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The failure of the uncountable non-commutative Specker Phenomenon

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Abstract. Higman proved in 1952 that every free group is non-commutatively slender, that is, for a free group G and for a homomorphism h from the free complete product $\circledast_\omega \mathbb{Z}$ of countably many copies of $\mathbb{Z}$ into G, there exists a finite subset $F \subseteq \omega$ and a homomorphism $\bar{h} : *_F \mathbb{Z} \to G$ such that $h = \bar{h}\rho_F$, where ρ_F is the natural map from $\circledast_\omega \mathbb{Z}$ into $*_F \mathbb{Z}$. Because of the corresponding phenomenon for abelian groups this is called the non-commutative Specker Phenomenon. In the present paper we shall show that Higman's result fails if one passes from countable to uncountable and thereby answer a question posed by K. Eda. In particular, we will see that, for an uncountable cardinal λ and for non-trivial groups G_α $(\alpha \in \lambda)$, there are 2^{2^λ} homomorphisms from the free complete product of the groups G_α into the integers.

Introduction

In 1952 Higman [9] proved that every free group G is non-commutatively slender, where slenderness means that any homomorphism h from the free complete product $\circledast_\omega \mathbb{Z}$ of countably many copies of the integers into G depends on finitely many coordinates only. A similar result was proven by Specker [12] in 1950 for abelian groups. Specker showed that any homomorphism from the product $\Pi_\omega \mathbb{Z}$ of countably many copies of $\mathbb{Z}$ into the integers is determined by only finitely many entries. These two phenomena are called the commutative and the non-commutative Specker Phenomenon, respectively. Eda [3] extended Higman's result by showing that, for any non-commutatively slender group S, for any non-trivial groups G_α $(\alpha \in I)$ and for any homomorphism h from the free σ-product of the groups G_α into S, there exist a finite subset F of I and a homomorphism $\bar{h} : *_{i \in F} G_i \to S$ such that $h = \bar{h}\rho_F$ where ρ_F is the natural map from $\circledast^\sigma_{i \in I} G_i$ to $*_{i \in F} G_i$ (for the definition of σ-product

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see 1.2). Motivated by this result Eda [3, Question 3.8] asked whether or not the non-commutative Specker Phenomenon still holds if one passes from countable to uncountable cardinals, replacing $\circledast_{\omega}\mathbb{Z}$ by the free complete product $\circledast_{\lambda}\mathbb{Z}$ for some uncountable cardinal λ (see 1.2). Here we shall give a negative answer to Eda's question by constructing, for a given uncountable cardinal λ and for non-trivial groups G_α $(\alpha \in \lambda)$, a homomorphism h from the free complete product of the groups G_α into $\mathbb{Z}$ for which the non-commutative Specker Phenomenon fails. In fact, we will show that there are 2^{2^λ} of these homomorphisms and so, in particular, the cardinality of the set of all homomorphisms from $\circledast_{\alpha\in\lambda}G_\alpha$ into the additive group of $\mathbb{Z}$ is the largest one possible. This contrasts with the countable case and also the abelian case.

Basics and notation

Let I be an arbitrary set. For groups G_i $(i \in I)$, the free product is denoted by $*_{i\in I}G_i$ (for details on free products, see [10]).

Given arbitrary subsets $X \subset Y$ of I we write $\rho_{XY} : *_{i\in Y}G_i \to *_{i\in X}G_i$ for the canonical homomorphism. Moreover, we use the notation $X \Subset I$ for finite subsets X of I. Then the set $\{*_{i\in X}G_i : X \Subset I\}$ together with the homomorphisms ρ_{XY} $(X \subset Y \Subset I)$ form an inverse system; its inverse limit $\varprojlim(*_{i\in X}G_i, \rho_{XY} : X \subset Y \Subset I)$ is called the *unrestricted free product* of the groups G_i (see [9]).

Eda [3] introduced an infinite version of free products and defined the *free complete product* $\circledast_{i\in I}G_i$ of the groups G_i (for the exact definition see 1.2); it is isomorphic to the subgroup $\bigcap_{F\Subset I}\{*_{i\in F}G_i * \varprojlim(*_{i\in X}G_i, \rho_{XY} : X \subset Y \Subset I)\}$ of the unrestricted free product.

For the convenience of the reader unfamiliar with free complete products we recall the definition of words of infinite length and also the definition of and some basic facts about $\circledast_{i\in I}G_i$ as can be found in [3].

Definition 1.1. Let G_i $(i \in I)$ be non-trivial groups such that $G_i \cap G_j = \{e\}$ for $i \neq j \in I$. The elements of $\bigcup_{i\in I}G_i$ are called *letters*.

A *word* W is a function $W : \overline{W} \to \bigcup_{i\in I}G_i$ from a linearly ordered set $\overline{W}$ into the set of all letters $\bigcup_{i\in I}G_i$ such that $W^{-1}(G_i)$ is finite for any $i \in I$. If the domain $\overline{W}$ of the word W is countable then we say that W is a *σ-word*.

The class of all words is denoted by $\mathcal{W}(G_i : i \in I)$ (abbreviated to $\mathcal{W}$) and the class of all σ-words is denoted by $\mathcal{W}^\sigma(G_i : i \in I)$ (abbreviated to $\mathcal{W}^\sigma$).

Two words U and V are said to be *isomorphic* $(U \cong V)$ if there exists an order-isomorphism $\varphi : \overline{U} \to \overline{V}$ between the linearly ordered sets $\overline{U}$ and $\overline{V}$ such that $U(\alpha) = V(\varphi(\alpha))$ for all $\alpha \in \overline{U}$. Identifying isomorphic words it is easily seen that $\mathcal{W}$ is, in fact, a set. Moreover, for words of finite length (i.e. with finite domain) the above definition obviously coincides with the usual definition of words.

For a subset X of I, the *restricted word* (or *subword*) W_X of W is given by the function $W_X : \overline{W}_X \to \bigcup_{i\in X}G_i$ with $\overline{W}_X = \{\alpha \in \overline{W} : W(\alpha) \in \bigcup_{i\in X}G_i\}$ and $W_X(\alpha) = W(\alpha)$ for all $\alpha \in \overline{W}_X$. Therefore $W_X \in \mathcal{W}$. Using restricted words with respect to finite subsets of I we define an equivalence relation on $\mathcal{W}$ by saying that two words

U and V are *equivalent* ($U \sim V$) if $U_F = V_F$ for all $F \Subset I$, where we consider U_F and V_F as elements of the free product $*_{i\in F} G_i$. The equivalence class of a word W is denoted by $[W]$ and the composition of two words as well as the inverse of a word are defined naturally. Thus $\mathscr{W}/\sim\, = \{[W] : W \in \mathscr{W}\}$ together with the representative-wise defined composition forms a group.

Definition 1.2. Given groups G_i ($i \in I$), the *free complete product* $\times_{i\in I} G_i$ is defined to be the group $\mathscr{W}(G_i : i \in I)/\sim$ as described above. Moreover, the *free σ-product* $\times^{\sigma}_{i\in I} G_i$ is the group $\mathscr{W}^{\sigma}(G_i : i \in I)/\sim$, which is a subgroup of $\times_{i\in I} G_i$.

If G_i is isomorphic to a fixed group G for all $i \in I$ then we write $\times_I G$ and $\times^{\sigma}_I G$ instead of $\times_{i\in I} G_i$ and $\times^{\sigma}_{i\in I} G_i$.

Note that for a finite set I both $\times_{i\in I} G_i$ and $\times^{\sigma}_{i\in I} G_i$ are obviously isomorphic to $*_{i\in I} G_i$. In general, by [3, Proposition 1.8], the free complete product $\times_{i\in I} G_i$ is isomorphic to the subgroup $\bigcap_{F\Subset I}\{*_{i\in F} G_i * \varprojlim(*_{i\in X} G_i, \rho_{XY} : X \subset Y \Subset I)\}$ of the unrestricted free product. Moreover, Eda [3] proved that each equivalence class $[W]$ is determined uniquely by a reduced word; a word $W \in \mathscr{W}(G_i : i \in I)$ is said to be *reduced* if $W \cong UXV$ implies $[X] \neq e$ for any non-empty word X, where e is the identity, and it never occurs that the letters $W(\alpha)$ and $W(\beta)$ belong to the same G_i for neighbouring elements α and β of $\overline{W}$.

Lemma 1.3 (Eda [3]). *For any word $W \in \mathscr{W}(G_i : i \in I)$ there exists a reduced word $V \in \mathscr{W}(G_i : i \in I)$ such that $[W] = [V]$, and V is unique up to isomorphism.*

Furthermore, Eda [3] proved the following lemma; a word $W \in \mathscr{W}(G_i : i \in I)$ is called *quasi-reduced* if the reduced word of W can be obtained by multiplying neighbouring elements without cancellation.

Lemma 1.4 (Eda [3]). *For any two reduced words $W, V \in \mathscr{W}(G_i : i \in I)$ there exist reduced words $V_1, W_1, M \in \mathscr{W}(G_i : i \in I)$ such that $W \cong W_1 M$, $V \cong M^{-1} V_1$ and $W_1 V_1$ is quasi-reduced.*

We would like to remark that the free σ-product $\times^{\sigma}_I \mathbb{Z}$ is isomorphic to the fundamental group (see [3]) and the free complete product $\times_I \mathbb{Z}$ is isomorphic to the big fundamental group of the Hawaiian earring with $|I|$ circles (see [1]). Hence free complete products are also of topological interest.

The uncountable Specker Phenomenon

In 1950, E. Specker [12] proved that, for any homomorphism h from the direct product $\mathbb{Z}^{\omega}$ of countably many copies of $\mathbb{Z}$ into the additive group of the ring of integers $\mathbb{Z}$, there exist a finite subset F of ω and a homomorphism $\bar{h} : \mathbb{Z}^F \to \mathbb{Z}$ satisfying $h = \bar{h}\rho_F$ where $\rho_F : \mathbb{Z}^{\omega} \to \mathbb{Z}^F$ is the canonical projection. This result is called the Specker Phenomenon. It can be easily seen that Specker's result still holds for homomorphisms into any free abelian group instead of homomorphisms into $\mathbb{Z}$, i.e., free abelian groups are slender. In general, an abelian group G is said to be *slender* if G satisfies the above property for any homomorphism $h : \mathbb{Z}^{\omega} \to G$. For generalizations to products of uncountably many copies of $\mathbb{Z}$ within the category of abelian groups we refer to [6] or [7].

In [4], Eda introduced the non-commutative version of slenderness that we shall consider here.

Definition 2.1. A group G is *non-commutatively slender* if, for any homomorphism $h : \divideontimes_{\omega}\mathbb{Z} \to G$, there exists a natural number n such that $h(\divideontimes_{\omega\setminus\{1,\dots,n\}}\mathbb{Z}) = \{e\}$, where e denotes the identity element of G.

Eda proved that non-commutatively slender groups are torsion-free and that non-commutative slenderness for abelian groups is the same as ordinary (commutative) slenderness (see [3, Theorem 3.3. and Corollary 3.4.]). Moreover, he proved that non-commutatively slender groups have the following nice property:

Proposition 2.2 (Eda [3]). *Let G_i $(i \in I)$ be non-trivial groups, let S be a non-commutatively slender group, and let $h : \divideontimes^{\sigma}_{i\in I} G_i \to S$ be a homomorphism. Then there exist a finite subset F of I and a homomorphism $\bar{h} : *_{i\in F} G_i \to S$ such that $h = \bar{h}\rho_F$, where ρ_F is the canonical map from $\divideontimes^{\sigma}_{i\in I} G_i$ to $*_{i\in F} G_i$.*

Another interesting result is that the restricted direct product and the free product of non-commutatively slender groups S_j $(j \in J)$ are non-commutatively slender (see [3, Theorem 3.6.]). However, the first fundamental result on the class of non-commutatively slender groups was obtained by Higman [9] in 1952:

Theorem 2.3 (Higman [9]). *Every free group is non-commutatively slender.*

In contrast to Higman's result, we will show that the non-commutative Specker Phenomenon fails if one replaces the product of countably many groups by a product of uncountably many groups. To be more precise, we show that, for an uncountable cardinal λ, there are $2^{2^{\lambda}}$ homomorphisms from the free complete product $\divideontimes_{\alpha\in\lambda} G_{\alpha}$ of non-trivial groups G_{α} $(\alpha \in \lambda)$ into the additive group of the ring of integers.

To make the proof more transparent we first construct one homomorphism for which the Specker Phenomenon fails and then modify the construction to obtain our main result.

Theorem 2.4. *Let λ be any uncountable cardinal and G_{α} $(\alpha \in \lambda)$ non-trivial groups. Then there exists a homomorphism $\varphi : \divideontimes_{\alpha\in\lambda} G_{\alpha} \to \mathbb{Z}$ for which the Specker Phenomenon fails.*

Proof. Let G_{α} $(\alpha \in \lambda)$ be a collection of non-trivial groups with identity elements e_{α} and choose elements $g_{\alpha} \neq e_{\alpha}$ of G_{α} for each $\alpha \in \lambda$. For any regular uncountable cardinal $\kappa \leqslant \lambda$ we define the word $M_{\kappa} \in \divideontimes_{\alpha\in\lambda} G_{\alpha}$ as follows:

$$M_\kappa : (\kappa, <) \to \bigcup_{\alpha \in \lambda} G_\alpha \text{ via } \beta \mapsto g_\beta$$

where $<$ is the natural ordering of λ. Note that M_κ is a word of uncountable cofinality (i.e., the domain of M_κ has uncountable cofinality) since κ is regular and uncountable. For $\beta < \kappa$ we define $M_{\kappa,\beta}$ to be the subword $M_\kappa \restriction_{[\beta,\kappa)}$ of M_κ.

Now let X be any reduced word in $\circledast_{\alpha \in \lambda} G_\alpha$. We recall that a subset $J \subseteq (\overline{X}, <)$ is called convex if $x < y < z$ and $x, z \in J$ imply $y \in J$. We define

$$\mathrm{Occ}_\kappa^+(X) := \{J \subseteq (\overline{X}, <) : J \text{ is convex and } X \restriction_J \cong M_{\kappa,\beta} \text{ for some } \beta < \kappa\}.$$

Thus $\mathrm{Occ}_\kappa^+(X)$ counts the occurencies of end segments of M_κ in X. Similarly we let

$$\mathrm{Occ}_\kappa^-(X) := \{J \subseteq (\overline{X}, <) : J \text{ is convex and } X \restriction_J \cong M_{\kappa,\beta}^{-1} \text{ for some } \beta < \kappa\}.$$

In order to avoid counting subsets of $(\overline{X}, <)$ more often than necessary we define the following equivalence relation on $\mathrm{Occ}_\kappa^+(X)$ and $\mathrm{Occ}_\kappa^-(X)$.

Dealing with Occ^+, we call two convex subsets J_1, J_2 of $(\overline{X}, <)$ equivalent ($J_1 \sim_\kappa J_2$) if they have a common end segment; in other words $J_1 \sim_\kappa J_2$ if there exist $j_1 \in J_1$, $j_2 \in J_2$ such that $X \restriction_{S_1} \cong X \restriction_{S_2}$ where $S_i = \{j \in J_i : j \geqslant j_i\}$ $(i = 1, 2)$. Similarly, for Occ^-, we define the equivalence relation substituting end segments by initial segments. For simplicity we denote both equivalence relations by $\sim_\kappa$ but the reader should keep in mind that $\sim_\kappa$ is defined differently for Occ^+ and Occ^-.

First we prove that two subsets $J_1, J_2 \in \mathrm{Occ}_\kappa^+(X)$ are either disjoint or equivalent. To do so assume that $J_1, J_2 \in \mathrm{Occ}_\kappa^+(X)$ are not disjoint and let $j^* \in J_1 \cap J_2$ $(\neq \emptyset)$. Moreover, there are ordinals $\beta_1, \beta_2 < \kappa$ $(\leqslant \lambda)$ and isomorphisms $h_i : M_{\kappa,\beta_i} \to X \restriction_{J_i}$ $(i = 1, 2)$ since J_1, J_2 are elements of $\mathrm{Occ}_\kappa^+(X)$. Thus we can find $\gamma_i \geqslant \beta_i$ such that $h_i(\gamma_i) = j^*$ and therefore $X(j^*) = g_{\gamma_i}$ for $i = 1, 2$. Hence $\gamma_1 = \gamma_2$ and by transfinite induction we conclude that $X \restriction_{T_1} \cong X \restriction_{T_2}$, where $T_i = \{j \in J_i : j \geqslant j^*\}$. Note that h_i is an isomorphism of linearly ordered sets and hence h_i commutes with limits and the successor function.

Similarly, two subsets J_1, J_2 of $\mathrm{Occ}_\kappa^-(X)$ are either disjoint or equivalent.

Next we show that the set $\mathrm{Occ}_\kappa^+(X)/\sim_\kappa$ is finite; by similar arguments it then also follows that $\mathrm{Occ}_\kappa^-(X)/\sim_\kappa$ is finite. Let us assume the contrary, that is, there exist infinitely many pairwise non-equivalent $J_n \in \mathrm{Occ}_\kappa^+(X)$ $(n \in \omega)$. Then J_n and J_m are disjoint for any $n \neq m$ from above. For each $n \in \omega$ let $X \restriction_{J_n} \cong M_{\kappa,\beta_n}$ for some $\beta_n < \kappa$. Thus $\beta = \bigcup_{n \in \omega} \beta_n$ is strictly less than κ as κ is regular uncountable and hence $\mathrm{cf}(\kappa) > \aleph_0$. Since $\beta \in [\beta_n, \kappa)$ for all $n \in \omega$ we can find $j_n \in J_n$ such that

$$X(j_n) = M_{\kappa,\beta_n}(\beta) = M_{\kappa,\beta}(\beta)$$

for all $n \in \omega$. But all J_n are pairwise disjoint and therefore $X^{-1}(G_\beta)$ is infinite, which contradicts the definition of a word (see 1.1). Thus $\mathrm{Occ}_\kappa^+(X)/\sim_\kappa$ and also $\mathrm{Occ}_\kappa^-(X)/\sim_\kappa$ are finite sets.

We now define $\varphi_\kappa : \ast_{\alpha\in\lambda} G_\alpha \to \mathbb{Z}$ as follows:

$$W \mapsto |\mathrm{Occ}_\kappa^+(X)/\sim_\kappa| - |\mathrm{Occ}_\kappa^-(X)/\sim_\kappa|$$

where X is the reduced word corresponding to W. Note that φ_κ is well defined by Lemma 1.3. Moreover, it follows immediately from the definition that $\varphi_\kappa(X^{-1}) = -\varphi_\kappa(X)$ and also the Specker Phenomenon obviously fails for φ_κ. Note that in general the sets $\mathrm{Occ}_\kappa^+(X)$ and $\mathrm{Occ}_\kappa^-(X)$ are not of the same size, e.g. $\varphi(M_{\kappa,\beta}) = 1$. It remains to show, however, that φ_κ is a homomorphism. Therefore let X and Y be reduced words. By Lemma 1.4 there exist reduced words X_1, Y_1 and M such that $X \cong X_1 M$ and $Y \cong M^{-1} Y_1$ and $X_1 Y_1$ is quasi-reduced. Now it is easy to check that $\varphi_\kappa(XY) = \varphi_\kappa(X_1 Y_1)$ by definition and the fact that $XY = X_1 M M^{-1} Y_1$. Hence

$$\varphi_\kappa(XY) = \varphi_\kappa(X_1 Y_1) = \varphi_\kappa(X_1) + \varphi_\kappa(Y_1) = \varphi_\kappa(X) + \varphi_\kappa(Y),$$

as $X_1 Y_1$ is quasi-reduced and thus the reduced word of $X_1 Y_1$ is obtained without cancellation.

We remark that the uncountability of κ in Theorem 2.4 is essential for the definition of the homomorphism φ_κ and cannot be omitted because of Higman's theorem. Modifying the proof of Theorem 2.4 we obtain

Theorem 2.5. *Let λ be any uncountable cardinal and G_α ($\alpha \in \lambda$) be non-trivial groups. Then there are 2^{2^λ} homomorphisms from the free complete product of the groups G_α into the additive group of the ring of integers. In fact there is an epimorphism from $\ast_{\alpha\in\lambda} G_\alpha$ onto the free abelian group of 2^λ copies of the integers.*

Proof. Let λ be uncountable and $\{G_\alpha : \alpha \in \lambda\}$ be given as stated. We choose the following family of reduced words M_α for $\alpha \in 2^\lambda$. First we choose non-trivial elements $e_\gamma \neq g_\gamma \in G_\gamma$ for $\gamma \in \lambda$. Let $\{I_\varepsilon : \varepsilon \in \omega_1\}$ be a family of pairwise disjoint subsets of λ each of which has cardinality λ. It is well known (see e.g. [5]) that for every $\varepsilon \in \omega_1$ we can find a family $\{I_{\varepsilon,\alpha} \subseteq I_\varepsilon : \alpha \in 2^\lambda\}$ of subsets of I_ε such that any finite Boolean combination of them is of cardinality λ and moreover there is $\gamma_\varepsilon \in I_\varepsilon$ that belongs to each $I_{\varepsilon,\alpha}$ for every $\alpha \in 2^\lambda$. For every $\varepsilon \in \omega_1$ and for every $\alpha \in 2^\lambda$ we choose a word $M_{\varepsilon,\alpha}$ such that its domain $\overline{M}_{\varepsilon,\alpha}$ equals $I_{\varepsilon,\alpha}$ and $M_{\varepsilon,\alpha}(\sigma) = g_\sigma$ for $\sigma \in I_{\varepsilon,\alpha}$.

Then the composition $M_\alpha = \sum_{\varepsilon\in\omega_1} M_{\varepsilon,\alpha}$ is a well-defined reduced word in $\mathscr{W}(G_\gamma : \gamma \in \lambda)$ for every $\alpha \in 2^\lambda$. Before defining the claimed homomorphism let us first state the crucial condition satisfied by the groups M_α ($\alpha \in 2^\lambda$):

> the domain $\overline{M}_\alpha$ of M_α is well ordered of order type $\lambda\omega_1$ (the ordinal product) for each $\alpha \in 2^\lambda$ and therefore has uncountable cofinality.

Now we repeat the construction given in Theorem 2.4 replacing κ by $\overline{M}_\alpha$ and for a reduced word X and $\alpha \in 2^\lambda$ we define

$$\mathrm{Occ}_\alpha^+(X) = \{J \subseteq (\overline{X}, <) : J \text{ convex}, \ X \restriction_J \cong M_{\alpha,\sigma} \text{ for some } \sigma \in \overline{M}_\alpha\}$$

and

$$\mathrm{Occ}_\alpha^-(X) := \{J \subseteq (\overline{X}, <) : J \text{ convex}, \ X\restriction_J \cong M_{\alpha,\sigma}^{-1} \text{ for some } \sigma \in \overline{M}_\alpha\},$$

where $M_{\alpha,\sigma}$ is the end segment of M_α starting with the element σ, i.e. $M_{\alpha,\sigma} = M_\alpha\restriction_{\{\rho\in\overline{M}_\alpha : \rho\geqslant\sigma\}}$. As in the proof of Theorem 2.4 two equivalence relations are defined on the sets $\mathrm{Occ}_\alpha^+(X)$ and $\mathrm{Occ}_\alpha^-(X)$, both denoted by $\sim_\alpha$.

We are now able to define homomorphisms φ_α from the free complete product of the groups G_γ to the integers for each $\alpha \in 2^\lambda$. As in the proof of Theorem 2.4 we can see that the sets $\mathrm{Occ}_\alpha^+(X)/{\sim_\alpha}$ and $\mathrm{Occ}_\alpha^-(X)/{\sim_\alpha}$ are finite for any reduced word X and $\alpha \in 2^\lambda$. Moreover, the maps $\varphi_\alpha : \divideontimes_{\beta\in\lambda} G_\beta \to \mathbb{Z}$ defined by

$$V \mapsto |\mathrm{Occ}_\alpha^+(X)/{\sim_\alpha}| - |\mathrm{Occ}_\alpha^-(X)/{\sim_\alpha}|,$$

where X is the reduced word corresponding to V, are well-defined homomorphisms since $\gamma_\varepsilon \in I_{\varepsilon,\alpha}$ for every $\alpha \in 2^\lambda$. To obtain 2^{2^λ} homomorphisms we will show that there is a surjection onto the free abelian group of 2^λ copies of $\mathbb{Z}$. Define

$$\Phi : \divideontimes_{\alpha\in\lambda} G_\alpha \to \prod_{\alpha\in 2^\lambda} \mathbb{Z}$$

via

$$\Phi(V)(\alpha) = \varphi_\alpha(X)$$

for a word V, where X is the reduced word corresponding to V. As all the mappings φ_α $(\alpha \in 2^\lambda)$ are homomorphisms, so is Φ. We claim that Φ is a homomorphism from the free complete product of the groups G_α onto the direct sum $\bigoplus_{\alpha\in 2^\lambda} \mathbb{Z}$ of 2^λ copies of $\mathbb{Z}$. First assume that this mapping is not into. Then there exist a reduced word X and a sequence of pairwise distinct ordinals α_n $(n \in \omega)$ such that $\Phi_{\alpha_n}(X) \neq 0$ for all $n \in \omega$. Thus for each $n \in \omega$ there is a convex subset $J_n \subseteq \overline{X}$ such that without loss of generality

$$X\restriction_{J_n} \cong M_{\alpha_n,\sigma_n}$$

for some $\sigma_n \in \overline{M}_{\varepsilon_n,\alpha_n} \subseteq \overline{M}_{\alpha_n}$ $(\varepsilon_n \in \omega_1)$. But now, if $\kappa \in \omega_1$ such that $\varepsilon_n < \kappa$ for all $n \in \omega$, then the element g_κ (since γ_κ belongs to all sets I_{κ,α_n}) appears infinitely many times in X, i.e. $\overline{X}^{-1}(G_\kappa)$ is infinite, a contradiction. Thus the image of Φ is contained in the direct sum of 2^λ copies of the integers.

On the other hand, by the choice of the sets $I_{\varepsilon,\alpha}$ we certainly have

$$\mathrm{Occ}_\alpha^+(M_\beta) = 0 = \mathrm{Occ}_\alpha^-(M_\beta)$$

for distinct $\alpha, \beta \in 2^\lambda$. Moreover,

$$\mathrm{Occ}^+_\alpha(M_\alpha) = 1 \quad \text{and} \quad \mathrm{Occ}^-_\alpha(M_\alpha) = 0$$

for any $\alpha \in 2^\lambda$. Thus we obtain

$$\Phi(M_\alpha) = (0, \ldots, 0, 1_\alpha, 0, \ldots) \in \bigoplus_{\beta \in 2^\lambda} \mathbb{Z},$$

and therefore Φ is obviously surjective. Since there are 2^{2^λ} homomorphisms from the direct sum of copies of $\mathbb{Z}$ to $\mathbb{Z}$ itself we are done.

Remark 2.6. Note that the above proof gives us that the free complete product $G = \circledast_{\alpha \in \lambda} G_\alpha$ contains a free subgroup H (the group generated by the words M_α), and there is a projection onto H.

The following theorem gives us a complete description of all 'interesting' homomorphisms from $G = \circledast_{\alpha \in \lambda} G_\alpha$ to the integers for uncountable λ and groups G_α $(\alpha \in \lambda)$. By 'interesting', we mean interesting with respect to the Specker Phenomenon, i.e., if W is a finite subset of λ, then all homomorphisms from the subproduct $G_W = \circledast_{\alpha \in W} G_\alpha$ to the integers extend naturally to a homomorphism from G to $\mathbb{Z}$. However these homomorphisms are not of particular interest to us and are well understood, and hence we will restrict ourselves to homomorphisms from G to $\mathbb{Z}$ which are zero on every finite subproduct of G. First note that the definition of φ_{M_α} in the proof of Theorem 2.5 did not really depend on the particular word M_α but only on the fact that $\overline{M}_\alpha$ had uncountable cofinality. One sees immediately that for any word M whose domain $\overline{M}$ has uncountable cofinality we can define such a homomorphism $\varphi_M : G \to \mathbb{Z}$. Hence we define the following set:

$$I_G = \{M \in G : \mathrm{cf}(\overline{M}) \geqslant \aleph_1\}$$

and let

$$\Phi_G : G \to \prod_{M \in I_G} \mathbb{Z}_M$$

be defined by $\Phi_G(V) = (\varphi_M(X) : M \in I_G)$ where X is the reduced word corresponding to V. Now Φ_G is well defined and we have the following theorem.

Theorem 2.7. *Let $G = \circledast_{\alpha \in \lambda} G_\alpha$ for some uncountable cardinal λ and groups G_α $(\alpha \in \lambda)$. Then any homomorphism $\psi : G \to \mathbb{Z}$ which is zero on every finite subproduct of G factors through Φ_G.*

Proof. For simplicity we assume that all words are already in reduced form. First we will show that the kernel of Φ_G is exactly

$$\mathrm{Ker}(\Phi_G) = \{M \in G : M \text{ contains no monotonic sequence of length } \omega_1\},$$

where a monotonic sequence of length ω_1 is just a subset of $\overline{M}$ which is isomorphic to ω_1 or its inverse. Clearly

$$\mathrm{Ker}(\Phi_G) \subseteq \{M \in G : M \text{ contains no monotonic sequence of length } \omega_1\}.$$

Conversely, if M is a reduced word that contains no monotonic sequence of length ω_1 and $\Phi_G(M) \neq 0$, then there exists a reduced word $N \in I_G$, i.e. $\overline{N}$ has uncountable cofinality, such that $\varphi_N(M) \neq 0$. But then $\mathrm{Occ}_N^+(M)$ or $\mathrm{Occ}_N^-(M)$ is non-trivial and hence M must contain a monotonic sequence of length ω_1, a contradiction. It is now enough to prove that any homomorphism $f : G \to \mathbb{Z}$ which is zero on any finite subproduct of G acts trivially on $\mathrm{Ker}(\Phi_G)$. For this, assume that M is a reduced word which contains no monotonic sequence of length ω_1 such that $f(M) \neq 0$ for some homomorphism $f : G \to \mathbb{Z}$. We distinguish three cases.

Case (a). There exist subwords N_n of M $(n \in \omega)$ such that

(i) $\overline{N}_n$ is a convex subset of $\overline{M}$;

(ii) the subsets $\overline{N}_n$ $(n \in \omega)$ are pairwise (almost) disjoint;

(iii) $f(N_n) \neq 0$ (without loss of generality $f(N_n) > 0$).

Hence the composition N of the words N_n $(n \in \omega)$ is a well-defined word in G and applying Theorem 2.3 together with [3, Proposition 1.9] leads to a contradiction.

Case (b). There is an initial segment $\overline{M}^*$ of $\overline{M}$ such that

(i) $f(M^*) \neq 0$, where $M^* = M \restriction_{\overline{M}^*}$;

(ii) for every proper initial segment $\overline{N}$ of $\overline{M}^*$ we have $f(N) = 0$, where $N = M \restriction_{\overline{N}}$;

(iii) $\overline{M}^*$ has no largest element or for every $t \in \overline{M}^*$ there exists a convex subset $\overline{N}_t \subseteq \{m \in \overline{M}^* : m \geqslant t\}$ such that $f(N_t) \neq 0$, where $N_t = M \restriction_{\overline{N}_t}$.

Then the cofinality of $\overline{M}^*$ has to be $\aleph_0$ by the assumptions and we choose an increasing, unbounded sequence $\{t_n : n \in \omega\}$ in $\overline{M}^*$ and put

$$N_n = \{m \in \overline{M}^* : m \geqslant t_n\} \quad \text{or} \quad N_n = N_{t_n} \quad (\text{hence } f(N_n) \neq 0).$$

In both cases we easily obtain a contradiction. The same arguments apply for the inverse of M and give a contradiction.

Case (c). Neither Case (a) nor Case (b) applies. Then it is easy to see that the set

$$J = \{t \in \overline{M} : f(M_t) \neq 0\}$$

is finite, where $M_t = M \restriction_{\{t\}}$ (e.g. use Ramsey's Theorem). So without loss of generality we may assume that J is the empty set. We let I be the set of convex subsets of $\overline{M}$ such that $f(M \restriction_B) = 0$ for each convex $B \subseteq A$. Then I contains all singletons $t \in \overline{M}$, the empty set and it is downwards closed. Moreover, if A and B are elements of I and $A \cup B$ is convex, then $A \cup B \in I$. Finally every initial segment of $\overline{M}$ with no largest element has an end segment in I and hence $M \in I$, a contradiction. Similarly we obtain a contradiction using the inverse M^{-1} instead of M. This finishes the proof.

For completeness let us make the following remark.

Remark 2.8. If $h : G \to \mathbb{Z}$ is any homomorphism, then an application of Theorem 2.3 shows that the set $\{\alpha \in \lambda : h(G_\alpha) \neq \{0\}\} = F$ is finite. Hence, regarding $*_{\alpha \in F} G_\alpha$ as a subgroup of G we let $h_0 = (h \restriction_{*_{\alpha \in F} G_\alpha})\rho_F$. Then $h - h_0$ satisfies the assumptions of Theorem 2.7 and thus factors through Φ_G.

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