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END EXTENSIONS AND NUMBERS OF COUNTABLE MODELS

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Abstract. We prove that every model of $T = \mathrm{Th}(\omega, <, \dots)$ (T countable) has an end extension; and that every countable theory with an infinite order and Skolem functions has $2^{\aleph_0}$ nonisomorphic countable models; and that if every model of T has an end extension, then every $|T|$-universal model of T has an end extension definable with parameters.

§0. Introduction. Let us review this paper. You can think of a countable theory T which "says" $<$ orders the universe of the model and has Skolem functions.

In §1 we present the definitions and theorems known on end extensions and definable end extensions: in particular (Keisler [K1]) T has an $\aleph_1$-like model iff it has a model with an end extension *iff* it satisfies the regularity schema (Theorem 1.2); and the basic results of Gaifman [G3] (see [G1], [G2] too) on end-extension schemas and the definable end extension they define; the fact that every end-extension type defines an almost minimal end extension (2.3–1.8, see Gaifman [G3], in particular 2.8, 2.17).

In §2 we first prove a theorem of Rubin, that if T satisfies the regularity schema, every countable model has (i) a minimal end extension and (ii) an $\aleph_1$-like end extension with any possible order as the order type of the intermediate end extension.

We then notice that if T satisfies the inaccessability schema, every model with cofinality $\aleph_0$ has an end extension.

Now Gaifman [G1], [G2], [G3], following MacDowell and Specker [MS], proved that any theory $T = \mathrm{Th}(\omega, +, \cdot, <, \dots)$ has definable end extensions, minimal ones, rigid ones, etc. He uses the fact that *definitions by induction* are allowable. We show that for some of the results *proof by induction* only is sufficient, so $T = \mathrm{Th}(\omega, <, \dots)$ is sufficient (for almost minimal end-extension types). This seems the maximum we can get, but we do not have counter-examples.

Lastly, all theorems of the form "every model of T has an end extension" here are proved via definable end extensions. One may wonder whether this is really necessary. But we prove that if every model of T has an end extension then T has a definable end extension with *parameters.* For the class of $|T|$-universal models this gives a necessary and sufficient condition. Note that a countable model will have end extensions by Keisler's theorem whereas maybe every uncountable model necessarily has enough parameters.

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In §3 we prove that T has $2^{\aleph_0}$ nonisomorphic countable models. Benda and Rubin and Silver asked the question and Benda proved there are $\geq \aleph_0$ nonisomorphic countable models. After some reductions we show we can concentrate on

(i) $T = \mathrm{Th}(\omega, <, \dots)$; here for any model M we look at the models it is an end extension of, order by inclusion. As we showed every complete order with first and last element is possible we are finished (this was proved by Rubin).

(ii) $T = \mathrm{Th}(\omega + \omega^*, <, \dots)$ where ω is not definable; we define when two elements in a model of M of T are "near" (the closure of $\{z : z < x \equiv z < y\}$ is M) the model is decomposed to convex components (closed under nearness) and show we can get any reasonable order type for them.

The results were announced in [SR] and we thank Rubin for many discussions and for his kind permission to include here 2.1, 3.5.

§1. Preliminaries. T will be (in §1 and §2) a complete first-order theory in the language L, P, $<$ one- and two-place relations in L, and for every model M of T, $P^M = P(M)$ is infinite, and $<^M$ (or $<$) an order of $P(M)$ (so $(\forall x, y)(x < y \to P(x) \wedge P(y))$. Usually $<$ has no last element, T is countable with Skolem function and $|M| = P(M)$, but we shall mention such assumptions explicitly. A model will mean a model of T. $(\forall x \in P)$, $(\forall x > y)$, etc., are the obvious abbreviations.

Definition 1.1. (1) N is an end extension of M, $M \leq N$ *if* N is an elementary extension of M, (i.e., $M \prec N$) and $a \in P(M)$, $b \in P(N) - P(M) \Rightarrow a < b$ (i.e. $N \models a < b$).

(2) $M < N$ (N is a proper end extension of M) means $M \leq N$ and $P(M) \neq P(N)$.

(3) N is λ-like if $|P(N)| = \lambda$ but every initial segment of $P(N)$ is of cardinality $< \lambda$.

Theorem 1.2. *For a countable T, the following conditions are equivalent*:

(1) *It has a model with a proper end extension.*

(2) *It has an $\aleph_1$-like model.*

(3) *T satisfies the regularity schema, i.e.* $(\forall x)(\exists y)(y > x) \in T$ *and for every formula* $\varphi(x, y; \bar{z})$: *if* $\{x \in P : (\exists y < y_0)\varphi(x, y, \bar{z}_0)\}$ *is unbounded then for some* $y_1 < y_0$, $\{x \in P : \varphi(x, y_1, \bar{z}_0)\}$ *is unbounded i.e.*,

$$(\forall \bar{z})[(\forall x_1 \in P)(\exists x > x_1)(\exists y < y_0)\varphi(x, y, \bar{z}) \to (\exists y < y_0)(\forall x_1 \in P)(\exists x > x_1)\varphi(x, y, \bar{z})] \in T.$$

(4) *Every countable model of T has a countable proper end extension.*

Proof. Well known, due to Keisler (see [K1], [K2]). (2) $\Rightarrow$ (1) $\Rightarrow$ (3) are easy; for (3) $\Rightarrow$ (4) $\Rightarrow$ (2) by omitting types argument, every countable model of T has a proper elementary end extension, so we can build a chain of length ω_1, whose union is as desired.

Definition 1.3. For a theory T with Skolem functions and a model of it, M, we define:

(1) $\mathrm{tp}_\varphi(a, A, M) = \{\varphi(x, \bar{c}) : \bar{c} \in A, M \models \varphi[a, \bar{c}]\}$ where $a \in M$ and $A \subseteq M$.

(1′) Such a type is definable if there is a $\psi(\bar{y})$ such that for $\bar{c} \in A$, $M \models \psi[\bar{c}]$ iff $\varphi(x, \bar{c}) \in \mathrm{tp}_\varphi(a, A, M)$. We say that ψ defines the type.

(2) $\mathrm{tp}(a, A, M) = \bigcup_\varphi \mathrm{tp}_\varphi(a, A, M)$.

(2′) Such a type is definable if there is a schema $\langle \psi_\varphi(\bar{y}_\varphi): \varphi = \varphi(x, \bar{y}_\varphi) \in L \rangle$ such that $\psi_\varphi(\bar{y}_\varphi)$ defines $\mathrm{tp}_\varphi(a, A, M)$.

We say that the schema defines the type.

(3) N is a definable extension of M if

(i) $M < N$,

and for some $a \in P(N) - P(M)$

(ii) N is the Skolem hull of $|M| \cup \{a\}$,

(iii) $\mathrm{tp}(a, M, N)$ is definable.

(4) N is an almost minimal end extension of M if $M < N$ and there is no N' such that $M < N' < N$.

(5) N is a minimal extension of M if $M < N$ and for no $N' \neq M, N$, $M < N' < N$.

(6) N is rigid [strongly rigid] over M if there is no nontrivial automorphism [elementary endomorphism] of it over M.

(7) A schema $\Psi = \langle \psi_\varphi : \varphi \in L \rangle$ is "end-extension", "almost-minimal", "minimal" "rigid", or "strongly rigid" if every extension defined by it satisfies the corresponding property. It is strongly minimal if: for every n, and $M_0 < \cdots < M_n$, M_{l+1} the Skolem hull of $|M_l| \cup \{a_l\}$, $\mathrm{tp}(a_l, |M_l|, M_{l+1})$ defined by Ψ, and $M_0 < N < M_n$ implies N is the Skolem hull of $|M_0| \cup \{a_l : l \in I\}$ for some $I \subseteq \{0, 1, \ldots, n-1\}$.

(8) We can replace "definable" by "definable with parameters" when we allow the ψ_φ's to contain parameters from M (or A, in (2), (3)). As we can add those elements as individual constants, all results on "definable" can be translated to results on "definable with parameters".

Lemma 1.4. *If $\Psi = \langle \psi_\varphi : \varphi \in L \rangle$ is a defining schema and for some model of T it defines an end extension then for every model of T it defines a unique end extension. The type it defines will be denoted by $t(M, \Psi)$.*

Proof. Easy.

Definition 1.5. (1) Let p be a 1-type over M, $P(x) \in p$. We call p unbounded if for some a and N, $M < N$, $a \in N$ realizes p and $P(M) < a$.

(2) An unbounded type p over M is an end-extension type if for every N, $M < N$ and φ and $\bar{b} \in N$ exactly one of $p \cup \{\varphi(x, \bar{b})\}$, $p \cup \{\neg \varphi(x, \bar{b})\}$ is unbounded over N (always at least one of them is).

Lemma 1.6. *For p an unbounded type over M*

(A) *p is an end-extension type iff*

(B) *for every $\varphi(x, \bar{y})$ there is a finite $p_\varphi \subseteq p$ such that for any $\bar{b} \in M$, not both $p_\varphi \cup \{\varphi(x, \bar{b})\}$ and $p_\varphi \cup \{\neg \varphi(x, \bar{b})\}$ are unbounded (so easily one is).*

Remark. Note that if p is pure (i.e. without parameters) the choice of the model does not matter.

Proof. Easy (see Gaifman [G3]).

Definition 1.7. For any p as in 1.6, we attach a schema $\Psi_p = \langle \psi^p_\varphi : \varphi \in L \rangle$ where $\psi^p_\varphi = (\forall x_1 \in P)(\exists x > x_1)[\bigwedge p_\varphi \wedge \varphi(x, \bar{y})]$ (so Ψ_p has parameters iff p has).

LEMMA 1.8 (GAIFMAN [G3]). *Suppose*

(A) *T has Skolem functions,*

(B) *T satisfies the regularity schema* (*see* 1.2(3)),

(C) *p is a pure end-extension type.*

Then:

(1) *Ψ_p is an end extension defining schema.*

(2) *Moreover, it is an almost minimal defining schema.*

PROOF. (1) Let M be a model of T; it is easy to check $q = t(M, \Psi_p)$ (see 1.4) is finitely satisfiable in M, and is the only complete type over M extending p which is unbounded. So there is N_1, $M < N_1$ and $a \in N_1$ which realizes q: so $a \notin |M|$, and let $N < N_1$ be the Skolem hull of $|M| \cup \{a\}$; so $M < N$. Suppose not $M < N$, so for some Skolem function $F(x, \bar{y})$ and $\bar{b} \in |M|$, $F(a, \bar{b}) \notin |M|$ but $F(a, \bar{b}) < c$ for some $c \in |M|$ (hence $[F(x, \bar{b}) < c] \in q$). Let $\varphi(x; z, \bar{y}) = (F(x, \bar{y}) = z)$, p_φ as in 1.6(B), so as $p_\varphi \cup \{F(x, \bar{b}) < c\} \subseteq q$ it is unbounded (in M). Hence by the regularity schema for some $d < c$, $d \in |M|$, $p_\varphi \cup \{F(x, \bar{b}) = d\}$ is unbounded. But also $p_\varphi \cup \{F(x, \bar{b}) \neq d\} \subseteq q$ (as $F(a, \bar{b}) \notin |M| \Rightarrow (F(x, \bar{b}) = d) \notin q$) so it is unbounded; contradiction to the definition of p_φ.

(2) So suppose $M < N^* < N$ and choose $a^* \in P(N^*) - P(M)$; then necessarily $a^* = F(a, \bar{b})$, $\bar{b} \in M$ (for some Skolem function F of T). Let $\varphi(x; z, \bar{y}) = (F(x, \bar{y}) = z)$, p_φ as in 1.6(B). So $p_\varphi \cup \{F(x, \bar{b}) = a^*\}$ is unbounded in N^*, as $N^* < N$, and a realizes it. On the other hand $p_\varphi \cup \{F(x, \bar{b}) \neq a^*\}$ is unbounded in N^* because for every $c \in M$, $p_\varphi \cup \{F(x, \bar{b}) \neq c\}$ is unbounded. Clearly $M \models (\forall z_0)$("$p_\varphi \cup \{F(x, \bar{b}) \neq z_0\}$ is unbounded"); so also N^* satisfies this, contradiction.

THEOREM 1.9 (GAIFMAN [G3]). *Let T be with Skolem function, Ψ be an end-extension schema, M be a model of T, and I an order type. Then there is a model N such that*:

(1) *N is the Skolem hull of $|M| \cup \{a_i : i \in I\}$ and is an end extension of M.*

(2) *a_i realizes $t(M_i, \Psi)$ where M_i is the Skolem hull of $|M| \cup \{a_j : j < i\}$* (*see* 1.4).

(3) *Part* (2) *implies $\{a_i : i \in I\}$ is an indiscernible sequence over $|M|$.*

(4) *If the schema is strongly minimal* [*almost minimal*] *and $M < N^* < N$* [*$M \leq N^* \leq N$*] *then N^* is the Skolem hull of $|M| \cup \{a_i : i \in J\}$, for some $J \leq I$* [*some J an initial segment of I*].

(5) *If the schema is strongly rigid any automorphism of N over M is induced by an order-preserving permutation of I.*

PROOF. Left to the reader.

§2. Existence of end extensions.

THEOREM 2.1 (M. RUBIN). *Suppose T is a countable theory with Skolem functions and satisfies the regularity schema. For any countable model M of T*:

(1) *M has a minimal end extension N* (*i.e.* *$M < N^* < N \Rightarrow N^* = M$ or $N^* = N$*).

(2) *For every countable* [*$\aleph_1$-like*] *order I there is a countable* [*$\aleph_1$-like*] *N, $M < N$, and a_i $(i \in I)$ such that $M \leq N^* \leq N$ implies N^* is the Skolem hull of $|M| \cup \{a_j : j \in J\}$ for some initial segment J of I.*

PROOF. This proof is due to the author. (1) In the proof of 1.2(3) $\Rightarrow$ (4) the added fact we need is:

(*) if in M, $\varphi(x, \bar{b})$ is unbounded, $\bar{b}, c \in |M|$, $F(x, \bar{y})$ a function symbol, *then* for some $\psi(x, \bar{b}')$ ($\bar{b}' \in M$) the following hold:

(i) $\psi(x, \bar{b}') \vdash \varphi(x, \bar{b})$,

(ii) $\psi(x, \bar{b}')$ is unbounded,

(iii) on $\{x \in M: \psi(x, \bar{b}')\}$, $F(x, \bar{c})$ is constant or one-to-one.

(For we define inductively $\varphi_n(x, \bar{b}_n)$ whose union is a complete type and N generated by $|M|$ and an a realizing $\{\varphi_n(x, \bar{b}_n): n < \omega\}$. If $d \in |N| - |M|$, then $d = F(a, \bar{c})$ for some $\bar{c} \in |M|$, $F \in L$ so for some n (if we construct properly the φ_n's) $F(x, \bar{c})$ is one-to-one on $\{x \in |M|: \varphi_n(x, \bar{b}_n)\}$. Hence a is in the Skolem hull of $|M| \cup \{c\}$. So all N is in it. In defining the φ_n's we have to take care of "deciding" every formula, making the extension an end extension, and minimality.)

PROOF OF (*). If for some d, $\varphi(x, \bar{b}) \wedge F(x, \bar{c}) = d$ is unbounded, we are finished. Otherwise let $G(y, \bar{z}_0, \bar{z}_1)$ be a Skolem function for $(\exists x)[\varphi(x, \bar{z}_0) \wedge F(x, \bar{z}_1) = y]$, then $\psi = \varphi(x, \bar{b}) \wedge x = G(F(x, \bar{c}), \bar{b}, \bar{c})$ is as required.

(2) We can assume $|I| = \aleph_0$ (and then by iteration prove for $\aleph_1$ like I). Let $I = \{s(l): l < \omega\}$ and define inductively formulas $\varphi_l(x_{s(0)}, \ldots, x_{s(l)}, \bar{b}_l)$ ($\bar{b}_l \in |M|$) such that:

(i) φ_l is a conjunct of φ_{l+1}, every formula (with variables among $\{x_i: i \in I\}$ and parameters) it or its negation is a conjunct of some φ_l.

(ii) $s(n) < s(k)$ implies $x_{s(n)} < x_{s(k)}$ is a conjunct of some φ_l.

(iii) For the permutation σ of $\{0, \ldots, l\}$ such that $s(\sigma(0)) < s(\sigma(1)) < \cdots < s(\sigma(l))$,

$$M \models (\forall y_0 \in P)(\exists x_{s(\sigma(0))} > y_0)(\forall y_1 \in P)(\exists x_{s(\sigma(1))} > y_1) \cdots (\forall y_l \in P)(\exists x_{s(\sigma(l))} > y_l)\varphi_l(x_{s(0)}, \ldots, x_{s(l)}, \bar{b}_l).$$

This is equivalent to: "there are models $M = M_0 < M < \cdots < M_l$ such that M_{i+1} is the Skolem hull of $|M_i| \cup \{a_i\}$ and $M_{l+1} \models \varphi[a_{\sigma^{-1}(0)}, \ldots, a_{\sigma^{-1}(l)}, \bar{b}_l]$.

(iv) Each end extension $M_i < M_{i+1}$-form (iii) is minimal".

(v) Given a term $\tau = \tau(x_{s(0)}, \ldots, x_{s(m)}, \bar{c})$, for some e either $\tau < x_{s(\sigma(e))} \wedge x_{s(\sigma(e))} = \tau_2(\tau, x_{s(\sigma(0))}, \ldots, x_{s(\sigma(e-1))}, \bar{c})$ or $\tau \geqslant x_{s(\sigma(e))} \wedge \tau = \tau_2(x_{s(\sigma(0))}, \ldots, x_{s(\sigma(e))}, \bar{c})$ is a disjunct of some φ_n where τ_2 is a term in $\bar{b} \in M$. We arrange the assignments in such a way that each of the possible formulas and terms will be dealt in the construction of one of the φ_l's. (This is possible as we have only $\aleph_0$ terms and formulas with our $\aleph_0$ variables and parameters from M). Having built φ_l we use part (1) of the theorem l times to get the chain $M_0 < \cdots < M_{l+1}$, and then we decide for (i) and (v) according to the situation in M_{l+1}, (v) is possible as each of the extensions is minimal. Now $\Gamma = \{\varphi_l: l < \omega\}$ is finitely satisfiable in M so we have an elementary extension N^1, $M < N^1$ with a set $\langle a_i: i < \omega \rangle$ realizing Γ (when a_i stands for $x_{s(i)}$) and we take N to be the Skolem hull of this sequence and $|M|$ in N^1. Given N^*, $M \leq N^* \leq N$, we look at $\{a_i: i < \omega\} \cap |N^*|$; as N is an end extension of N^* this is an initial segment and condition (v) assures us that N^* is generated by it and M.

DEFINITION 2.2. T satisfies the inaccessibility schema if $(\forall x)(\exists y)(y > x) \in$

T and there is a rank function R into P (i.e. $(\forall x)P(R(x)) \in T$) which is the identity over P and for every $\varphi(x;\bar{z})$, for every $z_0 \in P$, for some z_1 any φ-type over $\{x: R(x) < z_0\}$ realizes in P, is realized by some $z < z_1$.

$$(\forall z_0 \in P)(\exists z_1 > z_0)[(\forall y_0 \in P)(\exists y_1 < z_1)E_\varphi(y_0, y_1; z_0)]$$

where

$$E_\varphi(y_0, y_1; x) = P(y_0) \wedge P(y_1) \wedge P(x)$$
$$\wedge (\forall z_0, z_1, \dots)\left[\bigwedge_l R(z_l) < x \rightarrow \varphi(y_0; z_0, z_1, \dots) \equiv \varphi(y_1; z_0 z_1, \dots)\right].$$

Remark. The inaccessibility schema implies the regularity schema by the following lemma and 1.3.

Lemma 2.3 (Folk?). *Suppose T satisfies the inaccessibility schema, is countable and has Skolem functions. Then every model of T cofinality $\aleph_0$ has an end extension (in fact, not only $M < N$ but also $a \in |N| - |M| \Rightarrow R^N(a) \notin |M|$).*

Proof. First we prove, for any model M of T that:

$(*)$ if $\varphi(x, \bar{a})$ is unbounded (in P of course), then for any z_0, ψ for some x_0, $\varphi(x, \bar{a}) \wedge E_\psi(x, x_0; z_0)$ is unbounded.

Let us prove $(*)$. For let z_1 be such that $M \models (\forall x \in P)(\exists x_1 < z_1)E_\psi(x, x_1; z_0)$ (exists by 2.2). Let F be a function (of L) such that if $\varphi(x, \bar{y}_0) \wedge E_\psi(x, x_0; z_0)$ is bounded, then $F(\bar{y}_0, x_0, \bar{z}_0)$ is such a bound. If there is no x_0 as required in $(*)$, then $\{F(\bar{a}, x_1, z_0); x_1 < z_1\}$ is unbounded. But then we get a contradiction to the assumption that T satisfies the inaccessibility schema.

Let us continue the proof of 2.3. Let $\{\varphi_n(x, \bar{z}_n): n < \omega\}$ be a list of all formulas of L, each appearing infinitely many times, and $a_n \in |M|$ $(n < \omega)$ an unbounded sequence in $P(M)$. We then define inductively $\psi_n(x, \bar{b}_n)$ such that $\psi_n(x, \bar{b}_n)$ is unbounded and

$$\psi_{n+1}(x, \bar{b}_{n+1}) = \psi_n(x, \bar{b}_n) \wedge E_{\varphi_n}(x, c_n, a_n)$$

for some c_n. We let ψ_0 be $P(x)$. Then $p = \{\psi_n(x, \bar{b}_n): n < \omega\}$ determines the extension: p is finitely satisfiable in M, hence realized by an element a^* in some N^*, $M < N^*$, clearly $N^* \models a_n < a^*$ for each n, so $a^* \notin M$. Let N be the Skolem hull of $M \cup \{a^*\}$ so $M < N$ and $M \neq N$. We now prove $M \leq N$. Suppose not, so for some function symbol F and $\bar{d} \in M$ and n, $F^N(a, \bar{d}) \notin M$, $N \models F^N(a, \bar{d}) < a_n$, but for some $n, \varphi_n = [F(x, \bar{y}) = z]$ and we get a contradiction to the construction.

Definition 2.4. (1) T satisfies the wellordering schema, if in every (definable) nonempty set of elements, there is a first one:

$$(\forall \bar{y})[(\exists x)\varphi(x, \bar{y}) \rightarrow (\exists x_0)(\varphi(x_0, y) \wedge (\forall x < x_0) \neg \varphi(x, \bar{y}))].$$

(2) T satisfies the induction schema if also there is no limit element. We then denote by 0, $z + 1$ the first element and the successor of z (when it exists) resp.

Theorem 2.5. *If T satisfies the induction schema $\forall x P(x) \in T$ then*:

(1) *T has definable Skolem functions (here only the wellordering schema is needed).*

(2) *If* $(\forall x)(\exists y)(y > x) \in T$ *then T satisfies the inaccessibility schema (hence the regularity schema, by 2.3). [Notice* $(\forall x)P(x) \in T$, *so let R be the identity.]*

(3) *If* $(\forall x)(\exists y)(y > x) \in T$, *T is countable, then in addition T has an end-extension type; moreover each unbounded formula* $\psi(x)$ *belongs to such a type.*

Conclusion 2.6. If T is countable, satisfies the induction schema and $(\forall x)(\exists y)(y > x) \in T$ *then* every model of T has an almost minimal end extension; moreover T has an almost minimal end-extension type.

PROOF OF 2.6. Immediate by 2.5 and 1.8.

PROOF OF 2.5. (1) Obvious (we choose the first element).

We now make some notations and observations.

Notation 2.6. Let $<^{lx}_m = <^{lx}_m(x_0, \ldots, x_{m-1}; y_0, \ldots, y_{m-1})$ be the formula expressing the lexicographic order among sequences of length m, $\bar{x} <^{lx}_m \bar{y}$ for convenience. Let $<_m = <_m (x_0, \ldots, x_{m-1}; y_0, \ldots, y_{m-1})$ be the formula saying $\max\{x_0, \ldots\} < \max\{y_0, \ldots\}$ *or* $\max\{x_0, \ldots\} = \max\{y_0, \ldots\}$ and $\bar{x} <^{lx}_m \bar{y}$. Again we write $\bar{x} <_m \bar{y}$ for convenience and $\leq_m$, $\leq^{lx}_m$ have the obvious meaning. Let $x^m = \langle x, \ldots, x\rangle$ (m x's appear).

Observation 2.7. If T satisfies the induction schema, then also $<_m$ satisfies the induction schema, i.e. if a definable set of m-sequences include the first element and is closed under successor, then it includes all sequences [the same holds for the regularity, wellordering and inaccessibility schemas].

PROOF. For a counterexample $\varphi(x, \bar{c})$ in a model M, we choose a minimal a_0 for which there is $\bar{x}$, $\bar{x} \leq_m a_0^m \wedge \varphi(\bar{x}, \bar{c})$ then we continue to choose $a_0, a_1, \ldots$ till we get the $<_m$-minimal $\langle a_0, \ldots, a_{m-1}\rangle$.

Notation 2.8. For every $m < \omega$, $\varphi = \varphi(x, \bar{y})$ ($\bar{y}$ of length m) and m-sequences $\bar{z}$ we define the formula:

$$E^+_\varphi(u, v; \bar{z}) = (\forall \bar{y} <_m \bar{z})[\varphi(u, \bar{y}) \equiv \varphi(v, \bar{y})], \quad E^*_\varphi(u, v; z) = E^+_\varphi(u, v, z^m)$$

and an order, essentially on the family of $E^+_\varphi(u, v; \bar{z})$-equivalence classes

$$Lx_\varphi(u, v; \bar{z}) = (\exists \bar{y} <_m \bar{z})[\neg \varphi(u, \bar{y}) \wedge \varphi(v, \bar{y}) \wedge E^+_\varphi(u, v; \bar{y})]$$

(this is a lexicographic order on φ-types).

Observation 2.9. (1) For every φ and $\bar{z}_0$, $E^+_\varphi(-, -; \bar{z}_0)$ is an equivalence relation.

(2) For every φ and $\bar{z}_0$, $Lx_\varphi(-, -; \bar{z}_0)$ is an order on the family of $E_\varphi(-, -; \bar{z}_0)$-equivalence classes provided that T satisfies the wellordering schema.

(3) In (2) this order satisfies the induction schema and has a last element provided that T satisfies the induction schema.

PROOF. (1) and (2) are immediate and (3) follows by induction on $\bar{z}_0$ (in the order $<_m$).

PROOF OF 2.5(2). Let $\varphi = \varphi(x; z_0, \ldots, z_{m-1})$. We prove that

$$(\forall \bar{z})(\exists x_1)[(\forall y_0)(\exists y_1 < x_1)E^+_\varphi(y_0, y_1; \bar{z})] \in T.$$

As $\forall x P(x) \in T$, R is the identity and $E^*_\varphi(y_0, y_1, z)$, $E^+_\varphi(y_0, y_1, z^m)$ are equivalent so we shall finish.

So for each $\bar{z}_0$, we prove by induction on x^* that

$$(\exists x_1)(\forall y_0)[Lx_\varphi(y_0, x^*; \bar{z}_0) \vee E^+_\varphi(y_0, x^*; \bar{z}_0) \rightarrow (\exists y_1 < x_1) E^+_\varphi(y_0, y_1; \bar{z}_0)]$$

holds. This is possible by 2.9(3) and gives the desired assertion as $Lx_\varphi(-,-;\bar{z})$ has a first element (equivalence class, more exactly).

From now until the proof of 2.5(3), we assume the induction schema and that $(\forall x)(\exists y)(y > x) \in T$.

Observation 2.10. Suppose $\bar{z}_1 <_m \bar{z}_2$, $x_l / E^+_\varphi(-,-;\bar{z}_l)$ $(l = 0, 1)$ is an $Lx_\varphi(-,-;\bar{z}_l)$-minimal $E^+_\varphi(-,-;\bar{z}_l)$-equivalence class intersecting $\{x: \psi(x, \bar{z})\}$ by an unbounded set. *Then* $E^+_\varphi(x_1, x_2; \bar{z}_1)$.

PROOF OF 2.10. Otherwise there are two possibilities: (A) $Lx_\varphi(x_1, x_2; \bar{z}_1)$ but then $x_2 / E^+_\varphi(-,-;\bar{z}_2)$ is a subset of $x_2/E^+_\varphi(-,-;\bar{z}_1)$ hence the latter is unbounded too, contradicting the $Lx_\varphi(-,-;\bar{z}_1)$-minimality of $x_1/E^+_\varphi(-,-;\bar{z}_1)$. (B) $Lx_\varphi(x_1, x_2; \bar{z}_1)$ then as 2.5(2) was proved, and $\{u: E^+_\varphi(u, x_1; \bar{z}_2)\}$ is unbounded, by 2.3(*) for some $x'_1 E^+_\varphi(x'_1, x_1; \bar{z}_1)$ and $\{u: E^+_\varphi(u, x'_1; \bar{z}_2)\}$ is unbounded, contradicting the $Lx_\varphi(-,-;\bar{z}_2)$-minimality of $x_2/E^+_\varphi(-,-;\bar{z}_2)$.

Observation 2.11. For any formulas $\psi(u, \bar{z})$ and $\varphi(u, \bar{v})$ there is $\psi_1(u, \bar{z})$ such that: *if* $\{u: \psi(u, \bar{z})\}$ is unbounded, *then* $\{u: \psi_1(u, \bar{z})\}$ is unbounded and for every $\bar{v}_1$, exactly one of $\{u: \psi_1(u, \bar{z}) \wedge \varphi(u, \bar{v}_1)\}$, $\{u: \psi_1(u, \bar{z}) \wedge \neg \varphi(u, \bar{v}_1)\}$ is unbounded.

PROOF OF 2.11. $\psi_1(x_0, \bar{z})$ will say: there is $\bar{z}_1$ such that

(i) x_0 is the first element in its $E^+_\varphi(u, v; \bar{z}_1)$-equivalence class.

(ii) This class is the $Lx_\varphi(u, v, \bar{z}_1)$-minimal $E^+_\varphi(u, v; \bar{z}_1)$-equivalence class which has an unbounded intersection with $\{u: \psi(u, \bar{z})\}$ (there is such an element for every $\bar{z}_1$ by 2.3(*) and the wellordering schema; ψ_1 satisfies the conclusion on 2.11 by 2.10).

PROOF OF 2.5(3). Let $\{\varphi_n(u, \bar{v}_n): n > \omega\}$ be a list of all formulas of $L(T)$. We define by induction unbounded formulas $\psi_n(x)$ such that $\psi_n = \psi_1$, $(\forall x)(\psi_{n+1}(x) \rightarrow \psi_n(x))$ and we get ψ_{n+1} from ψ_n, φ_n as in 2.11. Now $\{\psi_n(x): n > \omega\}$ is an end-extension type by 1.6 ($p_{\varphi_n} = \{\psi_n\}$).

Claim 2.12. Suppose M is a model of T.

(1) If M is κ-like, κ regular, then T satisfies the regularity schema.

(2) If M is κ-like, κ strongly inaccessible, $\forall x P(x) \in T$ *then* T satisfies the inaccessibility schema (instead $\forall x P(x) \in T$ we can assume $|\{b: R^M(b) = a_0\}| < \kappa$ for every a_0, R^M a one-place function into P, $(\forall x \in P)(R(x) = x)$).

(3) If M is $\aleph_0$-like (e.g. $M = (\omega, <, \dots)$) then T satisfies the induction schema (here $P^M = M$ if you want).

(4) If $P(M)$ is wellordered, T satisfies the wellordering schema.

PROOF. Easy.

Conclusion 2.13. Every model of a countable theory $T = \mathrm{Th}(\omega, <, \dots)$ has an end extension.

PROOF. See 2.12, 2.6.

Claim 2.14. For $A \subseteq |M|$, $M \models T$ let

$$T(A, M) = T \cup \{\varphi(a_1, \dots): a_1, \dots \in A, M \models \varphi[a_1, \dots]\},$$

let $T(\langle c_0, \dots \rangle, M)$ be $T(\{c_0, \dots\}, M)$. [So we extend the language by individual constants: the elements of A.]

(1) A defining end-extension schema for T is (essentially) a defining end-extension schema for $T(A, M)$. Similar assertions hold for a defining almost minimal [minimal] [strongly minimal] [rigid] schema.

(2) If p is an [almost minimal] [minimal] [strongly minimal] [rigid] end-extension type in T, them so is it in $T(A, M)$.

PROOF. Immediate.

THEOREM 2.15. (1) (*Gaifman* [G3]). *If T satisfies the induction schema and also definition by induction is possible (i.e. T contains Peano arithmetic, $\forall x P(x) \in T$ and T is countable) then for each unbounded $\psi(x)$, there is a rigid strongly minimal end-extension type p, $\psi(x) \in p$.*

(2) *If T satisfies the wellordering schema, and also definition by induction is possible. $M \models T$, M, T are countable, I a countable order, then there are N, a_s $(s \in I)$ such that $M \leq N$, N is the Skolem hull of $|M| \cup \{a_s : s \in I\}$, and $M < N^* < N$ implies N^* is the Skolem hull of $|M| \cup \{a_1 : N \in J\}$ for some $J \subseteq I$.*

PROOF. (1) See [G3].

(2) Left to the reader.

Claim 2.16. Every finite end-extension type is minimal.

PROOF. Easy.

EXAMPLES 2.17. (1) $T = \mathrm{Th}(\omega, <)$. T has (essentially) only one unbounded (pure) type, the empty one and it is strongly minimal but not rigid (as any model of T has the form $\omega + ZI$).

(2) $T = \mathrm{Th}(\omega, <, +)$. Every complete pure unbounded type (consistent with T) is an end-extension type (and vice versa) and any such type is minimal but not strongly minimal nor rigid. (We add names to every natural n (which is definable, of course) and to $P_r^n(x) = (\exists y)(x = ny + r)$. Now we have elimination of quantifiers and a complete unbounded type is $p = \{P_{r(n)}^n(x) : n < \omega\}$ (when it is consistent). If in M there is an element divisible by every n, the extension by p is not rigid.)

(3) For T = true arithmetic, there is an almost minimal not minimal end-extension type.

(4) For T = true arithmetic, there is a strongly rigid not almost minimal end-extension type (essentially extend twice and use a type of the pair of generators).

Question 2.18. (1) Give an example of a countable $T = \mathrm{Th}(\omega, <, \dots)$ with no minimal end-extension type.

(2) Give an example of a countable $T = \mathrm{Th}(\omega, <, \dots)$ with a finite not strongly minimal end-extension type.

THEOREM 2.19. (1) *Suppose T has Skolem functions, $\forall x P(x)$ and T satisfies the inaccessibility schema. If every model of T has an end extension then for some $A \subseteq |M|$, $M \models T$, $T(A, M)$ has a definable end-extension schema (so every model of $T(A, M)$ hence every $|T|$-universal model of T (note that w.l.o.g. $|A| \leq |T| + \aleph_0$) has an end extension).*

(2) *In* (1) *we can replace the second clause by: for some regular $\lambda \geq |T|$, every model of T of cardinality λ^+ and cofinality λ^+ has an end extension.*

PROOF. For every formula $\varphi(x, \bar{y})$ and $M \models T$ we correspond a tree $\mathrm{Tr}(\varphi, M)$:

(i) the set of levels is P ordered by $<$;

(ii) the elements of the tree of level a are of the form $\langle a, b/E^*_\varphi(u, v; a)\rangle$;

(iii) the tree is ordered by $\langle a_1, b_1/E^*_\varphi(u, v; a_1)\rangle \leq \langle a_2, b_2/E^*_\varphi(u, v, a_2)\rangle$ if $a_1 \leq a_2$ and $E^*_\varphi(b_1, b_2, a_1)$.

Suppose $M < N \models T$, $b \in |N| - |M|$; then for every $a \in M$, for some $b_a \in M$, $N \models E^*_\varphi[b_a, b, a]$ (by the inaccessibility schema), and b_a $(a \in M)$ form a branch of $\mathrm{Tr}(\varphi, M)$, and $\mathrm{tp}_\varphi(a, |M|, N)$ is definable in M, with parameters, iff this branch is. But by Shelah [S] T has a model M such that no tree $\mathrm{Tr}(\varphi, M)$ has an undefined (with parameters) branch. By the hypothesis for some N, $M < N$, so each $a \in |N| - |M|$ gives us the required schema.

(2) Same proof.

LEMMA 2.20. *Suppose* $\forall x P(x) \in T$ *(for simplicity) and every model of T of cardinality* $\mu \geq \lambda$, $\mu \leq 2^\lambda$, *has an end extension,* $\lambda \geq |T|$. *Then T satisfies the inaccessibility schema.*

PROOF. Let $M \models T$ have cardinality λ, $a \in |M|$, $\varphi(x, \bar{y}) \in L$. We define inductively models M_i $(i \in (2^\lambda)^+)$ such that $\alpha < i \Rightarrow M_\alpha < M_i$, $M_0 = M$, and $\|M_i\| \leq \lambda + |i|$, and let $a_i \in |M_{i+1}| - |M_i|$. Let $N = \bigcup M_i$. The number of $E^*_\varphi(u, v, a)$-equivalence classes is $\leq 2^{|\{b : b < a\}|} \leq 2^\lambda$; hence for some $i < (2^\lambda)^+$, for every $b \in |N|$ there is $b' \in |M_i|$ $N \models E^*_\varphi[b, b', a]$. So a_i exemplifies the satisfaction of $(\exists x_1)(\forall y_0)(\exists y_1 < x_1)E^*_\varphi(y_0, y_1, a)$ in N hence in M. As $a \in |M|$ was arbitrary, the schema is satisfied.

Conclusion 2.21. If T has Skolem functions, $(\forall x)P(x) \in T$ *then*

(A) every $|T|$-universal model of T has an end extension iff

(B) for some $A \subseteq |M|$, $M \models T$, $T(A, M)$ has an end-extension schema.

REMARK. In 1.2 we characterize when every countable model has an end extension; in 2.21 we characterize when every model has an end extension. "Every model of cofinality $\aleph_0$ has an end extension" is not clear. The case "for every $M \models T$ and I there is N, $M < N$, and a_s $(s \in I)$, satisfying 1.9(4)" ("strong minimal" case) is not clear either.

§3. The number of countable nonisomorphic models.

In this section T is always complete first-order theory in a language L, L countable, and we are interested in $I(\aleph_0, T)$—the number of nonisomorphic models of T of cardinality $\aleph_0$. We make some observations and then use them to get some conclusions. Of course, we assume T has only infinite models.

Observation 3.1. Let T^{df} be the definitional closure of T, i.e.

$$T^{\mathrm{df}} = T \cup \{(\forall \bar{x})(\varphi(\bar{x}) \equiv R_\varphi(\bar{x})) : \varphi(\bar{x}) \in L\}$$

(R_φ = new and distinct predicates) then $I(\aleph_0, T) = I(\aleph_0, T^{\mathrm{df}})$.

Claim 3.2 (Vaught). For a one-place predicate P, let $T_P = \{\psi : \psi^P \in T\}$, where ψ^P is ψ relativized to P. For a model M let M_P be the submodel of M with universe P^M. Then

(1) if M is a model of T, M_P is a model of T_P.

(2) If N is a countable model of T_P^{df} [i.e. $(T^{\mathrm{df}})_P$] then $N = N_P$ for some model M of T^{df}.

PROOF. (1) is trivial and (2) is a straightforward application of the theorem on omitting nonprincipal type [the theory is $T^{\mathrm{df}} \cup \{\varphi(a_1, \dots): N \models_\varphi^P [a_1, \dots],$ $\varphi \in L(T^{\mathrm{df}})$, the type $\{P(x) \wedge x \neq a: a \in |N|\}\}$].

Observation 3.3. $I(\aleph_0, T_P^{\mathrm{df}}) \leq I(\aleph_0, T^{\mathrm{df}})$.

(Immediate by 3.2).

Observation 3.4. Let $\bar{c} \in M$, M a model of T, then $I(\aleph_0, T(\bar{c})) > \aleph_0 \Rightarrow I(\aleph_0 T(\bar{c})) = I(\aleph_0, T)$.

THEOREM 3.5 (RUBIN). *If* $T = \mathrm{Th}(\omega, <, R, \dots)$, $|T| \leq \aleph_0$, $I(\aleph_0, T) = 2^{\aleph_0}$.

PROOF. Clearly T satisfies the induction schema $(\forall x)(\exists y)(y > x) \in T$ (2.12(3)). So T has Skolem function (2.5(1)) so it has a prime minimal model M_0 which is the Skolem hull of the empty set and also an end-extension type p (2.5(3)). Clearly M_0 is countable and for every order I it has a model M_I, $\|M_I\| \leq |I| + \aleph_0$, such that $\{M: M_0 \leq M \leq M_I\}$ ordered by inclusion has order type I^c, the completion of I (with a last element added, and if there is no first—a first element added) [see 1.9]. As $M \leq M_I \Rightarrow M_0 \leq M$, from the isomorphism type of M_I we can reconstruct that of I^c. As there are $2^{\aleph_0}$ pairwise nonisomorphic complete countable orders, $I(\aleph_0, T) = 2^{\aleph_0}$.

REMARK. In fact, Rubin proved that if T has Skolem functions, is countable, and has a model with end extension, then $I(\aleph_0, T) = 2^{\aleph_0}$; he used 2.1(2) of course.

Observation 3.6. If T has Skolem functions, $<$ an order, M_0 the prime model of T, and $<^{M_0}$ contains a copy of the order of the rationals then $I(\aleph_0, T) = 2^{\aleph_0}$.

PROOF OF 3.6. All elements of M_0 are interpretations of terms, so every Dedekind kind type of $<^M$ corresponds to a different type. So the number of types $\mathrm{tp}(a, \emptyset, M)$, $M \models T$, is $2^{\aleph_0}$, and as is well known, this implies $I(\aleph_0, T) = 2^{\aleph_0}$.

LEMMA 3.7. *If* $T = \mathrm{Th}(M_0)$, $M_0 = (\omega + \omega^*, <, R_1, \dots)$ *and* ω *is not definable in* M_0, *then* $I(\aleph_0, T) = 2^{\aleph_0}$.

PROOF. Notice that:

(i) Every element of M_0 is definable, so for M_0 "definable", "definable with parameters" are equivalent.

(ii) In M_0 (hence in T) every definable (with parameters) nonempty set has a last element and a first element (otherwise ω is definable).

(iii) T has definable Skolem functions (by (ii)); let us enumerate them $\{F_n: n < \omega\}$, F_0 the identity.

Let us define $\mathrm{cl}_m(x, y, M_0) = \{F_l^M(y_1, \dots): l \leq m$, and for every i, $y_i \leq x, y$ or $y_i \geq x, y\}$ (if $M = M_0$ we omit it). (So this closure always includes the outside of the interval x, y determined; clearly $z \in \mathrm{cl}_m(x, y)$ is definable by a formula in $L(T)$.

Define inductively on $l < \omega$, $a_l^m \in \omega$ such that $a_l^m < a_{l+1}^m$ and $\mathrm{cl}_m(a_l^m, a_{l+1}^m) \neq |M_0|$. This is possible as for each $b \in \omega^*$, $\mathrm{cl}_m(a_l^m, b)$ is finite hence $\neq |M_0|$. So the first $b \in M_0$ such that $a_l^m < b$, $\mathrm{cl}_m(a_l^m, b) \neq |M_0|$ should be in ω by (ii), and we call it a_{l+1}^m.

Now choose $b_l^m \in |M_0| - \mathrm{cl}_m(a_l^m, a_{l+1}^m)$, so $a_l^m < b_l^m < a_{l+1}^m$, and clearly $b_l^m <$

$b^m_{l+1} \in \omega$, $b^m_{l+1} \notin \mathrm{cl}_m(b^m_l, b^m_{l+2})$. By Ramsey's theorem and compactness for every order I, T has a model M_I, which is the Skolem hull of the indiscernible sequence $\{a_s : s \in I\}$, and for $s_1 < s_2 < s_3 \in I$, $a_{s_1} < a_{s_2} < a_{s_3}$ and $a_{s_2} \notin \mathrm{cl}_m(a_{s_1}, a_{s_3}, M_I)$ for every m.

We define a relation E_M on models M of T: aE_Mb if there are m, n and $c_0 \leq \cdots \leq c_n \in M$ such that $c_0 \leq a, b \leq c_n$ and $\bigcup_{m<\omega} \mathrm{cl}_m(c_l, c_{l+1}, M) = |M|$. Clearly E_M is an equivalence relation and each equivalence class is convex. In M_I, clearly if s_2 is the successor of s_1 in I then $a_{s_1} E a_{s_2}$, and if $\{s \in I : s_1 < s < s_2\}$ is infinite, then not $a_{s_1} E a_{s_2}$. Note that if $c \in M_I$, $c = F_m(a_{s_1}, \ldots, a_{s_n})$ for some m, $s_1 < \cdots < s_n \in I$, hence by the indiscernibility, for some l, for every $s \in I$, $c < a_s \Leftrightarrow s < s_l$ or $\tau < a_s \Leftrightarrow s \leqslant s_e$ or $(\forall s) c < a_s$ or $(\forall s) c > a_s$. So if I_1 is a converse subset of I of type Z, necessarily the convex hull of $\{a_s : s \in I_1\}$ is an E_{M_I}-equivalence class. Assume $I = ZJ$, then from the isomorphism type of M_I we can reconstruct J, except that we do not know what occurs before and after all the a_s. Either by looking more at the theory or taking $I = Z(\{s_0\} + J + \{s_1\})$ and add a_{s_0}, a_{s_1} as individual constants and using 3.4, we see that $I(\aleph_0, T) = 2^{\aleph_0}$.

THEOREM 3.8. *$I(\aleph_0, T) = 2^{\aleph_0}$ provided that:*

(A) *T has Skolem functions and in some $M \models T$, for some $\varphi \in L$, $\bar{c} \in M$, $< \in L$, $<^M$ is an order on the infinite set $\{b \in M : M \models \varphi[b, \bar{c}]\}$ (instead $<^M$ we can use $<(x, y, \bar{z})$) or*

(B) *there are formulas $\varphi(x, z), <(x, y, \bar{z})$ such that for every n there are $M_n \models T$, $\bar{c}_n \in M_n$, such that $<(x, y, \bar{c}_n)$ is an order of $\{b \in M_n : M_n \models \varphi[b, \bar{c}_n]\}$, which is a finite set of cardinality $\geq n$.*

PROOF. *Possibility* B: Let $p = p(\bar{z})$ be the set of formulas saying

(i) $\{u : \varphi(u, \bar{z})\}$ is infinite;

(ii) $\{u : \varphi(u, \bar{z})\}$ is ordered by $<(u, v; \bar{z})$;

(iii)$_\psi$ for every $\bar{y}$, $\{u : \varphi(u, \bar{z}) \wedge \psi(u, \bar{y})\}$ is empty or has a first element by $<(u, v; \bar{z})$.

Clearly p is consistent with T, so in some model M of T, some $\bar{c}$ realized p. By 3.4 it suffices to prove $I(\aleph_0, T(\bar{c}, M)) = 2^{\aleph_0}$, so by 3.1 it suffices to prove $I(\aleph_0, T(\bar{c}, M)^{\mathrm{df}}) = 2^{\aleph_0}$, so by 3.3 it suffices to prove $I(\aleph_0, T(\bar{c}, M)^{\mathrm{df}}_{\varphi(x,\bar{c})}) = 2^{\aleph_0}$. The last theory we consider satisfies hypothesis (A) (every definable set has a first element so there are definable Skolem functions).

Possibility A: As in the previous case we can assume $<^M$ is an order of M for any $M \models T$. As T has Skolem functions, it has a prime model M_0.

Case I. If $<^{M_0}$ contains a copy of the rationals, we are finished by 3.6. Otherwise $<^{M_0}$ is scattered.

Case II. If some formula $\varphi(x, \bar{c})$, $\bar{c} \in M_0$, define a set of order type ω. The conclusion follows by 3.6 (using 3.3, 3.4).

Case III. Not I nor II. Then $<^{M_0}$ has an interval $[a_0, b_0]$ of order type $\omega + \omega^*$. By 3.4, 3.3, we can assume $|M_0| = [a_0, b_0]$, and by not II, the hypothesis of 3.7 holds, so we are finished.

A limit on the possibility of extending 3.8 is provided by

EXAMPLE 3.9. For every T, we define T^* which consists of

(i) the sentences of T, with equality replaced by E, Sh:66

(ii) the sentences saying E is an equivalence relation, $<$ an order, and each equivalence class is dense (in the model) without first and last element.

Then $I(\aleph_0, T) \models I(\aleph_0, T^*)$.

Question 3.10 (Rubin). Suppose T is countable complete, and "says" $<$ orders the universe; and $T \restriction \{=\ <\}$ is not $\aleph_0$-categorical. Is $I(\aleph_0, T) \equiv 2^{\aleph_0}$?

REFERENCES

[G1] H. GAIFMAN, *Results concerning models of Peano arithmetic*, **Notices of the American Mathematical Society**, vol. 12 (1965), Abstract 65T–195, p. 377.

[G2] ———, *Uniform extension operators for models and their applications*, **Sets, models and recursion theory** (Crossley, Editor), North-Holland, Amsterdam, 1967, pp. 122–155.

[G3] ———, *Models and types of Peano arithmetic*, **Annals of Mathematical Logic.**

[K1] H. J. KEISLER, *Some model theoretic results on ω-logic*, **Israel Journal of Mathematics**, vol. 4 (1966), pp. 249–261.

[K2] ———, *Models with tree structures*, **Proceedings of the Tarski Symposium**, *Berkeley*, **1971**, **Proceedings of Symposia in Pure Mathematics**, XXV (Henkin, Editor), American Mathematical Society, 1974, pp. 331–348.

[MS] R. MACDOWELL and E. SPECKER, *Modelle der arithmetic*, **Infinitistic methods, Proceedings of the Symposium on Foundations of Mathematics**, *Warsaw*, **1959**, Pergamon Press, Oxford, London, Paris, 1961, pp. 257–263.

[SR] S. SHELAH and M. RUBIN, *On linearly ordered models and their end extensions*, **Notices of the American Mathematical Society**, vol. 22 (1975), p. A-646.

[S] S. SHELAH, *Models with second order properties*, **Annals of Mathematical Logic** (to appear).

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