POSSIBLE ORDERINGS OF AN INDISCERNIBLE SEQUENCE

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Let X be a set, R an n-ary relation on X ($n \geqslant 1$), and $<$ a linear ordering of X. We say that $<$ is *R-indiscernible* if

for all $x_1 < \ldots < x_n$ and all $y_1 < \ldots < y_n$ in X, and every permutation π of $\{1, \ldots, n\}$, we have

$$Rx_{\pi(1)} \ldots x_{\pi(n)} \text{ if and only if } Ry_{\pi(1)} \ldots y_{\pi(n)}.$$

If $<$ and $\prec$ are *R-indiscernible* orderings of X, must $<$ and $\prec$ be related in some way? The Theorem below answers this question.

Some terminology will help. "Ordering" will always mean linear ordering; $<^1$ means $<$ and $<^{-1}$ means $>$. If $<$ is an ordering of X, and $Y \subseteq X$, then we write $< \mid Y$ for the restriction of $<$ to Y. If $(X, <_X)$, $(Z, <_Z)$ are ordered sets with X, Z disjoint, we form their ordinal sum $(X, <_X)+(Z, <_Z)$ in the obvious way. S_n is the set of permutations of $\{1, \ldots, n\}$.

THEOREM. *Let R be an n-ary relation on the set X, where $|X| \geqslant \max(2n-3, n+2)$; let $<$ be an R-indiscernible ordering of X. Then precisely one of the following holds*:

(*a*) *Every ordering of X is R-indiscernible.*

(*b*) *An ordering $\prec$ of X is R-indiscernible if and only if there are disjoint $Y, Z \subseteq X$ and $i \in \{1, -1\}$ such that*

$$(X, <) = (Y, < \mid Y) + (Z, < \mid Z),$$
$$(X, \prec^i) = (Z, < \mid Z) + (Y, < \mid Y).$$

(*c*) *There are $Y, Z \subseteq X$ with $|Y \cup Z| \leqslant n$ such that, writing $X' = X - (Y \cup Z)$, we have*

$$(X, <) = (Y, < \mid Y) + (X', < \mid X') + (Z, < \mid Z),$$

and an ordering $\prec$ of X is R-indiscernible if and only if there is an $i \in \{1, -1\}$ such that

$$(X, \prec^i) = (Y, \prec^i \mid Y) + (X', < \mid X') + (Z, \prec^i \mid Z).$$

Proof. If X is infinite, then it suffices to prove the result for all large enough finite subsets of X. We may therefore assume that $|X|$ is finite and $\geqslant \max(2n-3, n+2)$. Enumerate X as $x_1 < x_2 < \ldots < x_{|X|}$. Let G be the set of $\pi \in S_n$ such that $Rx_{\pi(1)} \ldots x_{\pi(n)}$. We may assume without loss that G is a group, since this can be ensured by first replacing R by R', where

$R' y_1 \ldots y_n$ if and only if $y_1, \ldots, y_n$ are distinct $\in X$, and for each $\pi \in S_n$,

$$[Ry_{\pi(1)} \ldots y_{\pi(n)} \Leftrightarrow Rx_{\pi(1)} \ldots x_{\pi(n)}].$$

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(It can be checked that precisely the same orderings of X are R-indiscernible as are R'-indiscernible.)

If $1 \leqslant i < j \leqslant |X|$, we write $x_i S x_j$ to mean

$[Ry_1 \dots y_k x_i y_{k+1} \dots y_{n-1} \Leftrightarrow Ry_1 \dots y_k x_j y_{k+1} \dots y_{n-1}]$ whenever $y_1, \dots, y_{n-1}$ are distinct members of $X - \{x_i, x_j\}$, and $0 \leqslant k \leqslant n-1$.

If $\prec$ is an R-indiscernible ordering of X, and x_i and x_j are adjacent in $\prec$, with $i < j$, then $x_i S x_j$; in particular $x_i S x_{i+1}$ for all i.

LEMMA. *If $x_i S x_j$, then $(i-\gamma, \dots, j-\delta) \in G$ whenever $0 \leqslant \gamma < \delta \leqslant |X| - n$ and $1 \leqslant i-\gamma \leqslant j-\delta \leqslant n$.*

Proof of Lemma. Let γ, δ meet the conditions. Since G is a group, we have $Rx_1 \dots x_n$. Hence by R-indiscernibility

$$Rx_1 \dots x_{i-\gamma-1} x_{i+1} \dots x_{j+\gamma-\delta} x_j x_{j+1} \dots x_{n+\delta}.$$

(The conditions ensure that such elements can be found as a $<$ – increasing sequence in X.) Since $x_i S x_j$, it follows that

$$Rx_1 \dots x_{i-\gamma-1} x_{i+1} \dots x_{j+\gamma-\delta} x_i x_{j+1} \dots x_{n+\delta};$$

so by R-indiscernibility again,

$$Rx_1 \dots x_{i-\gamma-1} x_{i-\gamma+1} \dots x_{j-\delta} x_{i-\gamma} x_{j-\delta+1} \dots x_n.$$

Therefore $(i-\gamma, \dots, j-\delta) \in G$, proving the lemma.

We now prove the theorem, splitting into cases A, B, C to correspond to (a), (b), (c). It suffices to prove that at least one of (a), (b), (c) holds, since there is clearly no overlap between them.

Case A. $G = S_n$.

Then every ordering of X is R-indiscernible, so that (a) holds.

Case B. $(1, \dots, n) \in G \neq S_n$.

To show that (b) holds, it suffices to prove that if $1 \leqslant i < j < |X|$ and $x_i S x_j$, then $j = i+1$. Suppose to the contrary that $1 \leqslant i < j < |X|$ and $x_i S x_j$, but $i+1 < j$. Recalling that $|X| \geqslant n+2$, the conditions of the lemma are met if $\gamma = 0$ and $\delta = \max(1, j-n)$ or $\max(1, j-n)+1$. Putting $k = \max(1, j-n)$, the lemma tells us that both $(i, \dots, j-k)$ and $(i, \dots, j-k-1)$ are in G, so that G contains the transposition $(j-k-1, j-k)$. This contradicts the case assumption.

Case C. $(1, \dots, n) \notin G$.

We define p to be the greatest k such that every permutation of $\{1, \dots, k\}$ is in G, and q to be the least k such that every permutation of $\{k, \dots, n\}$ is in G. Then $p < q$, or we should not be in Case C. We put $Y = \{x_1, \dots, x_p\}$, $Z = \{x_{|X|-n+q}, \dots, x_{|X|}\}$, $X' = X - (Y \cup Z)$. Then $|Y \cup Z| \leqslant n$. To prove that (c) holds, it suffices to show the following:

(1) If $i \leqslant p < j$ and $x_i S x_j$, then $j = p+1$.

(2) If $i < |X| - n + q \leqslant j$ and $x_i S x_j$, then $i = (|X| - n + q) - 1$.

Sh:60

(3) For all but at most one pair i, j, if $p < i < j < |X|-n+q$ and $x_i S x_j$, then $j = i+1$.

Proof of (1): suppose $i \leqslant p < j$ and $x_i S x_j$, but $j > p+1$. If $j < |X|$, then the argument of Case B shows that G contains $(i, ..., j-k)$ and $(j-k-1, j-k)$ where $k = \max(1, j-n)$. Since G also contains every permutation of $\{1, ..., i\}$, we infer that every permutation of $\{1, ..., j-k\}$ is in G. But $j-k > p$, contradicting the choice of p. On the other hand if $j = |X|$, then by the lemma with $\gamma = 0$ and $\delta = |X|-n$, G contains $(i, ..., n)$ and hence $(1, ..., n)$, contradicting the case assumption.

Proof of (2): the mirror image of (1).

Proof of (3): suppose $p < i < j < |X|-n+q$ and $x_i S x_j$, but $j > i+1$. We assert that

$$\text{either } i+n < |X|+p, \text{ or } q < j, \text{ or } j = i+2 = n. \tag{$*$}$$

For

$$\begin{aligned}(i+n-p)+(q-j) &\leqslant n+(q-p)-(j-i)\\ &\leqslant n+(n-1)-2 = 2n-3 \leqslant |X|. \end{aligned} \tag{$**$}$$

If $(i+n-p)+(q-j) < |X|$, then either $i+n < |X|+p$ or $q < j$. If on the other hand $(i+n-p)+(q-j) = |X|$, $i+n \geqslant |X|+p$ and $q \geqslant j$, then equalities hold throughout ($**$), whence $p = 1, q = n, j = i+2$, and so

$$n = q \geqslant j = i+2 \geqslant |X|+p-n+2 = (2n-3)+1-n+2 = n,$$

proving that $j = i+2 = n$.

We show that neither of the first two possibilities in ($*$) can occur. First, if $i+n < |X|+p$, then the conditions of the lemma are satisfied with

$$\delta = \max(i-p+1, j-n)$$

and $\gamma = i-p$ or $i-p-1$. Hence by the lemma, G contains the cycles $(p, ..., j-\delta)$ and $(p+1, ..., j-\delta)$, whence also $(p, p+1)$; this contradicts the choice of p. Likewise if $q < j$, we get a contradiction by putting $\gamma = \min(i-1, j-1-q)$ and $\delta = j-q$ or $j-q+1$. This completes the proof of (3).

The Theorem is proved.

All the cases (*a*), (*b*), (*c*) of the Theorem do occur, say with $X = \omega$. For (*a*), take R to be the empty n-ary relation. For (*b*), take R to be the n-ary relation defined on ω by the formula

$$[v_1 < \ldots < v_n] \vee [v_2 < \ldots < v_n < v_1] \vee \ldots \vee [v_n < v_1 < \ldots < v_{n-1}].$$

Examples for (*c*) can be constructed in the same way.

The lower bound on the cardinality of X is sharp. Case B requires that $|X| \geqslant n+2$. For a counterexample where $|X| = n+1$, let n be 5, define $(X, <)$ and $(X, \prec)$ respectively by

$$1 < 2 < 3 < 4 < 5 < 6, \qquad 1 \prec 4 \prec 3 \prec 6 \prec 5 \prec 2,$$

and take R so that G is the alternating group. (It may be checked that for every 5-element subset of X, the $\prec$ ordering is an even permutation of the $<$ ordering.)

Case C requires that $|X| \geqslant 2n-3$. For a counterexample where $|X| = 2n-4$, let $n = 6$, let $(X, <)$ and $(X, \prec)$ be respectively

$$1 < 2 < 3 < 4 < 5 < 6 < 7 < 8, \qquad 1 \prec 2 \prec 3 \prec 5 \prec 4 \prec 6 \prec 7 \prec 8,$$

and let R be defined by the formula

$$\bigwedge_{2 \leqslant i \leqslant 5} [v_1 < v_i < v_6].$$

A weak form of the theorem (without the finite bounds) was announced by Hodges in [3]. Lachlan and Shelah independently strengthened the result; the present proof is based on ideas of Shelah.

Graham Higman [2] has shown that a recent group-theoretic result of Peter J. Cameron [1] can be deduced from our theorem by use of Ramsey's theorem and compactness. Theorem 5.1 of [2] is essentially equivalent to our theorem without the finite bounds; Higman's proof is new.

References

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