y$, then $\leq$ linear orders Y, a contradiction. Thus for each Y we may assume $\exists x, y \in Y$ $(x E_Y y \wedge x \not\approx y)$.

Main claim. For $Y \subseteq W$ either (a) there are $a, b \in Y$ $\mathbf{P} \times_{E_Y} \mathbf{P}$ generic with a and b incomparable or (b) Y is linearly ordered. If (a) holds then by 2.9 $\leq$ is not thin and if b holds we reach a contradiction. Thus the main claim suffices. So far now we assume that any two $\mathbf{P} \times_{E_Y} \mathbf{P}$ generic elements of Y are comparable.

Let $A = \{a \in Y \colon \exists b \in Y\ bE_Y a \wedge b \not\approx a\}$. If $A = \varnothing$, Y is linearly ordered so we may assume $A \neq \varnothing$.

Claim 1. If b and c are $\mathbf{P} \times_{E_Y} \mathbf{P}$ generic then $b \not\approx c$.

Suppose $(B_0, C_0) \Vdash \mathring{b} \approx \mathring{c}$. Let $B = \{x \in B_0 \colon \exists y \in C_0\ xE_Y y\}$. $C = \{x \in C_0 \colon \exists y \in B\ xE_Y y\}$. Since $(B_0, C_0) \in \mathbf{P} \times_{E_Y} \mathbf{P} B$ and C are nonempty.

Subclaim. $\forall b \in B\ \forall c \in C\ bE_Y c \Rightarrow c \approx b$.

Assume there are $b \in B$, $c \in C$, such that $b < c$ (the case $b > c$ is symmetric). Let $D = \{(b,c) \colon b \in B, c \in C, bE_Y c$ and $b < c\}$. Let (b_0, c_0), $(b_1, c_1) \in D$ be $\mathbf{P}^2_{E_Y} \times_{E_y} \mathbf{P}^2_{E_Y}$ generic. Then (b_0, c_1) and (b_1, c_0) are $\mathbf{P} \times_{E_Y} \mathbf{P}$ generic so $b_0 < c_0 \approx b_1 < c_1 \approx b_0$, a contradiction.

Thus for all b, $b' \in B$ if $bE_Y b'$, then $\exists c \in C\ bE_Y cE_Y b'$. But then $b \approx c \approx b'$ so $\forall b, b' \in B\ bE_Y b' \Rightarrow b \approx b'$. Next consider $\{d \in Y \colon \exists b \in B\ bE_Y d$ and b, d are incomparable$\}$. This must be empty else we can find b, $d \in B \times D$ $\mathbf{P} \times_{E_Y} \mathbf{P}$ generic and incomparable, contradicting our assumption. Thus for any $d \in Y$ $b \in B$ if $bE_Y d$, then b is comparable with $d\}$. Let $B^- = \{x \colon \exists y \in B\ xE_Y y \wedge x \leq y\}$ and $B^+ = \{x \colon \exists y \in B\ xE_Y y \wedge x > y\}$. Since on B $E_Y = \approx$, $B^+ \cap B^- = \varnothing$. Let $\mathscr{A}(U, V) \Leftrightarrow \forall u \notin U\ \forall v \notin V\ (uE_Y v \vee v > u)$. Then $\mathscr{A}(\neg B^-, B^-)$ so by strong reflection we can find $\hat{B} \in \Delta^1_1$ such that $\hat{B} \supseteq B^-$, $\hat{B} \cap B^+ = \varnothing$ and $\forall u \in \hat{B}\ \forall v \notin \hat{B}$ $uE_Y v \Rightarrow u < v$. Let $\mathscr{D}(U) \Leftrightarrow \forall x, y \in Y\ (\exists F \in \Delta^1_1\ F \in U\ F(x) \neq F(y) \vee x \notin \hat{B} \vee y \in \hat{B} \vee x < y$. Then $\mathscr{D}$ is Π^1_1 on Π^1_1 and $\mathscr{D}(\mathscr{F}_Y)$. Hence by reflection there is $X \subseteq \mathscr{F}_Y$ s.t. X is Δ^1_1 and for all $x, y \in Y$ $(x \in \hat{B} \wedge y \notin \hat{B} \wedge x \not< y) \rightarrow \exists F \in X$ $F(x) \neq F(y)$. As in 3.1 we can find an $F^* \in \mathscr{F}_Y$ such that for all x, y $F^*(x) \neq F^*(y)$ iff there is an $F \in X$ such that $F(x) \neq F(y)$. Let

$$G(x) = \begin{cases} F(x)^\wedge 0, & x \in \hat{B}, \\ F(x)^\wedge 1, & x \notin \hat{B}. \end{cases}$$

Suppose $x, y \in Y$ if $G(x) < G(y)$, then either $F(x) < F(y)$, in which case $x < y$, or $F(x) = F(y)$ and $x \in \hat{B}$ and $y \notin \hat{B}$ so $x < y$. Thus $G \in \mathscr{F}_Y$. Moreover since $\hat{B} \cap B^+ = \varnothing$, G splits an E_Y class, a contradiction.

Claim 2. If $(B, C) \Vdash \mathring{b} < \mathring{c}$, then $\forall b, c \in B \times_{E_Y} C\ b < c$. (We abbreviate this as $B <_{E_Y} C$.)

Let $B' = \{b \colon \exists c \in C\ bE_Y c \wedge b \not< c\}$ and assume $B' \neq \varnothing$. Suppose b_0, $b_1 \in B' x_{E_y} B'$ and $\mathbf{P} \times_{E_Y} \mathbf{P}$ generic. By our assumption $b_0 \leq b_1$ or $b_1 \leq b_0$. By Claim 1 $b_0 \not\approx b_1$. Assume $b_0 < b_1$. Let B_0, $B_1 \subseteq B'$ s.t. $(B_0, B_1) \Vdash \mathring{b}_0 < \mathring{b}_1$. Let $D = \{(b_0, c) \in B_0 \times_{E_Y} C \colon b_0 \not< c\}$. Let $(b_0, c), b_1$ be $\mathbf{P}^2_{E_Y} \times_{E_Y} \mathbf{P}$ generic with $(b_0, c) \in D$, $b_1 \in B_1$. By 2.7 (b_0, b_1) are $\mathbf{P} \times_{E_Y} \mathbf{P}$ generic so $b_0 < b_1$. Further c and b_1 are $\mathbf{P} \times_{E_Y} \mathbf{P}$ generic. Thus since $b_1 \in B$, $b_1 < c$. But then $b_0 < b_1 < c$ a contradiction.

Suppose $(B,C) \Vdash \mathring{b} < \mathring{c}$. Then by Claim 2 $B <_{E_Y} C$. Let $\overline{B} = \{x\colon \exists b \in B\ xE_Y b \wedge x \leq b\}$. Let $\mathscr{A}(U,V) \Leftrightarrow \forall u \notin U\ \forall v \notin V\ (u \not{E}_Y v \vee v \nleq u) \wedge \forall z \notin Z\ \forall y \in C\ z < y$. Then $\mathscr{A}(\neg\overline{B},\overline{B})$ so by strong reflection there is a $\Delta^1_1 \hat{B} \supseteq \overline{B}$ s.t. $\hat{B} \cap C = \varnothing$ and $\hat{B}$ is downward closed in each E_Y class it intersects.

Claim 3. $\forall xy\ (x \in \hat{B} \wedge y \notin \hat{B} \wedge xE_Y h) \to x < y$.

Suppose not. Let $D = \{d \notin \hat{B}\colon \exists b \in \hat{B}\ bE_Y d \wedge b \nless d \wedge d \in Y\}$. We can find d_0, $d_1 \in D$ s.t. d_0 and d_1 are $\mathbf{P}\times_{E_Y} \mathbf{P}$ generic. By our assumption and Claim 1 $d_0 < d_1$ or $d_1 < d_0$. Say $(D_0, D_1) \Vdash d_0 < d_1$. By Claim 2 $D_0 <_{E_Y} D_1$. Let $B_1 = \{b \in \hat{B}\colon \exists d \in D_1\ (bE_Y d \wedge b \nless d_1)\}$. Let $(b, d_0) \in B \times_{E_Y} D_0$ be $\mathbf{P}$ generic. Then by assumption and Claim 1 b and d_0 are comparable. Since $\hat{B}$ is downward closed $b < b_0$. Thus for all $d_1 \in D_1$ if $bE_Y d_1$, then $b < d_0 < d_1$, a contradiction.

Thus $\hat{B}$ is Δ^1_1, downward closed and if $\exists b \in \hat{B}\ y \in Y\ yE_Y b$ and $y \notin \hat{B}\ y > b$. Now by reflection arguments similar to those in Claim 1, we can find a $G \in \mathscr{F}_Y$ which splits some E_Y class. This gives the contradiction which proves the main claim and the theorem. □

COROLLARY 5.2 (SHELAH [S]). *If $(\mathbf{X}, \leq)$ is a Borel order and there is an uncountable set of pairwise incomparable elements, then there is a perfect set of pairwise incomparable elements.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, ILLINOIS 60680

DEPARTMENT OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM, ISRAEL

DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, BURNABY, B. C., CANADA V5A 1S6

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903