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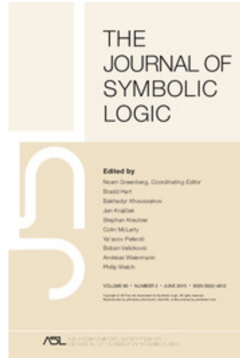

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DUAL BOREL CONJECTURE AND COHEN REALS

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Abstract. We construct a model of ZFC satisfying the Dual Borel Conjecture in which there is a set of size $\aleph_1$ that does not have measure zero.

§1. **Introduction.** Through this paper we will work in the space 2^ω. The definitions presented below can be adapted to many Polish spaces but the main construction given in the paper is specific to the Cantor space.

Let $\mathcal{N}$ be the ideal of measure zero subsets of 2^ω with respect to the standard product measure on 2^ω. Similarly, let $\mathcal{M}$ be the σ-ideal of meager sets in 2^ω. More precisely, $X \in \mathcal{M}$ if X there is a sequence $\langle F_n : n \in \omega\rangle$ such that $X \subseteq \bigcup_n F_n$ and for each n, $\mathsf{int}(\mathsf{cl}(F_n)) = \emptyset$.

DEFINITION 1. *For $X \subseteq 2^\omega$ we say that X is strongly meager ($X \in \mathcal{SM}$) if for every $H \in \mathcal{N}$, $X + H = \{x + h : x \in X, h \in H\} \neq 2^\omega$.*

Similarly, $X \subseteq 2^\omega$ is strong measure zero ($X \in \mathcal{SN}$) if $X + F \neq 2^\omega$ for all $F \in \mathcal{M}$.

Observe that $X + F \neq 2^\omega$ is equivalent to saying that there exists $z \in 2^\omega$ such that $(X + z) \cap F = \emptyset$.

Let BC (Borel Conjecture) stands for the statement $\mathcal{SN} = [2^\omega]^{\leq\aleph_0}$ and let DBC (Dual Borel Conjecture) stands for the statement $\mathcal{SM} = [2^\omega]^{\leq\aleph_0}$.

We will list here several background results, much more information can be found in [1].

- BC is consistent with ZFC (Laver, [5])
- DBC is consistent with ZFC (Carlson, [2])
- BC + $2^{\aleph_0} > \aleph_2$ is consistent with ZFC (Woodin, Judah-Woodin-Shelah, [4])

The question that motivates this work is whether BC + DBC is consistent with ZFC.

Recall that $\mathsf{cov}(\mathcal{M}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{M}, \bigcup\mathcal{A} = 2^\omega\}$. The number $\mathsf{cov}(\mathcal{N})$ is defined analogously. Note that

- If BC holds then $\mathsf{cov}(\mathcal{M}) = \aleph_1$,
- If DBC holds then $\mathsf{cov}(\mathcal{N}) = \aleph_1$.

Indeed, if $X \notin \mathcal{SN}$ then for some $F \in \mathcal{M}$, $X + F = \bigcup_{x\in X}(F + x) = 2^\omega$.

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Thus a model for BC + DBC, if one exists, must satisfy DBC + $\mathrm{cov}(\mathcal{M}) = \aleph_1$. There are several known constructions of models for DBC [7] but they all satisfy $\mathrm{cov}(\mathcal{M}) > \aleph_1$.

The goal of this paper is to show the following:

THEOREM 2. *DBC + $\mathrm{cov}(\mathcal{M}) = \aleph_1$ is consistent with* ZFC.

We will accomplish this goal by constructing a model for DBC where $\mathrm{non}(\mathcal{N}) = \aleph_1$, that is there is a non-null set of size $\aleph_1$. This suffices as $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}(\mathcal{N})$ (see [1]).

Observe that the coefficient $\mathrm{cov}(\mathcal{M})$ is connected with Cohen reals. In particular, obtaining a model by iterated forcing where $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_1$ is equivalent to not adding (too many) Cohen reals on the way.

The proof of the main theorem consists of two parts. In the first part we will define a forcing notion that will be used in the construction. In the second part we will use this forcing notion in the context of the general iteration framework of non-Cohen oracle ccc introduced by Shelah in [10]. We will make explicit references to this paper which is available from `http://shelah.logic.at/files/669.pdf`

§2. Getting DBC.

In this section we will describe how models for DBC are constructed.

DEFINITION 3. *Let $\mathbb{P}$ be a forcing notion.*

We say that $\mathbb{P}$ has caliber $(\aleph_1, \aleph_0)$ if for every uncountable subset $\mathcal{A} \subseteq \mathbb{P}$ there is an infinite set $\mathcal{B} \subseteq \mathcal{A}$ and a condition $q \in \mathbb{P}$ such that $q \geq p$ for all $p \in \mathcal{B}$.

We say that a forcing notion $\mathbb{P}$ has precaliber $\aleph_1$ if for every uncountable subset $\mathcal{A} \subseteq \mathbb{P}$ there is an uncountable set $\mathcal{B} \subseteq \mathcal{A}$ which is centered.

It is easy to see that $\mathbb{P}$ has caliber $(\aleph_1, \aleph_0)$ iff for every $X \in [\omega_1]^{\aleph_1} \cap \mathbf{V}^{\mathbb{P}}$ there is $Y \in [\omega_1]^{\aleph_0} \in \mathbf{V}$ such that $Y \subseteq X$.

The following notion leads to a standard way of building null sets witnessing that uncountable sets of reals in the ground model are not strongly meager.

DEFINITION 4. *Suppose that N is a model for a fragment of* ZFC$^-$. *We say that a sequence of clopen subsets of 2^ω, $\langle C_n : n \in \omega\rangle$ is big over N, if*

1. *C_n's have pairwise disjoint supports,*
2. *$\mu(C_n) \leq 2^{-n}$ for $n \in \omega$,*
3. *for every infinite set $X \subseteq 2^\omega$, $X \in N$, there exists infinitely many n such that $X + C_n = 2^\omega$.*

A forcing notion $\mathbb{P}$ has property R *if it adds a big sequence over the ground model.*

The following theorems are used in all constructions of the models for DBC (see original papers [2], [7], [3]).

THEOREM 5. *Suppose that $\mathbb{P}$ has caliber $(\aleph_1, \aleph_0)$ and property* R. *Then*

$$\mathbf{V}^{\mathbb{P}} \models \mathcal{SM} \cap \mathbf{V} \subseteq [2^\omega]^{\leq\aleph_0}.$$

Furthermore, if $\mathbb{Q}$ has precaliber $\aleph_1$ in $\mathbf{V}^{\mathbb{P}}$. Then

$$\mathbf{V}^{\mathbb{P}\star\mathbb{Q}} \models \mathcal{SM} \cap \mathbf{V} \subseteq [2^\omega]^{\leq\aleph_0}.$$

In particular,

$$\mathbf{V}^{\mathbb{P}\star\mathbf{C}} \models \mathcal{SM} \cap \mathbf{V} \subseteq [2^\omega]^{\leq\aleph_0},$$

where $\mathbf{C}$ is a countable (Cohen) forcing.

PROOF. Let $\langle \dot{C}_n : n \in \omega\rangle$ be $\mathbb{P}$-name for a sequence such that $\Vdash_{\mathbb{P}} \langle \dot{C}_n : n \in \omega\rangle$ is big over $\mathbf{V}$. Put $\dot{H}_n = \bigcup_{m\geq n} \dot{C}_m$ and let $\dot{H} = \bigcap_n \dot{H}_n$. It is easy to verify that $\Vdash_{\mathbb{P}} \dot{H} \in \mathscr{N}$. Let $X \in \mathbf{V}$ be an uncountable set of reals. We will show that $\mathbf{V}^{\mathbb{P}} \models X \notin \mathscr{SM}$, and more specifically that $\Vdash_{\mathbb{P}} X + \dot{H} = 2^\omega$. Towards the contradiction suppose that $\Vdash_{\mathbb{P}} \dot{z} \notin X + \dot{H}$. For each $x \in X$ there is a condition p_x and $n_x \in \omega$ such that $p_x \Vdash_{\mathbb{P}} \dot{z} \notin x + \dot{H}_{n_x}$. Let $n \in \omega$, $q \in \mathbb{P}$ and $Y \in \mathbf{V}$, Y infinite subset of X, be such that for all $x \in Y$, $n_x = n$ and $q \geq p_x$. As $Y \in \mathbf{V}$ and $q \Vdash_{\mathbb{P}} Y + \dot{H}_n \neq 2^\omega$, it follows that $q \Vdash \forall m \geq n\ \dot{C}_m + Y \neq 2^\omega$, a contradiction.

For the second part work in $\mathbf{V}^{\mathbb{P}}$ and let $X \subseteq 2^\omega \cap \mathbf{V}$ be an uncountable set of reals in $\mathbf{V}$ (or even in $\mathbf{V}^{\mathbb{P}}$). Suppose that $\Vdash_{\mathbb{Q}} \dot{z} \notin X + H$, where $H \in \mathbf{V}^{\mathbb{P}}$ is as above. For each $x \in X$ there is a condition q_x and $n_x \in \omega$ such that $q_x \Vdash_{\mathbb{Q}} \dot{z} \notin x + H_{n_x}$. Let $n \in \omega$, and Y be an uncountable subset of X be such that for all $x \in Y$, $n_x = n$ and $\{q_x : x \in Y\}$ is centered. For each finite set $a \subseteq Y$ let $q_a \geq p_x$, $x \in a$. Thus $q_a \Vdash a + H_n \neq 2^\omega$. By absoluteness, $a + H_n \neq 2^\omega$ in $\mathbf{V}^{\mathbb{P}}$. Since this holds for each finite subset of Y, by compactness, $Y + H_n \neq 2^\omega$. Since $Y \in \mathbf{V}^{\mathbb{P}}$, $Y \subset \mathbf{V}$ and $\mathbb{P}$ has caliber $(\aleph_1, \aleph_0)$ there is an infinite set $Z \subseteq Y$ such that $Z \in \mathbf{V}$. It follows that $Z + C_m \neq 2^\omega$ for $m \geq n$, a contradiction. ⊣

THEOREM 6 (Carlson [2]). *Suppose that* $\mathbf{V} \models \mathsf{GCH}$ *and* $\mathbf{C}_{\omega_2}$ *is a finite support iteration of Cohen forcing. Then* $V^{\mathbf{C}_{\omega_2}} \models \mathsf{DBC}$.

PROOF. Clearly Cohen forcing has caliber $(\aleph_1, \aleph_0)$ and the ω_2-iteration with finite support of Cohen forcing has precaliber $\aleph_1$.

To see that the Cohen forcing $\mathbf{C}$ has property R, interpret Cohen forcing as the family $\mathbf{C}$ of finite sequences $\{\langle (I_0, C_0), \dots, (I_k, C_k)\rangle : k \in \omega\}$ where

(1) for each $j \leq k$, $I_j \subset \omega$ is finite and $C_j \subseteq 2^{I_j}$ with $\dfrac{|C_j|}{2^{|I_j|}} \leq 2^{-j}$,
(2) $I_j \cap I_i = \emptyset$ for $i \neq j$.

Order $\mathbf{C}$ by extension.

To see that $\mathbf{C}$ has property R we use the following theorem:

THEOREM 7 (Lorenz, [6], [1]). *For every* $\varepsilon > 0$ *there exists* $N_\varepsilon \in \omega$ *such that for a sufficiently large finite set* $I \subset \omega$, *if* $X \subseteq 2^I$, $|X| \geq N_\varepsilon$ *then there exists a set* $C \subseteq 2^I$, $\dfrac{|C|}{2^{|I|}} \leq \varepsilon$ *and* $C + X = 2^I$.

Observe that if $X \subseteq 2^\omega$ is infinite and $\varepsilon > 0$ then there exists a clopen set $C \subseteq 2^\omega$ of measure at most ε such that $X + C = 2^\omega$. To see this choose I as in Theorem 7 so that $|X \restriction I| \geq N_\epsilon$.

This shows that any $\mathbf{C}$-generic sequence is big over the ground model. ⊣

Adding a big sequence over $\mathbf{V}$ is a natural method for making uncountable sets from $\mathbf{V}$ to become not strongly meager. In the sequel we will look for forcings adding such sequences and we will try to weaken the requirement about the caliber $(\aleph_1, \aleph_0)$.

For the construction of the forcing iteration that produces a model for $\mathsf{DBC} + \mathsf{cov}(\mathscr{N}) = \mathsf{cov}(\mathscr{M}) = \aleph_1$ we would like to have a forcing notion $\mathbb{P}$ such that

(1) $\mathbb{P}$ has property R,
(2) $\mathbb{P}$ does not add Cohen, or even more strongly: $\mathbf{V}^{\mathbb{P}} \models 2^\omega \cap \mathbf{V} \notin \mathscr{N}$.

We will show first how to construct a proper forcing notion having both properties. For our purpose we will need the following strenghtening of the Lorenz theorem, which will be proved in the next section.

THEOREM 8. *For every $\varepsilon, \delta > 0$ there exists $N_{\varepsilon,\delta} \in \omega$ such that for a sufficiently large finite set $I \subseteq \omega$ there is a family $\mathscr{A}_I$ consisting of sets $C \subseteq 2^I$, $\frac{|C|}{2^{|I|}} \leq \varepsilon$ such that if $X \subseteq 2^I$, $|X| \geq N_{\varepsilon,\delta}$ then*

$$\frac{\left|\left\{C \in \mathscr{A}_I : C + X = 2^I\right\}\right|}{|\mathscr{A}_I|} \geq 1 - \delta.$$

§3. Combinatorics. Let $I \subseteq \omega$ be a finite set. For $A, B \subseteq I$ let $A + B = (A \setminus B) \cup (B \setminus A)$ be the symmetric difference of A and B. Recall that $+$ is associative and commutative. If we identify subsets of I with their characteristic functions then $(A + B)(i) = A(i) + B(i) \mod 2$. We treat 2^I as a vector space over $\mathbb{Z}_2$.

For $|I| \geq k \geq 1$ let

$$Y_{k,I} = \left\{\bar{s} = \langle s_0, \ldots, s_{k-1}\rangle \in [2^I]^k : \bar{s} \text{ is linearly independent}\right\}.$$

Note that $s_0, \ldots, s_{k-1} \subseteq I$ are linearly independent if for any $u \subseteq k$ there is $x \in I$ such that x belongs to the odd number of $\{s_i : i \in u\}$, or equivalently if $0 \neq \sum_{i \in u} s_i \mod 2$. Observe that

$$\frac{|Y_{l,I}|}{|(2^I)^l|} = \frac{2^{|I|} - 1}{2^{|I|}} \cdot \frac{2^{|I|} - 2}{2^{|I|}} \cdots \frac{2^{|I|} - 2^{l-1}}{2^{|I|}} \geq 1 - \frac{1}{2^{|I|-l-1}}.$$

For $s, t \in 2^I$ define $s * t = \sum_{i \in I} s(i) \cdot t(i) \mod 2$. Note that $s * t = 0$ iff $|s \cap t|$ is even.

For $\bar{s} \in Y_{k,I}$ let $\bar{s}^{\perp} = \{t \in 2^I : \forall j < k \; s_j * t = 0\}$.

LEMMA 9. *For $\bar{s} \in Y_{k,I}$, $\frac{|\bar{s}^{\perp}|}{2^{|I|}} = \frac{1}{2^k}$.*

PROOF. Suppose that $\bar{s} = \{s_0, \ldots, s_{k-1}\}$ is given. Let $\{a_0, \ldots, a_{m-1}\}$ be atoms in the Boolean algebra generated by sets $\{s_0, \ldots, s_{k-1}\}$. In other words, sets a_i are pairwise disjoint and for every $i < k$ there is a set $u_i \subset m$ such that $s_i = \bigcup_{j \in u_i} a_j$. Let B be a $k \times m$ matrix such that $b_{i,j} = 1$ if $j \in u_i$ and $b_{i,j} = 0$ if $j \notin u_i$. Let $\bar{x} = \langle x_0, \ldots, x_{m-1}\rangle$ and let R be the set of solutions to the system of equations $B\bar{x} = 0 \mod 2$. It is not hard to see that $\frac{|R|}{2^m} = \frac{|\bar{s}^{\perp}|}{2^I}$, as the solutions to this system of equations correspond to the elements of $\bar{s}^{\perp}$.

Carry out the Gauss-Jordan algorithm on the system $B\bar{x} = 0 \mod 2$. Note that the equations are independent if and only if $\bar{s} \in Y_{k,I}$. Thus, if $\bar{s} \in Y_{k,I}$ then the system has exactly k dependent variables and the result follows. ⊣

The following is a specific form of Theorem 8.

THEOREM 10. *For every $\delta > 0$ and $l \in \omega$ there exists $N_{l,\delta} \in \omega$ such that for a sufficiently large finite set $I \subset \omega$, if $X \subseteq 2^I$, $|X| \geq N_{l,\delta}$ then*

$$\frac{\left|\left\{\bar{s} \in Y_{l,I} : \bar{s}^{\perp} + X = 2^I\right\}\right|}{|Y_{l,I}|} \geq 1 - \delta.$$

PROOF. Let us start with the following:

LEMMA 11. *Suppose that $l \in \omega$ and $m > l$ is sufficiently large and $r \in 2^l$. For every $\bar{t} = \langle t_0, \dots, t_{m-1} \rangle \in Y_{m,I}$,*

$$\frac{\left|\left\{\bar{s} \in Y_{l,I} : \left|\frac{|\{j < m : \forall i < l\ t_j * s_i = r(i)\}|}{|\bar{t}|} - \frac{1}{2^l}\right| \geq \left(\frac{1}{2}\right)^{l+1}\right\}\right|}{|Y_{l,I}|} \leq 2l \exp\left(-\frac{m}{4^{l+3}}\right).$$

PROOF. For $j < m$ let $r_j = \{t_j * s_i : i < l\}$. Observe that when $\bar{s}$ is randomly selected then possible values of r_j are equidistributed, each with probabilty 2^{-l}. Thus for any particular $r \in 2^l$, the fraction $\frac{|\{j < m : \forall i < l\ t_j * s_i = r(i)\}|}{|\bar{t}|}$ is approximately 2^{-l}. The lemma says that the percentage of those $\bar{s}$ for which this value is less than 2^{-l-1}, which is half of the expected value, is less than $2l \exp\left(-\frac{m}{4^{l+3}}\right)$.

Put $l = 1$. In this case r is constant, equal to 0 or 1. Without loss of generality we can assume that $r = 0$. For $j < m$ let X_j be the random variable defined on 2^I as $X_i(s) = 1$ iff $s * t_i = 0$. Note that $P(X_i = 1) = \frac{1}{2}$ and X_j's are mutually independent (since t_j's are algebraically independent). Note that $\frac{|\{j < m : t_j * s = 0\}|}{m} = \frac{|\{s\}^\perp \cap \bar{t}|}{|\bar{t}|} = \frac{\sum_{j<m} X_j(s)}{m}$.

Let S_m be the number of successes in m independent Bernoulli trials with probability of success p. For every $\delta > 0$,

$$P\left(\left|\frac{S_m}{m} - p\right| \geq \delta\right) \leq 2\exp\left(-m\delta^2/4\right).$$

Putting $\delta = \frac{1}{4}$ and $p = \frac{1}{2}$ we get that

$$\frac{\left|\left\{s \in 2^I : \left|\frac{|\{s\}^\perp \cap \bar{t}|}{|\bar{t}|} - \frac{1}{2}\right| \geq \frac{1}{4}\right\}\right|}{|2^I|} = P\left(\left|\frac{\sum_{j<m} X_j}{m} - \frac{1}{2}\right| \geq \frac{1}{4}\right) \leq 2\exp\left(-\frac{m}{64}\right).$$

Thus, given $r(0) = 0, 1$ the fraction of those $s \in 2^I$ for which

$$\frac{1}{4}m \leq |\{j < m : t_j * s = r(0)\}| \leq \frac{3}{4}m,$$

is $1 - 2\exp\left(-\frac{m}{64}\right)$. Now fix an s_0 belonging to this set and let $A_{s_0} = \{j < m : t_j * s_0 = r(0)\}$. Given $r(1) = 0, 1$, and applying the same argument, we get that the fraction of $s \in 2^I$ such that

$$\frac{1}{4^2}m \leq \frac{1}{4}|A_{s_0}| \leq |\{j \in A_{s_0} : t_j * s = r(1)\}| \leq \frac{3}{4}|A_{s_0}| \leq \frac{3^2}{4^2}m,$$

is at least $1-2\exp\left(\frac{|A_{s_0}|}{64}\right) \geq 1-2\exp\left(-\frac{m/4}{64}\right)$. Thus for $l=2$ we get that

$$\frac{\left|\left\{\bar{s}\in Y_{2,I}\colon \left|\frac{|\{j<m\colon \forall i<2\, t_j * s_i = r(i)\}|}{|\bar{t}|}-\frac{1}{2^2}\right| \leq \left(\frac{1}{2}\right)^3\right\}\right|}{|Y_{2,I}|}$$

$$\geq \frac{\left|\left\{\bar{s}\in (2^I)^2\colon \left|\frac{|\{j<m\colon \forall i<2\, t_j * s_i = r(i)\}|}{|\bar{t}|}-\frac{1}{2^2}\right| \leq \left(\frac{1}{2}\right)^3\right\}\right|}{|(2^I)^2|}\cdot\left(1-\frac{1}{2^{|I|-1}}\right)$$

$$\geq \left(1-2\exp\left(-\frac{m}{64}\right)\right)\left(1-2\exp\left(-\frac{m/4}{64}\right)\right)\left(1-\frac{1}{2^{|I|-1}}\right).$$

Similarly for arbitrary l and large enough $|I|$ we get

$$\frac{\left|\left\{\bar{s}\in Y_{l,I}\colon \left|\frac{|\{j<m\colon \forall i<l\, t_j * s_i = r(i)\}|}{|\bar{t}|}-\frac{1}{2^l}\right| \leq \left(\frac{1}{2}\right)^{l+1}\right\}\right|}{|Y_{l,I}|}$$

$$\geq \left(1-2\exp\left(-\frac{m}{64}\right)\right)\left(1-2\exp\left(-\frac{m/4}{64}\right)\dots\left(1-2\exp\left(-\frac{m/4^l}{64}\right)\right)\right.$$
$$\cdot\left(1-\frac{1}{2^{|I|-1}}\right)\left(1-\frac{1}{2^{|I|-2}}\right)\dots\left(1-\frac{1}{2^{|I|-l-1}}\right)$$
$$\geq 1-2l\exp\left(-\frac{m}{4^{l+3}}\right),$$

which finishes the proof. ⊣

Now we are ready to prove Theorem 10. Observe that for $\bar{t}\in Y_{m,I}$ and $\bar{s}\in Y_{l,I}$, $\bar{t}+\bar{s}^\perp = 2^I$ if for every $u\in 2^I$ there is $t_u\in\bar{t}$ such that

$$(u+t_u)*s=0 \text{ for all } s\in\bar{s}.$$

For every $u\in 2^I$ let $r_u\in 2^l$ be such that $u\star s_i = r_u(i)$ for all $i<l$. Note that $(u+t)*s=(u*s)+(t*s) \mod 2$. In particular, for $\bar{s}=\langle s_0,\dots,s_{l-1}\rangle\in Y_{l,I}$,

$$\forall i<l\,(u+t)*s_i=0 \iff \forall i<l\; t*s_i=r_u(i).$$

Notice that there are $2^{|I|}$ possible u's but only 2^l possible functions r_u.

By Lemma 11 we get that for any u and thus for the corresponding function r_u,

$$P\left(\exists j<m\,\forall i<l\,(u+t_j)*s_i=0\right)=P\left(\exists j<m\,\forall i<l\; t_j*s_i=r_u(i)\right)$$
$$\geq 1-2l\exp\left(-\frac{m}{4^{l+3}}\right).$$

We conclude that

$$P\left(\bar{s}^\perp+\bar{t}=2^I\right)=P\left(\forall u\in 2^I\,\exists j<m\,\forall i<l\,(u+t_j)*s_i=0\right)$$
$$=P\left(\forall r\in 2^l\,\exists j<m\,\forall i<l\; t_j*s_i=r(i)\right)$$
$$\geq 1-2^l\cdot 2l\exp\left(-\frac{m}{4^{l+3}}\right).$$

Since every set $X \subset 2^I$ contains an independent subset of size at least $\log_2(|X|)$ we get that for a set $X \subseteq 2^I$ of size m,

$$P\left(\bar{s}^{\perp} + X = 2^l\right) \geq 1 - 2^{l+1} l \exp\left(-\frac{\log_2 m}{4^{l+3}}\right) \overset{m\to\infty}{\longrightarrow} 1. \qquad \dashv$$

§4. **Definition of the forcing.** Let Ω consist of strictly increasing functions $g \in \omega^\omega$ such that for every k, $g(k+1) \geq 2^{2^{2^{g(k)}}}$. Fix a sequence $\langle \varepsilon_j^k : j \leq k \rangle$ such that

(1) $\forall k\, 0 < \varepsilon_0^k < \varepsilon_1^k < \cdots < \varepsilon_k^k < 2^{-k}$,

(2) $\forall l < k\, 2^{4^k} \varepsilon_l^k < \varepsilon_{l+1}^k$.

Suppose that $g \in \Omega$ is given and let $I_k = [g(k), g(k+1))$ for $k \in \omega$. For $A \subseteq Y_{k,I_k}$ let

$$\|A\|_k = \max\left\{\ell : \frac{|A|}{|Y_{k,I_k}|} \geq \varepsilon_\ell^k\right\}.$$

Consider the tree

$$p^{\max} = \bigcup_k \prod_{j=0}^{k} Y_{j,I_j}.$$

For a tree $p \subseteq p^{\max}$ let $p{\restriction}n = p \cap \prod_{j=0}^{n-1} Y_{k,I_k}$. For $t \in p{\restriction}k$ let $\mathsf{succ}_p(t) = \{\bar{s} \in Y_{k,I_k} : t^\frown \bar{s} \in p\}$ be the set of all immediate successors of t in p, and let $p_t = \{s \in p : s \subseteq t \text{ or } t \supseteq s\}$ be the subtree determined by t. Let $\mathsf{stem}(p)$ be the shortest $t \in p$ such that $|\mathsf{succ}_p(t)| > 1$.

Let $\mathbb{P}_g$ be the forcing notion which consists of perfect subtrees $p \subseteq p^{\max}$ such that

$$\inf_{n\in\omega} \min(\|\mathsf{succ}_p(s)\|_n : s \in p{\restriction}n) = \infty.$$

For $p, q \in \mathbb{P}_g$ and $n \in \omega$ define $p \geq q$ if $p \subseteq q$. For $n \in \omega$ and $p \in \mathbb{P}_g$ let

$$A_p^n = \left\{s \in p : \|\mathsf{succ}_p(s)\|_{|s|} \leq n\right\}.$$

Let $p \geq_n q$ be defined as $p \geq q$ and $A_p^n = A_q^n$.

Suppose that $G \subseteq \mathbb{P}_g$ is a generic filter over $\mathbf{V}$. Let $\bigcap G = \langle s_k : k \in \omega\rangle \in \prod_k Y_{k,I_k}$. The set $\langle s_k^{\perp} : k \in \omega\rangle$ can be identified with a sequence of clopen sets in 2^ω. This will be a prototype of a big sequence that we want to construct. As we will see below, bigness will require that function g is not bounded by any function from $\mathbf{V} \cap \omega^\omega$.

We will adopt the following (nonstandard) definition of the rational perfect forcing, $\mathbb{M}$. A condition p belongs to $\mathbb{M}$ if $p \subseteq \omega^{<\omega}$ is a perfect tree whose all branches belong to Ω and such that for all $s \in p$ there exists an extension $t \in p$ so that $|\mathsf{succ}_p(t)| = \aleph_0$. We can assume that for $p \in \mathbb{M}$, $|\mathsf{succ}_p(t)| = \aleph_0$ or $|\mathsf{succ}_p(t)| = 1$.

If $G \subseteq \mathbb{M}$ is a generic filter over $\mathbf{V}$ let $\mathbf{m} = \bigcap_{p\in G}[p]$ be the generic real.

Lemma 12. *The forcing $\mathbb{M} \star \mathbb{P}_{\dot{\mathbf{m}}}$ adds a a sequence $\langle C_k : k \in \omega\rangle$ which is big over $\mathbf{V}$.*

PROOF. The forcing $\mathbb{M}\star\mathbb{P}_{\dot{\mathbf{m}}}$ adds a sequence $\langle \bar{s}_k^\perp : k\in\omega\rangle \in \prod_k Y_{k.I_k}$, where $I_k=[\mathbf{m}(k),\mathbf{m}(k+1))$ and $\mathbf{m}$ is a Miller real. We claim that the sequence $\langle \bar{s}_k^\perp : k\in\omega\rangle$ is big over $\mathbf{V}$.

Suppose that $X\in\mathbf{V}$ is an infinite set of reals and $(p,\dot{q})\in\mathbb{M}\star\mathbb{P}_{\dot{\mathbf{m}}}$. By passing to a stronger condition we can assume that

(1) for every $s\in \mathsf{split}(p)$ there exists $q_s\in\prod_{k<|s|} Y_{k.I_k}$ such that
   (a) $p_s \Vdash \dot{q}\restriction|s| = q_s$
   (b) for $n\in\mathsf{succ}_p(s)$, n minimal, $[s(|s|-1,n)\subseteq I_{|s|-1}$,
   (c) for each $n\in\mathsf{succ}_p(s)$, $|X\restriction[s(|s|-1),n)|>N_{|s|,\varepsilon_0^{|s|}}$.

(2) if $s\in\mathsf{split}(p)$ is in the ℓ-splitting level and $n\in\mathsf{succ}_p(s)$ then $p_{s^\frown n}$ forces that for every $t\in\dot{q}\restriction|s|=q_s$ the set $\mathsf{succ}_{\dot{q}}(t)=\mathsf{succ}_{q_{s^\frown n}}(t)$ has norm at least ℓ.

Observe that $p\Vdash \exists^\infty k\, |X\restriction[\mathbf{m}(k),\mathbf{m}(k+1))|\geq N_{k,\varepsilon_0^k}$.

Now we will describe how to modify $\dot{q}$ so that $(p,\dot{q})\Vdash_{\mathbb{M}\star\mathbb{P}_{\dot{\mathbf{m}}}} \exists^\infty k\; \bar{s}_k^\perp + X = 2^\omega$.

Let $s_0=\mathsf{stem}(p)$ and $n\in\mathsf{succ}_p(s_0)$. We know that $p_{s_0}\Vdash \dot{q}\restriction|s_0|=q_{s_0}$ and $p_{s_0^\frown n}\Vdash \dot{q}\restriction|s_0|+1=q_{s_0^\frown n}$. Furthermore, for every $t\in q_{s_0}\restriction|s_0|$,

$$\left\|\left\{\bar{s}\in\mathsf{succ}_{q_{s_0^\frown n}}(t)\colon \bar{s}^\perp + X\restriction[s(|s|-1),n)=2^{[s(|s|-1),n)}\right\}\right\|_{|s_0|}$$
$$\geq \left\|\mathsf{succ}_{q_{s_0^\frown n}}(t)\right\|_{|s_0|}\left(1-\varepsilon_0^{|s_0|}\right)\geq \left\|\mathsf{succ}_{q_{s_0^\frown n}}(t)\right\|_{|s_0|}-1.$$

Thus we shrink $\dot{q}$ by making sure that $\bar{s}^\perp + X\restriction[s(|s|-1),n)=2^{[s(|s|-1),n)}$ for each $\bar{s}\in\mathsf{succ}_{\dot{q}}(t)$. We repeat this construction for each splitting node of p. It is clear that the resulting condition has the required properties. ⊣

We will need to verify that $\mathbb{P}_g$ has some additional properties. First we show that $\mathbb{P}_g$ is ω^ω-bounding. The arguments below are rather standard, we reconstruct them here for completeness but the reader familiar with [8] will see that they are a part of a much more general scheme.

LEMMA 13. *Suppose that $A\subseteq\mathbf{V}$ is a countable set, $n\in\omega$ and $p\Vdash_{\mathbb{P}_g}\dot{a}\in A$. There exists $q\geq_n p$ and $k\in\omega$ such that for every $t\in q\restriction k$ there exists $a_t\in A$ such that $q_t\Vdash_{\mathbb{P}_g}\dot{a}=a_t$.*

PROOF. Let $S\subseteq p$ be the set of all $t\in p$ such that p_t satisfies the lemma. In other words

$$S=\{t\in p\colon \exists k_t\in\omega\,\exists q'\geq_n p_t\,\forall s\in q'\restriction k_t\,\exists a_s\in A\; q'_s\Vdash_{\mathbb{P}_g}\dot{a}=a_s\}.$$

Note that for every condition $q\geq p$, $q\cap S\neq\emptyset$. We want to show that $\mathsf{stem}(p)\in S$. Notice that if $s\notin S$ then $\|\mathsf{succ}_p(s)\cap S\|_{|s|}\leq n$. This follows from the the fact that if $\|X\|_{|s|}\leq n$ and $\|Y\|_{|s|}>n$ then $\|Y\setminus X\|_{|s|}\geq n$.

Suppose that $\mathsf{stem}(p)\notin S$ and by induction on levels build a tree $q\geq_n p$ such that for $s\in q$,

$$\mathsf{succ}_q(s)=\begin{cases}\mathsf{succ}_p(s) & \text{if } \|\mathsf{succ}_p(s)\|_{|s|}\leq n\\ \mathsf{succ}_p(s)\setminus S & \text{otherwise}\end{cases}.$$

Clearly $q\in\mathbb{P}_g$ since $\|\mathsf{succ}_q(s)\|_{|s|}\geq\|\mathsf{succ}_p(s)\|_{|s|}-1$ for s containing $\mathsf{stem}(p)$. This is a contradiction since $q\cap S=\emptyset$ which is impossible. ⊣

In our case we have even stronger fact (see [8] for more general treatment):

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LEMMA 14. *Suppose that* $p \Vdash_{\mathbb{P}_g} \dot{A} \subseteq 2^{<\omega}$. *There exists* $q \geq p$ *such that for all* n, *for every* $t \in q\restriction n$ *there exists* $A_t \subseteq 2^n$ *such that* $q_t \Vdash_{\mathbb{P}_g} \dot{A}\restriction n = A_t$.

In particular, if $p \Vdash_{\mathbb{P}_g} \dot{x} \in 2^\omega$ *then there exists* $q \geq p$ *such that for all* n, *for every* $t \in q\restriction n$ *there exists* $s_t \in 2^n$ *such that* $q_t \Vdash_{\mathbb{P}_g} \dot{x}\restriction n = s_t$.

PROOF. This again is a fairly standard construction. It is enough to prove the first part. By applying lemma 13 we can assume that there exists an increasing sequence $\langle k_n : n \in \omega\rangle$ such that for every $t \in p\restriction k_n$ there exists $A_t \subseteq 2^{<n}$ such that $p_t \Vdash_{\mathbb{P}_g} \dot{A}\restriction n = A_t$. We will show how to shrink p to $q \geq p$ so that for this q, $k_n = n$ for each n. Let $n_0 = |\mathsf{stem}(p)|$. We will illustrate this process by showing how to shrink p to p' so that the new value of k_{n_0} becomes $k_{n_0} - 1$. Suppose that $t \in p\restriction k_{n_0} - 1$. For each $s \in \mathsf{succ}_p(t)$ there is $A_{t^\frown s}$ such that $p_{t^\frown s} \Vdash \dot{A}\restriction n_0 = A_{t^\frown s}$. There are only $2^{2^{n_0}}$ possible values of $A_{t^\frown s}$. Call the most frequent of them $\bar{A}_t$ and let

$$\mathsf{succ}_{p'}(t) = \{s \in \mathsf{succ}_p(t) : A_{t^\frown s} = \bar{A}_t\}.$$

Observe that p' has the required property. Now get q by applying this procedure $k_{n_0} - n_0$ times to collapse k_{n_0} to n_0, then k_{n_1} to n_0+1, etc. Note that the n_0+k-level of p gets modified k times. Since $\varepsilon_\ell^{k_{n_0}+k} \cdot 2^{-2^{n_0}} \cdot 2^{-2^{n_0+1}} \cdot \cdots \cdot 2^{-2^{n_0+k}} \geq \varepsilon_\ell^{k_{n_0}+k} \cdot 2^{-4^{k_{n_0}+k}} \geq \varepsilon_{\ell-1}^{k_{n_0}+k}$, it follows that $\|\mathsf{succ}_q(t)\|_{|t|} \geq \|\mathsf{succ}_p(t)\|_{|t|} - 1$. ⊣

Next we show that $\mathbb{P}_g$ preserves outer measure.

THEOREM 15. *If* $X \subseteq 2^\omega$, $X \in \mathbf{V}$ *and* $\mathbf{V} \models X \notin \mathcal{N}$ *then* $\mathbf{V}^{\mathbb{P}_g} \models X \notin \mathcal{N}$.

PROOF. The sketch of the proof presented here is a special case of a more general theorem (theorem 3.3.5 of [8]).

Fix $1 > \delta > 0$ and a strictly increasing sequence $\langle \delta_n : n \in \omega\rangle$ of real numbers such that

(1) $\sup_n \delta_n = \delta$.
(2) $\forall^\infty n\, \delta_{n+1} - \delta_n > \varepsilon_n^n$.

Suppose that $X \notin \mathcal{N}$. Without loss of generality we can assume that X has outer measure one. For contradiction, assume that $\mathbf{V}^{\mathbb{P}_g} \models X \in \mathcal{N}$. Let $\dot{A}$ be a $\mathbb{P}_g$-name for a tree of positive measure such that $\Vdash_{\mathbb{P}_g} \dot{A} \subseteq 2^{<\omega}$ & $\mu([\dot{A}]) \geq \delta$ and suppose that $p \Vdash_{\mathbb{P}_g} X \cap [\dot{A}] = \emptyset$. Let $n_0 = |\mathsf{stem}(p)|$. By lemma 14, we can assume that

$$\forall n > n_0\, \forall t \in p\restriction n\, \exists A_t \subseteq 2^n\; p_t \Vdash_{\mathbb{P}_g} \dot{A}\restriction n = A_t.$$

Fix $n > n_0$ and define by induction sets $\{A_t^n : t \in p\restriction m,\ n_0 \leq m \leq n+1\}$ such that

(1) $A_t^n \subseteq 2^{n+1}$ for $t \in p$,
(2) $|A_t^n| \cdot 2^{-n-1} \geq \delta_m$ for $t \in p\restriction m$.

For $t \in p\restriction n+1$ let $A_t^n = A_t$. Suppose that sets A_t^n are defined for $t \in p\restriction m$, $m > n_0$. Let $t \in p\restriction m-1$ and consider the family $\{A_{t^\frown s}^n : s \in \mathsf{succ}_p(t)\}$. By the induction hypothesis, $|A_{t^\frown s}^n| \cdot 2^{-n-1} \geq \delta_m$. Let

$$A_t^n = \left\{u \in 2^{n+1} : \|\{s : u \in A_{t^\frown s}\}\| \geq \|\mathsf{succ}_p(t)\| - 1\right\}.$$

Use Fubini's theorem to show that the requirement that we put on the sequence $\langle \delta_n : n \in \omega\rangle$ implies that $|A_t^n| \cdot 2^{-n-1} \geq \delta_{m-1}$. In particular, $A_{\mathsf{stem}(p)}^n \cdot 2^{-n-1} \geq \delta_{n_0}$ for all n. Let $B = \{x \in 2^\omega : \exists^\infty n\, x\restriction n+1 \in A_{\mathsf{stem}(p)}^n\}$. As $\mu(B) \geq \delta_{n_0}$, by the assumption that X has outer measure one, $B \cap X \neq \emptyset$. Fix $x \in B \cap X$. We will find $q \geq p$ such that $q \Vdash_{\mathbb{P}_g} x \in [\dot{A}]$, which will give a contradiction.

For each n such that $x \in A^n_{\mathsf{stem}(p)}$ let $q_n \subseteq p{\restriction}n$ be a finite tree such that

(1) $\mathsf{stem}(q_n) = \mathsf{stem}(p)$,
(2) for every $t \in q_n$, $n_0 < |t| < n$, $\|\mathsf{succ}_{q_n}(t)\|_{|t|} \geq \|\mathsf{succ}_p(t)\|_{|t|} - 1$,
(3) for every $t \in q_n$, $|t| = n$, $x \in A_t$.

The existence of q_n follows from the inductive definition of A^n_t's. By König's lemma, there exists $q \subseteq p$ such that for infinitely many n, $q{\restriction}n = q_n$. It follows that $q \in \mathbb{P}_g$ and $q \Vdash_{\mathbb{P}_g} \exists^\infty n\, x{\restriction}n \in \dot{A}{\restriction}n$. Since $\dot{A}$ is a tree we conclude that $s \Vdash_{\mathbb{P}_g} x \in [\dot{A}]$. ⊣

Since Miller forcing preserves outer measure (see 7.3.47 of [1]) it follows that $\mathbb{M} \star \mathbb{P}_{\dot{m}}$ preserves outer measure. Furthermore:

THEOREM 16. *Suppose that $N \prec \mathbf{H}(\chi)$ is a countable model containing $\mathbb{M} \star \mathbb{P}_{\dot{m}}$, $p \in N \cap \mathbb{M} \star \mathbb{P}_{\dot{m}}$ and x is a random real over N. Then there exists $q \geq p$ such that q is N-generic and*

$$q \Vdash_{\mathbb{M}\star\mathbb{P}_{\dot{m}}} x \text{ is random over } N[\dot{G}].$$

PROOF. Compare with 7.3.40 and 7.3.42 of [1] where the same theorem is proved for the Laver forcing. The only forcing specific properties used in the proof are definability and preservation of outer measure. For a more general setting see [10]. ⊣

§5. Forcing for the single task. The forcing notion $\mathbb{M} \star \mathbb{P}_{\dot{m}}$ adds a big sequence over $\mathbf{V}$ and preserves outer measure but it does not have the other properties (caliber $(\aleph_1, \aleph_0)$) that will guarantee that it kills uncountable strongly meager sets from the ground model. We have to modify it so that it becomes ccc and it acquires additional properties that allow the former proofs to go through. In this section we will describe how to do this for for a single step in the construction. In other words, suppose that $X \subseteq 2^\omega$ is an uncountable set of reals. We will describe how to construct a forcing notion $\mathbb{P}_X$ such that

(1) $\mathbb{P}_X \subseteq \mathbb{M} \star \mathbb{P}_{\dot{m}}$,
(2) $\mathbb{P}_X$ is ccc,
(3) $\mathbf{V}^{\mathbb{P}_X} \models \mathbf{V} \cap 2^\omega \notin \mathcal{N} \cup \mathcal{M}$,
(4) $\mathbf{V}^{\mathbb{P}_X} \models X \notin \mathcal{SM}$.

Note that (3) will guarantee that $\mathbb{P}_X$ does not add Cohen or random reals.

We say that $\mathbb{P} \subseteq_{ic} \mathbb{Q}$ if $\mathbb{P} \subseteq \mathbb{Q}$ and whenever $p, q \in \mathbb{P}$ are compatible in $\mathbb{Q}$ then they are compatible in $\mathbb{P}$. A set $\mathcal{D} \subseteq \mathbb{P}$ is pre-dense if for every $p \in \mathbb{P}$ there is $q \in \mathcal{D}$ such that p and q are compatible. Clearly, every maximal antichain is pre-dense.

Let AP be the set of all pairs $\langle \mathbb{P}, I \rangle$ where

(1) $\mathbb{P} \subseteq_{ic} \mathbb{M} \star \mathbb{P}_{\dot{m}}$ is countable,
(2) $I = \{\mathcal{D}_n : n \in \omega\}$, where each $\mathcal{D}_n$ is pre-dense in $\mathbb{P}$.

We have $\langle \mathbb{P}_1, I_1 \rangle \geq \langle \mathbb{P}_0, I_0 \rangle$ if $\mathbb{P}_0 \subseteq \mathbb{P}_1$ and $I_0 \subseteq I_1$.

We will build the required forcing as an increasing chain of approximations $\{\langle \mathbb{P}_\alpha, I_\alpha \rangle : \alpha < \omega_1\}$ and put $\mathbb{P}_X = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha$. In order to guarantee that $\mathbb{P}_X$ satisfies ccc we will use an oracle that will tell us that whenever $\mathcal{A}$ is a maximal antichain in $\mathbb{P}_X$ then for some $\alpha < \omega_1$, $\mathcal{A} \in I_\alpha$.

Suppose that $S \subseteq \omega_1$ is a stationary set of limit ordinals. Assume that $\diamond(S)$ holds. In particular we can assume that $\diamond(S)$ is witnessed by a sequence $\bar{M} = \{M_\alpha \colon \alpha \in S\}$ such that

(1) M_α is a countable transitive model of ZFC*,
(2) $\alpha + 1 \subseteq M_\alpha$ and $M_\alpha \models \alpha$ is countable.
(3) $\forall X \subseteq \omega_1 \{\alpha \in S \colon X \cap \alpha \in M_\alpha\}$ is stationary.

Let NS_{ω_1} be the ideal of non-stationary subsets of ω_1.

DEFINITION 17. *Let $D_{\bar{M}}$ be the filter which is generated by sets*

(1) $A_U = \{\alpha \in \omega_1 \colon U \cap \alpha \in M_\alpha$ *or* $\alpha \notin S\}$ *for* $U \subseteq \omega_1$,
(2) *club subsets of* ω_1.

LEMMA 18. *$D_{\bar{M}}$ is is a normal filter containing all closed unbounded sets.*

PROOF. See chapter IV claim 1.4 in [9]. ⊣

DEFINITION 19. *For two oracles $\bar{M}_0 = \{M^0_\alpha \colon \alpha \in S^0\}$ and $\bar{M}_1 = \{M^1_\alpha \colon \alpha \in S^1\}$ we say that $\bar{M}_0 \leq \bar{M}_1$ if*

(1) $S^0 \setminus S^1 \in \mathrm{NS}_{\omega_1}$ *and*
(2) $\{\alpha \in S^0 \colon M^0_\alpha \neq M^1_\alpha\} \in \mathrm{NS}_{\omega_1}$.

It follows from Lemma 18 that if $\bar{M}_0 \leq \bar{M}_1$ then $D_{\bar{M}_0} \subseteq D_{\bar{M}_1}$.

We will identify the forcing notion $\mathbb{P}$ that we are constructing with ω_1. Suppose that $f \colon \mathbb{P} \xrightarrow{1\text{-}1} \omega_1$ and let $\mathscr{D} \subseteq \mathbb{P}$ be a pre-dense set in $\mathbb{P}$. Let $\mathbb{P}^f_\alpha = \{p \in \mathbb{P} \colon f(p) < \alpha\}$ and let $\mathscr{D}^f_\alpha = \{p \in \mathscr{D} \colon f(p) < \alpha\}$.

LEMMA 20. $\{\alpha \in S \colon \mathscr{D}^f_\alpha$ *is pre-dense in* $\mathbb{P}^f_\alpha\} \in D_{\bar{M}}$.

Furthermore, if $g \colon \mathbb{P} \xrightarrow{1\text{-}1} \omega_1$ *then*

$$\{\alpha \in S \colon \mathbb{P}^f_\alpha = \mathbb{P}^g_\alpha \;\&\; \mathscr{D}^g_\alpha = \mathscr{D}^f_\alpha \;\&\; \mathscr{D}^f_\alpha \text{ is pre-dense in } \mathbb{P}^f_\alpha\} \in D_{\bar{M}}.$$

PROOF. Note that the appropriate sets are closed and unbounded. ⊣

Thus the choice of function f identifying $\mathbb{P}$ with ω_1 does not matter for the properties defined below which require that certain conditions are met on a set which belongs to $D_{\bar{M}}$. Thus we will suppress the superscript f in the sequel.

DEFINITION 21. *We say that $\mathbb{P}$ is $(\bar{M}, D)$-cc if*

$$\{\alpha < \omega_1 \colon \text{If } \mathscr{D} \in M_\alpha \text{ is pre-dense in } \mathbb{P}_\alpha \text{ then } \mathscr{D} \text{ is pre-dense in } \mathbb{P}\} \in D.$$

We say that $\mathbb{P}$ is $\bar{M}$-cc if $D = D_{\bar{M}}$.

LEMMA 22. *Suppose that $\bar{M}$ is an oracle.*

(1) *If $\mathbb{P}$ is $(\bar{M}, D)$-cc and $D_{\bar{M}} \subseteq D$ then $\mathbb{P}$ is ccc. In particular, if $\mathbb{P}$ is $\bar{M}$-cc then it is ccc.*
(2) *If $D_{\bar{M}} \subseteq D_0 \subseteq D_1$ then $(\bar{M}, D_0)$-cc implies $(\bar{M}, D_1)$-cc.*
(3) *If $\bar{M}_0 \leq \bar{M}_1$ and $\mathbb{P}$ satisfies $(\bar{M}_1, D)$-cc and $\mathbb{Q} < \mathbb{P}$ then $\mathbb{Q}$ satisfies the $(\bar{M}_0, D)$-cc.*

PROOF. (1) If $\mathscr{A}$ is a maximal antichain in $\mathbb{P}$ then there exists $\alpha \in S$ such that $\mathscr{A}_\alpha \in M_\alpha$, and $\mathscr{A}_\alpha$ is maximal in $\mathbb{P}_\alpha$. In this case $\mathscr{A}_\alpha$ is also maximal in $\mathbb{P}$. It follows that $\mathscr{A} = \mathscr{A}_\alpha$ and so it is countable.

(2) is obvious and (3) is proved in [10] (fact 1.4). ⊣

The following definition is a special case of Definition 1.6 in [10].

DEFINITION 23. *Given an oracle $\bar{M}$ with domain S let $Y_S^{random} = \{x_\alpha : \alpha \in S\}$ be a sequence of reals such that x_α is random over M_α for $\alpha \in E \in D_{\bar{M}}$. We call Y_S^{random} a 0-commitment. We say that $\mathbb{P}$ satisfies a 0-commitment Y_S^{random} if*

$$\{\alpha \in S : \Vdash_{\mathbb{P}} x_\alpha \text{ is random over } M_\alpha[\mathbb{P}_\alpha \cap \dot{G}]\} \in D_{\bar{M}}.$$

In other words we can think of $\mathbb{P}$ being $\bar{M}$-cc as about a *trivial* commitment, that is all that needs to be preserved is the maximality of antichains. This corresponds to classical oracle ccc as described in chapter IV of [9].

In the sequel we will be using only two types of commitments, Y_S^{random}, and $Y_S^{trivial}$ for the trivial case. The notation $Y_S^{trivial}$ is introduced only for notational consistency as in this case there are no reals whose genericity needs to be preserved.

Now we go back to the construction of forcing $\mathbb{P}_X$ for a fixed uncountavle sets of reals X. Fix a stationary set $S \subseteq \omega_1$. Since we will need two types of commitments let us assume that $S = S_0 \dot{\cup} S_1$, where both S_0, S_1 are stationary. Furthermore, let $\bar{M} = \{M_\alpha : \alpha \in S\}$ be an oracle witnessing $\diamondsuit(S_0)$ and $\diamondsuit(S_1)$. In addition, we will require that for each $\alpha \in S$,

(1) $X \cap M_\alpha \in M_\alpha$,
(2) $M_\alpha \models X \cap M_\alpha$ is uncountable.

This can be accomplished by increasing the M_α's using the following easy observation. More specifically, let $N \prec \mathbf{H}(\chi)$ be a countable model containing M_α and X. The transitive collapse of N is the object we are looking for.

For each $\alpha \in S_0$ choose $x_\alpha \in \mathbf{V} \cap 2^\omega$ which is random over M_α. Then $Y_{S_0}^{random}$ is a commitment on $\{M_\alpha : \alpha \in S_0\}$.

The following construction is a special case of Claim 1.18 of [10].

For a forcing notion $\mathbb{P}$ and $p \in \mathbb{P}$ let $\mathbb{P}(p) = \{q \in \mathbb{P} : q \geq p\}$. We will build a continuous increasing sequence of approximations $\{\langle \mathbb{P}_X^\alpha, I^\alpha \rangle : \alpha \in S\}$ such that

(1) $\mathbb{P}_X^\alpha \subseteq_{ic} \mathbb{M} \star \mathbb{P}_{\dot{\mathrm{m}}}$,
(2) $\mathbb{P}_X$ satisfies both commitments $Y_{S_0}^{random}$ and $Y_{S_1}^{trivial}$,
(3) I^α is a countable family of countable pre-dense subsets of $\mathbb{P}_X^\alpha$,
(4) if $\mathscr{D} \in I^\alpha$, $n \in \omega$ and $p \in \mathbb{P}_X^\alpha$ then for some $q \in \mathbb{P}_X^\alpha$, $q \geq_n p$ and $\mathscr{D}$ is predense above q in $\mathbb{P}_X$.

Fix a bijection f between $\mathbb{M} \star \mathbb{P}_{\dot{\mathrm{m}}}$ and ω_1. We will suppress f but we will always think that $\mathbb{P}_X$ is a subset of ω_1 which is the image of the forcing constructed below under f. As we noted earlier the choice of f does not matter.

Suppose that $\{\mathbb{P}_X^\beta : \beta \in S \cap \alpha\}$ have already been constructed. We will describe how to construct $\mathbb{P}_X^\alpha$.

CASE 1 α is limit in S. Let $\mathbb{P}_X^\alpha = \bigcup_{\beta \in S \cap \alpha} \mathbb{P}_X^\beta$ and $I^\alpha = \bigcup_{\beta \in S \cap \alpha} I^\beta$.

CASE 2 $\alpha = \beta^+$ and either β is a successor in S or $(\mathbb{P}_X^\beta, I^\beta) \notin M_\beta$. Let $\mathbb{P}_X^\alpha = \mathbb{P}_X^\beta$ and $I^\alpha = I^\beta$.

CASE 3 $\alpha = \beta^+$, $\beta \in S_0$ is limit and $(\mathbb{P}_X^\beta, I^\beta) \in M_\beta$.

Note that according to the convention mentioned at the beginning, we are really requiring that the image of $\mathbb{P}_X^\beta$ under f belongs to M_β and is contained in β.

Let I be the collection of all pre-dense sets in $\bigcup_{p \in \mathbb{P}_X^\beta} \mathbb{M} \star \mathbb{P}_{\dot{\mathrm{m}}}(p) \cap M_\beta$ which belong to M_β and let

$$D_\beta = \{q \in \mathbb{M} \star \mathbb{P}_{\dot{\mathrm{m}}} : q \text{ is } M_\beta\text{-generic and } q \Vdash_{\mathbb{M} \star \mathbb{P}_{\dot{\mathrm{m}}}} x_\beta \text{ is random over } M_\beta\}.$$

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By Theorem 16, for every condition $p \in M_\beta \cap (\mathbb{M} \star \mathbb{P}_{\dot{\mathfrak{m}}})$, $\mathbb{M} \star \mathbb{P}_{\dot{\mathfrak{m}}}(p) \cap D_\beta \neq \emptyset$.

For each $p \in \mathbb{P}_X^\beta$ and $n \in \omega$ choose $q_{p,n} \in D_\beta$ such that $q_{p,n} \geq_n p$ and let $D = \{q_{p,n} : p \in \mathbb{P}_X^\beta, n \in \omega\}$. Define $\mathbb{P}_X^\alpha = \bigcup_{p \in \mathbb{P}_X^\beta}((\mathbb{M} \star \mathbb{P}_{\dot{\mathfrak{m}}})(p) \cap M_\beta) \cup D$.

Let $I^\alpha = I^\beta \cup I \cup \{D\}$.

CASE 4 $\alpha = \beta^+$, $\beta \in S_1$ is limit and $(\mathbb{P}_X^\beta, I^\beta) \in M_\beta$.

In this case, let I be the collection of all pre-dense sets in $\mathbb{P}_X^\beta$ which belong to M_β and let $\mathbb{P}_X^\alpha = \mathbb{P}_X^\beta$ and let $I^\alpha = I^\beta \cup I$.

This concludes the construction, and it remains to show that $\mathbb{P}_X$ has the required properties. First of all note that $\mathbb{P}_X$ is indeed ccc, as guaranteed by the oracle.

Observe, and this is the main point of the construction, that $M_\beta[G] = M_\beta[G \cap \mathbb{P}_X^{\beta^+}]$. If $\beta \in S_0$ then forcing with $\mathbb{P}_X$ is equivalent to forcing with $\mathbb{M} \star \mathbb{P}_{\dot{\mathfrak{m}}}$ over M_β. To see that, suppose that $G \subseteq \mathbb{P}_X$ is a generic filter. It follows that $G \cap D_\beta \cap \mathbb{P}_X^{\beta^+} \neq \emptyset$ so let $q \in G \cap D_\beta$. Note that $q \Vdash_{\mathbb{P}_X} M_\beta[G \cap \mathbb{P}_X^{\beta^+}] = M_\beta[G \cap (\mathbb{M} \star \mathbb{P}_{\dot{\mathfrak{m}}})]$, since q is M_β-generic for $\mathbb{M} \star \mathbb{P}_{\dot{\mathfrak{m}}}$.

If $\beta \in S_1$ then $\mathbb{P}_X^{\beta^+} = \mathbb{P}_X^\beta$ is a countable forcing in M_β. Thus it locally looks like Cohen forcing.

It is clear that $\mathbb{P}_X$ satisfies both commitments, preservation of x_β being random over M_β is explicit in the definition of $\mathbb{P}_X^{\beta^+}$ and similarly for the trivial commitment.

LEMMA 24. $\mathbf{V}^{\mathbb{P}_X} \models 2^\omega \cap \mathbf{V} \notin \mathscr{N}$.

PROOF. We will show that $\mathbf{V}^{\mathbb{P}_X} \models Y_{S_0}^{random} \notin \mathscr{N}$, and since $Y_{S_0}^{random} \subseteq 2^\omega \cap \mathbf{V}$ this will suffice. Suppose that $\Vdash_{\mathbb{P}_X} \dot{H} \in \mathscr{N}$. Since $\mathbb{P}_X$ is ccc the name $\dot{H}$ is encoded by a real. Thus there is $\alpha \in S_0$ such that $\dot{H}$ is a $\mathbb{P}_X^\alpha$-name and $\dot{H} \in M_\alpha$. By the construction,

$$\Vdash_{\mathbb{P}_X} x_\alpha \text{ is random over } M_\alpha[G \cap \mathbb{P}_X^\alpha].$$

Thus $M_\alpha^{\mathbb{P}_X} \models x_\alpha \notin \dot{H}[G]$ and by absoluteness $\mathbf{V}^{\mathbb{P}_X} \models x_\alpha \notin \dot{H}[G]$. ⊣

THEOREM 25. *If $\mathbb{P}$ satisfies the commitment $Y_S^{trivial}$ and for $\beta \in S$, $M_\beta \models X \notin \mathscr{M}$ then $\mathbf{V}^{\mathbb{P}} \models X \notin \mathscr{M}$.*

In particular, $\mathbf{V}^{\mathbb{P}_X} \models 2^\omega \cap \mathbf{V} \notin \mathscr{M}$.

PROOF. This is Fact 1.7 of [10] which is really the Example 2.2 in chapter IV of [9]. We will sketch the proof here as this construction will be relevant later on.

The following is well-known.

LEMMA 26. *If $X \notin \mathscr{M}$ then $\mathbf{V}^{\mathbf{C}} \models X \notin \mathscr{M}$, where $\mathbf{C}$ is countable (Cohen) forcing.*

For simplicity we will prove the Theorem 25 for the forcing $\mathbb{P}_X$. The general case is the same. Suppose that $\dot{F}$ is a $\mathbb{P}_X$-name such that $\Vdash_{\mathbb{P}_X} \dot{F} \in \mathscr{M}$. Find a limit $\beta \in S_1$ such that

(1) $\dot{F}$ is a $\mathbb{P}_X^\beta$-name,

(2) $\mathbb{P}_X^{\beta^+} = \mathbb{P}_X^\beta \in M_\beta$ (as in case 4 above),

(3) if $D \subseteq \mathbb{P}_X^\beta$ is predense in $\mathbb{P}_X^\beta$ then it is predense in $\mathbb{P}_X$.

It follows that if $G \subseteq \mathbb{P}_X$ is a generic filter then $\dot{F}[G] = \dot{F}[G \cap \mathbb{P}_X^\beta]$.

Since forcing with $\mathbb{P}_X^\beta$ over M_β is the same as forcing with $\mathbf{C}$ we conclude that there is $x \in X$ such that $M_\beta[G \cap \mathbb{P}_X^\beta] \models x \notin \dot{F}[G \cap \mathbb{P}_X^\beta]$. By absoluteness, $\mathbf{V}[G] \models x \notin \dot{F}[G]$ and so $\mathbf{V}[G] \models X \not\subseteq \dot{F}[G]$. It follows that $\mathbf{V}[G] \models X \notin \mathscr{M}$. ⊣

Note that the above proof does not show that $\mathbb{P}_X$ preserves being not null or not meager, the construction is taylored for a particular set.

It remains to show that

THEOREM 27. $\mathbf{V}^{\mathbb{P}_X} \models X \notin \mathcal{SM}$. *Furthermore,* $\mathbf{V}^{\mathbb{P}_X \star \mathbf{C}} \models X \notin \mathcal{SM}$, *where* $\mathbf{C}$ *is countable forcing notion.*

PROOF. Since $\mathbb{P}_X \subseteq \mathbb{M} \star \mathbb{P}_{\dot{\mathfrak{m}}}$ the generic object is a sequence $\langle \bar{s}_k : k \in \omega \rangle$.

LEMMA 28. $\langle \bar{s}_k^{\perp} : k \in \omega \rangle$ *is big over* $\mathbf{V}$.

PROOF. Suppose that $Z \subseteq 2^{\omega}$ is a countable infinite set. Find $\alpha \in S_0$ such that $Z \in M_\alpha$ and α witnesses commitment $Y_{S_0}^{random}$. Since $M_\alpha[G] = M_\alpha[G \cap \mathbb{P}_X^{\alpha^+}] = M_\alpha[G \cap (\mathbb{M} \star \mathbb{P}_{\dot{\mathfrak{m}}})]$, it follows by Lemma 12 that $M_\alpha[G] \models \exists^\infty k\, Z + \bar{s}_k^{\perp} = 2^{\omega}$. By absoluteness, the same holds in $\mathbf{V}$. ⊣

The rest of the proof follows the argument in Lemma 5 and the fact that our forcing is locally Cohen. Put $\dot{H}_n = \bigcup_{k \geq n} \bar{s}_k^{\perp}$ and let $\dot{H} = \bigcap_n \dot{H}_n$. Suppose that $\Vdash_{\mathbb{P}_X} \dot{z} \notin X + \dot{H}$. Let $G \subseteq \mathbb{P}_X$ be generic filter. Find $\alpha \in S_1$ such that $\dot{z}$ is a $\mathbb{P}_X^{\alpha}$-name and $\dot{z} \in M_\alpha$ and α witnesses the commitment $Y_{S_1}^{trivial}$. Recall that in this case

(1) $M_\alpha[G] = M_\alpha[G \cap \mathbb{P}_X^{\alpha^+}] = M_\alpha[G \cap \mathbb{P}_X^{\alpha}]$,
(2) $\mathbb{P}_X^{\alpha}$ is countable (i.e., isomorphic to $\mathbf{C}$) forcing notion in M_α,
(3) $M_\alpha \models X$ is uncountable.

Work in M_α. For each $x \in X$ there is a condition $p_x \in \mathbb{P}_X^{\alpha}$ and $n_x \in \omega$ such that $p_x \Vdash_{\mathbb{P}_X^{\alpha}} \dot{z} \notin x + \dot{H}_{n_x}$. Since $\mathbb{P}_X^{\alpha}$ is countable let $n \in \omega$, $q \in \mathbb{P}_X^{\alpha}$ and $Z \subset X$ infinite be such that for all $x \in Z$, $n_x = n$ and $q = p_x$. Now $q \Vdash_{\mathbb{P}_X} Z + \dot{H}_n \neq 2^{\omega}$, it follows that $q \Vdash \forall m \geq n\, \bar{s}_k^{\perp} + Z \neq 2^{\omega}$. Thus $M_\alpha[G] \models \forall m \geq n\, \bar{s}_k^{\perp} + Z \neq 2^{\omega}$, and by absoluteness, $\mathbf{V}[G] \models \forall m \geq n\, \bar{s}_k^{\perp} + Z \neq 2^{\omega}$, contradiction since $\{\bar{s}_k^{\perp} : k \in \omega\}$ is big.

The second part of the argument is the same, as it relies on the fact that the $\mathbb{P}_X$-generic sequence is big over $\mathbf{V}$ and that forcing is locally countable. $\mathbb{P}_X \star \mathbf{C}$ is locally countable as well.

To summarize, for some models M_α, $\alpha \in S_0$ the generic sequence $\{\bar{s}_k^{\perp} : k \in \omega\}$ appears as if added by the Cohen forcing and for some $\alpha \in S_1$, $\{\bar{s}_k^{\perp} : k \in \omega\}$ looks like the generic object added by $\mathbb{M} \star \mathbb{P}_{\dot{\mathfrak{m}}}$. The construction done on S_0 will quarantee two things: (1) that this sequence is big over $\mathbf{V}$ and that a given non-null set remains not null, and (2) that the construction done for $\alpha \in S_1$ yields forcing which is close enough to Cohen forcing so that bigness implies that uncountable sets of reals from $\mathbf{V}$ are not strongly meager in the extension. The key point is that the bigness of sequence $\{\bar{s}_k^{\perp} : k \in \omega\}$ is absolute between models M_α and $\mathbf{V}$. ⊣

§6. Non-Cohen oracle ccc.

In the remaining section we will show that:

THEOREM 29. *Assume* $\mathbf{V} = \mathbf{L}$. *There exists a ccc forcing notion* $\mathbb{P}$ *such that*

(1) $\mathbf{V}^{\mathbb{P}} \models 2^{\aleph_0} = \aleph_2$,
(2) $\mathbf{V}^{\mathbb{P}} \models 2^{\omega} \cap \mathbf{V} \notin \mathcal{N}$. *In particular,* $\mathbb{P}$ *does not add Cohen reals.*
(3) $\mathbf{V}^{\mathbb{P}} \models \mathsf{DBC}$.

The construction is an application of the non-Cohen oracle ccc method described in [10]. The basic ingredient of the proof which is specific for this construction is forcing $\mathbb{P}_X$ defined in the previous section.

Before we go further we will introduce the main ideas used in the construction. The forcing $\mathbb{P}$ will be constructed from $\omega_1 \times \omega_2$ countable pieces. The ω_2 axis will correspond to the ω_2-iteration while the ω_1 axis will correspond to the single task of making a given $\aleph_1$-set not strongly meager. In other words, $\mathbb{P} = \bigcup_{\alpha<\omega_2} \mathbb{P}_\alpha$ and for each $\alpha < \omega_2$, $\mathbb{P}_\alpha = \bigcup_{\xi<\omega_1} \mathbb{P}^\alpha_\xi$. Furthermore, the sequence $\langle \mathbb{P}^\alpha_\xi : \xi < \omega_1 \rangle$ will be increasing and continuous for each α (that is $\mathbb{P}^\alpha_\lambda = \bigcup_{\eta<\lambda} \mathbb{P}^\alpha_\eta$ for limit λ). On the other hand the sequence $\langle \mathbb{P}_\alpha : \alpha < \omega_2 \rangle$ will be increasing but generally not continuous. In general, $\mathbb{P}_{\alpha+1}$ will be of the form $\mathbb{P}_\alpha \star \mathbb{P}_X$, where $\mathbb{P}_X$ will be forcing defined in the previous section and $X \subset 2^\omega \cap \mathbf{V}^{\mathbb{P}_\alpha}$ is an $\aleph_1$ set of reals. A suitable bookkeping will guarantee that every $\aleph_1$ set of reals will be captured.

In a typical finite or countable support iteration construction we accomplish the goals of the iteration at the successor steps and then we preserve them at the subsequent successor and limit steps. For example, in our case, a single forcing notion would add a witness that a given set is not strongly meager and a preservation theorem for countable/finite support iteration would guarantee that it remains so through the iteration. The scheme used in this construction is different. Instead of a preservation theorem, forcings $\langle \mathbb{P}_\alpha : \alpha < \omega_2 \rangle$ will satisfy the commitments as in the definition of $\mathbb{P}_X$. In other words, the commitments imposed on the $\mathbb{P}_\alpha$'s will guarantee that the sets that were made not strongly meager by the iterands $\mathbb{P}_X$ remain so. Thus, instead of showing that an algebraic construction such as direct or inverse limit preserves certain properties, we will work to show that given a sequence $\langle \mathbb{P}_\alpha : \alpha < \lambda \rangle$ (λ limit) satisfying appropriate commitments, there is an object $\mathbb{P}_\lambda$ which also satisfy the commitments and $\mathbb{P}_\alpha \lessdot \mathbb{P}_\lambda$ for all $\alpha < \lambda$. $\mathbb{P}_\lambda$ will be defined inductively, but for the purpose of the argument all we need to know is that it exists. A good illustration of this phenomenon is Theorem 1.17 of [10].

The "iterands" in the construction will be forcing notions defined in the previous section. Each of these forcing notions requires two disjoint stationary sets and an oracle on them. In our case these stationary sets will come from a family of $\aleph_2$ almost disjoint stationary subsets of ω_1.

Lemma 30. *Assume* $\mathbf{V} = \mathbf{L}$ *and let* $\{S^0_\alpha, S^1_\alpha : \alpha < \omega_2\}$ *be a family of almost disjoint stationary subsets of* ω_1 . *There exists a family* $\{A^\alpha : \alpha < \omega_2\}$ *of stationary subsets of* ω_1 *such that*

(1) $A^\beta \setminus A^\alpha \in \mathrm{NS}_{\omega_1}$ *for* $\beta \leq \alpha < \omega_2$,
(2) $A^{\alpha+1} = A^\alpha \cup S^0_\alpha \dot\cup S^1_\alpha$,
(3) $S^i_\alpha \setminus A^\alpha \notin \mathrm{NS}_{\omega_1}$.

Proof. Since NS_{ω_1} is not $\aleph_2$-saturated we can find a family $\{S^0_\alpha, S^1_\alpha : \alpha < \omega_2\}$ as above. Furtermore, we can assume that $S^0_\alpha \cap S^1_\alpha = \emptyset$ for all α.

Build tower $\{A^\alpha : \alpha < \omega_2\}$ by putting $A^{\alpha+1} = A^\alpha \cup S^0_\alpha \dot\cup S^1_\alpha$. For limit λ order $\{A^\alpha : \alpha < \lambda\}$ as $\{T_\xi : \xi < \omega_1\}$ and define $A^\lambda = \bigcup_{\xi<\omega_1} T_\xi \setminus \xi$. It is straightforward to check that this works and (1) and (3) are satisfied. $\dashv$

In the same way we can extend oracles.

Lemma 31. *Suppose that* $\{\bar{M}_\alpha : \alpha < \lambda\}$, $\lambda < \omega_2$ *is a* $\leq$*-increasing sequence of oracles. Suppose that* $A \setminus \mathsf{dom}(\bar{M}_\alpha) \notin \mathrm{NS}_{\omega_1}$ *for each* $\alpha < \lambda$. *Then there is an oracle* $\bar{M}$ *such that* $\mathsf{dom}(\bar{M}) = A$ *and* $\bar{M}_\alpha \leq \bar{M}$ *for* $\alpha < \lambda$.

Recall that $Y^{random}_{S_0}$ and $Y^{trivial}_{S_1}$ were commitments on the forcing $\mathbb{P}_X$ that were supposed to be satisfied on a stationary subset of $S_0 \cup S_1$. Here we intend that on the step $\alpha+1$ of the construction we will use forcing $\mathbb{P}_{X_\alpha}$ defined using sets $S^0_\alpha \cup S^1_\alpha$ and a suitably chosen $\aleph_1$-set of reals X_α in $\mathbf{V}^{\mathbb{P}_\alpha}$.

In order to carry out the construction we will need to define commitments for the future iterands. More precisely, given a forcing $\mathbb{P}$ satisfying some 0-commitment, we would like to extend the oracle and define a $\mathbb{P}$-name for a 0-commitment that some forcing $\mathbb{Q} \in \mathbf{V}^{\mathbb{P}}$ will satisfy.

DEFINITION 32. *Suppose that $\mathbb{P}$ satisfies $\bar{M}$-cc. We say that $\dot{Y}^{random}_S$ is a 1-commitment if $\dot{Y}^{random}_S = \{\dot{x}_\alpha : \alpha \in S\}$ is a sequence of $\mathbb{P}$-names such that*

$$\{\alpha \in S : \Vdash_{\mathbb{P}} \dot{x}_\alpha \text{ is random over } M^{\mathbb{P}}_\alpha\} \in D_{\bar{M}}.$$

The case $\dot{Y}^{trivial}_S$ is defined as before.

Note that if $\dot{Y}^{random}_S$ is a 1-commitment and $G \subset \mathbb{P}$ is a generic filter then $\dot{Y}^{random}_S[G]$ is a 0-commitment.

Consider the set IS consisting of triples $\langle \mathbb{P}, \dot{Y}, \bar{M} \rangle$, where

(1) $\mathbb{P}$ is a $\bar{M}$-cc forcing notion of size $\aleph_1$,
(2) $\dot{Y}$ is a 1-commitment on $\mathbb{P}$ of form $\dot{Y}^{random} \cup \dot{Y}^{trivial}$.

For $\langle \mathbb{P}_0, \dot{Y}_0, \bar{M}_0 \rangle, \langle \mathbb{P}_1, \dot{Y}_1, \bar{M}_1 \rangle \in$ IS we say $\langle \mathbb{P}_0, \dot{Y}_0, \bar{M}_0 \rangle \leq \langle \mathbb{P}_1, \dot{Y}_1, \bar{M}_1 \rangle$ if

(1) $\mathbb{P}_0 \lessdot \mathbb{P}_1$,
(2) $\bar{M}_0 \leq \bar{M}_1$,
(3) for $E \in D_{\bar{M}_0}$, $\mathrm{dom}(\dot{Y}_0) \cap E \subset \mathrm{dom}(\dot{Y}_1) \cap E$ and $\dot{Y}_0 \restriction E = \dot{Y}_1 \restriction E$.

Given $\langle \mathbb{P}, \dot{Y}, \bar{M} \rangle \in$ IS we can perform the following operations:

(1) Increase the oracle, that is replace $\bar{M}$ with $\bar{M}'$ such that $\bar{M} \leq \bar{M}'$.
(2) Increase the commitment, that is, extend $\dot{Y}$.
(3) Increase the forcing by replacing $\mathbb{P}$ with $\mathbb{P}'$ such that $\mathbb{P} \lessdot \mathbb{P}'$ and $\mathbb{P}'$ is $\bar{M}$-cc and satisfies $\dot{Y}$.

In our case we will construct a tower $\langle (\mathbb{P}_\alpha, \dot{Y}_\alpha, \bar{M}_\alpha) : \alpha < \omega_2 \rangle$ such that for each $\alpha < \omega_2$:

(1) $\mathbb{P}_\alpha$ is $\bar{M}_\alpha$-cc,
(2) $\mathrm{dom}(\bar{M}_\alpha) = A^\alpha$,
(3) $\mathrm{dom}(\dot{Y}_\alpha) = A^\alpha$ and $\dot{Y}_\alpha = \dot{Y}^{random}_\alpha \cup Y^{trivial}_\alpha$.
(4) $\dot{Y}_\alpha \cup \dot{Y}^{random}_{S^0_\alpha} \cup Y^{trivial}_{S^1_\alpha} \leq \dot{Y}_{\alpha+1}$ (in the sense of $\leq$ on IS.)

In our case, given $(\mathbb{P}_\alpha, \dot{Y}_\alpha, \bar{M}_\alpha)$ we will do the following steps:

(1) select an $\aleph_1$ set of $\mathbb{P}_\alpha$-names for the reals. This could be done using $\diamondsuit_{\omega_2}$ or some other bookkeeping mechanism, since every $\mathbb{P}_{\omega_2}$-name for a set of $\aleph_1$ many reals has a $\mathbb{P}_\alpha$-name for some $\alpha < \omega_2$.
(2) Extend the oracle $\bar{M}_\alpha$ to $\bar{M}_{\alpha+1} = \langle M^{\alpha+1}_\delta : \delta \in A^{\alpha+1} = A^\alpha \cup S^0_\alpha \cup S^1_\alpha \rangle$ so that for all $\delta \in A^{\alpha+1} \setminus A^\alpha$, $M^{\alpha+1}_\delta \models X_\alpha$ is uncountable,
(3) Extend the commitment $\dot{Y}_\alpha$ by putting $\dot{Y}_{\alpha+1} \restriction S^0_\alpha = \dot{Y}^{random} = \dot{Z}_\alpha$ to be any set of $\mathbb{P}_\alpha$-names for random reals over $M^{\alpha+1}_\delta$ for $\delta \in S^0_\alpha$. Let $\dot{Y}_{\alpha+1} \restriction S^1_\alpha = \dot{Y}^{trivial}$.
(4) Build $\mathbb{P}_{X_\alpha}$ such that
  (a) $\mathbb{P}_{X_\alpha} \subseteq \mathbb{M} \star \mathbb{P}_{\dot{\mathbf{m}}}$,
  (b) $\mathbb{P}_\alpha \star \mathbb{P}_{X_\alpha}$ is $\bar{M}_{\alpha+1}$-cc,

(c) $\mathbf{V}^{\mathbb{P}_\alpha\star\mathbb{P}_{X_\alpha}} \models Z_\alpha \notin \mathscr{N}$,
(d) $\mathbf{V}^{\mathbb{P}_\alpha\star\mathbb{P}_{X_\alpha}} \models X_\alpha \notin \mathscr{SM}$, as witnessed by a null set $\dot{H}_\alpha$ obtained from the generic sequence added by $\mathbb{P}_{X_\alpha}$.

(5) Set $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \star \mathbb{P}_{X_\alpha}$.

These steps are covered by the general theorems in [10]; in particular, the limit steps are handled by 1.9 and 1.11, and step (3) by 1.12 of [10].

Let $\mathbb{P} = \bigcup_{\alpha<\omega_2} \mathbb{P}_\alpha$. Clearly, $\mathbb{P}$ satisfies ccc. We need to check that $\mathbf{V}^{\mathbb{P}}$ has the required properties.

LEMMA 33. $\mathbf{V}^{\mathbb{P}} \models 2^\omega \cap \mathbf{V} \notin \mathscr{N}$.

PROOF. Suppose that $\dot{H}$ is a $\mathbb{P}$-name for a null set. Since $\mathbb{P}$ is ccc and $\dot{H}$ is encoded by a real, it follows that $\dot{H}$ is a $\mathbb{P}_\alpha$-name for some $\alpha < \omega_2$. Consider 0-commitment $Y^{random}_{S^0_0}$ which consists of reals from $2^\omega \cap \mathbf{V}$ and the oracle $\bar{M}_0 = \langle M^0_\xi : \xi \in S^0_0 \cup S^0_1\rangle$. (In general, if we were interested in showing that some set Z_β considered in the iteration is not null in $\mathbf{V}^{\mathbb{P}}$ then we would carry the argument in a model $\mathbf{V}^{\mathbb{P}_\beta}$.) Since $\mathbb{P}_\alpha$ satisfies this commitment we can find $\delta \in S^0_0$ such that

(1) $\dot{H} \in M^0_\delta$,
(2) x_δ is random over $(M^0_\delta)^{\mathbb{P}_\alpha}$.

This is possible since both (1) and (2) hold on a set from $D_{\bar{M}_0}$. Thus for a generic filter $G \subset \mathbb{P}_\alpha$, $M^0_\delta[G] \models x_\delta \notin \dot{H}[G]$. Since $\dot{H}$ evaluated using G in M^0_δ is the same as evaluated in $\mathbf{V}[G]$ we conclude that $\mathbf{V}[G] \models x_\delta \notin \dot{H}[G]$ and consequently $\mathbf{V}[G] \models 2^\omega \cap \mathbf{V} \notin \mathscr{N}$. ⊣

THEOREM 34. $\mathbf{V}^{\mathbb{P}} \models$ DBC.

PROOF. Suppose that $X \subset \mathbf{V}^{\mathbb{P}} \cap 2^\omega$ is an $\aleph_1$ set of reals. Find $\alpha < \omega_2$ such that at the step α of the construction the following holds:

(1) $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha \star \mathbb{P}_{X_\alpha}$,
(2) $\mathbf{V}^{\mathbb{P}_\alpha} \models X_\alpha \subset X$.

Such α can be found because of bookkeping. Recall that forcing $\mathbb{P}_{X_\alpha}$ introduced a measure zero set H_α such that $\mathbf{V}^{\mathbb{P}_{\alpha+1}} \models X_\alpha + H_\alpha = 2^\omega$ and furthermore $\mathbf{V}^{\mathbb{P}_{\alpha+1}\star\mathbf{C}} \models X_\alpha + H_\alpha = 2^\omega$, where $\mathbf{C}$ is countable forcing (lemma 27).

Let $\bar{M}_{\alpha+1}$ be the oracle for $\mathbb{P}_{\alpha+1}$. This time we will be interested in the commitments $Y^{trivial}$ that we made on the set S^1_α which is contained (modulo $D_{\bar{M}}$) in $A^{\alpha+1}$.

Suppose that there is a $\mathbb{P}$-name $\dot{z}$ such that $\Vdash_{\mathbb{P}} \dot{z} \notin X_\alpha + H_\alpha$. We can assume that $\dot{z}$ is a $\mathbb{P}_\beta$-name for some $\beta > \alpha$. Let $\eta > \beta$, $\eta \in S^1_\alpha$ be such that the following holds:

(1) $\mathbb{P}^\eta_{X_\alpha} = \mathbb{P}^{\eta^+}_{X_\alpha}$ is a countable forcing in M^α_η,
(2) if $\mathscr{D}$ is predense in $\mathbb{P}^\beta_\eta = \mathbb{P}_\beta \cap \eta$ then it is predense in $\mathbb{P}_\beta$,
(3) $\dot{z} \in M^\alpha_\eta$ and $\dot{z}$ is a $\mathbb{P}^\beta_\eta$-name,
(4) $\mathbb{P}^\eta_{X_\alpha} \lessdot \mathbb{P}^\beta_\eta$, or in other words, $\mathbb{P}^\beta_\eta \simeq \mathbb{P}^\eta_{X_\alpha} \star \mathbf{C}$ for some countable forcing notion $\mathbf{C}$.

Note that (1) and (2) follow from the fact that $\mathbb{P}_\beta$ and $\mathbb{P}_{X_\alpha}$ satisfy the commitments. the rest is straightforward - over the model M^α_η, H_α is a $\mathbb{P}^\eta_{X_\alpha}$-name. Thus on one hand, if $G \subset \mathbb{P}$ is a generic filter then $M^\alpha_\eta[G\cap\mathbb{P}_\beta] = M^\alpha_\eta[G\cap\mathbb{P}^\beta_\eta] \models \dot{z}[G] \notin X_\alpha + H_\alpha$.

On the other hand, $M^\alpha_\eta[G \cap \mathbb{P}^\eta_{X_\alpha}] \models X_\alpha + H_\alpha = 2^\omega$, and by Lemma 16 the same must hold in the model $M^\alpha_\eta[G \cap \mathbb{P}^\beta_\eta]$ which (by (4)) is an extension of $M^\alpha_\eta[G \cap \mathbb{P}^\eta_{X_\alpha}]$ obtained by a countable forcing notion. Contradiction. $\dashv$

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