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UNIQUENESS AND CHARACTERIZATION OF PRIME MODELS OVER SETS FOR TOTALLY TRANSCENDENTAL FIRST-ORDER THEORIES

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THEOREM. *If T is a complete first-order totally transcendental theory then over every T-structure A there is a prime model unique up to isomorphism over A. Moreover M is a prime model over A iff*: (1) *every finite sequence from M realizes an isolated type over A, and* (2) *there is no uncountable indiscernible set over A in M.*

The existence of prime models was proved by Morley [3] and their uniqueness for countable A by Vaught [9]. Sacks asked (see Chang and Keisler [1, question 25]) whether the prime model is unique. After proving this I heard Ressayre had proved that every two strictly prime models over any T-structure A are isomorphic, by a strikingly simple proof. From this follows

THEOREM. *If T is totally transcendental, M a strictly prime model over A then every elementary permutation of A can be extended to an automorphism of M.* (*The existence of M follows by* [3].)

By our results this holds for any prime model. On the other hand Ressayre's result applies to more theories. For more information see [6, §0A]. A conclusion of our theorem is the uniqueness of the prime differentially closed field over a differential field. See Blum [8] for the total transcendency of the theory of differentially closed fields.

We can note that the prime model M over A is minimal over A iff in M there is no indiscernible set over A (which is infinite).

In order to help the reader, §§1 and 2 contain known results which are from Morely [3] (except 2.3, 2.4), with a variation of the definition of rank type. If we define T as totally transcendental iff $R(x = x) < \infty$, then the restriction "T is countable" is superfluous.

The result of this paper was announced in [6, §0A.5B] (in more general form) and in [7, Theorem 6].

Notation. Let T be a fixed first-order countable complete theory in the language L. For simplicity all the sets and models we shall deal with, will be of cardinality $< \bar{\kappa}$, for some high enough cardinal $\bar{\kappa}$; and let $\overline{M}$ be a $\bar{\kappa}$-saturated model of T. As every model of T of cardinality $< \bar{\kappa}$ is isomorphic to an elementary submodel of $\overline{M}$, we can deal with them only. (See Morley and Vaught [5], or Chang and Keisler [1] for $\bar{\kappa}$-saturated models.) So let M, N denote elementary submodels of $\overline{M}$ (of cardinality $< \bar{\kappa}$), A, B, C sets of elements of $\overline{M}$ (of cardinality $< \bar{\kappa}$), a, b, c elements of $\overline{M}$, $\bar{a}$, $\bar{b}$, $\bar{c}$ finite sequences of elements of $\overline{M}$. Let $|M|$ be the set of elements of M, $|A|$ the cardinality of A, so $\|M\|$ is the cardinality of M. Let φ, ψ, θ denote formulas

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of L, x, y, z variables, $\bar{x}, \bar{y}, \bar{z}$ finite sequences of variables. $M \vDash \varphi[a_1 \cdots a_n]$ means M satisfies $\varphi[a_1 \cdots a_n]$, (so $a_1, \cdots, a_n \in M$). As the satisfaction does not depend on the particular model we omit it. Rang $\bar{a}$ is the set of elements appearing in $\bar{a}$; we write $\bar{a} \in A$ instead of Rang $\bar{a} \subset A$, and $\bar{a} \frown \bar{b}$ for concatenation of the sequences $\bar{a}, \bar{b}$. An m-type over A is a set p of formulas $\varphi(x_1, \cdots, x_m, \bar{a})$, $\bar{a} \in A$, such that:

$$\varphi_i(\bar{x}, \bar{a}^i) \in p, \qquad i = 1, \cdots, n \Rightarrow \vDash (\exists \bar{x}) \bigwedge_{i=1}^{n} \varphi_i(\bar{x}, \bar{a}^i).$$

If $m = 1$ we omit it. Types are denoted by p, q.

§1. Rank of types and formulas.

DEFINITION 1.1. (A) The type $\bar{c}$ realizes over A is

$$p(\bar{c}, A) = \{\varphi(\bar{x}, \bar{a}) : \bar{a} \in A, \vDash \varphi[\bar{c}, \bar{a}]\}.$$

(B) $S^m(A) = \{p(\bar{c}, A) : l(\bar{c}) = m \wedge \bar{c} \in |\bar{M}|\}$ [clearly for any m-type p over A, there is $q \in S^m(A) p \subset q$]. $S(A) = S^1(A)$.

(C) T is $\aleph_0$-stable (=totally transcendental) iff $|A| \leq \aleph_0$ implies $|S(A)| \leq \aleph_0$.

DEFINITION 1.2. We define $R[\varphi(x, \bar{a})]$ (the rank of $\varphi(x, \bar{a})$) by induction:

(A) $R[\varphi(x, \bar{a})] = -1$ *iff* $\vDash \neg(\exists x)\varphi(x, \bar{a})$;

(B) $R[\varphi(x, \bar{a})] = \alpha$ *iff*

(1) $\vDash (\exists x)\varphi(x, \bar{a})$,

(2) for no $\beta < \alpha$, $R[\varphi(x, \bar{a})] = \beta$,

(3) for no $\psi(x, \bar{b})$, both $\varphi(x, \bar{a}) \wedge \psi(x, \bar{b})$,
$\varphi(x, \bar{a}) \wedge \neg\psi(x, \bar{b})$ satisfies (1) and (2).

(C) $R[\varphi(x, \bar{a})] = \infty$ if $R[\varphi(x, \bar{a})]$ is not defined by (A) and (B). We stipulate $-1 < \alpha < \infty$ for any ordinal α.

DEFINITION 1.3. For a type p, the rank of p is

$$R[p] = \min\{R[\varphi_1(x, \bar{a}^1) \wedge \cdots \wedge \varphi_n(x, \bar{a}^n)] : n < \omega, \varphi_i(x, \bar{a}^i) \in p\}.$$

THEOREM 1.1. (A) *If* $\vDash (\forall x)[\varphi(x, \bar{a}) \rightarrow \psi(x, \bar{b})]$ *then* $R[\varphi(x, \bar{a})] \leq R[\psi(x, \bar{b})]$.

(B) *If* $\alpha = R[\varphi(x, \bar{a})] < \infty$, *then for no* $\psi(x, \bar{b})$

$$\alpha = R[\varphi(x, \bar{a}) \wedge \psi(x, \bar{b})] = R[\varphi(x, \bar{a}) \wedge \neg\psi(x, \bar{b})].$$

(C) *If* $\alpha < R[\varphi(x, \bar{a})]$ *then there is* $\psi(x, \bar{b})$ *such that*

$$R[\varphi(x, \bar{a}) \wedge \psi(x, \bar{b})] \geq \alpha, \qquad R[\varphi(x, \bar{a}) \wedge \neg\psi(x, \bar{b})] \geq \alpha.$$

(D) *If* $\bar{a}, \bar{b}$ *realize the same type*, $R[\varphi(x, \bar{a})] = R[\varphi(x, \bar{b})]$.

(E) *There is* $\alpha_0 < (2^{\aleph_0})^+$ *such that no* $\varphi(x, \bar{a})$ *has rank* α_0.

THEOREM 1.2. (A) $p \subset q$ *implies* $R[p] \geq R[q]$ (*by Theorem* 1.1, *Definition* 1.3).

(B) *Every type has a finite subtype of the same rank* (*so if* $p \in S(A)$, *we can take one formula*) (*by Definition* 3.1).

(C) *If* p *is a type over* A, $R[p] < \infty$, *then there is at most one* q, $p \subset q \in S(A)$, $R[p] = R[q]$ (*by* 1.1B).

THEOREM 1.3. *T is $\aleph_0$-stable iff* $R[x = x] < \infty$ (*by* Theorem 1.1. *This means* $R[p] < \infty$ *for every* p.)

PROOF. If $R(x = x) = \infty$, then $R[x = x] > \alpha_0$, so by repeated use of Theorem 1.1(C), T is not $\aleph_0$-stable. If $R(x = x) < \infty$, then every $p \in S(A)$ has a finite subtype p^* of the same rank. By Theorem 1.2(C) $|S(A)| = |\{p^* : p \in S(A)\}| \leq |A| + \aleph_0$.

Remark. We can define, similarly, ranks for m-types, and the same theorems hold; as T is $\aleph_0$-stable iff $|S^m(A)| \leq |A| + \aleph_0$ for every $m < \omega$. From now on T is totally transcendental.

§2. Prime models and indiscernible sets.

DEFINITION 2.1. A function f; Rang f, Dom $f \subset |\bar{M}|$, is elementary if for $a_1, \cdots, a_n \in \text{Dom} f$, $\varphi \in L \vDash \varphi[a_1, \cdots, a_n] \equiv \varphi[f(a_1), \cdots, f(a_n)]$.

DEFINITION 2.2. M is a prime model over A, *if* every elementary function from A into a model N *can be* extended to an elementary function from M into N.

DEFINITION 2.3. A type $p \in S^m(A)$ is isolated if there is a finite $p' \subset p$ such that p is the unique extension of p' in $S(A)$. We say then that p is isolated over $\wedge p'$.

DEFINITION 2.4. M is strictly prime over A by $|M| = A \bigcup \{a_i : i < \alpha\}$ if each a_j realizes an isolated type over $A_j = A \bigcup \{a_i : i < j\}$.

THEOREM 2.1. (A) *For every formula $\varphi(x, \bar{a})$, $\bar{a} \in A$, there is an isolated $p \in S(A)$, $\varphi(x, \bar{a}) \in p$.*

(B) *Over every A there is a strictly prime model; and every strictly prime model over A is a prime model over A.*

(C) *Every finite sequence in a model, M, prime over A realizes over A an isolated type.*

PROOF. (A) We choose p with minimal rank. (B) follows by repeated use of (A). (C) by Definition 2.2, Theorem 2.1(B), it suffices to prove it for a model strictly prime over A by, say, $|M| = A \bigcup \{a_i : i < i_0\}$ which is proved by induction on i_0.

DEFINITION 2.5. The set $\{a_i : i < \alpha\}$ is indiscernible over A, *if* $a_0 \neq a_1$ and for every distinct $\beta_1, \cdots, \beta_n < \alpha$, distinct $\gamma_1, \cdots, \gamma_n < \alpha$, $\varphi \in L$ and $\bar{a} \in A$, $\vDash \varphi[a_{\beta_1}, \cdots, a_{\beta_n}, \bar{a}] \equiv \varphi[a_{\gamma_1}, \cdots, a_{\gamma_n}, \bar{a}]$.

THEOREM 2.2. *If $A_j = A \bigcup \{a_i : i < j\}$, $p(a_0, A_0) \subset p(a_j, A_j)$, $R[p(a_0, A_0)] = R[p(a_j, A_j)]$ for any $j < \alpha \geq \omega$, then $\{a_i : i < \alpha\}$ is indiscernible over A. (See* [3].)

THEOREM 2.3. *For any $\varphi(x, \bar{y})$ there is $r = r_\varphi < \omega$ such that for any indiscernible $\{a_i : i < \alpha\}$, and $\bar{b}$,*

$$|\{i < \alpha : \vDash \varphi[a_i, \bar{b}]\}| < r \quad \text{or} \quad |\{i < \alpha : \vDash \neg\varphi[a_i, \bar{b}]\}| < r.$$

PROOF. Otherwise $\Gamma = \{\varphi(x_I, \bar{y}_n), \neg\varphi(x_I, \bar{y}_m) : I \subset \omega, n \in I, m \notin I\}$ is consistent. Let $\bar{b}_m$ realize $\bar{y}_m$, a_I realize x_I, $A = \bigcup_{m<\omega} \text{Rang } \bar{b}_m$. So $|S(A)| \geq |\{p(a_I, A) : I \subset \omega\}| = 2^{\aleph_0} > |A|$, contradiction.

THEOREM 2.4. *If $\{a_i : i < \alpha\}$ is indiscernible over A, then for any $\bar{b}$ there is a finite $I \subset \alpha$ such that $\{a_i : i < \alpha, i \notin I\}$ is indiscernible over $A \bigcup \{a_i : i \in I\} \bigcup$ Rang $(\bar{b})$.*

Theorem 2.4 can be proved like Theorem 2.3, see [6, Theorem 6.13, Theorem 5.9], compare with [2, Theorem 1.3], Theorem 2.4 was independently noted by V. Harnik and the author.

§3. Indiscernible sets in a prime model.

LEMMA 3.1. *If $a^1, \cdots, a^n \in |M|$, M is a strictly prime model over A; [by $|M| = A \cup \{a_i : i < i_0\}$] then M is strictly prime over $A \cup \{a^1, \cdots, a^n\}$ [by $|M| = (A \cup \{a^1, \cdots, a^n\}) \cup \{a_i : i < i_0\}$].*

PROOF. Let $j < i_0$, and we should prove only that a_j realizes an isolated type over $A \cup \{a^1, \cdots, a^n\} \cup \{a_i : i < j\}$. By the Definition 2.4 M is strictly prime over $A_j = A \cup \{a_i : i < j\}$ by $|M| = A_j \cup \{a_i : j \leq i < i_0\}$, So by Theorem 2.1(c)

$\langle a_j, a^1, \cdots, a^n\rangle$ realizes an isolated type over A_j, say isolated over $\varphi(x_1, x_2, \cdots, x_{n+1}, \bar{c})$, $\bar{c} \in A_j$. Clearly the type a_j realizes over $A_j \cup \{a^1, \cdots, a^n\}$ is isolated by $\varphi(x_1, a^1, \cdots, a^n, \bar{c})$.

LEMMA 3.2. *If $B = A \cup \{a_i: i < \alpha\}$, $\alpha \geq \omega$, and $\{a_i: i < \alpha\}$ is an indiscernible set over A, then $\{a_i: i < \alpha\}$ is a maximal indiscernible set over A, in any prime model M over A.*

PROOF. Let $p = \{\varphi(x, \bar{c}): \bar{c} \in B$, and for infinitely many $i < \omega$, $\models\varphi[a_i, \bar{c}]\}$. By Theorem 2.3 p is consistent and complete, so $p \in S(B)$. It is easy to see that p is not isolated—*as* if it is isolated over $\varphi(x, \bar{c})$, $\bar{c} \in B$, *then* for infinitely many a_i's $\{\varphi(x, \bar{c}) \wedge x = a_i\}$ is consistent. So by Theorem 2.1(C) p is not realized in M. Now *if* the conclusion of the lemma fails, for some $c \in |M|$, $\{a_i: i < \alpha\} \cup \{c\}$ is an indiscernible set over A, *then* clearly c realizes p, contradiction.

THEOREM 3.3. *In any strictly prime model M over A there is no uncountable indiscernible set over A.*

PROOF. Suppose $\{b_i': i < \aleph_1\}$ is an indiscernible set over A in M. Let $p = \{\varphi(x, \bar{c}): \bar{c} \in |M|, \{i: \models\varphi[b_i', \bar{c}]\}$ is infinite$\}$. As before by Theorem 2.3, $p \in S(|M|)$. By Theorem 1.2(B) there is $\varphi(x, \bar{c}^\circ) \in p$, $R[p] = R[\varphi(x, \bar{c}^\circ)]$. By Lemma 3.1 M is strictly prime over $A_1 = A \cup \operatorname{Rang} \bar{c}^\circ$, and by Theorem 2.4 for some finite I, $\{b_i': i < \aleph_1, i \notin I\}$ is an indiscernible set over A_1. Let $\{b_i: i < \aleph_1\} = \{b_i': i < \aleph_1, i \notin I\}$.

Let $B = A_1 \cup \{b_i: i < \omega\}$. Then by Theorem 2.1(B) there is a prime model N over B, and as $B \subset M$, using Definition 2.2 (with F the identity on B) we can assume $|N| \subset |M|$. Using the definition again, as $A_1 \subset B \subset N$, M a prime model over A_1, there is an elementary function F from $|M|$ into N, such that for $a \in A_1$, $F(a) = a$. Let $b^i = F(b_i)$ for every $i < \aleph_1$. Clearly $\{b^i: i < \aleph_1\}$ is an indiscernible set over A_1. By Theorem 2.4 there is $\alpha < \omega_1$, such that $\{b^i: \alpha \leq i < \aleph_1\}$ is an indiscernible set over B. Let q be the type b^α realizes over B. It is easy to see that for any c, c realizes $p \mid B$ iff $\{b_i: i < \omega\} \cup \{c\}$ is an indiscernible set over A_1. So by Lemma 3.2 b^α does not realize $p \mid B$. But by Theorem 1.2(A) and the definition of $\varphi(x, \bar{c}^\circ)$, and as $\bar{c}^\circ \in A_1$

$$R[\varphi(x, \bar{c}^\circ)] \geq R[p \mid A_1] \geq R[p \mid B] \geq R[p] = R[\varphi(x, \bar{c}^\circ)].$$

So $q \neq p \mid B$, $q \in S(B)$, $q \mid A_1 = p \mid A_1$, so by Theorem 1.2(A) and $C \subset R[q] < R[p]$. So by Theorem 1.2(B) there is $\varphi_1(x, \bar{c}^1) \in q$, (so $\bar{c}^1 \in B$), $R[\varphi_1(x, \bar{c}^1)] = R[q] < R[p]$. Let $r = r_{\varphi_1}$ (from Theorem 2.3); $\psi(\bar{x}, y_1, \cdots, y_r, \bar{a})(\bar{a} \in A_1)$ isolate the type $\bar{c}^1{}^\frown\langle b^{\alpha+1}, \cdots, b^{\alpha+r}\rangle$ realize over A_1, so $\models(\exists\bar{x})\psi(\bar{x}, b^{\alpha+1}, \cdots, b^{\alpha+r}, \bar{a})$. As F was an elementary function, $f(b_i) = b^i$ and $a \in A \Rightarrow F(a) = a$, clearly $\models(\exists\bar{x})\psi(\bar{x}, b_1, \cdots, b_r, \bar{a})$. So for some $\bar{c}^2 \in |M|$, $\models\psi[\bar{c}^2, b_1, \cdots, b_r, \bar{a}]$. Now by the definition of α, b^α, $b^{\alpha+1}, \cdots, b^{\alpha+r}$ realize the same type over B, so $\models\varphi_1(b^{\alpha+1}, \bar{c}^1), \cdots, \models\varphi_1(b^{\alpha+r}, \bar{c}^1)$. So by the definition of ψ, $\models\varphi_1(b^i, \bar{c}^2)$, $i = 1, \cdots, r$, and $\bar{c}^1, \bar{c}^2$ realize the same type over A_1. Hence by Theorem 1.1(D), $R[\varphi_1(x, \bar{c}^2)] = R[\varphi_1(x, \bar{c}^1)] < R(p)$, and $|\{i < \aleph_1: \models\varphi_1(b_i, x, \bar{c}^2)\}| \geq r$. So by Theorem 2.3 $|\{i < \aleph_1: \models\varphi_1[b_i, \bar{c}^2]\}| = \aleph_1$, so $\varphi_1(x, \bar{c}^2) \in p$, so $R[p] \leq R[\varphi_1(x, \bar{c}^2)]$ contradiction.

THEOREM 3.4. *If M is a prime model of A then* (A) *every finite sequence from M realizes over A an isolated type,* (B) *in M, there is no uncountable indiscernible set over A.*

PROOF. (A) is Theorem 2.1(C). For (B) let N be a strictly prime model over A, and F an elementary function from M into N, such that for $a \in A$, $F(a) = a$ (which exists by Definition 2.2). If $\{a_i : i < \aleph_1\} \subset |M|$ is indiscernible over A, then $\{F(a_i) : i < \aleph_1\} \subset |N|$ is also indiscernible over A, contradition to Theorem 3.3.

§4. The uniqueness theorem.

THEOREM 4.1. *Suppose $A \subset |M|$, $B \subset N$, F is an elementary function from A onto B. Moreover assume*

(1) *every finite sequence from $M(N)$ realizes over $A(B)$ an isolated type,*

(2) *in $M(N)$ there is no uncountable indiscernible set over A.*

Then F can be extended to an isomorphism from M onto N.

Remark. Together with Theorems 2.1(B), 3.4, this proves the main theorem.

We shall extend gradually F, such that (1), (2) remain true with A, B replaced by Dom F, Rang F. Condition (2) clearly remains true, but we should be careful about (1). We shall need the following two lemmas:

LEMMA 4.2. *Suppose $A_1 \subset |M_1|$, every finite sequence from M_1 realizes over A_1 an isolated type, $a_1, \cdots, a_n \in |M_1|$, $A_2 = A_1 \cup \{a_1, \cdots, a_n\}$. Then every finite sequence from M_1 realizes over A_2 is an isolated type.*

PROOF. Let $\bar{c}$ be a sequence from M_1. By hypothesis, $\bar{c}^\frown\langle a_1, \cdots, a_n\rangle$ realizes over A_1 an isolated type, say isolated over $\varphi(\bar{x}, y_1, \cdots, y_n, \bar{b})$, $\bar{b} \in A_1$. So the type $\bar{c}$ realizes over A_2 is isolated by $\varphi(\bar{x}, a_1, \cdots, a_n, \bar{b})$.

LEMMA 4.3. *Suppose $A_1 \subset |M_1|$, every finite sequence from M_1 realizes over A_1 an isolated type; $p_i \in S(A_1)$ for $i < i_0$. Let $A_2 = A_1 \cup \bigcup_{i<i_0}\{a \in |M_1| : a$ realizes $p_i\}$.*

Then every sequence from M_1 realizes over A_2 an isolated type.

PROOF. Let $\bar{c}$ be any sequence from M_1 and p be the type $\bar{c}$ realizes over A_2. Let $\varphi_1(\bar{x}, \bar{b}^1) \in p$ be such that $R[\varphi_1(\bar{x}, \bar{b}^1)] = R[p]$ (it exists by Theorem 1.2(B)). By the hypothesis and Lemma 4.2, the type $p \mid (A_1 \cup \text{Rang}\, \bar{b}^1)$ is isolated, so let it be isolated over $\varphi_2(\bar{x}, \bar{b}^2)$, $\bar{b}^2 \in (A \cup \text{Rang}\, \bar{b}^1)$, and let $\varphi_3(\bar{x}, \bar{b}^3) = \varphi_1(\bar{x}, \bar{b}^1) \wedge \varphi_2(\bar{x}, \bar{b}^2)$. So by Theorem 1.1(A) $R[\varphi_3(\bar{x}, \bar{b}^3)] = R[p]$ and $p \mid (A_1 \cup \text{Rang}\, \bar{b}^1)$ is isolated. Suppose this fails, so p is not isolated over $\varphi_3(\bar{x}, \bar{b}^3)$, so there are $\bar{b}^4 \in A_2$, $\varphi_4 \in L$ such that $\varphi_4(\bar{x}, \bar{b}^4) \in p$, but $\vDash (\exists \bar{x})[\varphi_3(\bar{x}, \bar{b}^3) \wedge \neg\varphi_4(\bar{x}, \bar{b}^4)]$, so there is $\bar{c}^* \in |M_1|$ such that $\vDash \varphi_3(\bar{c}^*, \bar{b}^3) \wedge \neg\varphi_4(\bar{c}^*, \bar{b}^4)$. But the definition of φ_3, $\bar{c}^*$ realizes $p \mid (A_1 \cup \text{Rang}\, \bar{b}^1)$. As in the proof of Theorem 3.3, as $\bar{c}$, $\bar{c}^*$ realize the same type over $A \cup \text{Rang}\, \bar{b}^1$, and as every finite sequence from M_1 realizes an isolated type over $A_1 \cup \text{Rang}\, \bar{b}^1$ there is $\bar{b}^5 \in |M_1|$ such that $\bar{c}^{*\frown}\bar{b}^4$, $\bar{c}^\frown\bar{b}^5$ realize the same type over $A_1 \cup \text{Rang}\, \bar{b}^1$. By the definition of A_2, as $\bar{b}^4 \in A_2$, also $\bar{b}^5 \in A_2$. As $\vDash \neg\varphi_4(\bar{c}^*, \bar{b}^4)$ clearly $\vDash \neg\varphi_4(\bar{c}, \bar{b}^5)$, so $\neg\varphi(\bar{x}, \bar{b}^5) \in p$. So we can conclude: $\bar{b}^4$, $\bar{b}^5$ realizes the same type over $A_1 \cup \text{Rang}\, \bar{b}^1$, $\varphi_4(\bar{x}, \bar{b}^4) \in p$, $\neg\varphi_4(\bar{x}, \bar{b}^5) \in p$.

Now $\{\varphi_3(x, \bar{b}^3)\} \subset \varphi_3(x, \bar{b}^3)$, $\varphi_4(x, \bar{b}^4)\} \subset p$, so

$$R[\varphi_3(x, \bar{b}^3)] \geq R[\varphi_3(x, \bar{b}^3) \wedge \varphi_4(x, \bar{b}^4)] \geq R[p] = R[\varphi_1(x, \bar{b}^1)] = R[\varphi_3(x, \bar{b}^3)].$$

But this implies $R[\varphi_3(x, \bar{b}^3) \wedge \neg\varphi_4(x, \bar{b}^4)] < R[p]$. As $\bar{b}^3 \in A_1 \cup \text{Rang}\, \bar{b}^1$, $\bar{b}^{3\frown}\bar{b}^4$, $\bar{b}^{3\frown}\bar{b}^5$ realizes the same type, so $R[\varphi_3(x, \bar{b}^3) \wedge \neg\varphi_4(x, \bar{b}^5)] = R[\varphi_3(x, \bar{b}^3) \wedge \neg\varphi_4(x, \bar{b}^4)] < R[p]$ but $\varphi_3(x, \bar{b}^3) \wedge \neg\varphi_4(x, \bar{b}^5) \in p$, contradiction.

PROOF OF THEOREM 4.1. We shall prove by induction on α that:

(*) If $\bar{c} \in A$, $R[\varphi(x, \bar{c})] = \alpha$, then we can extend F to an elementary function F'.

$$\operatorname{Dom}(F') = A \cup \{b \in |M| : \models \varphi[b, \bar{c}]\},$$

$\operatorname{Rang}(F') = A \cup \{b \in |M| : \models \varphi[b, F(\bar{c})]\}$ where $F\langle a_1, \cdots, a_n\rangle = \langle F(a_1), \cdots, F(a_n)\rangle$.

This is sufficient because for $\varphi(x, \bar{c}) = (x = x)$ this is the theorem.

Suppose that (*) holds for every $\beta < \alpha$, and $\bar{c} \in A$, $R[\varphi(x, \bar{c})] = \alpha$.

Let us define by induction $a_i \in |M|$: a_i will be any element in M_1, which realizes over $A \cup \{a_j : j < i\}$ a type p_i, $\varphi(x, \bar{c}) \in p_i$, $R[\varphi(x, \bar{c})] = R[p_i]$. Let α_0 be the first ordinal for which a_{α_0} is not defined. By Theorem 2.2, if $\alpha_0 \geq \omega$ then $\{a_j : j < \alpha_0\}$ is an indiscernible set over A. So by assumption (2), $\alpha_0 < \omega_1$. So we can rename the a_j's so that $\alpha_0 \leq \omega$, and clearly all the demands in their definition remain valid. Similarly we can find b_i, $i < \beta_0 \leq \omega$ in N, such that $\models\varphi[b_i, F(\bar{c})]$ for $i < \beta_0$, and the type b_i realizes over $B \cup \{b_j : j < i\}$ has rank $R[\varphi(x, F(\bar{c})]$, but b_{β_0} is not defined. (It is easy to prove $\alpha_0 = \beta_0$, but we do not need this.)

Now we define by induction F_n $n < \omega$, such that:

$$(**) \qquad A_n = \operatorname{Dom} F_n \subset |M|, \qquad B_n = \operatorname{Rang} F_n \subset |N|,$$

every sequence from M realizes an isolated type over A_n, and every sequence from N realizes an isolated type over B_n.

(1) Let $F_0 = F$.

(2) Suppose F_n is defined, $n = 3r + 1$. If $r \geq \beta_0$, $F_{n+1} = F_n$. If $r < \beta_0$, as $a_r \in |M|$, it realizes an isolated type over A_n; say isolated over $\psi(x, c^1, \cdots, c^n)$. Thus $\models(\exists x)\psi(x, c^1, \cdots, c^n)$, so $\models(\exists x)\psi(x, F(c^1), \cdots, F(c^n))$ and for some $a'_r \in N$ $\models\psi[a'_r, F(c^1), \cdots, F(c^n)]$. Extend F_n to F_{n+1} by defining $F_{n+1}(a_r) = a'_r$. By Lemma 4.2 (**) is satisfied for $n+1$; as clearly F_{n+1} is elementary.

(3) Suppose F_n is defined $n = 3r + 2$. Define F_{n+1} as in the previous case, transposing the roles of M, N; so $b_r \in \operatorname{Rang} F_{n+1}$.

(4) Suppose F_n is defined $n = 3r + 3$. Let $\{p_i : i < i_0\}$ be the list of types in $S(A_n)$ which are realized in M, $\varphi(x, \bar{c}) \in p_i$ and their rank is $< \alpha$. By (**) each p_i is isolated, say over $\psi_i(x, \bar{c}^i)$, and without loss of generality $R[\psi_i(x, \bar{c}^i)] = R[p_i] < \alpha$, $\models(\forall x)[\psi_i(x, \bar{c}^i) \to \varphi(x, \bar{c})]$. Define by induction F_n^i, $i \leq i_0$ such that: $\operatorname{Dom} F_n^i = A_n \cup \bigcup_{j<i}\{c \in |M| : \models\psi_i[c, \bar{c}^i]\}$, $\operatorname{Rang} F_n^i = B_n \cup \bigcup_{j<i}\{c \in |N| : \models\psi_j[c, F(\bar{c}^i)]\}$, and F_n^i is an elementary function. For $i=0$, $F_n^i = F_n$, for a limit ordinal i, $F_n^i = \bigcup_{j<i} F_n^i$; for a successor ordinal $i = j + 1$; we can use the induction hypothesis (on α, not on i) because $R[\psi_j(x, \bar{c}^j)] < \alpha$, and because by Lemma 4.3 the hypothesis of (*) is satisfied; this means there exists F_n^i as required. Let $F_{n+1} = F_n^{i_0}$.

So we have defined F_n, and let $F' = \bigcup_{n<\omega} F_n$. Clearly F' is an elementary function, $\operatorname{Rang} F' \subset |N|$, $\operatorname{Dom} F' \subset |M|$. It suffices to show

$$\operatorname{Rang} F' = |B| \cup \{a \in |N| : \models\varphi[a, F(\bar{c})]\}, \qquad \operatorname{Dom} F' = A \cup \{a \in |M| : \models\varphi[a, \bar{c}]\}.$$

As the proofs are similar let us prove for $\operatorname{Dom} F'$. Let $a \in |M|$, $\models\varphi[a, \bar{c}]$, and let a realize $p \in S(A \cup \{a_i : i < \alpha_0\})$. By the definition of α_0, $R[p] < \alpha$, so for some $j < \alpha_0$, $R[p \mid (A \cup \{a_i : i < j < \alpha_0 \leq \omega\})] < \alpha$, so for some $n < \omega R[p \mid A_n] < \alpha$, so $\{a' \in |M| : a'$ realizes $p \mid A_{3n}\} \subset A_{3n+1}$, so $a \in \operatorname{Dom} F_{3n} \subset \operatorname{Dom} F'$.

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