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A note on extensions of infinitary logic*

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Abstract. We show that a strong form of the so called Lindström’s Theorem [4] fails to generalize to extensions of $L_{\kappa\omega}$ and $L_{\kappa\kappa}$: For weakly compact κ there is no strongest extension of $L_{\kappa\omega}$ with the (κ, κ) -compactness property and the Löwenheim-Skolem theorem down to κ . With an additional set-theoretic assumption, there is no strongest extension of $L_{\kappa\kappa}$ with the (κ, κ) -compactness property and the Löwenheim-Skolem theorem down to $< \kappa$.

By a well-known theorem of Lindström [4], first order logic $L_{\omega\omega}$ is the strongest logic which satisfies the compactness theorem and the downward Löwenheim-Skolem theorem. For weakly compact κ , the infinitary logic $L_{\kappa\omega}$ satisfies both the (κ, κ) -compactness property and the Löwenheim-Skolem theorem down to κ . In [1] Jon Barwise pointed out that $L_{\kappa\omega}$ is not maximal with respect to these properties, and asked what is the strongest logic based on a weakly compact cardinal κ which still satisfies the (κ, κ) -compactness property and some other natural conditions suggested by κ . We prove (Corollary 5) that for weakly compact κ there is no strongest extension of $L_{\kappa\omega}$ with the (κ, κ) -compactness property and the Löwenheim-Skolem theorem down to κ . This shows that there is no extension of $L_{\kappa\omega}$ which would satisfy the most obvious generalization of Lindström’s Theorem. A stronger result (Theorem 11) is proved under an additional assumption.

We use the notation and terminology of [2, Chapter II] as much as possible. We will work with concrete logics such as first order logic $L_{\omega\omega}$, infinitary logic $L_{\kappa\lambda}$ and their extensions $L_{\omega\omega}(\{Q_i : i \in I\})$ and $L_{\kappa\lambda}(\{Q_i : i \in I\})$ by generalized quantifiers. Therefore it is not at all critical which definition of a logic one uses as long as these logics are included and some basic closure properties are respected. We use $\mathcal{L} \leq \mathcal{L}'$ to denote the sublogic relation. Let \mathcal{P} be a property of logics. A logic \mathcal{L}^* is *strongest extension of \mathcal{L} with \mathcal{P}* , if

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1. $\mathcal{L} \leq \mathcal{L}^*$,
2. \mathcal{L}^* has property \mathcal{P} ,

and whenever \mathcal{L} is a sublogic \mathcal{L}' and \mathcal{L}' has property \mathcal{P} , then $\mathcal{L}' \leq \mathcal{L}^*$.

Let \mathcal{L} be a logic and κ and λ infinite cardinals. \mathcal{L} is (κ, λ) -compact if for all $\Phi \subseteq \mathcal{L}$ of power κ , if each subset of Φ of cardinality $< \lambda$ has a model, then Φ has a model. \mathcal{L} is κ -compact if it is (κ, ω) -compact. \mathcal{L} is weakly κ -compact if \mathcal{L} is (κ, κ) -compact. \mathcal{L} is fully compact if it is κ -compact for all κ . \mathcal{L} has the Löwenheim-Skolem property down to κ , denoted by $LS(\kappa)$ if every $\phi \in \mathcal{L}$ which has a model, has a model of cardinality $\leq \kappa$. If every sentence $\phi \in \mathcal{L}$ which has a model, has a model of cardinality $< \kappa$, we say that \mathcal{L} satisfies $LS(< \kappa)$. The order-type of the well-ordering R is denoted by $otp(R)$.

Theorem 1. [4] *The logic $L_{\omega\omega}$ is strongest extension of $L_{\omega\omega}$ with \aleph_0 -compactness and $LS(\aleph_0)$.*

Let C be a class of cardinals. Let

$$\mathfrak{A} \models \mathcal{Q}_C^{cf} xy\phi(x, y, \vec{z}) \iff \{(a, b) : \mathfrak{A} \models \phi(a, b, \vec{c})\}$$

is a linear order with cofinality in C .

By [9], $L_{\omega\omega}(\mathcal{Q}_C^{cf})$ is always fully compact. For C an interval we use the notation $\mathcal{Q}_{[\kappa, \lambda)}^{cf}$ and $\mathcal{Q}_{[\kappa, \lambda]}^{cf}$.

Proposition 2. *There is no strongest κ -compact extension of $L_{\omega\omega}$. In fact:*

1. *there are fully compact logics \mathcal{L}_n , $n < \omega$, such that $\mathcal{L}_n \leq \mathcal{L}_{n+1}$ for all $n < \omega$, but no \aleph_0 -compact logic extends each \mathcal{L}_n .*
2. *There is an \aleph_0 -compact logic \mathcal{L}_1 and a fully compact logic \mathcal{L}_2 such that no \aleph_0 -compact logic extends both \mathcal{L}_1 and \mathcal{L}_2 .*

Proof. Let $\mathcal{L}_n = L_{\omega\omega}(\{\mathcal{Q}_{[\aleph_0, \infty]}^{cf}\} \cup \{\mathcal{Q}_{\aleph_l}^{cf} : l < n\})$. By [9], each \mathcal{L}_n is fully compact. Clearly, no \aleph_0 -compact logic can extend each \mathcal{L}_n .

For the second claim, let \mathcal{L}_1 be the logic $L_{\omega\omega}(Q_1)$, where Q_1 is the quantifier “there exists uncountable many” introduced by Mostowski [8]. This logic is \aleph_0 -compact [3], see [2, Chapter IV] for more recent results. Let \mathcal{L}_2 be the logic $L_{\omega\omega}(Q_B)$, where Q_B is the quantifier “there is a branch” introduced by Shelah [10]. More exactly,

$$Q_B xytuM(x)T(y)(t \leq u)$$

if and only if \leq_T is a partial order of $T \subseteq M$ and there are D, \leq_D , f and B such that:

1. \leq_D is a total order of $D \subseteq M$
2. $f : \langle T, \leq_T \rangle \rightarrow \langle D, \leq_D \rangle$ is strictly increasing
3. $\forall s \in D \exists p \in T (f(p) = s)$
4. $B \subseteq T$ is totally ordered by \leq_T
5. $\forall b \in B ((p \in T \& p \leq_T b) \rightarrow p \in B)$
6. $\forall s \in D \exists b \in B (s \leq_D f(b))$.

The reader is referred to [10] for a proof of the full compactness of \mathcal{L}_2 .

Suppose there were an \aleph_0 -compact logic \mathcal{L} containing both \mathcal{L}_1 and \mathcal{L}_2 as a sublogic. It is easy to see that the class of countable well-orders can be expressed as a relativized pseudoelementary class in \mathcal{L} . This contradicts \aleph_0 -compactness of \mathcal{L} . \square

Lauri Hella pointed out that by elaborating the proof of claim (2) of the above proposition, we can make \mathcal{L}_1 fully compact. It was proved in [11] that, assuming GCH, there is no strongest extension of $\mathcal{L}_{\omega\omega}$ which is \aleph_0 -compact. Our proof of (1) of the above proposition essentially occurs in a note, based on a suggestion of Paolo Lipparini, added after Theorem 8 of [11].

Proposition 3. *Suppose $\kappa > \aleph_0$. There is no strongest extension of $L_{\kappa+\omega}$ with $LS(\kappa)$.*

Proof. Let $\mathcal{L}_1 = L_{\kappa+\omega}(Q_{\aleph_0}^{\text{cf}})$ and $\mathcal{L}_2 = L_{\kappa+\omega}(Q_{[\aleph_1, \kappa]}^{\text{cf}})$. By using standard arguments with elementary chains of submodels, it is easy to see that both \mathcal{L}_1 and \mathcal{L}_2 have $LS(\kappa)$, but the consistent sentence

$$R \text{ is a linear order with no last element } \wedge$$

$$\neg Q_{\aleph_0}^{\text{cf}} xy R(x, y) \wedge \neg Q_{[\aleph_1, \kappa]}^{\text{cf}} xy R(x, y)$$

has no models of size $\leq \kappa$. \square

It was proved in [11] that there is no strongest extension of $\mathcal{L}_{\omega\omega}$ with $LS(\omega)$.

Lemma 4. *Suppose κ is weakly compact. Then $L_{\kappa\omega}(Q_{\{\aleph_0\}}^{\text{cf}})$ and $L_{\kappa\omega}(Q_{[\aleph_1, \kappa]}^{\text{cf}})$ are weakly κ -compact. Moreover, if $\kappa > \omega$, these logics satisfy $LS(\kappa)$.*

Proof. The claim concerning $LS(\kappa)$ is proved with a standard elementary chain argument. We prove the weak compactness of $L_{\kappa\omega}(Q_{[\aleph_1, \kappa]}^{\text{cf}})$. The case of $L_{\kappa\omega}(Q_{\{\aleph_0\}}^{\text{cf}})$ is similar, but easier. For this end, suppose T is a set of sentences of $L_{\kappa\omega}(Q_{[\aleph_1, \kappa]}^{\text{cf}})$ and $|T| = \kappa$. We may assume $T \subseteq \kappa$. If $\alpha < \kappa$, then we assume that there is a model $\mathfrak{M}_\alpha \models T \cap \alpha$. In view of $LS(\kappa)$, it is not a loss of generality to assume that $\mathfrak{M}_\alpha = \langle \kappa, R_\alpha \rangle$, where $R_\alpha \subseteq \kappa \times \kappa$. Let $R(\alpha, \beta, \gamma) \iff R_\alpha(\beta, \gamma)$. By weak compactness there is a transitive M of cardinality κ such that

$$\langle H(\kappa), \epsilon, T, R \rangle <_{L_{\kappa\kappa}} \langle M, \epsilon, T^*, R^* \rangle$$

and $\kappa \in M$. Let $\mathfrak{M} = \langle M, S \rangle$, where $S(x, y) \iff R^*(\kappa, x, y)$. We claim that $\mathfrak{M} \models T$. We need only worry about the cofinality-quantifier. Cofinalities $< \kappa$ can be expressed in $L_{\kappa\kappa}$, so they are preserved both ways. Therefore also cofinality κ is preserved, and no other cofinalities can occur as the models have cardinality κ . \square

Since the logics $L_{\kappa\omega}(Q_{\aleph_0}^{\text{cf}})$ and $L_{\kappa\omega}(Q_{[\aleph_1, \kappa]}^{\text{cf}})$ cannot both be sublogics of a logic with $LS(\kappa)$, we get from the above lemma:

Corollary 5. *Suppose $\kappa > \omega$ is weakly compact. Then there is no strongest weakly κ -compact extension of $L_{\kappa\omega}$ with $LS(\kappa)$.*

The logic $L_{\kappa\omega}$ actually satisfies the property $\text{LS}(< \kappa)$ which is stronger than $\text{LS}(\kappa)$. To prove a result like the above corollary for the property $\text{LS}(< \kappa)$ we have to work a little harder. At the same time we extend the proof to extensions of $L_{\kappa\kappa}$. Here the cofinality quantifiers Q_C^{cf} will not help as $Q_{\{\lambda\}}^{\text{cf}}$ is definable in $L_{\kappa\kappa}$ for $\lambda < \kappa$. Therefore we use more refined order-type quantifiers.

Definition 6. Let $L_{\kappa\lambda}(Q)$ denote the formal extension of $L_{\kappa\lambda}$ by the generalized quantifier symbol $Qxy\phi(x, y, \vec{z})$. If \mathcal{Y} is a class of ordinals, we get a logic $L_{\kappa\lambda}(Q, \mathcal{Y})$ from $L_{\kappa\lambda}(Q)$ by defining the semantics by

$$\mathfrak{A} \models Qxy\phi(x, y, \vec{c}) \iff \text{otp}(\{\langle a, b \rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}) \in \mathcal{Y}.$$

If $\phi \in L_{\kappa\lambda}(Q, \mathcal{Y})$ and $\mathfrak{A} \models \phi$, we say that $\mathfrak{A} \models \phi$ holds in the \mathcal{Y} -interpretation.

If \mathfrak{A} is a model, then

$$o(\mathfrak{A}, \mathcal{Y}, \kappa, \lambda)$$

is the supremum of all $\text{otp}(\{\langle a, b \rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\})$ where $\phi \in L_{\kappa\lambda}(\mathcal{Y})$, $\vec{c} \in A^{<\lambda}$ and $\{\langle a, b \rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}$ is well-ordered.

Lemma 7. Suppose $\kappa \geq \lambda$, $\phi \in L_{\kappa\lambda}(Q)$, \mathfrak{A} is a model, $\vec{a} \in A^{<\lambda}$, and $\mathcal{Y}' \cap o(\mathfrak{A}, \mathcal{Y}, \kappa, \lambda) = \mathcal{Y}$. Then $\mathfrak{A} \models \phi(\vec{a})$ in the \mathcal{Y} -interpretation if and only if $\mathfrak{A} \models \phi(\vec{a})$ in the \mathcal{Y}' -interpretation.

Proof. This is a straightforward induction of the length of the formula ϕ . \square

Lemma 8. 1. Suppose $\kappa > \omega$, $\phi \in L_{\kappa\kappa}(Q)$, and ϕ has a model \mathfrak{A} in the \mathcal{Y} -interpretation. Then there is a submodel \mathfrak{B} of \mathfrak{A} of cardinality $\leq 2^\kappa$ and $\mathcal{Y}' \subseteq (2^\kappa)^+$ such that $\mathcal{Y}' \cap \kappa = \mathcal{Y}$ and $\mathfrak{B} \models \phi$ in the \mathcal{Y}' -interpretation.

2. Suppose $\kappa = \kappa^{<\kappa}$, $T \in L_{\kappa\kappa}(Q)$, $|T| \leq \kappa$ and T has a model \mathfrak{A} in the \mathcal{Y} -interpretation. Then for all $\xi < \kappa^+$ there is a submodel \mathfrak{B} of \mathfrak{A} of cardinality $\leq \kappa$ and $\mathcal{Y}' \subseteq \kappa^+$ such that $\mathcal{Y} \cap \xi = \mathcal{Y}' \cap \xi$ and $\mathfrak{B} \models T$ in the \mathcal{Y}' -interpretation.

Proof. We may assume $|A| \geq 2^\kappa$. Let us expand \mathfrak{A} by

1. A well-ordering \prec the order-type of which exceed all the order-types of well-orderings definable by subformulas of ϕ with parameters in A .
2. A new predicate P which contains those elements d of A for which $\text{otp}(\{\langle a, b \rangle : a < b < d\}) \in \mathcal{Y}$.
3. A predicate F which codes an isomorphism from each well-ordering, definable by a subformula of ϕ with parameters in A , onto an initial segment of \prec .

Let $\langle \mathfrak{A}, \prec, P, F \rangle$ be the expanded structure and $\langle \mathfrak{B}, \prec^*, P^*, F^* \rangle$ an $L_{\kappa\kappa}$ -elementary substructure of it of cardinality $\leq 2^\kappa$. Let

$$\mathcal{Y}' = \{\text{otp}(\{\langle a, b \rangle \in B^2 : a \prec^* b \prec^* d\}) : d \in P^*\}$$

It is easy to see that $\mathfrak{B} \models \phi$ in the \mathcal{Y}' -interpretation. The second claim is proved similarly. \square

Let π be a canonical bijection from triples of ordinals to ordinals such that $\pi[\kappa^3] = \kappa$ for each infinite cardinal κ . We say that a pair (δ_1, Z_1) codes a pair (δ_2, Z_2) if δ_1, δ_2 are ordinals, $Z_1 \subseteq \delta_1, Z_2 \subseteq \delta_2$ and there is a bijection $f : \delta_2 \rightarrow \delta_1$ such that

1. δ_1 is closed under π
2. $\pi(0, \alpha, \beta) \in Z_1 \iff f(\alpha) < f(\beta)$
3. $\pi(1, 0, \alpha) \in Z_1 \iff f(\alpha) \in Z_2$.

Suppose κ is an uncountable cardinal. The *weakly compact ideal* on κ is the ideal of subsets of κ generated by the sets $\{\alpha : (V(\alpha), \epsilon, A \cap V(\alpha)) \models \neg\phi\}$, where $A \subseteq V(\kappa)$ and ϕ is a Π_1^1 -sentence such that $(V(\kappa), \epsilon, A \cap V(\kappa)) \models \phi$.

Definition 9. A cardinal κ satisfies $\diamond(\text{WC})$ if it is weakly compact and there is a sequence $\langle A_\alpha : \alpha < \kappa \rangle$ such that

1. $A_\alpha \subseteq \alpha$ for $\alpha < \kappa$.
2. $(\forall A \subseteq \kappa)(\{\lambda < \kappa : A_\lambda = A \cap \lambda\} \notin \mathcal{I})$, where \mathcal{I} is the weakly compact ideal on κ .

We make the following remarks without proof (See note at the end): If κ is measurable $> \omega$, then κ satisfies $\diamond(\text{WC})$. If κ is weakly compact $> \omega$, then there is a generic extension which preserves all cardinals and in which κ satisfies $\diamond(\text{WC})$. If $V=L$, then every weakly compact cardinal $> \omega$ satisfies $\diamond(\text{WC})$.

Theorem 10. Suppose $\kappa > \omega$ satisfies $\diamond(\text{WC})$ and $2^\kappa = \kappa^+$. Then there is no strongest extension of $L_{\kappa\kappa}$ for which κ is weakly compact and which has $\text{LS}(< \kappa)$.

Proof. We shall construct two sets $\mathcal{Y}^1, \mathcal{Y}^2 \subseteq \kappa^+$ such that the logics $L_{\kappa\kappa}(Q, \mathcal{Y}^i)$ are weakly κ -compact for and these logics satisfy $\text{LS}(< \kappa)$, but no logic containing both $L_{\kappa\kappa}(Q, \mathcal{Y}^1)$ and $L_{\kappa\kappa}(Q, \mathcal{Y}^2)$ satisfies $\text{LS}(< \kappa)$. The sets \mathcal{Y}^i are constructed by induction together with ordinals $\xi_\alpha^i < \kappa^+$ such that:

$$\begin{aligned} \mathcal{Y}^i &= \bigcup_{\alpha < \kappa^+} \mathcal{Y}_\alpha^i \\ \mathcal{Y}_0^i &= \emptyset & \xi_0^i &= 0 \\ \mathcal{Y}_\alpha^i &= \mathcal{Y}_\beta^i \cap \xi_\alpha^i & \text{for } \alpha < \beta \\ \xi_\alpha^i &\leq \xi_\beta^i & \text{for } \alpha < \beta \\ \mathcal{Y}_\nu^i &= \bigcup_{\alpha < \nu} \mathcal{Y}_\alpha^i, & \xi_\nu^i &= \bigcup_{\alpha < \nu} \xi_\alpha^i, \text{ for } \nu = \cup \nu \\ \mathcal{Y}_\alpha^1 \cap \mathcal{Y}_\alpha^2 &= \emptyset & \text{for } \alpha < \kappa \end{aligned}$$

First we define \mathcal{Y}_α^i for $\alpha < \kappa$ in such a way that $L_{\kappa\kappa}(\mathcal{Y}^i)$ will in the end have the property $\text{LS}(< \kappa)$.

Let S_1, S_2 be a partition of the set of cardinals $< \kappa$ into two stationary sets. Let $\{\phi_\nu^i : \nu \in S_i\}$ list all $L_{\kappa\kappa}(Q)$ -sentences so that each sentence is listed as ϕ_ν^i for stationary many $\nu \in S_i$.

Suppose $\alpha = \lambda + 1$ and $\xi_\lambda^i = \lambda$. Suppose $\lambda \in S_i$.

Case 1. Suppose that (λ, A_λ) codes some pair (ξ, Z) . In this case we let

$$\mathcal{Y}_\alpha^i = \mathcal{Y}_\lambda^i \cup (Z \setminus \lambda), \xi_\alpha^i = \xi$$

$$\mathcal{Y}_\alpha^{3-i} = \mathcal{Y}_\lambda^{3-i}.$$

Case 2. Otherwise we let $\xi_\alpha^i = \lambda$, $\mathcal{Y}_\alpha^i = \mathcal{Y}_\lambda^i$, $\mathcal{Y}_\alpha^{3-1} = \mathcal{Y}_\lambda^{3-i}$.

Suppose then $\alpha = \lambda + 2$, $\xi_\lambda^i = \lambda \in S_i$ and we have defined $\xi_{\lambda+1}^i$ and $\mathcal{Y}_{\lambda+1}^i$.

Case 3. The sentence ϕ_λ^i has a model in the \mathcal{Y} -interpretation for some $\mathcal{Y} \subseteq \kappa^+$ with $\mathcal{Y} \cap \xi_{\lambda+1}^i = \mathcal{Y}_{\lambda+1}^i$. By Lemma 8 part 2, ϕ_λ^i has a model \mathfrak{A} of cardinality $< \kappa$ in the \mathcal{Y} -interpretation for some $\mathcal{Y} \subseteq \kappa$ of cardinality $< \kappa$ with $\mathcal{Y} \cap \xi_{\lambda+1}^i = \mathcal{Y}_{\lambda+1}^i$. Let μ be minimal such that $\phi_\lambda^i \in \mathcal{L}_{\mu\mu}(\mathcal{Y})$. Let $\xi_{\lambda+2}^i = o(\mathfrak{A}, \mathcal{Y}, \mu, \mu)$ and $\mathcal{Y}_{\lambda+2}^i = \mathcal{Y}$. Let $\mathcal{Y}_{\lambda+2}^{3-i} = \mathcal{Y}_{\lambda+1}^{3-i}$.

Case 4. Otherwise $\xi_{\lambda+1}^i = \xi_\lambda^i$, $\mathcal{Y}_\alpha^i = \mathcal{Y}_{\lambda+1}^i$, $\mathcal{Y}_\alpha^{3-1} = \mathcal{Y}_{\lambda+1}^{3-i}$.

Finally for all other $\alpha \leq \kappa$ we let ξ_α^i and \mathcal{Y}_α^i be defined canonically.

This ends the construction of \mathcal{Y}_α^i for $\alpha \leq \kappa$. Note that $\mathcal{Y}_\kappa^1 \cap \mathcal{Y}_\kappa^2 = \emptyset$. Moreover, if ϕ_ν^i has a model in the \mathcal{Y} -interpretation for some $\mathcal{Y} \supseteq \mathcal{Y}_\kappa^i$, then, by construction, ϕ_ν^i has a model of cardinality $< \kappa$ in the \mathcal{Y}_κ^i -interpretation.

Let $\mathcal{Y}_{\kappa+1}^i = \mathcal{Y}_\kappa^i \cup \{\kappa\}$ and $\xi_{\kappa+1}^i = \kappa + 2$. Next we shall define \mathcal{Y}_α^i and ξ_α^i for $\kappa + 1 < \alpha < \kappa^+$. For this, let $\langle T_\alpha : \kappa < \alpha < \kappa^+ \rangle$ enumerate all $L_{\kappa\kappa}(Q)$ -theories of cardinality $\leq \kappa$ in a language of cardinality $\leq \kappa$ which satisfy the condition that every subset of cardinality $< \kappa$ has a model in the \mathcal{Y}_κ^i -interpretation. Here we use the assumption $2^\kappa = \kappa^+$. We may assume $T_\alpha \subseteq \kappa$ for all α .

Suppose \mathcal{Y}_β^i and ξ_β^i have been defined for $\beta < \alpha$. If $\alpha = \cup \beta$, \mathcal{Y}_α^i and ξ_α^i are defined canonically. So assume $\alpha = \beta + 1$. Let $T : H(\kappa) \rightarrow H(\kappa)$ be the function $T(a) = T_\beta \cap a$. If $a \in H(\kappa)$, then $T(a)$ has a model \mathfrak{B}_a in the \mathcal{Y}_κ^i -interpretation. By construction, we may assume $\mathfrak{B}_a \in H(\kappa)$. Let $B : H(\kappa) \rightarrow H(\kappa)$ be the function $B(a) = \mathfrak{B}_a$. Let $Z \subseteq \kappa$ code $(\xi_\beta^i, \mathcal{Y}_\beta^i)$. By $\diamond(\text{WC})$, $W = \{\lambda < \kappa : A_\lambda = Z \cap \lambda\} \in \mathcal{I}^+$, where \mathcal{I} is the weakly compact ideal on κ . Let $A : \kappa \rightarrow H(\kappa)$ be the function $A(\alpha) = A_\alpha$. By the definition of \mathcal{I} , there are a transitive set M and Y^*, R^* such that

$$\langle H(\kappa), \epsilon, A, W, \mathcal{Y}_\kappa^i, B, T \rangle \prec_{\kappa\kappa} \langle M, \epsilon, A^*, W^*, Y^*, B^*, T^* \rangle$$

and $\kappa \in W^*$. Now $A^*(\kappa) = Z$ and, by construction, $Y^* \cap \xi_\beta^i = \mathcal{Y}_\beta^i$.

It is clear now that $B(\kappa)$ is a model of T_α in the Y^* -interpretation. By Lemma 8 there is a model \mathfrak{B} of cardinality $\leq \kappa$ of T_β in the Y^{**} -interpretation for some Y^{**} with $Y^{**} \cap \xi_\beta^i = \mathcal{Y}_\beta^i$. Let $\xi_\alpha^i = o(\mathfrak{B}, Y^{**}, \kappa, \kappa)$ and $\mathcal{Y}_\alpha^i = Y^{**} \cap \xi_\alpha^i$.

Finally, let $\mathcal{Y}^i = \bigcup_{\alpha < \kappa^+} \mathcal{Y}_\alpha^i$.

Claim 1. $L_{\kappa\kappa}(\mathcal{Y}^i)$ satisfies the $\text{LS}(< \kappa)$ -property.

Suppose ϕ is a sentence of $L_{\kappa\kappa}(\mathcal{Y}^i)$ with a model. Let $\lambda \in S_i$ such that $\xi_\lambda^i = \lambda$ and $\phi_\lambda^i = \phi$. By the construction of $\mathcal{Y}_{\lambda+2}^i$ there is a model of ϕ of cardinality $< \kappa$.

Claim 2. $L_{\kappa\kappa}(\mathcal{Y}^i)$ is weakly κ -compact.

Suppose $T \subseteq L_{\kappa\kappa}(\mathcal{Y}^i)$ is given and every subset of T_α of cardinality $< \kappa$ has a model in the \mathcal{Y}^i -interpretation. Then $T = T_\alpha$ for some α . By construction, every subset of T_α of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i \cap \kappa$ -interpretation. Thus the

definition of \mathcal{Y}_α^i is made so that T_α has a model \mathfrak{B} in the \mathcal{Y} -interpretation for some \mathcal{Y} such that $\mathcal{Y} \cap o(\mathfrak{B}, \mathcal{Y}, \kappa, \kappa) = \mathcal{Y}^i \cap o(\mathfrak{B}, \mathcal{Y}, \kappa, \kappa)$. Thus by Lemma 7, $\mathfrak{B} \models T_\alpha$ in the \mathcal{Y}^i -interpretation. The Claim is proved.

We can now finish the proof of the theorem. In a logic in which both the quantifier $Q_{\mathcal{Y}^1}$ and $Q_{\mathcal{Y}^2}$ are definable, we can say that the order-type of a well-ordering is in $\mathcal{Y}^1 \cap \mathcal{Y}^2$. Thus such a logic cannot satisfy $\text{LS}(< \kappa)$. \square

It is interesting to note that a proof like above would not be possible for the following stronger Löwenheim-Skolem property: A *filter-family* is a family $\mathcal{F} = (\mathcal{F}(A))_{A \neq \emptyset}$, where $\mathcal{F}(A)$ is always a filter on the set A . Luosto [6] defines the concept of a (κ^+, ω) -*neat* filter family. We will not repeat the definition here, its elements are closure under bijections, fineness, κ^+ -completeness, normality and upward relativizability (all defined in [6]). Suppose \mathcal{L} is a logic of the form $L_{\kappa\lambda}(\bar{Q})$ for some sequence \bar{Q} of generalized quantifiers. We say that \mathcal{L} has the \mathcal{F}, κ -*persistence property*, if for all models \mathfrak{A} and $B \in \mathcal{F}(A)$, we have $\mathfrak{A} \upharpoonright B < \mathfrak{A}$. Luosto proves that if \mathcal{L}_1 and \mathcal{L}_2 both satisfy the \mathcal{F}, κ -persistence property, then there is \mathcal{L}_3 such that $\mathcal{L}_1 \leq \mathcal{L}_3$, $\mathcal{L}_2 \leq \mathcal{L}_3$ and \mathcal{L}_3 satisfies the \mathcal{F}, κ -persistence property. Lipparini [5] proves a similar result for families of limit ultrafilters related closely to compactness.

Tapani Hyttinen pointed out that the assumption $2^\kappa = \kappa^+$ is not needed in Theorem 10, if κ is assumed to be measurable.

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Note added in proof: The principle $\diamond(\text{WC})$ has been considered in the literature by W. Sun (*Arch. Math. Logic* **32**, 1993), A. Hellsten (*Ann. Acad. Sci. Fenn. Math. Diss.* **134**, 2003) and J. Hamkins (A class of strong diamond principles, 2003).