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THE SPECTRUM PROBLEM I: $\aleph_\varepsilon$-SATURATED MODELS, THE MAIN GAP

BY

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ABSTRACT

The main goal of this paper is to complete the classification of those first-order theories such that $I^a_{\aleph_0}(\lambda, T) = 2^\lambda$. We introduce two notions, the *dimensional order property* and *deepness*. Our main theorem asserts that for a superstable theory T, $I^a_{\aleph_0}(\lambda, T) = 2^\lambda$ iff T has the dimensional order property or is deep. In a sense made precise in §4 this provides a syntactical characterization of theories with the maximum number of $\aleph_\varepsilon$-saturated models in each power.

Introduction

We deal with the number $I^a_{\aleph_0}(\aleph_\alpha, T)$ of non-isomorphic $\aleph_\varepsilon$-saturated models of a complete first-order theory T of cardinality $\aleph_\alpha$. (Note that for $\aleph_0$-categorical countable T, every model of T is $\aleph_\varepsilon$-saturated:[††] $\aleph_\varepsilon$-saturation is a slight strengthening of $\aleph_0$-saturated, denoted by $F^a_{\aleph_0}$-saturation.) For this we continue the classification of first-order theories (see [4]) introducing the dop (dimensional order property) and deepness.

We conclude that either $I^a_{\aleph_0}(\aleph_\alpha, T) \geqq 2^{\aleph_\alpha}$ for every $\aleph_\alpha \geqq \lambda(T) + \aleph_1$ *or* $I^a_{\aleph_0}(\aleph_\alpha, T) \leqq \beth_{\delta(|T|)}(|\alpha| + \aleph_0)$ for every α (remember that $\delta(|T|) < (2^{|T|})^+$).

As the name indicates, the author feels that this essentially solves the classification problem for $\aleph_\varepsilon$-saturated models (of first-order theories). The main results were announced in [3] and [4].

We deal also with the number of models no one elementarily embeddable into another. So we advance our knowledge on $I(\aleph_\alpha, T)$ (the number of models of T of power $\aleph_\alpha$), and, we think, help to deal with classes which do not strictly fall

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[††] But any such T, if it is superstable without the dop, then by the work of Cherlin, Harrington and Lachlan, $R[x = x, L, \aleph_0]$ is finite and then we can show its depth is finite (in the notation of the proof of 3.2, we prove that for every $b \in M_\eta$, $l(\eta) + R[\mathrm{tp}(b, N_\eta), L, \infty] \leqq R[x = x, L, \infty]$).

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into our context. We end by introducing and writing down some properties of trivial types.

§1. Notation and preliminaries

Our notation is from [4], and as we refer to it often we omit the [4] (so 4.1 means 4.1 in this paper, III 4.1 means [4] III 4.1). The following lemmas are just corollaries to [4] (1.1–1.4 to V §1, 1.7 to IV), however 1.1–1.4 will be used often in sections 3, 4, 5 and 1.5, 1.6 will be used only for proving the equivalences between various definitions of the dop in §2, and 1.7 will be used in section 5. One deviation from the notation of [4]: $\boldsymbol{I}, \boldsymbol{J}$ are sets of sequences from $\mathfrak{C}$, I, J are sets of indexes (ordered sets, or trees or sets of sequences of ordinals, usually).

(Hyp) In this section T will be stable.

1.1. Claim. *If* $A \subseteq B_l$ $(l = 0,1)$, $\mathrm{tp}(B_1, B_0)$ *does not fork over* A, $p \in S^m(B_0)$ *is orthogonal to* A, then *it is orthogonal to* B_1.

Proof. W.l.o.g. we work in $\mathfrak{C}^{\mathrm{eq}}$ and w.l.o.g. $B_l = \mathrm{acl}\, B_l$, $A = \mathrm{acl}\, A$. Choose any $q \in S(B_1)$. Construct an infinite indiscernible set $\boldsymbol{I}$ of elements realizing q such that $\mathrm{tp}_*(\boldsymbol{I}, B_0 \cup B_1)$ does not fork over B_1 and $\mathrm{Av}(\boldsymbol{I}, \boldsymbol{I})$ is a stationarization of q. Similarly choose a set $\boldsymbol{J}$ of indiscernibles over B_0 realizing p. Since $\mathrm{tp}_*(\boldsymbol{I}, B_0 \cup B_1)$ does not fork over B_1, and $\mathrm{tp}(B_1, B_0)$ does not fork over A, by III 0.1 $\mathrm{tp}_*(B_1 \cup \boldsymbol{I}, B_0)$ does not fork over A, hence $\mathrm{tp}_*(\boldsymbol{I}, B_0)$ does not fork over A. As $p \in S^m(B_0)$ is orthogonal to A by V 1.5(1) $\mathrm{tp}_*(\boldsymbol{J}, B_0)$ is orthogonal to A. Hence $\mathrm{tp}_*(\boldsymbol{J}, B_0)$, $\mathrm{tp}_*(\boldsymbol{I}, B_0)$ are orthogonal, hence by V 1.2(1) (since we work in $\mathfrak{C}^{\mathrm{eq}}$) weakly orthogonal, hence by V 2.7 $\mathrm{Av}(\boldsymbol{I}, \cup \boldsymbol{I})$, $\mathrm{Av}(\boldsymbol{J}, \cup \boldsymbol{J})$ are orthogonal, so p, q are orthogonal.

1.2. Definition. For $A \subseteq B \subseteq C$ we say $B <_A C$ iff for every $\bar{c} \in C$, $\mathrm{tp}(\bar{c}, B)$ is orthogonal to A.

1.3. Lemma. (1) *Let* $N \subseteq M \subseteq A$, $M <_N A$, M *and* N *are* $\boldsymbol{F}^a_\kappa$*-saturated,* $\kappa \geqq \kappa_r(T)$ *and* M' *is* $\boldsymbol{F}^a_\kappa$*-prime over* A, then $M <_N M'$.

(2) *If* $B <_A C$, $A \subseteq A_0$, $B \subseteq B_0$, $A_0 \subseteq B_0 \subseteq C_0$, $C_0 = C \cup B_0$, $\mathrm{tp}_*(C, A_0)$ *does not fork over* A, *and* $\mathrm{tp}_*(C, B_0)$ *does not fork over* B, then $B_0 <_{A_0} C_0$.

Proof. (1) By V 3.2 and V 1.2(3).

(2) By 1.1 it is easy.

1.4 Claim. *Let* $\kappa \geqq \kappa_r(T)$. *If* N *is* $\boldsymbol{F}^a_\kappa$*-saturated,* $p \in S^m(B)$ *is regular, stationary, not orthogonal to* N, then p *is not orthogonal to some regular* $q \in S^m(N)$.

PROOF. W.l.o.g. $p \in S^m(M)$, $N \subseteq M$ and M is $\boldsymbol{F}^a_\kappa$-saturated. Let $C \subseteq M$, $|C| < \kappa$, p does not fork over C, $p \restriction C$ stationary. Let $A \subseteq N$, $|A| < \kappa$, $\mathrm{tp}_*(C,N)$ does not fork over A, $\mathrm{tp}_*(C,A)$ stationary.

Choose by induction on $i < \omega$ elementary mappings f_i, $\mathrm{Dom} f_i = A \cup C$, $f_i \restriction A = \mathrm{id}$, $\mathrm{stp}_*(f_i(C),A) \equiv \mathrm{stp}_*(C,A)$, and $f_0 = \mathrm{id}$, $\mathrm{Rang}(f_1) \subseteq N$, and for $i < \omega$, $i \geqq 2$, $\mathrm{tp}_*(f_i(C), M \cup \bigcup_{j<i} f_j(C))$ does not fork over A.

Let $p_i = f_i(p \restriction (A \cup C))$.

Clearly for $i < j$, p_i is orthogonal to p_j iff p_0 is orthogonal to p_1 (by indiscernability).

Extend the domain of f_2 to $N \cup C$ by fixing $N - C$. By V 3.4, p_0, p_2 are not orthogonal ($p_0, f, f(p)$, A, B there, stand for p_0, f_2, p_2, N, $N \cup C$ here). Hence p_0, p_1 are not orthogonal, so p is parallel to p_0, not orthogonal to p_1, parallel to the stationarization q of p_1 in $S^m(N)$, q regular. So we finish.

1.5. DEFINITION. Let $A \subseteq B$, $p \in S^m(B)$, then we say that p is almost orthogonal to A *if* for every $\bar{c}$, s.t. $\mathrm{tp}(\bar{c},B)$ does not fork over A and every $\bar{b}$ realizing p, $\mathrm{tp}(\bar{b}, B \cup \bar{c})$ does not fork over B.

1.6. CLAIM. (1) *Let* $A \subseteq B$, then $\mathrm{tp}(\bar{b},B)$ *is almost orthogonal to* A *iff* $\mathrm{tp}(\bar{b}, \mathrm{acl}\, B)$ *is orthogonal to* $\mathrm{acl}\, A$ *if* $\mathrm{tp}(\bar{b},B)$ *is orthogonal to* A.

(2) *If* $A \subseteq B$, $p = \mathrm{tp}(\bar{b},B)$, then p *is almost orthogonal to* A iff *for any* $\bar{c}$, $\mathrm{tp}(\bar{c},B)$ *does not fork ovęr* A *implies* $\mathrm{stp}(\bar{b},B) \vdash \mathrm{stp}(\bar{b}, B \cup \bar{c})$ iff *for any* C, $\mathrm{tp}_*(C,B)$ *does not fork over* A *implies* $\mathrm{stp}(\bar{a},B) \vdash \mathrm{stp}(\bar{a}, B \cup C)$.

(3) *Let* $A \subseteq B$, $p = \mathrm{tp}(\bar{b},B)$, then p *is almost orthogonal to* A iff *for every* $\bar{a}$, $\mathrm{tp}(\bar{a},B)$ *does not fork over* A *implies* $\mathrm{tp}(\bar{a}, B \cup \bar{b})$ *does not fork over* A.

(4) *For every* $\bar{a}$, $\mathrm{tp}(\bar{a},B)$ *does not fork over* A *implies* $\mathrm{tp}(\bar{b},B) \vdash \mathrm{tp}(\bar{b}, B \cup \bar{a})$ iff $\mathrm{tp}(\bar{b},B)$ *is almost orthogonal to* A and in $\mathfrak{C}^{\mathrm{eq}}$, $\mathrm{tp}(\bar{b},B) \vdash \mathrm{tp}(\bar{b}, B \cup \mathrm{acl}\, A)$.

(5) *If* $A = |M| \subseteq B$, M *is* $\boldsymbol{F}^a_\lambda$-*saturated*, $p \in S^m(B)$ *is* $\boldsymbol{F}^a_\lambda$-*isolated*, $\lambda \geqq \kappa_r(\tau)$, then p *is almost orthogonal to* A.

(6) *If* $A = |M| \subseteq B$, M *is* $\boldsymbol{F}^l_\lambda$-*saturated* (*i.e.* λ-*compact*), $p \in S^m(B)$ *is* $\boldsymbol{F}^l_\lambda$-*isolated*, then p *is almost orthogonal to* A.

1.7. CLAIM. *If* N *is* $\boldsymbol{F}^a_\lambda$-*prime over* $\emptyset$, $\lambda \geqq \kappa_r(T)$, $|A| < \lambda$, $\mathrm{cf}\,\lambda \geqq \kappa(T)$ *and* M *is* $\boldsymbol{F}^a_\lambda$-*prime over* $N \cup A$, then M *is* $\boldsymbol{F}^a_\lambda$-*prime over* $\emptyset$.

PROOF. We concentrate on the case $\lambda = \aleph_0$, so $A = \bar{a}$. W.l.o.g. M is $\boldsymbol{F}^a_\lambda$-*constructible over* $N \cup A$. By IV 4.9(3), IV 4.10(2), IV 4.18 we can find $\boldsymbol{I} \subseteq N$, $\boldsymbol{I}$ an infinite indiscernible set, $\mathrm{Av}(\boldsymbol{I},N) = \mathrm{tp}(\bar{a},N)$, and $\mathrm{Av}(\boldsymbol{I}, \bigcup \boldsymbol{I}) \vdash \mathrm{Av}(\boldsymbol{I},N)$. Now $\|\boldsymbol{I}\| = \lambda$ as N is $\boldsymbol{F}^a_\lambda$-prime over $\emptyset$ (see IV 4.9(2)). By IV 4.18, N is $\boldsymbol{F}^a_\lambda$-prime over $\bigcup \boldsymbol{I}$, and by IV 4.10(2) $\boldsymbol{F}^a_\lambda$-constructible over $\boldsymbol{I} \cup \{\bar{a}\}$. Hence M is

F^a_λ-constructible over $I \cup \{\bar{a}\}$, but as $|I \cup \{\bar{a}\}| = \lambda$, $I \cup \{\bar{a}\}$ is F^a_λ-constructible over $\emptyset$. Hence M is F^a_λ-constructible over $\emptyset$, hence M is F^a_λ-prime over $\emptyset$.

1.8. DEFINITION. (1) $I(\lambda, F, T)$ is the number of F-saturated models of T of power λ, up to isomorphism. If $F = F^x_\kappa$ we write $I^x_\kappa(\lambda, T)$.

(2) $IE(\lambda, F, T)$ is the maximal number of pairwise nonelementarily embeddable F-saturated models of T of power λ. If a maximum is not obtained, and the supremum is a limit cardinal μ, we write the value as μ^-. If $F = F^x_\kappa$ we write $IE^x_\kappa(\lambda, T)$. We omit λ if we do not restrict the cardinality (so the value may be ∞ or ∞^-).

§2. The dimensional order property

(Hyp) In this section T will be stable and κ be $\kappa_r(T)$.

Remember that T is unstable iff it has the order property, and unstable theories are complicated in some respects, e.g., they have many non-isomorphic models. However, a stable T may have an order hidden in it. For example, consider for $\lambda > \aleph_0$, $A \subseteq \lambda^2$, the theory T of the model (B, F_1, F_2) where

$$B = \lambda \cup \{\langle \alpha, \beta, \gamma \rangle : \alpha, \beta, \gamma < \lambda, \text{ and } \langle \alpha, \beta \rangle \in A \Rightarrow \gamma < \omega\},$$

$$F_1(\alpha) = \alpha, \quad F_1(\langle \alpha, \beta, \gamma \rangle) = \alpha, \quad F_2(\alpha) = \alpha, \quad F_2(\langle \alpha, \beta, \gamma \rangle) = \beta.$$

Clearly T is not only stable, but even $\aleph_0$-stable, and $\aleph_0$-categorical; however, by cardinality quantifiers we can define an order (if A is an order).

We shall consider here a property, which clearly means there is a hidden order property. From later sections we can see that for T superstable and $F^a_{\aleph_0}$-saturated models it is the only one. Note that here the order and independence properties coincide.

2.1. DEFINITION. T has the dimensional order property (dop in short) *if* there are models M_l $(l = 0, 1, 2)$, each F^a_κ-saturated, $M_0 \subseteq M_1, M_2$, and $\{M_1, M_2\}$ is independent over M_0, *and* the F^a_κ-prime model over $M_1 \cup M_2$ is not F^a_κ-minimal over $M_1 \cup M_2$.

In this section we first develop a number of equivalent forms of the dimensional order property. These are summarized in Lemma 2.4. Condition 2.4 $(d)_{\aleph_0}$ is the form of the dimensional order property used in Theorem 2.5 to show that if a superstable T has dop then $I^a_{\aleph_0}(T, \lambda) = 2^\lambda$ (when T is stable in λ).

2.2. LEMMA. *Let* $M_0 < M_1, M_2$, *each* M_l F^a_λ*-saturated*, $\lambda \geqq \mu \geqq \kappa$, $\{M_1, M_2\}$

independent over M_0, M F^a_μ-atomic over $M_1 \cup M_2$ and M is F^a_μ-saturated. Then the following conditions are equivalent:

(a) *M is not F^a_μ-minimal over $M_1 \cup M_2$.*

(b) *There is an infinite indiscernible $I \subseteq M$ over $M_1 \cup M_2$.*

(c) *There is $p \in S^m(M)$ orthogonal to M_1 and to M_2, p not algebraic.*

(d) *There is an infinite $I \subseteq M$ indiscernible over $M_1 \cup M_2$ such that* $\mathrm{Av}(I, M)$ *is orthogonal to M_1 and M_2. Hence by* 1.6(5), $\mathrm{tp}_*(I, M_1 \cup M_2)$ *is almost orthogonal to M_1 and to M_2.*

PROOF. The equivalence of (a) and (b) is the content of IV 4.21, and (d) $\Rightarrow$ (b), (c) are trivial; we now prove (b) $\Rightarrow$ (d), (c) $\Rightarrow$ (d). The "hence" in (d) can be proved as in the proof of (c) $\Rightarrow$ (d).

(b) $\Rightarrow$ (d). We can assume $|I| = \aleph_0$, and suppose $I \subseteq M$ is indiscernible over $M_1 \cup M_2$ but not orthogonal to M_1 (by symmetry). So by definition, $\mathrm{Av}(I, I)$ is not orthogonal to some $r \in S^m(M_1)$.

Let $I = J \cup \{\bar a_n : n < \omega\}$, $|J| < \kappa$, $\mathrm{Av}(I, I)$ does not fork over J, $\mathrm{Av}(I, J)$ is stationary, and let $\{\bar b_n : n < \omega\}$ be an independent set over $(M_1 \cup M_2 \cup I, M_1)$ of sequences realizing r (hence indiscernible over $M_1 \cup M_2 \cup I$). By V 2.7 for some k, $\mathrm{tp}(\bar b_0{}^\wedge \cdots {}^\wedge \bar b_k, M_1 \cup M_2 \cup J)$, $\mathrm{tp}(\bar a_0{}^\wedge \cdots {}^\wedge \bar a_k, M_1 \cup M_2 \cup J)$ are not weakly orthogonal.

As M is F^a_μ-atomic over $M_1 \cup M_2$, for some $B_1 \subseteq M_1 \cup M_2$, $|B_1| < \mu$, $\mathrm{tp}_*(I', B_1) \vdash \mathrm{tp}_*(I', M_1 \cup M_2)$ where $I' = J \cup \{\bar a_l : l \leq k\}$. Hence for some $C \subseteq M_1 \cup M_2$, $|C| < \aleph_0$, $\mathrm{tp}(\bar b_0{}^\wedge \cdots {}^\wedge \bar b_k,\ B_1 \cup C \cup J)$ and $\mathrm{tp}(\bar a_0{}^\wedge \cdots {}^\wedge \bar a_k, B_1 \cup C \cup J)$ are not weakly orthogonal. Now $\mathrm{tp}(\bar b_0{}^\wedge \cdots {}^\wedge \bar b_k, M_1 \cup M_2 \cup J)$ does not fork over M_1 (by the choice of the $\bar b$'s) and M_1 is F^a_μ-saturated and $|B_1 \cup C \cup J| < \mu + \aleph_0 + \aleph_0 = \mu$, so some $\bar c_0{}^\wedge \cdots {}^\wedge \bar c_k \in M_1$ realizes $\mathrm{tp}(\bar b_0{}^\wedge \cdots {}^\wedge \bar b_k, B_1 \cup C \cup J)$ (see III 0.1). So $\mathrm{tp}(\bar a_0{}^\wedge \cdots {}^\wedge \bar a_k, B_1 \cup C \cup J)$, $\mathrm{tp}(\bar c_0{}^\wedge \cdots {}^\wedge \bar c_k, B_1 \cup C \cup J)$ are not weakly orthogonal, hence

$$\mathrm{tp}(\bar a_0{}^\wedge \cdots {}^\wedge \bar a_k, B_1 \cup C \cup J) \nvdash \mathrm{tp}(\bar a_0{}^\wedge \cdots {}^\wedge \bar a_k, B_1 \cup C \cup J \cup \bar c_0{}^\wedge \cdots {}^\wedge \bar c_k),$$

hence (by *Ax* V2 for F^s_∞) $\mathrm{tp}_*(I', B_1 \cup C) \nvdash \mathrm{tp}_*(I', B_1 \cup C \cup (\bar c_0{}^\wedge \cdots {}^\wedge \bar c_k))$, hence by monotonicity $\mathrm{tp}_*(I', B_1) \nvdash \mathrm{tp}(I', M_1 \cup M_2)$ contradicting the choice of B_1.

(c) $\Rightarrow$ (d). So let $r \in S^m(M)$ be (not algebraic) and orthogonal to M_1, M_2. It suffices to prove that $r \restriction (B \cup M_1 \cup M_2)$ is realized in M for every $B \subseteq M$, $|B| < \kappa$. [This is because then we can choose $B_0 \subseteq M$, $|B_0| < \kappa$, r does not fork over B, $r \restriction B_0$ stationary, and $\bar b_n \in M$ realizing $r \restriction (B_0 \cup \{\bar b_l : l < n\} \cup M_1 \cup M_2)$, and then $I = \{\bar b_n : n < \omega\}$ is as required — indiscernible by III 1.10(1).]

We can, of course, increase B as long as $|B| < \kappa$. So w.l.o.g. r does not fork over B, $r \restriction B$ is stationary, for $l = 0,1,2$, $\mathrm{tp}_*(B, M_l)$ does not fork over $B \cap M_l$, and let $\bar{c}$ realize r, $B_l = B \cap M_l$.

As $|B| < \kappa$, M is $\boldsymbol{F}^a_\kappa$-saturated, it suffices to prove

$$\mathrm{stp}(\bar{c}, B) \vdash r \restriction (B \cup M_1 \cup M_2).$$

For this let $\bar{b}_l \in M_l (l = 1,2)$ and it suffices to prove:

$$\mathrm{stp}(\bar{c}, B) \vdash r \restriction (B \cup \bar{b}_1 \cup \bar{b}_2).$$

We can find $C \subseteq M_0$, $|C| < \kappa$ such that $\mathrm{tp}(\bar{b}_l, M_0 \cup B_l)$ does not fork over $B_l \cup C$ for $l = 1,2$.

Now $\mathrm{tp}_*(B, M_0)$ does not fork over $B \cap M_0 = B_0$ and $C \subseteq M_0$ so $\mathrm{tp}_*(B, B_0 \cup C)$ does not fork over B_0. By symmetry, $\mathrm{tp}_*(C, B_0 \cup B) = \mathrm{tp}_*(C, B)$ does not fork over B_0. Extend $\mathrm{tp}_*(C, B)$ to a type q over M which does not fork over B_0. Then $q \restriction M_1$ is orthogonal to r and parallel to $\mathrm{stp}_*(C, B)$ so by V 1.2(4), $\mathrm{stp}_*(C, B)$ is orthogonal to r. Hence,

(1) $\mathrm{stp}_*(\bar{c}, B) \vdash \mathrm{stp}_*(\bar{c}, B \cup C)$.

Now $\mathrm{tp}(\bar{b}_1 \cup B_1, M_2)$ does not fork over M_0 [as it is $\subseteq \mathrm{tp}_*(M_1, M_2)$] hence $\mathrm{stp}_*(\bar{b}_1, M_2 \cup B_1)$ does not fork over $M_0 \cup B_1$. Also $\mathrm{tp}(\bar{b}_1, M_0 \cup B_1)$ does not fork over $B_1 \cup C$ by the choice of C, hence by III 0.1 (2), $\mathrm{tp}(\bar{b}_1, M_2 \cup B_1)$ does not fork over $B_1 \cup C$. Now $B_1 \cup C \subseteq M_1$, $B_1 \cup C \subseteq B \cup C \subseteq M_2 \cup B$, hence $\mathrm{stp}_*(\bar{b}_1, C \cup B)$ is parallel to some complete type over M_1, hence is orthogonal to r, hence to $\mathrm{stp}_*(\bar{c}, C \cup B)$. So

(2) $\mathrm{stp}_*(\bar{c}, C \cup B) \vdash \mathrm{stp}_*(\bar{c}, C \cup B \cup \bar{b}_1)$.

Now $\mathrm{tp}_*(\bar{b}_2 \cup B_2, M_1)$ does not fork over M_0 [as it is $\subseteq \mathrm{tp}_*(M_2, M_1)$], hence $\mathrm{tp}(\bar{b}_2, M_1 \cup B_2)$ does not fork over $M_0 \cup B_2$.

Also $\mathrm{tp}(\bar{b}_2, M_0 \cup B_2)$ does not fork over $B_2 \cup C$ (by C's choice), hence $\mathrm{tp}(\bar{b}_2, M_1 \cup B_2)$ does not fork over $B_2 \cup C$. As $B_2 \cup C \subseteq M_2$, and $B_2 \cup C \subseteq B \cup C \cup \bar{b}_1 \subseteq M_1 \cup B$, clearly $\mathrm{stp}(\bar{b}_2, C \cup B \cup \bar{b}_1)$ is parallel to some complete type over M_2 hence is orthogonal to r, hence to $\mathrm{stp}(\bar{c}, C \cup B)$. So

(3) $\mathrm{stp}(\bar{c}, C \cup B \cup \bar{b}_1) \vdash \mathrm{stp}(\bar{c}, C \cup B \cup \bar{b}_1 \cup \bar{b}_2)$.

By (1), (2), (3) clearly

(4) $\mathrm{stp}(\bar{c}, B) \vdash \mathrm{stp}(\bar{c}, B \cup \bar{b}_1 \cup \bar{b}_2)$

which, as mentioned above, is sufficient.

2.3. CLAIM. *Suppose* $\lambda \geqq \kappa$, M_l $(l=0,1,2)$ *are* $\boldsymbol{F}^a_\lambda$*-saturated,* $M_0 < M_1, M_2$, *and* $\{M_1, M_2\}$ *is independent over* M_0. *Then*

(1) *Every* $\boldsymbol{F}^a_\lambda$*-isolated* $p \in S^m(M_1 \cup M_2)$ *is* $\boldsymbol{F}^a_\kappa$*-isolated.*

(2) *A model* M *is* $\boldsymbol{F}^a_\lambda$*-atomic over* $M_1 \cup M_2$ *iff it is* $\boldsymbol{F}^a_\kappa$*-atomic over* $M_1 \cup M_2$. *Hence if* $\kappa \leqq \mu$, $\chi \leqq \lambda$, *and an* $\boldsymbol{F}^a_\mu$*-prime model* M *over* $M_1 \cup M_2$ *is* $\boldsymbol{F}^a_\mu$*-minimal, then* M *is* $\boldsymbol{F}^a_\chi$*-prime over* $M_1 \cup M_2$ *(and* $\boldsymbol{F}^a_\chi$*-minimal).*

PROOF. (1) Suppose $B \subseteq M_1 \cup M_2$, $|B| < \lambda$, $\bar{c}$ realizes p and $\operatorname{stp}(\bar{c}, B) \vdash p = \operatorname{tp}(\bar{c}, M_1 \cup M_2)$. We can assume that $\operatorname{tp}_*(B, M_0)$ does not fork over $B \cap M_0$.

Let $C \subseteq B, |C| < \kappa$ be such that p does not fork over C, $\operatorname{tp}_*(C, M_0)$ does not fork over $C \cap M_0$. Now by III 4.22 it suffices to prove:

(*) for any $\bar{a} \in M_1 \cup M_2$, there is $\bar{a}' \in M_1 \cup M_2$ such that $\operatorname{stp}(\bar{a}', B)$ is a stationarization of $\operatorname{stp}(\bar{a}, C)$

We can find $A \subseteq M_0$ such that $\operatorname{tp}_*(\bar{a} \cup C, M_0)$ does not fork over A, its restriction to A is stationary, $|A| < \kappa$, and let $\bar{a} = \bar{a}_1{}^\wedge \bar{a}_2$, $\bar{a}_l \in M_l$ $(l = 1,2)$. Let $A = \{a_i : i < i(0)\}$, and we can find $a'_i \in M_0$, such that $\operatorname{stp}_*(\langle a'_i : i < i(0)\rangle, B \cap M_0)$ extends $\operatorname{stp}_*(\langle a_i : i < i(0)\rangle, C \cap M_0)$ and does not fork over $C \cap M_0$, hence over C. Now find $\bar{a}'_l \in M_l$ such that $\operatorname{stp}_*(\langle a_i : i < i(0)\rangle^\wedge \bar{a}'_l, B)$ extends $\operatorname{stp}_*(\langle a_i : i < i(0)\rangle^\wedge \bar{a}_l, C)$ and does not fork over C. It is easy to check that $\bar{a}' = \bar{a}'_1{}^\wedge \bar{a}'_2$ is as required.

(2) trivial from (1).

2.4. LEMMA. *The following properties of* T *are equivalent for* $\lambda \geqq \chi \geqq \kappa$:

(a) T *has the dop (= dimensional order property).*

(b)$_{\lambda,\chi}$ *There are* $\boldsymbol{F}^a_\lambda$*-saturated models* M_0, M_1, M_2, $M_0 < M_1, M_2$, $\{M_1, M_2\}$ *independent over* M_0, *such that the* $\boldsymbol{F}^a_\chi$*-prime model* M_3 *over* $M_1 \cup M_2$ is not $\boldsymbol{F}^a_\chi$*-minimal.*

(c)$_{\lambda,\chi}$ *There are* $\boldsymbol{F}^a_\lambda$*-saturated models* $M_0, M_1, M_2, M_0 < M_1, M_2, \{M_1, M_2\}$ *independent over* M_0, *and there is an* $\boldsymbol{F}^a_\chi$*-atomic non-*$\boldsymbol{F}^a_\chi$*-minimal model* M_3 *over* $M_1 \cup M_2$.

(d)$_\lambda$ *There are sets* A_0, A_1, A_2 *such that* $A_0 \subseteq A_1$, A_2, $|A_l| < \lambda$, $\{A_1, A_2\}$ *is independent over* A_0; and *there is an infinite* $\boldsymbol{I}$ *indiscernible over* $A_1 \cup A_2$, *orthogonal to* A_1 *and to* A_2, *and* $\operatorname{tp}_*(\boldsymbol{I}, A_1 \cup A_2)$ *is almost orthogonal to* A_1 *and to* A_2. *Moreover,* if $\bar{a}_l$ $(l = 0,1,2)$ *are such that* $\{A_1, A_2, \bar{a}_0\}$ *is independent over* A_0, *and* $\operatorname{tp}(\bar{a}_l, A_0 \cup A_1 \cup A_2 \cup \bar{a}_0 \cup \bar{a}_{3-l})$ *does not fork over* $A_l \cup \bar{a}_0$ *for* $l = 1,2$, then

$$\operatorname{stp}_*(\boldsymbol{I}, A_1 \cup A_2) \vdash \operatorname{tp}_*(\boldsymbol{I}, A_1 \cup \bar{a}_1 \cup A_2 \cup \bar{a}_2).$$

We can replace $\bar{a}_l$ *by* B_l.

PROOF. As (a) is $(b)_{\kappa,\kappa}$ it suffices to prove:
(i) $(c)_{\lambda,\chi} \Rightarrow (d)_\kappa$; (ii) $(d)_\kappa \Rightarrow (d)_\chi$; (iii) $(d)_\chi \Rightarrow (b)_{\lambda,\chi}$; (iv) $(b)_{\lambda,\chi} \Rightarrow (c)_{\lambda,\chi}$.

(i) $(c)_{\lambda,\chi} \Rightarrow (d)_\kappa$

So let M_0, M_1, M_2, M_3 exemplify $(c)_{\lambda,\chi}$. Now by 2.2(d) there is an infinite $\boldsymbol{I} \subseteq M_3$ indiscernible over $M_1 \cup M_2$ such that $\mathrm{Av}(\boldsymbol{I}, M_3)$ is orthogonal to M_1 and to M_2. First assume $\kappa > \aleph_0$, and w.l.o.g. $|\boldsymbol{I}| = \aleph_0$. By 2.3, $\mathrm{tp}_*(\boldsymbol{I}, M_1 \cup M_2)$ is $\boldsymbol{F}^a_\kappa$-isolated, and so for some $A \subseteq M_1 \cup M_2$, $|A| < \kappa$, with $\mathrm{stp}_*(\boldsymbol{I}, A) \vdash \mathrm{stp}_*(\boldsymbol{I}, M_1 \cup M_2)$. We can also assume that $\mathrm{tp}_*(A, M_0)$ does not fork over $A \cap M_0$ and $\mathrm{tp}_*(A, A \cap M_0)$ is stationary. Let $A_l = A \cap M_l$. It is easy to check that $\{A_1, A_2\}$ is independent over A_0 (as $\{M_1, M_2\}$ is independent over M_0, and use III 0.1). Clearly $\mathrm{tp}_*(\boldsymbol{I}, A_1 \cup A_2)$ is almost orthogonal to A_1 and to A_2, and even the stronger assertion in 2.4 (d) holds.

If $\kappa = \aleph_0$, T is superstable and so there is a finite $\boldsymbol{J} \subseteq \boldsymbol{I}$ such that $\mathrm{Av}(\boldsymbol{I}, \boldsymbol{I})$ does not fork over $\bigcup \boldsymbol{J}$ and $\mathrm{Av}(\boldsymbol{I}, \boldsymbol{J})$ is stationary, and continue as before with $\boldsymbol{J}$ instead of $\boldsymbol{I}$ noticing that $\mathrm{tp}_*(\boldsymbol{I}, A_1 \cup A_2 \cup \boldsymbol{J})$ does not fork over $\bigcup \boldsymbol{J}$, is stationary, and is orthogonal to M_1 and to M_2.

(ii) $(d)_\kappa \Rightarrow (d)_\chi$

As $\kappa \leqq \chi$, if A_l $(l = 1,2,3)$, $\boldsymbol{I}$ exemplify $(d)_\kappa$, then they exemplify $(d)_\chi$.

(iii) $(d)_\chi \Rightarrow (b)_{\lambda,\chi}$

Let $A_0, A_1, A_2, \boldsymbol{I}$ exemplify $(d)_\chi$.

Let M_0 be an $\boldsymbol{F}^a_\lambda$-saturated model of T, $A_0 \subseteq M_0$. By using automorphism of $\mathfrak{C}$ we can assume $\mathrm{tp}_*(A_l, M_0 \cup A_{3-l})$ does not fork over A_0. Next choose an $\boldsymbol{F}^a_\lambda$-saturated M_1, such that $M_0 \cup A_1 \subseteq M_1$, $\mathrm{tp}_*(M_1, M_0 \cup A_2)$ does not fork over M_0, and an $\boldsymbol{F}^a_\lambda$-saturated M_2, such that $M_0 \cup A_2 \subseteq M_2$ and $\mathrm{tp}_*(M_2, M_1)$ does not fork over M_0 (this is easy). So clearly $\{M_1, M_2\}$ is independent over M_0, and by the latter part of $(d)_\chi$

$$\mathrm{stp}_*(\boldsymbol{I}, A_1 \cup A_2) \vdash \mathrm{stp}_*(\boldsymbol{I}, M_1 \cup M_2).$$

So $\mathrm{tp}_*(\boldsymbol{I}, M_1 \cup M_2)$ is $\boldsymbol{F}^a_\chi$-isolated, so there is M_4 $\boldsymbol{F}^a_\chi$-prime over $M_1 \cup M_2$, such that $\boldsymbol{I} \subseteq M_4$.

As M_4 contains an infinite indiscernible set $\boldsymbol{I}$ over $M_0 \cup M_1$, it is not $\boldsymbol{F}^a_\chi$-minimal (by [4] IV 4.21), so we prove $(b)_{\lambda,\chi}$.

(iv) $(b)_{\lambda,\chi} \Rightarrow (c)_{\lambda,\chi}$

Trivial.

2.5. THEOREM. *Suppose T has the* dop, *T stable in λ and $\kappa \leqq \mu < \lambda$. Then T has 2^λ non-isomorphic $\boldsymbol{F}^a_\mu$-saturated models of cardinality λ.*

PROOF. Let $A_0, A_1, A_2, \boldsymbol{I}$ be as 2.2 $(\mathrm{d})_\kappa, |\boldsymbol{I}| = \aleph_0$ and let J be any set of indices. W.l.o.g. we shall work in $\mathfrak{C}^{eq}$ (see [4] III §6).

Let $A'_l = \operatorname{acl} A_l$; and we can define, for $l = 1,2$, $s \in J$, an elementary mapping f^l_s, such that:

(α) $\operatorname{Dom} f^l_s = A'_l$,

(β) $f^l_s \restriction A'_0 =$ the identity,

(γ) $\operatorname{tp}_*(f^l_s(A'_l), \bigcup \{f^k_t(A'_k) : (t,k) \neq (s,l), t \in I, k \in 3\})$ does not fork over A'_0.

Next, for every $s, t \in J$ (not necessarily distinct) we choose an elementary mapping $f_{s,t}$, whose domain is $A'_1 \cup A'_2 \cup \boldsymbol{I}$, and which extends f^1_s, f^2_t. (This is possible as $f^1_s \cup f^2_t$ is an elementary mapping, which is true because $\operatorname{tp}(A'_l, A'_0)$ is stationary (by III 6.9 (1) and the independence of $\{A'_1, A'_2\}$ and of $\{f^1_s(A_1), f^2_t(A_2)\}$ over A'_0).) Let $A^l_s = f^l_s(A_l)$ (so $|A^l_s| < \kappa$), $\boldsymbol{I}_{s,t} = f_{s,t}(\boldsymbol{I})$. Now:

$$\text{(st)} \qquad \operatorname{stp}_*(\boldsymbol{I}_{s,t}, A^1_s \cup A^2_t) \vdash \operatorname{stp}_*\Big(\boldsymbol{I}_{s,t}, \bigcup_{v \in I} A^1_v \cup \bigcup_{v \in I} A^2_v \cup \bigcup_{(v,u) \neq (s,t)} \boldsymbol{I}_{v,u}\Big).$$

This holds by 2.4(d) for

$$B_0 = \bigcup \{A^l_v : (v,l) \neq (s,1),(t,2)\} \cup \{\boldsymbol{I}_{u,v} : \{u,v\} \cap \{s,t\} = \varnothing\},$$

$$B_1 = B_0 \cup A^1_s \cup \bigcup \{\boldsymbol{I}_{s,v} : v \neq t, v \in J\},$$

$$B_2 = B_0 \cup A^2_t \cup \bigcup \{\boldsymbol{I}_{u,t} : u \neq s, u \in J\}.$$

Let $\boldsymbol{I}^*_{s,t}$ be any set indiscernible over $A^1_s \cup A^2_t$, extending $\boldsymbol{I}_{s,t}$, of power μ^+. Clearly (st) continues to hold for the $\boldsymbol{I}^*_{s,t}$'s.

Let R be any two place relation over J, and let $C_R = \bigcup_{l,s} A^l_s \cup \bigcup_{R(s,t)} \boldsymbol{I}^*_{s,t}$, and let M_R be an $\boldsymbol{F}^a_\mu$-prime model over C_R.

It is easy to check that $|J| = \lambda$ implies $\|M_R\| = \lambda$ (remember T is stable in λ). Now, using (st) we show:

$$\text{(st1)} \qquad \begin{array}{c}\text{for } s,t \in J,\\ \text{there is in } M_R \text{ an } \boldsymbol{I} \text{ of power } \mu^+ \text{ realizing } \operatorname{tp}_*(\boldsymbol{I}^*_{s,t}, A^1_s \cup A^2_t) \quad \text{iff} \quad R(s,t).\end{array}$$

If $R(s,t)$ holds, clearly there is in M_R an $\boldsymbol{I}$ (of power μ^+) realizing $\operatorname{tp}_*(\boldsymbol{I}^*_{s,t}, A^1_s \cup A^2_t)$: $\boldsymbol{I}^*_{s,t}$ itself. Suppose $R(s,t)$ fail, $\boldsymbol{I} \subseteq M_R, |\boldsymbol{I}| = \mu^+$ realizes $\operatorname{tp}_*(\boldsymbol{I}^*_{s,t}, A^1_s \cup A^2_t)$. We shall work in $\mathfrak{C}^{eq}$. If $\operatorname{stp}_*(\boldsymbol{I}, A^1_s \cup A^2_t) \equiv \operatorname{stp}_*(\boldsymbol{I}_{s,t}, A^1_s \cup A^2_t)$ we get an easy contradiction: by IV 4.9, $\dim(\boldsymbol{I}, C_R, M) \leqq \mu$, and by (st), $\dim(\boldsymbol{I}, C_R, M) = \dim(\boldsymbol{I}, A^1_s \cup A^2_t, M)$ which should be $\geqq |\boldsymbol{I}| = \mu^+ > \mu$, contradiction. However, if $\operatorname{stp}_*(\boldsymbol{I}, A^1_s \cup A^2_t) \neq \operatorname{stp}_*(\boldsymbol{I}^*_{s,t}, A^1_s \cup A^2_t)$, as $\operatorname{tp}(\boldsymbol{I}, A^1_s \cup A^2_t) = \operatorname{tp}(\boldsymbol{I}^*_{s,t}, A^1_s \cup A^2_t)$, still $A_0, A^1_s, A^2_t, \boldsymbol{I}$ satisfies $2.4(\mathrm{d})_\kappa$, so we can prove the assertion corresponding to (st).

So in M_R the relation R on $\langle A_t^1 \cup A_t^2 : t \in J \rangle$ is defined; and $|A_t^l| < \kappa, \lambda = \lambda^{<\kappa}$ (as T is stable in $\lambda, \kappa = \kappa_r(T)$). So as in VIII 3.2, we can prove that there are 2^λ pairwise non-isomorphic $\boldsymbol{F}_\mu^a$-saturated models of T of cardinality λ. The case which most interests us, $\mu = \aleph_0$, T superstable (i.e. $\kappa(T) = \aleph_0$), has the same proof as in VIII §2: we have just to choose the right Φ (see there).

Remarks. (1) For most λ we can also get 2^λ such models no one elementarily embeddable into the other, e.g., for λ regular $> |T|$; more generally, see also [6].

(2) In 2.5, if T is not stable in λ we can still get similar results.

For the reader unsatisfied with this, we give in more detail two cases.

2.5A. Fact. If T has the dop $\chi \geqq \lambda > \kappa$, $\chi > \mu \geqq \kappa$, λ regular, then there are sets A_i $(i < 2^\lambda)$, each of power χ, such that, letting M_i be $\boldsymbol{F}_\mu^a$-prime over A_i, the following holds:

(i) The M_i's are pairwise not isomorphic.

(ii) For $i \neq J$, M_i is not elementarily embeddable into M_j.

(iii) For $i \neq j$, there is no elementary mapping from A_i into M_j.

Proof. Let $S^* = \{\delta < \lambda : \operatorname{cf} \delta = \kappa\}$, and for each $\delta \in S$ let η_δ be an increasing sequence of successor ordinals of length κ converging to δ. For every $S \subseteq S^*$ let

$$J_S = \chi 2, \quad R_S = \{\langle \eta_\delta(i), \delta \rangle : \delta \in S, i < \kappa\}, \quad A_S = C_{R_S}, \quad M^S = M_{R_S},$$

$$A_S^l = \bigcup_{\substack{l<2 \\ i<\lambda}} A_i^l \cup \bigcup \{\boldsymbol{I}_{\langle i, \delta \rangle} : \langle i, \delta \rangle \in R\}.$$

Now suppose $S_1, S_2 \subseteq S$, $S_1 - S_2$ is stationary, $f : A_{S_1}^l \to M_{S_2}$ an elementary mapping, and we shall eventually get a contradiction. This clearly suffices. By renaming, we can assume that f is into

$$B = B_0 \cup \{a_i : i < \lambda\} \qquad \text{where } B_0 = A_{S_2} \cup \bigcup_{\substack{i<\lambda 2 \\ l<2}} A_i^l,$$

$\operatorname{tp}(a_i, B_i)$ is $\boldsymbol{F}_\mu^a$-isolated where $B_i = B_0 \cup \{a_j : j < i\}$.

Let $\lambda^* > 2^\chi$ be regular, and choose $N_i < (H(\lambda^*), \in)$, $\|N_i\| < \lambda$, $N_i \cap \lambda = \delta_i$, $\langle N_i : j \leqq i \rangle \in N_{i+1}$, and f, A_{s_1}', B, B_0, $\langle a_i : i < \lambda \rangle$, R_S, $\langle A_i^l : l < 2, i < \chi 2 \rangle$, $\langle \boldsymbol{I}_{s,t}^* : s, t \in J_{S_m} \rangle$ $(m = 1, 2)$ belongs to N_0. Choose $\zeta \in S_1 - S_2$, $\zeta = \delta_\zeta$.

Next choose $M < (H(\lambda^*), \in)$, $\zeta \in M$, $M \cap \kappa$ an ordinal ξ, $\langle N_i : i < \lambda \rangle \in M$, and all the elements which we demand to be in N_0 will be in M too.

Now $f(A_\zeta^0 \cup A_\xi^1 \cup \bigcup \boldsymbol{I}_{\langle \eta_\delta(\xi), \zeta \rangle})$ gives us the desired contradiction.

2.5B. FACT. There are Φ, T_1 such that $T \subseteq T_1$, $|T_1| \leqq \lambda(T)$ and Φ is proper for (ω, T_1) (see Def. VII 2.6 p. 393, and Lemma VIII 2.3) such that for every linear ordering I:

(1) $EM(I,\Phi)$ is $F^a_{\aleph_0}$-saturated.

(2) There is a formula

$$\psi(\bar{y},\bar{z}) = (\exists x_0,\cdots,x_i,\cdots)_{i<\mu^+} \bigwedge_{\alpha<\mu^+} \varphi_i(\cdots,x_{i(\alpha,l)},\cdots,\bar{y},\bar{z}),$$

φ_i first order, $\bar{y},\bar{z}$ of length $<\kappa$, such that, for $s,t \in I$,

$$EM(I,\Phi) \models \psi[\bar{b}_s,\bar{b}_t] \qquad \text{iff } s<t$$

where

$$\bar{b}_t = \langle f_i(\bar{a}_t) : i < |A_1 \cup A_2| < \kappa \rangle, \text{ and if } \kappa = \aleph_0, \bar{b}_t = \bar{a}_t.$$

PROOF. Let J be $\{i : i < \beth^+_{\delta_0}\}$, where $\delta_0 = \beth_8(|T|)$, $I^*_{s,t}$ has power λ, M^0_R be F^a_κ-prime over $\bigcup_{l,s} A^l_s \cup \bigcup \{I^*_{s,t} : s<t\}$ (so R is the natural ordering). Now expand M^0_R by $\lambda(T)$ functions and get M'_R such that:

(a) For every $\bar{b} \in \mathfrak{C}$, $\bar{a} \in M$, $\operatorname{stp}(\bar{b},\bar{a})$ is realized in the closure of $\bar{a}$ by the functions of M'_R.

(b) $P = \{\bar{a}_s : s \in J\}$ where $\bar{a}_s \subseteq A^0_s \cup A^1_s$, and if $\kappa = \aleph_0$ then equality holds (i.e., the range of $\bar{a}_s$ is $A^0_s \cup A^1_s$) and always $A^0_s \cup A^1_s \subseteq \{f_i(\bar{a}_s) : i\}$.

(c) $f(-,\bar{a}_s,\bar{a}_t)$ $(s<t)$ is a one-to-one function from M_R into $I^*_{s,t}$.

(d) M'_R has Skolem functions.

Now apply Morley's proof of the omitting type theorem to get $EM^1(\Phi,\omega)$, a model of $T_1 = \operatorname{Th}(M'_R)$, which realizes only types M'_R realizes.

From 2.5B, $I^a_{\aleph_0}(\lambda,T) = 2^\lambda$ for $\lambda \geqq \lambda(T) + \aleph_1$ follows from VIII 3.2 (for $\lambda > \lambda(T)$ it already follows from [1], [2] 2.6).

Remember that for T superstable, $|T| + \aleph_1 \leqq \lambda \leqq \lambda(T)$, $I(\lambda,T) = 2^\lambda$ was proved in IX 1.20.

Now we shall mention a topic, not necessary for the rest of the paper, but naturally connected to the dop. Just as we have looked at "hidden order" we can look for "hidden unstability," like the one caused by $\kappa(T) > \aleph_0$.

2.6. DEFINITION. T has the discontinuity dimensional property (didip in short) in the cardinal μ (μ regular) *if* there are F^a_κ-saturated models M_α $(\alpha<\mu)$ such that $\alpha<\beta \Rightarrow M_\alpha < M_\beta$ and the F^a_κ-prime model over $\bigcup_{i<\mu} M_i$ is not F^a_κ-minimal over $\bigcup_{i<\alpha} M_i$.

Now if T has the didip for μ, then $\mu < \kappa(T)$ and there are 2^λ F^a_χ-saturated non-isomorphic models of power λ if $\lambda > \chi \geqq \kappa$, T stable in λ, at least when $(\forall \lambda_1 < \lambda)\ \lambda_1^{<\mu} < \lambda$, $\mathrm{cf}\,\lambda = \lambda$.

The proofs are parallel to the proofs in VIII on the number of $\boldsymbol{F}^a_\chi$-saturated models of power λ, when $\chi < \kappa(T)$ regular.

Also, the parallel of 2.3 holds, and if $\chi > \kappa$ is singular, T does not have the didip for $\mathrm{cf}\,\chi$, then $\boldsymbol{F}^a_\chi$-prime model over any set A is unique. Also, *if* each M_i is $\boldsymbol{F}^a_\lambda$-saturated $(i < \alpha)$ $M < M_i$, $\{M_i : i < \alpha\}$ is independent over M, and N is $\boldsymbol{F}^a_\lambda$-prime over $\bigcup_{i<\alpha} M_i$, and in addition T does not have the dop nor any case of didip, *then* N is $\boldsymbol{F}^a_\lambda$-minimal over $\bigcup_{i<\alpha} M_i$.

§3. The decomposition lemma

(Hyp) In this section T is superstable without the dimensional order property.

The main result of this section, the decomposition lemma, states that (for superstable T without the dimensional order property) every $\boldsymbol{F}^a_{\aleph_0}$-saturated model is $\boldsymbol{F}^a_{\aleph_0}$-prime over a non-forking tree N_η $(\eta \in I)$ (i.e., $I \subseteq {}^{\omega>}\|M\|$ is closed under initial segments, and $\mathrm{tp}(N_\eta, \bigcup \{N_\nu : \eta \not\trianglelefteq \nu\})$ does not fork over $N_{\eta \restriction (l(\eta)-1)}$, when $l(\eta) > 0$, and $\eta \triangleleft \nu \Rightarrow N_\eta \subseteq N_\nu$). This is a kind of structure theorem, so this division line (superstable + not dop) is significant. Though we shall eventually prove that some such theories (the deep ones, see §5) have many non-isomorphic models, all of them do not have a family of κ $\boldsymbol{F}^a_{\aleph_0}$-saturated models no one elementarily embeddes into another, with κ of arbitrary cardinality (this is with the help of [5]). Recall that for $A \subseteq B \subseteq C$, $B <_A C$ means that for each $\bar{c} \in C$, $\mathrm{tp}(\bar{c}, B)$ is orthogonal to A.

3.1. The Atomic Decomposition Lemma. *Suppose $N_1 < M$ are $\boldsymbol{F}^a_{\aleph_0}$-saturated models. Then there are elements $a_i \in M$ $(i < \alpha)$ and models $N_{2,i}$, M_i such that:*

(a) $N_1 < N_{2,i} < M_i < M$,

(b) *M_i, $N_{2,i}$ are $\boldsymbol{F}^a_{\aleph_0}$-saturated,*

(c) *$\mathrm{tp}(a_i, N_1)$ is regular, and for $i \neq j$, $\mathrm{tp}(a_i, N_1)$, $\mathrm{tp}(a_j, N_1)$ are orthogonal or equal,*

(d) *$a_i \in N_{2,i}$, and $N_{2,i}$ is $\boldsymbol{F}^a_{\aleph_0}$-prime over $N_1 \cup \{a_i\}$,*

(e) *$N_{2,i} <_{N_1} M_i$, and M_i is maximal with respect to this property (in fact, for no $a \in M - M_i$ is $\mathrm{tp}_*(M_1 \cup \{a\}, N_{2,i})$ orthogonal to N_1),*

(f) *M is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_{i<\alpha} M_i$,*

(g) *$\{M_i : i < \alpha\}$ is independent over N_1, and $\mathrm{tp}_*(M_i, N_1 \cup \{a_i\})$ is almost orthogonal to N_1.*

Proof. Let $\boldsymbol{I} = \{a_i : i < \alpha\} \subseteq M$ be a maximal set, independent over N_1 of elements of M realizing over N_1 regular types, and w.l.o.g. satisfying (c).

Let $N_{2,i} \subseteq M$ be $\boldsymbol{F}^a_{\aleph_0}$-prime over $N_1 \cup \{a_i\}$. So (c) and (d) hold trivially. By V 3.2, we have:

FACT A. $\operatorname{tp}(N_{2,i}, \bigcup_{j<\alpha, j\neq i} N_{2,j})$ does not fork over N_1.

Now for each $i<\alpha$, we define by induction on $j<\|M\|^+$ an element $b_{i,j} \in M - N_{2,j} \cup \{b_{i,\gamma} : \gamma < j\}$ such that $\operatorname{tp}(b_{i,j}, N_{2,i} \cup \{b_{i,\gamma} : \gamma < j\})$ is $\boldsymbol{F}^a_{\aleph_0}$-isolated or is orthogonal to N_1 (we can assume that the second possibility occurs only if $N_{2,i} \cup \{b_{i,\gamma} : \gamma < j\}$ is the universe of an $\boldsymbol{F}^a_{\aleph_0}$-prime model). There is a first $\beta(i) < \|M\|^+$ such that $b_{i,\beta(i)}$ is not defined. Obviously:

FACT B. $N_{2,i} \cup \{b_{i,\beta} : \beta < \beta(i)\}$ is the universe of an $\boldsymbol{F}^a_{\aleph_0}$-saturated model which we denote by M_i (clearly $M_i < M$).

Now:

FACT C. $\operatorname{tp}_*(M_i, \bigcup_{j\neq i} M_j)$ does not fork over N_1.

To show Fact C, we just prove by induction on $\xi \leqq \Sigma_{i<\alpha}\beta(i)$ that if we let $A_i^\xi = N_{2,i} \cup \{b_{i,\beta} : \Sigma_{j<i}\beta(j) + \beta < \xi\}$, then for every $i < \alpha$, $\operatorname{tp}(A_i^\xi, \bigcup_{j\neq i} A_j^\xi)$ does not fork over N_1. The induction step is by V 3.2, III 0.1 (when we add an element realizing an $\boldsymbol{F}^a_{\aleph_0}$-isolated type) and V 1.2(3), III 0.1 (in the other case).

FACT D. $N_{2,i} <_{N_1} M_i$.

We prove by induction on $\beta \leqq \beta(i)$ that:

(*) $N_{2,i} <_{N_1} N_{2,i} \cup \{b_{i,\gamma} : \gamma < \beta\}$.

For $\beta = 0$ this is trivial, as well as for β limit. So let us prove it for $\beta + 1$. Let $\operatorname{tp}(\bar{c}, N_{2,i})$ be a type which does not fork over N_1. Since N_1 is $\boldsymbol{F}^a_{\kappa(T)}$-saturated, by V 1.2(3) it suffices to prove that $\operatorname{tp}(\bar{c}, N_{2,i})$, $\operatorname{tp}_*(\{b_{i,\gamma} : \gamma \leqq \beta\}, N_{2,i})$ are weakly orthogonal. This is equivalent to

$$\operatorname{tp}(\bar{c}, N_{2,i}) \vdash \operatorname{tp}(\bar{c}, N_{2,i} \cup \{b_{i,\gamma} : \gamma \leqq \beta)\}).$$

By the induction hypothesis on β

$$\operatorname{tp}(\bar{c}, N_{2,i}) \vdash \operatorname{tp}(\bar{c}, N_{2,i} \cup \{b_{i,\gamma} : \gamma < \beta\}).$$

Hence $\operatorname{tp}(\bar{c}, N_{2,i} \cup \{b_{i,\gamma} : \gamma < \beta\})$ does not fork over $N_{2,i}$, hence (by transitivity, see III 0.1) does not fork over N_1, and it suffices to prove it is weakly orthogonal to $\operatorname{tp}(b_{i,\beta}, N_{2,i} \cup \{b_{i,\gamma} : \gamma < \beta\})$. If the latter is $\boldsymbol{F}^a_{\aleph_0}$-isolated this holds by V 3.2 and if the latter is orthogonal to N_1, this holds by V 1.2(3).

FACT E. M is $\boldsymbol{F}^a_{\aleph_0}$-prime and $\boldsymbol{F}^a_{\aleph_0}$-minimal over $\bigcup_{i<\alpha} M_i$.

PROOF OF FACT E. Let $M' < M$ be $\boldsymbol{F}^a_\kappa$-prime over $\bigcup_{i<\alpha} M_i$ (we know that there is such M'). Suppose $M' \neq M$ which holds if M is not $\boldsymbol{F}^a_\kappa$-prime over $\bigcup_{i<\alpha} M_i$, and M' can be chosen so that it holds if M is not $\boldsymbol{F}^a_{\aleph_0}$-prime and $\boldsymbol{F}^a_{\aleph_0}$-minimal over $\bigcup_{i<\alpha} M_i$; we shall eventually get a contradiction. Choose $b \in M - M'$ with $R[\mathrm{tp}(b, M'), L, \infty]$ minimal (it is $< \infty$ as T is superstable).

By V 3.5, $\mathrm{tp}(b, M')$ is regular. Let us first assume that $\mathrm{tp}(b, M')$ is not orthogonal to N_1, then by 1.4 there is a regular type $p \in S^m(N_1)$ not orthogonal to it, hence by V 1.12 some $b' \in M - M'$ realizes the stationarization of p over M'. But this contradicts the maximality of $\boldsymbol{I} = \{a_i : i < \alpha\}$ as b' can serve as a_α.

So we can assume $\mathrm{tp}(b, M')$ is orthogonal to N_1. Choose a set $B \subseteq M'$, $|B| < \kappa(T) = \aleph_0$ such that $\mathrm{tp}(b, M')$ does not fork over B. Since M' is $\boldsymbol{F}^a_{\aleph_0}$-saturated we can also assume $\mathrm{tp}(b, B)$ is stationary. Also there is a finite $S \subseteq \alpha$ such that $\mathrm{tp}_*(B, \bigcup_{i\in S} M_i)$ is $\boldsymbol{F}^a_{\aleph_0}$-isolated [as $\mathrm{tp}_*(B, \bigcup_{i<\alpha} M_i)$ is $\boldsymbol{F}^a_{\aleph_0}$-isolated by IV 4.3 as $B \subseteq M'$, and M' is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_{i<\alpha} M_i$]. So there is $M^* \subseteq M'$ $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_{i\in S} M_i$ and $b \in M$ so that $\mathrm{tp}(b, M^*)$ is a regular type orthogonal to N_1. We prove that this is impossible by induction on $|S|$. If $|S| = 0$ then $\mathrm{tp}(b, M^*)$ is not orthogonal to N_1 as $M^* = N_1$, contradiction. If $|S| = 1$, $S = \{i\}$ we get a contradiction to the definition of M_i. Suppose $|S| = n + 1$ and assume by induction that for any $U \subseteq \alpha$ with $|U| \leqq n$, if $N^* < M$ is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_{i\in U} M_i$ and $r \in S(N^*)$ is a regular type which is realized in M, then r is not orthogonal to N_1. Let $i \in S$ and choose $M^+ \subseteq M^*$, $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_{j\in S, j\neq i} M_j$ such that M^* is $\boldsymbol{F}^a_{\aleph_0}$-prime over $M^+ \cup M_i$ (for some fixed $i \in S$). By V 3.2 $\{M^+, M_i\}$ is independent over N_1. Since T does not have dop, by 2.3, $\mathrm{tp}(b, M^*)$ is not orthogonal to one of M_i or M^+. Let N denote the model $\mathrm{tp}(b, M^*)$ is not orthogonal to. By Lemma 1.4 there is a regular $q \in S^m(N)$ such that q is not orthogonal to $\mathrm{tp}(b, M^*)$. But $\mathrm{tp}(b, M^*)$ is orthogonal to every (regular) complete type over N_1. Since non-orthogonality is transitive on regular types (V 1.13), it follows that q is orthogonal to every regular type in $S(N_1)$, i.e. by 1.4, q is orthogonal to N_1. But q is not orthogonal to $\mathrm{tp}(b, M^*)$ and b is in the $\boldsymbol{F}^a_{\aleph_0}$-saturated model M, so by V 1.12, the stationarization of q on M^* and hence q is realized in M^*. But then q and N contradict the hypothesis of induction.

So we prove Fact E and we can check that the only part of 3.1 to be proved is the last phrase of (e) which we leave to the reader. For (g) use 1.6(5), 1.6(1).

3.2. THE DECOMPOSITION LEMMA. *For any $\boldsymbol{F}^a_{\aleph_0}$-saturated model M we can find a set $I \subseteq {}^{\omega>}\|M\|$ (finite sequences of ordinals $< \|M\|$) closed under initial segments, and N_η, a_η for $\eta \in I$ and $p_\eta (\eta \in I - \{\langle\ \rangle\})$ such that:*

(1) $N_\eta < M$ *is* $\boldsymbol{F}^a_{\aleph_0}$*-saturated.*

(2) $N_{\langle\ \rangle}$ *is* $\boldsymbol{F}^a_{\aleph_0}$*-prime (over* $\emptyset$*).*

(3) $p_{\eta^\wedge\langle i\rangle} = \mathrm{tp}(a_{\eta^\wedge\langle i\rangle}, N_\eta)$ *is regular, and for* $\eta^\wedge\langle j\rangle \in I$, $p_{\eta^\wedge\langle i\rangle}$, $p_{\eta^\wedge\langle j\rangle}$ *are orthogonal or equal.*

(4) $N_{\eta^\wedge\langle i\rangle}$ *is* $\boldsymbol{F}^a_{\aleph_0}$*-prime over* $N_\eta \cup \{a_{\eta^\wedge\langle i\rangle}\}$.

(5) $\mathrm{tp}_*(N_{\eta^\wedge\langle i\rangle}, \bigcup\{N_\nu : \nu \in I$ *but not* $\eta^\wedge\langle i\rangle \trianglelefteq \nu\}$ *does not fork over* N_η *(for* $\eta^\wedge\langle i\rangle \in I$*).*

(6) M *is* $\boldsymbol{F}^a_{\aleph_0}$*-prime and* $\boldsymbol{F}^a_{\aleph_0}$*-minimal over* $\bigcup_{\eta\in I} N_\eta$.

(7) $\mathrm{tp}_*(\bigcup\{N_\nu : \eta \trianglelefteq \nu \in I\}, N_\eta)$ *is orthogonal to* $N_{\eta\restriction n}$ *when* $l(\eta) = n+1$.

Proof. We define by induction on n, a set I_n of sequences of ordinals of length n, models, N_η, M_η and elements a_η for each $n \in I_n$ such that:

(1) $\eta \in I_n$, $m < n$ implies $\eta\restriction m \in I_m$, and $I_0 = \{\langle\ \rangle\}$.

(2) N_η is $\boldsymbol{F}^a_{\aleph_0}$-saturated.

(3) $N_{\langle\ \rangle}$ is $\boldsymbol{F}^a_{\aleph_0}$-prime (over $\emptyset$).

(4) $N_{\eta^\wedge\langle i\rangle}$ is $\boldsymbol{F}^a_{\aleph_0}$-prime over $N_\eta \cup \{a_{\eta^\wedge\langle i\rangle}\}$.

(5) Let $p_{\eta^\wedge\langle i\rangle} = \mathrm{tp}(a_{\eta^\wedge\langle i\rangle}, N_\eta)$, then it is regular, and $p_{\eta^\wedge\langle i\rangle}$, $p_{\eta^\wedge\langle j\rangle}$ are equal or are orthogonal.

(6) M_η is $\boldsymbol{F}^a_{\aleph_0}$-saturated, $N_\eta \subseteq M_\eta \subseteq M$.

(7) $\eta \triangleleft \nu$ implies $N_\eta \subseteq N_\nu \subseteq M_\nu \subseteq M_\eta$.

(8) M_η is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_i M_{\eta^\wedge\langle i\rangle}$.

(9) $N_{\eta^\wedge\langle i\rangle} <_{N_\eta} M_{\eta^\wedge\langle i\rangle}$.

(10) $\{M_{\eta^\wedge\langle i\rangle} : \eta^\wedge\langle i\rangle \in I\}$ is independent over N_η.

(11) $A_\eta = \{a_{\eta^\wedge\langle i\rangle} : \eta^\wedge\langle i\rangle \in I\}$ is a maximal subset of M_η (or even set of sequences from M_η) independent over N_η.

(12) A_η is a maximal subset of M which is independent over N_η and every element of it realizes over N_η a type orthogonal to $\bigcup_{k<l(\eta)} N_{\eta\restriction k}$ if $\eta \neq \langle\ \rangle$.

The definition is easy: for $n = 0$ trivial, for $n+1$, for each $\eta \in I_n$ we apply 3.1 with N_η, M_η standing for N_1, M and get a_i, $N_{2,i}$, M_i and let $a_{\eta^\wedge\langle i\rangle} = a_i$, $N_{\eta^\wedge\langle i\rangle} = N_{2,i}$, $M_{\eta^\wedge\langle i\rangle} = M_i$ (so I_{n+1} is the set of ν's of length $n+1$ for which N_ν is defined).

Let $I = \bigcup_n I_n$. Now all the conditions of the lemma are obvious except "M is $\boldsymbol{F}^a_\kappa$-prime and $\boldsymbol{F}^a_\kappa$-minimal over $\bigcup_{\eta\in I} N_\eta$" and (11), (12).

Let us first prove (11), (12). If (11) or (12) fails for η, let $\bar a$ exemplify it; i.e. $\bar a \notin N_\eta$, $\mathrm{tp}(\bar a, N_\eta)$ is orthogonal to $N_{\eta\restriction(k-1)}$ if $k = l(\eta) > 0$, and $\bar a \notin N_\eta$ and $\mathrm{tp}(\bar a, N_\eta \cup \{a_{\eta^\wedge\langle i\rangle} : \eta^\wedge\langle i\rangle \in I\})$ does not fork over N_η; w.l.o.g. $\mathrm{tp}(\bar a, N_\eta)$ is regular.

As in the proof of 3.1,

$$\mathrm{tp}(\bar a, \bigcup\{N_{\eta^\wedge\langle i\rangle} : \eta^\wedge\langle i\rangle \in I\}) \vdash \mathrm{tp}(\bar a, \bigcup\{M_{\eta^\wedge\langle i\rangle} : \eta^\wedge\langle i\rangle \in I\}),$$

and as M_η is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup\{M_{\eta^\wedge\langle i\rangle}: \eta^\wedge\langle i\rangle \in I\}$, $\mathrm{tp}(\bar a, \bigcup_i M_{\eta^\wedge\langle i\rangle})$ does not fork over N_η, by V 3.2,

$$\mathrm{tp}\left(\bar a, \bigcup_i M_{\eta^\wedge\langle i\rangle}\right) \vdash \mathrm{tp}(\bar a, M_\eta).$$

This proves (11) (i.e., when $\bar a \in M_\eta$); for (12) note that by the above $\mathrm{tp}(\bar a, M_\eta)$ does not fork over N_η, hence is the stationarization of $\mathrm{tp}(\bar a, N_\eta)$ which is orthogonal to $N_{\eta\restriction(l(\eta)-1)}$ when it is defined. Let $k = l(\eta)$, and we now prove by induction on $l \leqq k$ that $\mathrm{tp}(\bar a, M_{\eta\restriction(k-l)})$ does not fork over N_η.

We have proved for $l = 0$ and for $l+1$ notice that as $\mathrm{tp}(\bar a, M_{\eta\restriction(k-l)})$ does not fork over N_η, it is parallel to $\mathrm{tp}(\bar a, N_\eta)$, hence orthogonal to $N_{\eta\restriction(k-1)}$ hence to $N_{\eta\restriction(k-l-1)}$. So for $j \neq \eta(k-l-1)$, $\mathrm{tp}(\bar a, M_{\eta\restriction(k-l)})$ is orthogonal to $\mathrm{tp}_*(M_{\eta\restriction(k-l-1)^\wedge\langle j\rangle}, N_{\eta\restriction(k-l-1)})$.

As $\{M_{\eta\restriction(k-l-1)^\wedge\langle j\rangle}: j < \alpha\}$ is independent over $N_{\eta\restriction(k-l-1)}$, $\mathrm{tp}(\bar a, M_{\eta\restriction(k-l)})$ is orthogonal to $\mathrm{tp}_*(\bigcup_{j\neq\eta(k-l-1)} M_{\eta\restriction(k-l-1)^\wedge\langle j\rangle}, N_{\eta\restriction(k-l-1)})$, and

$$\mathrm{tp}(\bar a, M_{\eta\restriction(k-l)}) \vdash \mathrm{tp}\left(\bar a, \bigcup_j M_{\eta\restriction(k-l-1)^\wedge\langle j\rangle}\right).$$

So as $M_{\eta\restriction(k-l-1)}$ is $F^a_{\aleph_0}$-prime over $\bigcup_j M_{\eta\restriction(k-l-1)^\wedge\langle j\rangle}$, by IV 4.10(2),

$$\mathrm{tp}(\bar a, M_{\eta\restriction(k-l)}) \vdash \mathrm{tp}(\bar a, M_{\eta\restriction(k-l-1)}),$$

so also the latter does not fork over N_η. For $l = k$ we get a contradiction to $\bar a \in M = M_{\langle\ \rangle}$, as $\mathrm{tp}(\bar a, M_{\langle\ \rangle})$ does not fork over N_η, and $\bar a \notin N_\eta$.

Now we shall prove that M is $\boldsymbol{F}^a_{\aleph_0}$-prime, $\boldsymbol{F}^a_{\aleph_0}$-minimal over $\bigcup_{\eta\in I} N_\eta$. So suppose $M' \subseteq M$, $M' \neq M$, M' is $\boldsymbol{F}^a_\kappa$-prime over $\bigcup_{\eta\in I} N_\eta$, and choose $b \in M - M'$, $R[\mathrm{tp}(b, M'), L, \infty]$ minimal, hence $\mathrm{tp}(b, M')$ is regular. Now $\mathrm{tp}(b, M')$ is orthogonal to each N_η. If not, choose η with minimal length, then by 1.4 for some regular $q \in S^m(N_\eta)$, $\mathrm{tp}(b, M')$, q are not orthogonal and $q = p_{\eta^\wedge\langle i\rangle}$ for some i, or q is orthogonal to every $p_{\eta^\wedge\langle i\rangle}$. Let $q' \in S^m(M')$ be the stationarization of q over M', so by V 1.12 there is $\bar a' \in M$ which realizes q', so $\mathrm{tp}(\bar a', M')$ does not fork over N_η. By choice of η (being of minimal length) (if $\eta \neq \langle\ \rangle$) $\mathrm{tp}(b, M')$ is orthogonal to $\bigcup_{k<l(\eta)} N_{\eta\restriction k}$, hence by V 1.13 also $\mathrm{tp}(\bar a', M')$ is, so we get a contradiction to (12). We conclude $\mathrm{tp}(b, M')$ is really orthogonal to each N_η.

Choose a finite $B \subseteq M'$ over which $\mathrm{tp}(b, M')$ does not fork, and a finite $I^* \subseteq I$ closed under initial segments such that $\mathrm{tp}_*(B, \bigcup_{\eta\in I} N_\eta)$ does not fork over $\bigcup_{\eta\in I^*} N_\eta$. As M' is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_{\eta\in I} N_\eta$, $\mathrm{tp}_*(B, \bigcup_{\eta\in I} N_\eta)$ is $\boldsymbol{F}^a_{\aleph_0}$-isolated, so $\mathrm{tp}_*(B, \bigcup_{\eta\in I^*} N_\eta)$ is $\boldsymbol{F}^a_{\aleph_0}$-isolated (see IV 4.3). Let $N < M'$ be $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup\{N_\eta : \eta \in I^*\}$ and $B \subseteq N$. We have found an $\boldsymbol{F}^a_{\aleph_0}$-prime model N over

$\bigcup_{\eta\in I^*} N_\eta$, so that there is a regular type in $S^m(N)$, orthogonal to each N_η ($\eta \in I^*$), realized in M, and I^* is finite, closed under initial segments. We get a contradiction to the statement in the last sentence by induction on $|I^*|$. If $I^* = \{\eta \restriction l : l < n\}$ this is trivial, and if not choose distinct η, ν in some $I^* \cap I_n$ with n minimal. Let $N^0 = N_{\eta\restriction(n-1)}$, $N' < N$ be $F^a_{\aleph_0}$-prime over $\bigcup\{N_\sigma : \eta \trianglelefteq \sigma \in I^*\}$ and $N^2 \subseteq N$ be $F^a_{\aleph_0}$-prime over $\bigcup\{N_\sigma$: not $\eta \trianglelefteq \sigma$, but $\sigma \in I^*\}$ such that N is $F^a_{\aleph_0}$-prime over $N^1 \cup N^2$. Now by 2.3 (and as T does not have the dop) for some $l \in \{1,2\}$, there is a regular complete type over N^l orthogonal to N_0, realized in M, and we get a contradiction to the induction hypothesis on $|I^*|$.

In fact our proofs also prove:

3.3. Lemma. *Suppose $I \subseteq {}^{\omega>}\lambda$ is closed under initial segments, $\{N_\eta : \eta \in I\}$ is a non-forking tree of F^a_κ-saturated models* [*i.e.*, $\eta \triangleleft \nu \Rightarrow N_\eta < N_\nu$, $\eta \in I - \{\langle\ \rangle\} \Rightarrow$ tp$(N_\eta, \bigcup\{N_\nu : \nu \in I$, *not* $\eta \trianglelefteq \nu\})$ *does not fork over* $N_{\eta\restriction(l(\eta)-1)}$, *and each* N_η *is* $F^a_{\aleph_0}$-*saturated*].

If T does not have the dop, M F^a_κ-prime over $\bigcup_{\eta\in I} N_\eta$, *then M is F^a_κ-minimal over* $\bigcup_{\eta\in I} N_\eta$ *and every $q \in S^m(M)$ is not orthogonal to some N_η.*

Proof. First note:

3.3A. Fact. If S_1, S_2 are non-empty subsets of I which are downward closed, then $\{\bigcup_{\nu\in S_1} N_\nu, \bigcup_{\nu\in S_2} N_\nu\}$ is independent over $\bigcup_{\nu\in S_1\cap S_2} N_\nu$.

Proof. By the local properties of forking, it suffices to restrict ourselves first to the case $S_1 - S_2$ finite, then to the case $S_1 - S_2$, $S_2 - S_1$ finite, and at last S_1, S_2 finite (using III 0.1).

Now we prove the statement by induction on $|S_1 \cup S_2|$; w.l.o.g. $S_1 \neq S_1 \cap S_2$, $S_2 \neq S_1 \cap S_2$, clearly $\langle\ \rangle \in S_1 \cap S_2$.

So choose $\eta \in S_1 - S_2$ of maximal length, and by the induction hypothesis and transitivity of forking (see III 0.1) it is enough that tp$_*(N_\eta, \bigcup\{N_\nu : \nu \in S_1 \cup S_2, \nu \neq \eta\})$ does not fork over $N_{\eta\restriction(l(\eta)-1)}$, but this follows from the hypothesis. Now return to 3.3.

If one of the conclusions fails we can reduce it to the case $I = I^*$ is finite; then, as in 2.2, the two conclusions are equivalent. If both fail, there is $M' < M$, M' $F^a_{\aleph_0}$-prime over $\bigcup_{\eta\in I^*} N_\eta$, $b \in M - M'$, tp(b, M') orthogonal to every N_η; and continue as in the last part of the proof of 3.2: from the choice of B.

§4. Deepness

(Hyp) In this section T is superstable without the dop.

As we know that the number of non-isomorphic $F^a_{\aleph_0}$-saturated models in every $\aleph_\alpha \geqq |T| + \aleph_1$ for unsuperstable T and for T with the dop, we concentrate on the case in the hypothesis. By the decomposition Lemma 3.2 we know every $F^a_{\aleph_0}$-saturated model is $F^a_{\aleph_0}$-prime over a non-forking tree of F^a_κ-prime models. Clearly, if there are few trees then there are few models, but the converse is less clear. Anyhow, clearly the most important distinction is whether the tree (I in 3.2) is always well-founded. If it is always well-founded, naturally some rank is defined (called here the *depth*), and if the rank is small the number of models is small.

In this section we introduce the basic relevant notions and the simple facts about them. Notice that we could have chosen some other variants of the notions, but by later sections we will show that they would be equivalent.

We define the depth so that the results on the number of models can be stated smoothly (this is why 4.1(iii), 4.2(1) are such that the depth is not a limit ordinal).

Let us make some more specific remarks. Note that if the tree is not well-founded there are $N_l < N_{l+1}$ N_{l+1} $F^a_{\aleph_0}$-prime over $N_l \cup \{a_l\}$, $\mathrm{tp}(a_l, N_l)$ regular orthogonal to N_{l-1}. So the rank is defined as an attempt to build such a sequence (i.e., the rank of (N, N', a) is ∞ iff there is such a sequence, $N = N_0, N' = N_1, a = a_0$).

In Definitions 4.1 and 4.3 we give some variants of this, in 4.2 we define the relevant property of a theory (deepness), in 4.4 we prove various facts, and in 4.5 the essential equivalence of some variants is given. Now 4.6 says that, looking for high depth, it is enough to look at types not orthogonal to $\varnothing$. For "canonical" examples see 4.9, 4.10.

4.1. Definition. Let $K' = \{(N, N', a) : \mathrm{tp}(a, N)$ is regular, and N' is $F^a_{\aleph_0}$-prime over $N \cup \{a\}\}$.

For every member of K' we define its depth, an ordinal (zero or successor but not limit) or infinity ∞, by:

(i) $\mathrm{Dp}(N, N', \bar a) \geqq 0$ iff $(N, N', \bar a) \in K'$;

(ii) $\mathrm{Dp}(N, N', \bar a) \geqq \alpha + 1$ (α zero or successor) *iff* for some $N'', \bar a'$: $(N', N'', \bar a') \in K'$, $N' <_N N''$ and $\mathrm{Dp}(N', N'', \bar a') \geqq \alpha$;

(iii) $\mathrm{Dp}(N, N', \bar a) \geqq \delta + 1$ (δ limit) *iff* $\mathrm{Dp}(N, N', \bar a) \geqq \beta$ for $\beta < \delta$;

(iv) $\mathrm{Dp}(N, N', \bar a) = \infty$ iff for every ordinal β $\mathrm{Dp}(N, N', \bar a) \geqq \beta$, $\mathrm{Dp}(N, N', \bar a) = \alpha$ iff $\mathrm{Dp}(N, N', \bar a) \geqq \alpha$ but not $\mathrm{Dp}(N, N', \bar a) \geqq \alpha + 1$.

4.2. Definition. (1) The depth of the theory $\mathrm{Dp}(T)$ is $\bigcup\{\mathrm{Dp}(N, N', a) : (N, N', a) \in K'\} + 1$.

(2) The theory T is *deep* if its depth is ∞; otherwise it is *shallow*.

4.3. DEFINITION. (1) $K_\lambda = \{(N,N',\bar a): N,N'$ are F^a_λ-saturated, $\bar a \in N'$, $\bar a \notin N$, N' F^a_λ-atomic over $N \cup \{\bar a\}\}$. $\mathrm{Dp}((N,N',\bar a),K)$ is defined as in 4.1 for any set K of triples. If $(N,N',\bar a) \notin K$ we interpret $\mathrm{Dp}((N,N',\bar a),K)$ as $\mathrm{Dp}(N,N',\bar a), K \cup \{(N,N',\bar a)\})$ (not closing under isomorphism) and if $K = K'$ we omit it.

Let $K'_\lambda = \{(N,N',\bar a) \in K_\lambda : \mathrm{tp}(\bar a, N)$ is regular$\}$.

(2) For a tree I, we define $\mathrm{Dp}_\lambda(\eta, I)$ (λ a cardinal, $\eta \in I$): $\mathrm{Dp}_\lambda(\eta,I) \geqq \alpha+1$ iff for λ ν's, $\eta \lessdot \nu$, $\mathrm{Dp}(\nu,I) \geqq \alpha$, and for $\alpha = 0$ or limit $\mathrm{Dp}_\lambda(\eta,I) \geqq \alpha$ iff $\mathrm{Dp}_\lambda(\eta,I) \geqq \beta$ for every $\beta < \lambda$. So $\mathrm{Dp}_\lambda(\eta,I) = \alpha$ if it is $\geqq \alpha$ but not $\geqq \alpha+1$. Let $\mathrm{Dp}_\lambda(I) = \sup\{\mathrm{Dp}_\lambda(\eta,I) : \eta \in I\}$. For $\lambda = 1$ we omit λ.

(3) $\mathrm{Dp}(T,K)$ is defined as in 4.2.

4.4. LEMMA. (1) *If* $(N,N',\bar a)$, $(N',N'',\bar a') \in K$, $N' <_N N''$ then $\mathrm{Dp}((N,N',\bar a),K) \geqq \mathrm{Dp}((N',N'',\bar a'),K)$ (*the inequality is strict except when both are* ∞) [*this holds for any class* K *of triples*].

(2) *If* $\alpha \leqq \mathrm{Dp}((N,N',\bar a),K) < \infty$, $(N,N',a) \in K$, α *not limit, then some* $(N_0,N'_0,\bar a_0) \in K$ *has depth* α (*in* K).

(3) $\mathrm{Dp}(N,N',\bar a) = \infty$ *iff there are* N_l, $\bar a_l$ ($l < \omega$), $N_{l+1} <_{N_l} N_{l+2}$, $N_0 = N$, $N_1 = N'$, $\bar a_0 = \bar a$, $(N_{l+1}, N_{l+2}, \bar a_l) \in K'$. (*Similarly for any "reasonable"* K.)

(4) *If* $\alpha = \mathrm{Dp}(T)$ *or* $\alpha = \mathrm{Dp}(N,N',\bar a)$, $\alpha < \infty$, *then* $\alpha < (2^{|T|})^+$ *and in fact* $\alpha < \delta(|T|)$ (*see below what is* $\delta(|T|)$).

(5) *The depth is preserved by automorphisms of* $\mathfrak{C}$.

(6) *If* $(N_l,N'_l,\bar a_l) \in K'_{\aleph_0}$ ($l = 0,1$) *and* $\mathrm{tp}(\bar a_l,N_l)$ *are parallel or not orthogonal, then* $\mathrm{Dp}(N_0,N'_0,\bar a_0) = \mathrm{Dp}(N_1,N'_1,\bar a_1)$. *If only* $(N_1,N'_1,\bar a_1) \in K'_{\aleph_0}$, $\mathrm{tp}(\bar a_1,N_1)$ *regular, still* $\mathrm{Dp}(N_0,N',\bar a_0) \leqq \mathrm{Dp}(N_1,N'_1,\bar a_1)$.

REMARK. Remember that $\beth_{\delta(\lambda)}$ is the Hanf number of omitting types for theories of power λ, i.e., it is the first cardinal μ such that if T is a (first-order) theory of cardinality λ, p a type of cardinality λ, and T has a model omitting p in every $\chi < \mu$, *then* T has a model omitting p is every cardinality $\geqq |T|$.

This is well investigated, e.g., $\delta(\aleph_0) = \aleph_1$, $\lambda^+ \leqq \delta(\lambda) < (2^\lambda)^+$, $\mathrm{cf}\,\lambda > \aleph_0 \Rightarrow \delta(\lambda) > \lambda^+$; for proofs and references see, e.g., VII §5.

The following lemma shows that there is no real difference between the various $\mathrm{Dp}(-,K)$'s, in particular, whether we use a or $\bar a$.

4.4A. DEFINITION. We let, for regular p, $\mathrm{Dp}(p)$ be the depth of $(N,N',\bar a) \in K'_{\aleph_0}$ when $\mathrm{tp}(\bar a,N)$, p *are parallel.*

PROOF OF LEMMA 4.4. Easy. Note for (3) we need (6).

(1) Trivial.

(2) We prove it by induction on $\beta = \mathrm{Dp}(N,N',\bar a)$; for $\beta=\alpha$ there is nothing to prove; for $\beta>\alpha$, use 4.1 (ii) applied to $\mathrm{Dp}(N,N',\bar a)\geqq\beta+1$, to get N'', a' with $\gamma=\mathrm{Dp}(N',N'',\bar a')\geqq\alpha$, but by 4.4(1), $\gamma<\beta$, and use the induction hypothesis.

(4) By (6), $\mathrm{Dp}(N,N',\bar a)$ depends only on $\mathrm{tp}(\bar a,N)$ up to parallelism, and by (5) it is preserved by automorphisms of $\mathfrak{C}$, hence there are $\leqq 2^{|T|}$ possible depths. But by (2) the ordinals which are the depth of some triple form an initial segment of the set of the non-limit ordinals, hence

$$\alpha=\mathrm{Dp}(N,N',\bar a)<\infty \Rightarrow \alpha<(2^{|T|})^+.$$

The first phrase has been proved above. We work as in [4] VII §5. For the second phrase, let $\mathfrak{B}=(H(\lambda),\in,T,\delta(|T|))$, $H(\lambda)$ — the family of sets of hereditary cardinality $<\lambda$, and w.l.o.g. $T\subseteq|T|^+$. Now consider $\mathfrak{B}'$ elementarily equivalent to $\mathfrak{B}$, with "T and $|T|$" standard: but non-well-ordered "ordinals" $<\delta(T)$. ($\mathfrak{B}'$ is known to exist.)

In this model we can consider various notions and check whether they are absolute, i.e., whether if $\mathfrak{B}'$ says something holds it really holds. Now this holds for

(a) being a model of T,

(b) $R(p,\Delta,\lambda)=n$ (Δ finite, $\lambda\leqq\aleph_0$),

(c) being orthogonal types,

(d) non-orthogonal types,

(e) $\mathrm{tp}_*(A,B)$ does not fork over C, and

(f) $A\subseteq B$, $p\in S^m(B)$; p has a unique extension in $S^m(N\cup B)$ when $\mathrm{tp}_*(N,B)$ does not fork over A.

We can conclude by 4.5 that also if $\mathfrak{B}'\models$"$\mathrm{Dp}(p)\geqq a^*$, a^* an ordinal" then p has depth $\geqq$ the order type of $\{a\in\mathfrak{B}'': a<a^*\}$. If $a^*\geqq\delta(|T|)$ this is not well-founded so $\mathrm{Dp}(p)=\infty$.

(5) Trivial.

(6) We prove by induction on γ that [$\mathrm{Dp}(N_0,N_0',\bar a_0)\leqq\gamma$ or $\mathrm{Dp}(N_1,N_1',\bar a_0)\leqq\gamma$] implies the equality. We can choose $\boldsymbol{F}^a_\kappa$-saturated N_2, $N_0\cup N_1\subseteq N_2$, and by 4.4(5) w.l.o.g. $\mathrm{tp}(N_l',N_2)$ does not fork over N_l (for $l=0,1$). By V 3.2, N_0' is $\boldsymbol{F}^a_\kappa$-constructible over $N_2\cup\bar a_0$, and let N_2' be $\boldsymbol{F}^a_\kappa$-primary over $N_2\cup N_0'=N_2\cup N_0'\cup\bar a_0$. In N_2' $\mathrm{tp}(\bar a_1,N_2)$ is realized, so w.l.o.g. $\bar a_1\in N_2'$, $N_1'\subseteq N_2'$, N_2' $\boldsymbol{F}^a_\kappa$-prime over $N_2\cup\bar a_1$ and over $N_2\cup N_1'$ (and over $N_2\cup\bar a_0$ and over $N_2\cup N_0'$).

By symmetry it is enough to prove $\mathrm{Dp}(N_0,N_0',\bar a_0)=\mathrm{Dp}(N_2,N_2',\bar a_0)$, because checking the definition we can observe that $\mathrm{Dp}(N_2,N_2',\bar a_0)=\mathrm{Dp}(N_2,N_2',\bar a_1)$. Now the inequality $\leqq$ is trivial (with the induction hypothesis for γ) and for the inequality $\geqq$ we have to replace parameters with others of the same type. For proving the second phrase act as before, observing that by the first phrase $\mathrm{Dp}(N_1, N_1', \bar{a}_1) = \mathrm{Dp}(N_2, N_2', \bar{a}_0)$.

(3) First suppose that there are N_l, $\bar{a}_l$, and let $\alpha_l = \mathrm{Dp}(N_l, N_{l+1}, \bar{a}_l)$. By 4.4(1), $\alpha_l \geqq \alpha_{l+1}$, and if $\alpha_l \neq \infty$, $\alpha_l > \alpha_{l+1}$. So if α_0 is $\neq \infty$, then α_l $(l < \omega)$ is a strictly decreasing sequence of ordinals, contradiction; so $\alpha_0 = \infty$, but $\alpha_0 = \mathrm{Dp}(N_0, N_1, \bar{a}_0) = \mathrm{Dp}(N, N', \bar{a})$, so we have proved the "if" part of 4.4(3).

Now suppose $\mathrm{Dp}(N, N', \bar{a}) = \infty$. We define by induction on l, N_{l+1}, $\bar{a}_l$ such that N_{l+1} is $\boldsymbol{F}^a_\kappa$-saturated, $\mathrm{Dp}(N_l, N_{l+1}, \bar{a}_l) = \infty$, $N_l <_{N_{l-1}} N_{l+1}$ (when $l > 0$). So let $N_0 = N$, $N_1 = N'$, $\bar{a}_0 = \bar{a}$. So the induction hypothesis holds. For $l+1$ as $\mathrm{Dp}(N_l, N_{l+1}, \bar{a}_l) = \infty$, there are N_{l+2}, $\bar{a}_{l+1} = \langle a_{l+2} \rangle$, such that $(N_{l+1}, N_{l+2}, a_{l+2}) \in K^r$, $\mathrm{Dp}(N_{l+1}, N_{l+2}, a_{l+1}) \geqq (2^{|T|})^+$, hence by 4.4(4), $\mathrm{Dp}(N_{l+1}, N_{l+2}, a_{l+2}) = \infty$. As we can carry the induction we have proved the "only if" part of 4.4(3).

4.5. Lemma. (1) *For any* $(N, N', \bar{a}) \in K_\lambda$

$$\mathrm{Dp}((N, N', \bar{a}), K_\lambda) = \mathrm{Dp}(N, N', \bar{a}).$$

(2) *If* $K_1^r = \{(N, N', a) \in K^r_{\aleph_0} : N$ *is* $\boldsymbol{F}^a_{\aleph_0}$*-atomic over* $N \cup \{a\}\}$, *then on* K_1^r, $\mathrm{Dp}(-, K^r)$, $\mathrm{Dp}(-, K_1^r)$ *are equal.*

Proof. (1) Remember that by III 4.22, if M is $\boldsymbol{F}^a_\lambda$-saturated, M' $\boldsymbol{F}^a_\lambda$-prime over $M \cup \bar{a}$, $\kappa < \lambda$ then M' is $\boldsymbol{F}^a_{\aleph_0}$-prime over $M \cup \bar{a}$. So trivially by the definitions and 4.4(6) $\mathrm{Dp}((N, N', \bar{a}), K_\lambda) \geqq \mathrm{Dp}(N, N', \bar{a})$. So it suffices to prove by induction on α that:

$$(*) \quad \mathrm{Dp}((N, N', \bar{a}), K_\lambda) \geqq \alpha \Rightarrow \mathrm{Dp}(N, N', \bar{a}) \geqq \alpha \qquad \text{for } (N, N', \bar{a}) \in K_\lambda.$$

For $\alpha = 0$, α limit and successor of limit this is trivial; for $\alpha = \beta + 1, \beta$ not limit, there is $(N', N'', \bar{a}') \in K_\lambda$, $N' <_N N''$, and $\mathrm{Dp}((N', N'', \bar{a}'), K_\lambda) \geqq \beta$, hence by the induction hypothesis $\mathrm{Dp}(N', N'', \bar{a}') \geqq \beta$. Apply Lemma 3.1 for N', N'' standing for N_1, M and get $a_i, N_{2,i}, M_i$ $(i < i(0))$. By V Def. 3.2, Th. 3.2, w.l.o.g. $N_{2,i} = M_i$ and $i(0)$ is finite $(= w(\bar{a}', N))$, and let $\gamma = \mathrm{Max}_{i < i(0)} \mathrm{Dp}(N', N_{2,i}, a_i)$ (so γ is not limit); clearly it suffices to prove $\gamma \geqq \beta$. Otherwise, as γ is not limit there is $(N'', N^*, \bar{a}^*) \in K^r$, $N'' <_N N^*$, $\mathrm{Dp}(N'', N^*, \bar{a}^*) \geqq \gamma$. As $\mathrm{tp}(\bar{a}^*, N'')$ is regular, orthogonal to N', clearly as in the proof of 3.1 (see Fact E) for some $i < i(0)$ and regular $q \in S^1(N_{2,i})$, $\mathrm{tp}(\bar{a}^*, N'')$ and q are not orthogonal. Let c realize q, $N_{2,i}'$ be $\boldsymbol{F}^a_{\aleph_0}$-prime over $N_{2,i} \cup \{c\}$. So by 4.4(6)

$$\text{(a)} \qquad \mathrm{Dp}(N'', N^*, \bar{a}^*) = \mathrm{Dp}(N_{2,i}, N_{2,i}', c).$$

Now as q is not orthogonal to $\mathrm{tp}(\bar{a}^*, N'')$, it is orthogonal to any regular complete type over N' (as non-orthogonality is an equivalence relation among regular types, see V 1.13) hence q is orthogonal to N'. Hence $N_{2,i} <_{N'} N'_{2,i}$, hence (by 4.4(1))

(b) $$\mathrm{Dp}(N', N_{2,i}, a_i) > \mathrm{Dp}(N_{2,i}, N'_{2,i}, \bar{c}).$$

As $\mathrm{Dp}(N'', N^*, \bar{a}^*) \geqq \gamma$ by (a), (b), $\mathrm{Dp}(N, N_{2,i}, a_i) > \gamma$, contradicting γ's definition.

(2) Easy

4.6. Lemma. *If* $(N_0, N'_0, \bar{a}_0) \in K'_{\aleph_0}$, $\mathrm{tp}(\bar{a}_0, N_0)$ *is orthogonal to* $\emptyset$ *and has depth* $< \infty$, then *there is* $(N_1, N'_1, \bar{a}_1) \in K'_{\aleph_0}$ *such that* $\mathrm{tp}(\bar{a}_1, N_1)$ *is not orthogonal to* $\emptyset$ *and*

$$\mathrm{Dp}(N_0, N'_0, \bar{a}_0) < \mathrm{Dp}(N_1, N'_1, \bar{a}_1).$$

4.6A. Remark. So clearly for any complete type p, $\mathrm{Dp}(p) = \mathrm{Max}\{\mathrm{Dp}(r) : r$ a complete regular type not orthogonal to $p\}$.

Proof. By 4.4(6), w.l.o.g. N_0 and N'_0 are $\boldsymbol{F}^a_{\aleph_0}$-prime over $\emptyset$. Let $B \subseteq N_0$ be finite, such that $\mathrm{tp}(\bar{a}_0, N_0)$ does not fork over B. There is B' realizing the stationarization of $\mathrm{stp}_*(B, \emptyset)$ over N_0, and let M' be $\boldsymbol{F}^a_{\aleph_0}$-prime over $N_0 \cup B'$. By 1.7, M' is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\emptyset$. Hence by IV 4.18, N_0, M' are $\boldsymbol{F}^a_{\aleph_0}$-prime over B, B' resp., hence there is an isomorphism from M' onto N_0 taking B' to B. So there is a model $N < N_0$ such that $\mathrm{tp}_*(B, N)$ does not fork over $\emptyset$ and N_0 is $\boldsymbol{F}^a_{\aleph_0}$-prime over $N \cup B$. Hence by V 3.9 there is a finite set $\boldsymbol{J} = \{b_i : i < n\} \subseteq N_0$ independent over N, of elements realizing regular types, such that N_0 is $\boldsymbol{F}^a_{\aleph_0}$-prime over $N \cup \boldsymbol{J}$. By V 3.2 there are $N^*_m < N_0$ $\boldsymbol{F}^a_{\aleph_0}$-prime over $N \cup b_m$ such that N_0 is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_{l<n} N^*_l$. As $\mathrm{tp}(\bar{a}_0, N_0)$ is orthogonal to $\emptyset$, is parallel to $\mathrm{stp}(\bar{a}_0, B)$, and $\mathrm{tp}_*(B, N)$ does not fork over $\emptyset$, by 1.1 $\mathrm{tp}(\bar{a}_0, N_0)$ is orthogonal to N. On the other hand, by 3.3 $\mathrm{tp}(\bar{a}_0, N)$ is not orthogonal to some N^*_l, hence by 1.4 some regular $q \in S^m(N^*_l)$ is not orthogonal to $\mathrm{tp}(\bar{a}_0, N_0)$, so by V 1.13 it is orthogonal to N.

Clearly $\mathrm{Dp}(N, N^*_l, b_l) > \mathrm{Dp}(q) = \mathrm{Dp}(N_0, N_1, \bar{a}_0)$ (see 4.4(1), 4.4(6) resp.) and $\mathrm{tp}(b_l, N)$ is not orthogonal to $\emptyset$ as it is not orthogonal to $\mathrm{tp}_*(B, N)$ (because $b_l \in N_0$, N_0 $\boldsymbol{F}^a_{\aleph_0}$-prime over $N \cup B$, $\mathrm{tp}(B, N)$ does not fork over $\emptyset$). Now put $(N_1, N'_1, \bar{a}_1) \overset{\mathrm{def}}{=} (N, N^*_l, b_l)$.

4.7. Theorem. $I^a_{\aleph_0}(\aleph_\alpha, T)$ (*the number of non-isomorphic* $\boldsymbol{F}^a_{\aleph_0}$*-saturated models of power* $\aleph_\alpha$) *is at most* $\beth_{(-1)+(\mathrm{Dp}(T))}(|\alpha|^{2^{|T|}})$ *for shallow* T, *so it is* $< \beth_{\delta(|T|)}(|\alpha|^{2^{|T|}}) = \beth_{\delta(|T|)}(|\alpha|) < \beth_{(2^{|T|})^+}(|\alpha|)$ *and if* T *is countable* $< \beth_{\omega_1}(|\alpha|)$.

Proof. Immediate by 3.2, and the bounds on $\delta(|T|)$ (see e.g. VII 5.5 and 5.5(2)).

4.8. Example. A very natural example of a superstable T without the dop which is deep, is the following $T: T_{\mathrm{dp}} = \mathrm{Th}({}^{\omega>}\omega, f)$ when $f(\eta)$ is η if $\eta = \langle\ \rangle$, and $\eta \restriction n$ if $\eta \in {}^{n+1}\omega$.

Notice that a model of T_{dp} consist of trees, exactly one with a root (i.e., $f(x) = x$), in which every element has infinitely many immediate predecessors (i.e., y's such that $f(y) = x$). A similar example is $T^*_{\mathrm{dp}} = \mathrm{Th}({}^{\omega>}\omega, \cdots, P_n, f_n, \cdots)$ where $P_n = {}^n\omega$, f_n is a partial function: $f \restriction P_n$.

Both theories are $\aleph_0$-stable, and by expanding a little we can get elimination of quantifiers.

4.9. Examples. Examples of shallow theories can be obtained similarly to 4.8; we prefer to use the T^{α^*} from II p. 22: the language consists just of the two-place relations E_i $(i < \alpha)$. The axioms of T state: each E_i is an equivalence relation, for $i < j$, E_i refines E_j, moreover each E_j-equivalence class is the union of infinitely many distinct E_i-equivalence classes. Also E_i has infinitely many equivalence classes and each E_i-equivalence class is infinite. It is not hard to check that $\mathrm{Dp}(N, N', a) = i$ *iff* $i = \gamma$ when $\gamma < \omega$, $i = \gamma + 1$ otherwise, where $\gamma = \min\{j$: there is $b \in N$, $bE_ja\}$ (and $\gamma = \alpha$ if there is no such j).

4.10. Conclusion. For every ordinal β, which is a positive natural number or a successor ordinal, for some $T = T^*_\beta$, $|T| = |\beta|$, and for infinite α, $I^a_{\aleph_0}(\aleph_\alpha, T) = \beth_\beta(|\alpha|)$.

Proof. For $\beta < \omega$ or $\beta = \alpha + 2$ we use the previous example; for $\beta = \delta + 1$, δ limit, take the sum of models of T^{i^*} $(i < \delta)$ with disjoint languages. The computation is easy, but it is a worthwhile exercise for the reader as we are proving in the paper that every shallow theory T is in some sense similar to $T_{\mathrm{Dp}(T)}$. Note also (see [5] and 5.5(2)).

4.11. Theorem. (1) *If T is shallow, then $IE^a_{\aleph_0}(T) < \beth_{\delta(|T|)}$.*

(2) *If T is deep, κ_1 the first beautiful cardinal $> |T|$, κ_0 the first beautiful cardinal, then $\kappa_0^- \leqq IE^a_{\aleph_0}(T) \leqq \kappa_1^-$.*

Remark. So when $|T|$ is big ($=$ greater than the first beautiful cardinal) the following is not yet proved.

4.12. Conjecture. *$IE^a_{\aleph_0}(T)$ is $\leqq \beth_{\delta(|T|)}$ or is κ^- for some beautiful cardinal* T (superstable without the dop, of course).

§5. Deep theories have many non-isomorphic models and trivial types

(Hyp) In this section, T is superstable without the dop.

Clearly, if T is deep, we can construct trees like the one we get in 3.2, and try to prove that we get many models. The freedom we have is to determine various dimensions. So when $\alpha = \aleph_\alpha$ this is easy. Generally notice that we have much less freedom than, e.g., in 2.5 (when proving that the dop implies there are many models).

This section is dedicated to the proof of:

5.1. THEOREM. *If T is deep, $\lambda(T) \leqq \aleph_\alpha$, $\aleph_\beta < \aleph_\alpha$ then $I^a_{\aleph_\beta}(\aleph_\alpha, T) = 2^{\aleph_\alpha}$, i.e., T has $2^{\aleph_\alpha}$ non-isomorphic $\boldsymbol{F}^a_{\aleph_\beta}$-saturated models of cardinality $\aleph_\alpha$.*

REMARK. But the lemmas will be used for shallow theories. We shall concentrate on the case $\aleph_\alpha > \lambda(T)$, where $\lambda(T)$ is the first cardinal in which T is stable.

5.2. DEFINITION. We call $\langle N_\eta, a_\nu : \eta \in I, \nu \in I^+ \rangle$ a *representation if*:

(1) I is a tree with root $\langle\ \rangle$ of height $\leqq \omega$, and we let η^- be the unique predecessor of η for $\eta \in I - \{\langle\ \rangle\} = I^+$ and $I^- = \{\eta^- : \eta \in I^+\}$.

(2) $N_{\langle\ \rangle}$ is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\emptyset$.

(3) If $\eta = \nu_1^- = \nu_2^-$, then $p_{\nu_1} = \text{tp}(a_{\nu_1}, N_\eta)$ is regular, and p_{ν_1}, p_{ν_2} are equal or are orthogonal. If all p_ν with $\nu^- = \eta$ are equal, q_η will denote their common value.

(4) For $\eta \in I^+$, N_η is $\boldsymbol{F}^a_{\aleph_0}$-prime over $N_{(\eta^-)} \cup \{a_\eta\}$.

(5) For $\eta \in I^-$, $\{a_\nu : \nu^- = \eta\}$ is independent over N_η.

(6) If $\eta \in I$, η^- is defined then $\text{tp}(a_\eta, N_{\eta^-})$ is orthogonal to $N_{\eta^{--}}$.

REMARK. We shall write in short $\langle N_\eta, a_\eta : \eta \in I \rangle$, though we do not need $a_{\langle\ \rangle}$, so $a_{\langle\ \rangle}$ is any element of $N_{\langle\ \rangle}$ or undefined.

5.3. DEFINITION. We say $\langle N_\eta, a_\eta : \eta \in I \rangle$ is an $\boldsymbol{F}$-representation of M if it is a representation and M is $\boldsymbol{F}$-primary over $\bigcup_{\eta \in I} N_\eta$. If $\boldsymbol{F} = \boldsymbol{F}^a_{\aleph_0}$, we omit it.

5.4. LEMMA. *Let $\langle N_\eta, a_\eta : \eta \in I \rangle$ be a representation, then:*

(1) $\text{tp}(\bigcup\{N_\nu : \eta \trianglelefteq \nu\} \cup \{N_\nu : \textit{not}\ \eta \trianglelefteq \nu\})$ *does not fork over* $N_{(\eta^-)}$ *(for $\eta \in I^+$).*

(2) *For $\eta \in I^+$, $\text{tp}_*(\bigcup_{\nu \trianglerighteq \eta} N_\nu, N_\eta)$ is orthogonal to N_{η^-}.*

(3) *For $\eta, \nu \in I$, p_η, p_ν are orthogonal or equal (and then $\eta^- = \nu^-$).*

(4) *Each N_η is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\emptyset$.*

(5) *if M is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_{\eta \in I} N_\eta$, $p \in S^m(M)$ is regular, then p is not orthogonal to some $p' \in S^{m'}(N_\eta)$ for some η.*

(6) *$\langle N_\eta, a_\eta : \eta \in I \rangle$ represent some model of power $\lambda(I) + |I|$.*

(7) $\text{Dp}(N_{\eta^-}, N_\eta, a_{\eta^-})$ *is at least* $\text{Dp}(\eta, I)$ *(see 4.1).*

PROOF. As in the proof of 3.2, or easy using 3.3, 1.7. (1.7 is needed for (4).)

5.5. LEMMA. (1) *Every $F^a_{\aleph_0}$-saturated model has a representation.*

(2) *If $\langle N^l_\eta, a^l_\eta : \eta \in I_l \rangle$ $F^a_{\aleph_0}$-represents M_l, $l = 0,1$, $F : I_0 \to I_1$ an isomorphism (for partially ordered sets so it preserves the level), $F' : \bigcup_{\eta \in I_0} N^0_\eta \to \bigcup_{\eta \in I_1} N^1_\eta$ is an elementary mapping, it maps N^0_η onto $N^1_{F(\eta)}$ (equivalently, for each $\eta \in I_0$, $F' \restriction N^0_\eta$ is an elementary mapping onto $N^1_{F(\eta)}$,* Dom $F' = \bigcup_{\eta \in I_0} N^0_\eta$), then *$M_0$, M_1 are isomorphic.*

(3) *If in (2) F, F' are not necessarily onto, then M_0 can be elementarily embedded into M_1.*

PROOF. (1) This is by 3.2 (and 5.4).

(2),(3) Trivial.

5.6. LEMMA. *For $\beta > 0$, $\langle N_\eta, \bar a_\eta : \eta \in I \rangle$ $F^a_{\aleph_0}$-represent an $F^a_{\aleph_\beta}$-saturated model* iff *for every $\eta \in I$, and regular $p \in S^m(N_\eta)$ orthogonal to N_{η^-} for some $q \in S^m(N_\eta)$ which is not orthogonal to p and for at least $\aleph_\beta$ ν's, $p_\nu = q$.*

PROOF. Easy by 5.4(5), and usual arguments.

Let M be $F^a_{\aleph_0}$-prime over $\bigcup_\eta N_\eta$, and suppose $A \subseteq M$, $|A| < \aleph_\beta$, $p \in S(A)$ is omitted. Let tp(a, M) be a stationarization of p over M; clearly w.l.o.g. p is stationary. By V 3.9 there are $\{a_i : i < n\}$, independent over M, realizing over it regular types such that tp$(a, M \cup \{a_i : i < n\})$ is $F^a_{\aleph_0}$-isolated: w.l.o.g. tp$(a, A \cup \{a_i : i < n\})$ is isolated, and tp(a_i, M) does not fork over A and tp(a_i, A) is stationary. We now try to define by induction on i, $b_i \in M$ realizing stp$(a_i, A \cup \{b_j : j < i\})$. We have to fail for some i, so w.l.o.g. p is regular.

By 5.4(4) p is not orthogonal to some regular $q \in S(N_\eta)$, which is not orthogonal to some $p_{\eta^\frown\langle i\rangle}$, so we know p, $p_{\eta^\frown\langle i\rangle}$ are not orthogonal. By V 2.3, 2.4, dim$(p, M) = $ dim$(p_{\eta^\frown\langle i\rangle}, M)$, which is $\geqq \aleph_\beta$ by hypothesis, so we finish.

5.7. LEMMA. *If T is deep, then for every tree I with root $\langle\ \rangle$ and height $\leqq \omega$, there is a representation $\langle N_\eta, a_\eta : \eta \in I \rangle$ (in fact q_η is well defined for every $\eta \in I^-$, i.e., $p_{\eta^\frown\langle i\rangle} = q_\eta$).*

PROOF. Easy.

By 4.4(3) there are $F^a_{\aleph_0}$-saturated N_l, N_{l+1} $F^a_{\aleph_0}$-prime over $N_l \cup \{a_l\}$, tp(a_l, N_l) regular, $N_{l+1} <_{N_l} N_{l+2}$. Complete the partial ordering of I to a well ordering and let $\{\eta_i : i < i^*\}$ be a list of the members of I in increasing order. Let $n(i) = l(\eta_i)$ and $m(i)$ be maximal such that $\eta_i \restriction m(i) \in \{\eta : j < i\}$, in fact $\eta_i \restriction m(i) = \eta_{j(i)}$ (for $i > 0$). Now we define by induction on i an elementary mapping F_i. F_0 is the identity on N_0, F_i is an elementary mapping with domain $N_{m(i)}$, extending $F_{j(i)}$ such that tp$_*(F_i(N_{n(i)}), \bigcup_{j<i} F_j(N_{n(i)}))$ does not fork over $F_{j(i)}(N_{m(i)})$. Let $N_{\eta_i} = F_i(N_{\eta_i})$, $a_{\eta_i} = F_i(a_{n(i)})$, and the checking is easy.

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REMARK. The following lemma will be needed during the proof of 5.1, so if the lemma does not make sense to the reader, it is recommended that he skip it and return when we use it.

5.8. MAIN LEMMA. *Suppose M^l is represented by $\langle N^l_\eta, a^l_\eta : \eta \in I_l \rangle$ for $l = 0,1$ and $F: M^0 \to M^1$ is an elementary embedding. Suppose further that $\eta \in I_0$, and $S_\eta \subseteq I_1$, $|S_\eta| \leqq \lambda(T)$, is closed downward (i.e., $\sigma \lessdot \nu \in S_\eta \Rightarrow \sigma \in S_\eta$) and such that $\mathrm{tp}_*(F(N^0_\eta), \bigcup_{\nu \in I_1} N^1_\nu)$ does not fork over $\bigcup_{\nu \in S_\eta} N^1_\nu$. Suppose further that p is regular, $p \in S^m(N^0_\eta)$, $J = \{\nu \in I_0 : \nu^- = \eta$, $p_\nu = p$, and there are $> \lambda(T)$, $\sigma \in I_0$, $\sigma^- = \nu\}$ has power $> \lambda(T)$. Then:*

(1) *There is $\nu^* = \nu^*_\eta \in S_\eta$ and $q^*_\eta \in S^m(N^1_{\nu^*})$ (no connection to Definition 5.2(3)) such that $F(p)$, q^*_η are not orthogonal (and p orthogonal to N_σ when $\sigma \lessdot \nu^*_\eta$). Now by 5.4(3), q^*_η is the unique p^1_σ which is not orthogonal to $F(p)$, hence it does not depend on the particular choice of S_η (but only on $F \restriction N_\eta$).*

(2) *If for every $\sigma_0 \in I_0$, $|\{\sigma \in I_0 : \sigma^- = \sigma_0^-,\ p_\sigma = p_{\sigma_0}\}| \geqq \lambda > \lambda(T)$ and ν^*_η is from (1), then $\mathrm{Dp}_\lambda(\eta, I_0) \leqq \mathrm{Dp}_\lambda(\nu^*_\eta, I_1)$.*

PROOF. (1) By II 4.2 there is $J_0 \subseteq J$,

$$|J_0| \leqq \left| \bigcup_{\sigma \in S_\eta} N'_\sigma \right| \leqq |S_\eta| + \lambda(T) \leqq \lambda(T) + \lambda(T) = \lambda(T)$$

such that $\{F(a_\tau) : \tau \in J - J_0\}$ is an independent set based on $(F(N^0_\eta) \cup \bigcup_{\tau \in S_\eta} N^1_\tau \cup \{F(a_\tau) : \tau \in J_0\}, F(N^0_\eta))$.

Let $M^* \subseteq M^1$ $F^a_{\aleph_0}$-prime over $F(N^0_\eta) \cup \bigcup_{\tau \in S_\eta} N^1_\tau$, so that clearly $\mathrm{tp}_*(M^*, \bigcup_{\tau \in I_1} N^1_\tau)$ does not fork over $\bigcup_{\tau \in S_\eta} N^1_\tau$. So by the symmetry lemma $\mathrm{tp}_*(\bigcup_{\tau \in I_1} N^1_\tau, M^*)$ does not fork over $\bigcup_{\tau \in S_\eta} N^1_\tau$.

Now $F(p)$ is a type over $F(N^0_\eta) \subseteq M^*$, and let $r \in S^m(M^*)$ be the stationarization of it. Let $J^* = \{\nu \in I_1 : \nu^- \in S_\eta, \nu \notin S_\eta\}$. As for no $\nu \in J^*$ does ν^-, p^1_ν serve as ν^*_η, q^*_η of (1), clearly, for every $\nu \in J^*$, q_ν is orthogonal to r. By the definition of a representation, $\{a^1_\nu : \nu \in J^*\}$ is independent over $\bigcup_{\tau \in S_\eta} N_\tau$. As $\mathrm{tp}_*(\{a^1_\nu : \nu \in J^*\}, M^*) \subseteq \mathrm{tp}(\bigcup_{\tau \in I_1} N_\tau, M^*)$ does not fork over $\bigcup_{\tau \in S_\eta} N_\tau$, clearly $\{a^1_\nu : \nu \in J^*\}$ is independent over M^*, and $\mathrm{tp}(a_\nu, M^*)$ is the stationarization of p^1_ν for every $\nu \in J^*$. As $\mathrm{tp}(a^1_\nu, M^*)$ is orthogonal to r for every $\nu \in J^*$, also $\mathrm{tp}_*(\{a^1_\nu : \nu \in J^*\}, M^*)$ is orthogonal to r. Let $\bar{c}$ realize r. By V 3.2,

$$\mathrm{tp}_*\left(\bigcup_{\nu \in J^*} N^1_\nu, M \cup \{a^1_\nu : \nu \in J^*\}\right) \vdash \mathrm{tp}\left(\bigcup_{\nu \in J^*} N^1_\nu, M^* \cup \{a^1_\nu : \nu \in J^*\} \cup \bar{c}\right),$$

but by what we said before $\mathrm{tp}_*(\{a^1_\nu : \nu \in J^*\}, M^*) \vdash \mathrm{tp}_*(\{a^1_\nu : \nu \in J^*\}, M^* \cup \bar{c})$, hence $\mathrm{tp}_*(\bigcup_{\nu \in J^*} N^1_\nu, M^*) \vdash \mathrm{tp}(\bigcup_{\nu \in J^*} N^1_\nu, M^* \cup \bar{c})$ hence $\mathrm{tp}_*(\bigcup_{\nu \in J^*} N^1_\nu, M^*)$, r are weakly orthogonal, hence orthogonal. If $\nu^- \in J^*$, clearly $\{N^1_\nu, M^* \cup$

$\bigcup_{\tau\in J^*} N^1_\tau\}$ is independent over N_{ν^-}, hence by 1.1 p^1_ν is orthogonal to $M^* \cup \bigcup_{\tau\in J^*} N^1_\tau$. So we can easily continue and get, eventually, that r is orthogonal to $\mathrm{tp}_*(\bigcup_{\tau\in I_1} N_\tau, M^*)$.

Clearly w.l.o.g. M^1 is $\boldsymbol{F}^a_{\aleph_0}$-prime over $\bigcup_{\tau\in I_1} N_\tau \cup M^*$, hence r has a unique extension in $S^m(M^1)$ and $\dim(F(p), M^*) = \dim(F(p), M^1)$. However

$$\lambda(T) < |J| \leqq \dim(p, M^0) \leqq \dim(F(p), M^1) = \dim(F(p), M^*)$$

$$\leqq \|M^*\| \leqq \lambda(T) + \Big| F(N^0_\eta) \cup \bigcup_{\tau\in S_\eta} N^1_\tau \Big| \leqq \lambda(T), \qquad \text{contradiction.}$$

Checking the proof we see that $F(p)$ is not orthogonal to some p^1_ν, $\nu^- \in S_\eta$. So w.l.o.g. q^*_η is like that and we know there is at most one p^1_ν not orthogonal to $F(p)$ by 5.4(3).

Before we continue to prove 5.8(2), we prove a sublemma which will do most of the work.

5.9. Sublemma. *For every $\nu \in J$ (by 5.8's notation and assumptions) except $\leqq \lambda(T)$ of them, there is $\sigma_\nu \in \{\sigma \in I_1 : \sigma^- = \nu^*_\eta, p^1_\sigma = q^*_\eta\}$ such that:*

(a) *$\mathrm{tp}_*(F(N^0_\nu), \bigcup\{N^1_\tau : \text{not } \sigma_\nu \leq \tau\})$ does not fork over $\bigcup_{\tau\in S_\eta} N^1_\tau$.*

(b) *$\{F(\bar a^0_\nu), a^1_{\sigma_\nu}\}$ is not independent over $F(N^0_\eta) \cup \bigcup_{\tau\in S_\eta} N^1_\tau$.*

(c) *$\mathrm{tp}_*(F(\bigcup_{\tau\geqq\sigma_\nu} N^0_\tau), F(N^0_\nu) \cup \bigcup_{\tau\in S_\eta} N^1_\tau)$ and $\mathrm{tp}_*(\{N^1_\tau : \text{not } \sigma_\nu \leq \tau\}, F(N^0_\nu) \cup \bigcup_{\tau\in S_\eta} N^1_\tau)$ are weakly orthogonal.*

(d) *For every $\tau_0 \in I_0$, $\nu \lessdot \tau_0$, $F(p^0_{\tau_0})$ not orthogonal to $p^1_{\tau_1}$, implies $\sigma_\nu \lessdot \tau_1$.*

(e) *The mapping $\nu \mapsto \sigma_\nu$ is one-to-one.*

Proof of 5.9. Let $J_1 = \{\sigma \in I_1 : \sigma^- = \nu^*_\eta,\ p^1_\sigma = q^*_\eta\}$. Let $\nu \in J - J_0$. Now by V 3.1 there is $J^1_\nu \subseteq J_1$, $|J_1 - J^1_\nu| \leqq 1$ such that $\{\bar a^1_\sigma : \sigma \in J^1_\nu\} \cup \{F(\bar a^0_\nu)\}$ is independent over $F(N^0_\eta) \cup \bigcup_{\tau\in S_\eta} N^1_\tau$. Now it is easy to see that $\{F(N^0_\nu),\ \bigcup\{N^1_\tau : \tau \text{ is not } \trianglerighteq \text{ than a member of } J^1 - J^1_\nu\}\}$ is independent over $\bigcup_{\tau\in S_\eta} N^1_\tau$. If $J_1 = J^1_\nu$, by the definition of J we can get a contradiction as in the proof of (1). So let $J_1 - J^1_\nu = \{\sigma_\nu\}$.

Now (a) (of 5.9) has already been proved (as $J^1_\nu = J^1 - \{\sigma_\nu\}$) and (e) follows by V 3.1. Now (d) follows from (c) which is quite easy. Part (b) can be proved easily: by the definition of J, there is $\rho \in I_0$, $\rho^- = \nu$, $\dim\{p^0_\rho, M'\} > \lambda(T)$, hence $F(p^0_\rho)$ is not orthogonal to some $p^1_{\rho_1}$, clearly $\nu^*_\eta \leq \rho_1$, but if $\sigma \leq \rho_1$, $\sigma \in J_1$, $\{F(a^0_\nu), a^1_\sigma\}$ are independent over $F(N^0_\eta \cup \bigcup_{\tau\in S_\eta} N^1_\tau)$, then by the parallel of (c), $F(p^0_\rho)$, $p^1_{\rho_1}$ are orthogonal, so if (b) fails we get a contradiction. Alternatively use 5.10, 5.11.

Proof of 5.8(2). (2) We prove by induction on α that $\mathrm{Dp}_\lambda(\eta, I_0) \geqq \alpha$ implies $\mathrm{Dp}_\lambda(\nu^*_\eta, I_1) \geqq \alpha$. For $\alpha = 0$, α limit, this is trivial. For $\alpha = \beta + 1$, $\beta > 0$, use 5.9. In particular, for parts (d) and (e), and for $\alpha = 1$, look at the proof of 5.9.

The following lemma is useful in the proof of 5.1 and interesting by itself. The lemma holds for stable T without the dop.

5.10. LEMMA. *Suppose* $\mathrm{Dp}(N,N',\bar a)>0$, $p\in S^m(B)$ *stationary and parallel to* $\mathrm{tp}(\bar a,N)$ *(which is regular). If* $\boldsymbol{I}$ *is an independent set of sequences realizing* p, $\bar b$ *realizes* p *but* $\mathrm{tp}(\bar b,B\cup\bigcup\boldsymbol{I})$ *forks over* B, then *for some* $\bar c\in\boldsymbol{I}$, $\mathrm{tp}(\bar b,B\cup\bar c)$ *forks over* B. *(Such* p *will be called trivial.)*

PROOF. W.l.o.g. $B=|N|$, N $\boldsymbol{F}^a_{\aleph_0}$-saturated, $\boldsymbol{I}$ is finite and let $\boldsymbol{I}=\{\bar a_l : l<n\}$. Suppose there is no such $\bar c$, i.e. $\{\bar b,\bar a_l\}$ is independent over N for each l. Let M be $\boldsymbol{F}^a_{\aleph_0}$-prime over $N\cup\boldsymbol{I}\cup\bar b$. As $\boldsymbol{I}\cup\bar b$ is finite by Definition V 3.2, Theorem V 3.6(2), V 3.9(1) there is a finite set $\boldsymbol{J}\subseteq M$ independent over N, of sequences realizing regular types, such that M is $\boldsymbol{F}^a_{\aleph_0}$-prime over $N\cup\boldsymbol{J}$.

FACT. There is $\bar b^*\in\boldsymbol{I}\cup\{\bar b\}$ such that for no $\bar d\in\boldsymbol{J}$, $\mathrm{tp}(\bar b^*,N\cup\bar d)$ forks over N.

Otherwise there are $\bar a^0_l\in\boldsymbol{J}$, $\bar b^0\in\boldsymbol{J}$, such that each of the pairs $\{\bar a^0_l,\bar a_l\}$ $(l<n)$ and $\{\bar b^0,\bar b\}$ is not independent over N.

So for each l, $\mathrm{tp}(\bar a_l,N\cup\{\bar a^0_l : l<n\})$ forks over N and $\mathrm{tp}(\bar b,N\cup\{\bar a_l : l<n\})$ forks over N, so by V 1.14 $\mathrm{tp}(\bar b,N\cup\{a^0_l : l<n\})$ forks over N. By the choice of $\bar b^0$, $\mathrm{tp}(\bar b^0,N\cup\bar b)$ forks over N, hence by V 1.14 again $\mathrm{tp}(\bar b^0,N\cup\{\bar a^0_l : l<n\})$ forks over N. As $\boldsymbol{J}$ is independent $\bar b^0\in\{\bar a^0_l : l<n\}$, so let $\bar b^0=\bar a^0_l$. So, by our choice, $\mathrm{tp}(\bar b,N\cup\bar a^0_l)$ forks over N, and $\mathrm{tp}(\bar a^0_l,N\cup\bar a_l)$ forks over N, hence by V 1.14 $\mathrm{tp}(\bar b,N\cup\bar a_l)$ forks over N; contradiction to the assumption "$\boldsymbol{I},\bar b$ is a counterexample" to 5.10. So we have proved the fact.

Let $\boldsymbol{J}=\{\bar d_l : l<m_0\}$. As M is prime over $N\cup\boldsymbol{J}$, there are N_l $\boldsymbol{F}^a_{\aleph_0}$-prime over $N\cup\bar d_l$, such that N is $F^a_{\aleph_0}$-prime over $\bigcup_{l<m}N_l$ (see V 3.2). Let $N^*<M$ be $\boldsymbol{F}^a_{\aleph_0}$-prime over $N\cup\bar b^*$. As $\bar b^*$ realizes p (as all members of $\boldsymbol{I}\cup\{\bar b\}$ do), by 4.4(6) $D(N,N^*,\bar b^*)>0$, hence there is a regular $q^*\in S^m(N^*)$ orthogonal to N, and let q be its stationarization over M. For each $l<m_0$, $\{\bar b^*,\bar d_l\}$ is independent over N (by the fact), hence by V 3.2 $\mathrm{tp}_*(N_l,N^*)$ does not fork over N. Hence by 1.1 q^* is orthogonal to N_l. Hence q is orthogonal to each N_l. But as T has the dimensional order property q is not orthogonal to some N_l. This contradiction proves the lemma.

PROOF OF THEOREM 5.1 FOR $\aleph_\alpha>\lambda(T)$. As T is a deep theory, by Lemma 5.7 (and 4.5) there is a representation $\langle N^0_\eta,a^0_\eta:\eta\in I^0\rangle$ such that $I^0={}^{\omega>}\aleph_\alpha$, q_η is defined. By adding more elements and models we can get a representation $\langle N_\eta,\bar a_\eta:\eta\in I\rangle$ such that (the only role of $I-I^0$ is to make the models $\boldsymbol{F}^a_{\aleph_\beta}$-saturated):

(a) $I^0 \subseteq I$, $|I| = \aleph_\alpha$, and every element has $\geqq \aleph_\beta$ immediate successors, and if $\eta \in I^0$ then $|\{\nu \in I : \eta \leqq \nu,\ \nu \restriction (l(\eta)+1) \notin I^0\}| \leqq \aleph_\beta + \lambda(T)$;

(b) for every $\eta \in I$, and regular $p \in S^m(N_\eta)$, $\{\sigma \in I - I^0 : p_\sigma$ is not orthogonal to $p\}$ has cardinality $\geqq \aleph_\beta$.

Now for each $\xi < \aleph_\alpha$, we can find a set $I_\xi \subseteq I$ such that:

(α) $\langle \xi \rangle, \langle\ \rangle \in I_\xi$, $I_\xi \subseteq I$ and is downward closed, and $\eta \in I_\xi$, $\eta \in I^0$, $\eta \neq \langle\ \rangle$, implies $\langle \xi \rangle \leq \eta$,

(β) for $\eta \in I - I^0$, $\eta \in I_\xi$ iff $\langle \xi \rangle \lhd \eta$, and for some m, $\eta \restriction m \in I_\xi \cap I^0$, $\eta \restriction (m+1) \notin I^0$,

(γ) $\mathrm{Dp}_{\aleph_\alpha}(\langle \xi \rangle, I_\xi) = \xi$,

(δ) if $\eta \in I_\xi \cap I^0$ then $\{\nu \in I_\xi \cap I^0 : \nu^- = \eta^-\}$ has power $\aleph_\alpha$.

For any $S \subseteq \aleph_\alpha$ let $I^S = \bigcup_{\xi \in S} I_\xi$, and M^S be the model represented by $\langle N_\eta, \bar{a}_\eta : \eta \in I^S \rangle$.

Suppose $q_{\langle\ \rangle}$ does not fork over $B \subseteq N_{\langle\ \rangle}$, $|B| < \aleph_0$, $q_{\langle\ \rangle} \restriction B$ is stationary. By VIII 1.2 w.l.o.g. all members of B are individual constants. Now it suffices to prove:

$$(*)\qquad \text{if } S, U \subseteq \aleph_\alpha, |S - U| = |U - S| = \aleph_\alpha \text{ then } (M^S, B), (M^U, B) \text{ are not isomorphic.}$$

This is because we can find $2^{\aleph_\alpha}$ subsets S_i of $\lambda = \aleph_\alpha$ such that for $i \neq j$, $|S_i - S_j| = |S_j - S_i| = \lambda$, so by 5.4(6) $\|M^{S_i}\| = \aleph_\alpha$, and by $(*)$, they are not isomorphic.

Proof of $(*)$. Suppose F is an isomorphism from M^S onto M^U. It is easy to find $S_1 \subseteq I^S$, $U_1 \subseteq I^U$, $|S_1| + |U_1| = \lambda(T) + \aleph_\beta$ such that

(i) $\langle\ \rangle \in U_1, S_1$,

(ii) $\mathrm{tp}_*(F(\bigcup_{\eta \in S_1} N_\eta), \bigcup_{\eta \in I^U} N_\eta)$ does not fork over $\bigcup_{\eta \in U_1} N_\eta$,

(iii) $\mathrm{tp}_*(\bigcup_{\eta \in U_1} N_\eta, F(\bigcup_{\eta \in I^S} N_\eta))$ does not fork over $F(\bigcup_{\eta \in S_1} N_\eta)$,

(iv) $\eta^- \in S_1$, $\eta \notin I^0$ implies $\eta \in S_1$,

(v) $\eta^- \in U_1$, $\eta \notin I^0$ implies $\eta \in U_1$.

Remember $q_{\langle\ \rangle} = \mathrm{tp}(\bar{a}_{\langle \xi \rangle}, N_{\langle\ \rangle})$ for any $\langle \xi \rangle \in I^0$. Now as F is the identity on B, $q_{\langle\ \rangle}$ is parallel to its image by F, $F(q_{\langle\ \rangle})$. Remember $q_{\langle\ \rangle}$ is regular.

Clearly $\{a_{\langle \xi \rangle} : \xi \in S\}$ is a maximal set $\subseteq M^S$, independent over $N_{\langle\ \rangle}$, of elements realizing $q_{\langle\ \rangle}$. By the choice of S_1, U_1 III 0.1 (see Def. III 4.4) $\{a_{\langle \xi \rangle} : \xi \in U, \langle \xi \rangle \notin U_1\}$ is an independent set over $(\bigcup_{\eta \in U_1} N_\eta \cup \bigcup_{\eta \in S_1} F(N_\eta), B)$. (Remember $q_{\langle\ \rangle}$ does not fork over B.) All of the $a_{\langle \xi \rangle}$ realize $q_{\langle\ \rangle} \restriction B$. Moreover, $\langle a_{\langle \xi \rangle} : \xi \in U, \langle \xi \rangle \notin U_1 \rangle$ is (in M^U) maximal independent over $(\bigcup_{\xi \in U_1} N_\xi \cup \bigcup_{\xi \in S_1} F(N_\xi), B)$ since $\langle a_{\langle \xi \rangle} : \xi \in U \rangle$ is maximal independent over $N_{\langle\ \rangle}$. Similarly

$\{F(a_{\langle\xi\rangle}): \xi \in S, \langle\xi\rangle \notin S_1\}$ is a maximal set independent over $(\bigcup_{\eta\in U_1} N_\eta \cup \bigcup_{\eta\in S_1} F(N_\eta), B)$ of elements of M^U realizing $q_{\langle\ \rangle}\restriction B$. As $q_{\langle\ \rangle}$ is regular, of depth >0, by 5.10 there is a one-to-one function h from $V_0 = \{\langle\xi\rangle : \xi \in S, \langle\xi\rangle \notin S_1\}$ onto $V_1 = \{\langle\xi\rangle : \xi \in U, \langle\xi\rangle \notin U_1\}$ such that the set $\{F(a_{\langle\xi\rangle}), a_{h(\langle\xi\rangle)}\}$ is not independent over $(\bigcup_{\eta\in U_1} N_\eta \cup \bigcup_{\eta\in S_1} F(N_\eta), B)$.

As $|U_1| + |S_1| \leqq \aleph_\beta + \lambda(T) < \aleph_\alpha$, and $|V - S| = |S - V| = \aleph_\alpha$ for some $\xi \in S, \langle\xi\rangle \notin S_1$, $\xi \geqq \omega$, $h(\langle\xi\rangle) \neq \langle\xi\rangle$, and let $h(\langle\xi\rangle) = \langle\zeta\rangle$. By symmetry (as we can use F^{-1}), $\xi > \zeta$. Now we apply the main lemma 5.8, and get a contradiction by part (2). Since $\mathrm{Dp}_{\aleph_\alpha}(\langle\zeta\rangle, I^S) = \zeta$ and $\mathrm{Dp}_{\aleph_\alpha}(\langle\zeta\rangle, I^U) = \zeta$, I^S and I^U are constructed so that the conditions in 5.8(1) and 5.8(2) are satisfied.

Proof of Theorem 5.1 for $\aleph_\alpha = \lambda(T)$. Let N_n, a_n be as in 4.4(3). Now as in 5.7 we can find for $\eta \in I^0 = {}^{\omega>}\aleph_\alpha$ an elementary mapping f_η such that $\langle f_\eta(N_{l(\eta)}), f_\eta(\bar a_{l(\eta)-1}) : \eta \in I_0\rangle$ is a representation. We can choose countable $A_n \subseteq N_n$ such that $a_n \in A_{n+1}$, $\mathrm{tp}_*(A_{n+1}, N_n)$ does not fork over A_n, $\mathrm{tp}_*(A_{n+1}, A_n)$ is stationary and $\mathrm{tp}_*(A_{n+1}, A_n \cup \{a_{n+1}\}) \vdash \mathrm{tp}(A_{n+1}, N_n \cup \{a_{n+1}\})$.

We let $A_\eta = f_\eta(A_{l(\eta)})$, let $I^S = \bigcup_{\xi\in S} I_\xi$ $(S \subseteq \lambda)$ (I_ξ is defined as $I_\xi \cap I^0$ for I_ξ as in the proof of 5.1 for $\aleph_\alpha > \lambda(T)$ above) and let M^S be $\mathbf{F}^a_{\aleph_\beta}$-prime over $\bigcup_{\eta\in I^S} A_\eta$. We have to prove the parallel of (*). Notice that if some regular type q over M^S has dimension $>\aleph_\beta$, then it is not orthogonal to some $\mathrm{tp}(a_\eta, A_\eta)$.

We leave the details to the reader.

5.11. Lemma. *Let T be stable.*

(1) *Suppose $r_l \in S^{m(l)}(A_l)$ for $l = 0,1$, r_0 parallel to r_1, r_0 is stationary, regular and trivial, then so is r_1.*

(2) *Suppose $r \in S^m(A)$ is a (stationary) regular trivial type and $\{\bar a, \bar b\}$ is independent over A. Then for any $\bar c$ realizing r, $\mathrm{tp}(\bar c, A \cup \bar a \cup \bar b)$ forks over A iff $\mathrm{tp}(\bar c, A \cup \bar a)$ forks over A or $\mathrm{tp}(\bar c, A \cup \bar b)$ forks over A.*

(3) *Suppose $r \in S^n(A)$ is stationary regular and trivial. For any $\bar a$ there are $\bar d_0, \cdots, \bar d_{n-1} \in r(\mathfrak{C})$ such that: $\{\bar d_0, \cdots, \bar d_{n-1}\}$ is independent over A, $\mathrm{tp}(\bar d_l, A \cup \bar a)$ forks over A for $l < n$, and $n = w_r(\bar a, A)$. So for any $\bar d \in r(\mathfrak{C})$, $\mathrm{tp}(\bar d, A \cup \bar a)$ forks over A iff $\mathrm{tp}(\bar d, A \cup \bar d_l)$ forks over A for some $l < n$.*

(4) *Suppose r_0, r_1 are stationary regular and not orthogonal. Then r_0 is trivial iff r_1 is trivial.*

(5) *Suppose $A \subseteq B$, $\mathrm{tp}(\bar a, B)$ does not fork over A, $r \in S^m(B)$ is stationary regular and trivial, not orthogonal to $\mathrm{stp}(\bar a, A)$. Then for some $e \in \mathrm{acl}(A \cup \bar a)$, $e \notin \mathrm{acl}\, A$, $\mathrm{stp}(e, A)$ is r-simple of weight 1. In fact there are $e_0, \cdots, e_{n-1}$ as above, independent over A, $\mathrm{tp}(\bar a, A \cup \{e_l : l < n\})$ orthogonal to r. If $\mathrm{stp}(\bar a, A)$ is semi-regular, $\mathrm{stp}(e, A)$ is regular.*

(6) *If for* $l = 0,1$, $p_l = \mathrm{stp}(\bar{a}_l, A)$ *is not orthogonal to* r, r *a stationary regular trivial type, then* p_0, p_1 *are not weakly orthogonal; in fact* p_1 *has an extension over* $A \cup \bar{a}_0$ *which forks over* A.

PROOF. (1) By the definition of parallel, r_1 is stationary and by V 1.8(1), r_1 is regular. Let $A_0 \cup A_1 \subseteq M$, M $\boldsymbol{F}^a_\lambda$-saturated, $\lambda > (A_0 \cup A_1)$ and r_2 be the stationarization of r_0 (and r_1) over M. Clearly r_2 is regular and stationary.

Suppose r_1 is not trivial, then there are $\bar{b}$, $\bar{a}_0, \cdots, \bar{a}_{n-1}$ realizing r_1, $\mathrm{tp}(\bar{b}, A_1 \cup \bar{a}_0 \cup \cdots \cup \bar{a}_{n-1})$ forks over A_1, but $\mathrm{tp}(\bar{b}, A_1 \cup \bar{a}_m)$ does not fork over A_1 for $m < n$. W.l.o.g. $\mathrm{tp}(\bar{b}{}^\wedge \bar{a}_0{}^\wedge \cdots {}^\wedge \bar{a}_{n-1}, M)$ does not fork over A_1, and then clearly $\bar{b}$, $\bar{a}_0, \cdots, \bar{a}_{n-1}$ exemplify r_2 is not trivial (use III 0.1). So $\mathrm{tp}(\bar{b}, M \cup \bar{a}_0 \cup \cdots \cup \bar{a}_{n-1})$ forks over M, hence over A_0, and $\bar{b}, \bar{a}_0, \cdots, \bar{a}_{n-1}$ realizes r_2 and $r_0 \subseteq r_2$. So by 1.11, clearly $\mathrm{tp}(\bar{b}, A_0 \cup \boldsymbol{I})$ forks over A_0 where $\boldsymbol{I} = r_0(M) \cup \{\bar{a}_0, \cdots, \bar{a}_{n-1}\}$. Obviously for every $\bar{c} \in M$, $\mathrm{tp}(\bar{b}, A_0 \cup \bar{c})$ does not fork over A_0.

So $\bar{b}$, $\boldsymbol{I}$ exemplify r_0 is not trivial (except that $\boldsymbol{I}$ is not independent, but this can be discarded by r_0's regularity); contradiction, hence r_1 is trivial as required.

(2) The implication $\Leftarrow$ is trivial. So suppose $\bar{c}$ is a counterexample to the other direction. Let M be an $\boldsymbol{F}^a_\kappa$-saturated model, $A \subseteq M$, $\mathrm{tp}_*(M, A \cup \bar{a} \cup \bar{b} \cup \bar{c})$ does not fork over A. By some application of III 0.1 clearly $\{\bar{a}, \bar{b}\}$ is independent over M, $\mathrm{tp}(\bar{c}, M \cup \bar{a})$ and $\mathrm{tp}(\bar{c}, M \cup \bar{b})$ do not fork over M, $\mathrm{tp}(\bar{c}, M \cup \bar{a} \cup \bar{b})$ forks over M, and $\mathrm{tp}(\bar{c}, M)$ is a stationarization of r, hence by (1) is a stationary regular trivial type. So w.l.o.g. $M = B$.

Let $\{\bar{d}_m : m < n^0\}$ be a maximal set of sequences realizing $\mathrm{tp}(\bar{c}, M)$ independent over M such that $\mathrm{tp}(\bar{d}_m, M \cup \bar{a})$ forks over M, and similarly let $\{\bar{e}_m : m < n^1\}$ be a maximal set of sequences realizing $\mathrm{tp}(\bar{c}, M)$, independent over M, such that $\mathrm{tp}(\bar{e}_m, M \cup \bar{b})$ forks over M. By V 3.9(1) $n^0 = w_r(\bar{a}, M)$, $n^1 = w_r(\bar{b}, M)$ and w.l.o.g. $\mathrm{tp}(\bar{d}_0{}^\wedge \cdots {}^\wedge \bar{d}_{n_0-1}, M \cup \bar{a})$ and $\mathrm{tp}(\bar{e}_0{}^\wedge \cdots {}^\wedge \bar{e}_{n_1-1}, M \cup \bar{b})$ are $\boldsymbol{F}^a_\kappa$-isolated. So clearly $\{\bar{d}_0, \cdots, \bar{d}_{n_0-1}, \bar{e}_0, \cdots, \bar{e}_{n_1-1}\}$ is independent over M and $\mathrm{tp}(\bar{d}_0{}^\wedge \cdots {}^\wedge \bar{d}_{n_0-1}{}^\wedge \bar{e}_0{}^\wedge \cdots {}^\wedge \bar{e}_{n_1-1}, M \cup \bar{a} \cup \bar{b})$ is $\boldsymbol{F}^a_\kappa$-isolated.

So let N be $\boldsymbol{F}^a_\kappa$-prime over $M \cup \bar{a} \cup \bar{b}$, $\{\bar{d}_0, \cdots, \bar{d}_{n_0-1}, \bar{e}_0, \cdots, \bar{e}_{n_1-1}\} \subseteq N$. By V 3.11(1) $w_r(\bar{a}{}^\wedge \bar{b}, M) = w_r(\bar{a}, M) + w_r(\bar{b}, M)$, and $\{\bar{d}_0, \cdots, \bar{d}_{n_0-1}, \bar{e}_0, \cdots, \bar{e}_{n-1}\}$ is a maximal subset of $r(N)$ independent over M.

Now what about $\bar{c}$? $\bar{c}$ realizes $r \in S^m(M)$, $\bar{c}$ depends on $\bar{a}{}^\wedge \bar{b}$ (i.e., $\mathrm{tp}(\bar{c}, M \cup \bar{a} \cup \bar{b})$ forks over M), hence by V 1.16(1) $\mathrm{tp}(\bar{c}, M \cup r(N))$ forks over M, hence $\mathrm{tp}(\bar{c}, M \cup \{\bar{d}_0 \cup \cdots \cup \bar{d}_{n_0-1} \cup \bar{e}_0 \cup \cdots \cup \bar{e}_{n_1-1}\})$ forks over M. By the definition of triviality, $\mathrm{tp}(\bar{c}, M \cup \bar{d}_l)$ forks over M or $\mathrm{tp}(\bar{c}, M \cup \bar{e}_l)$ forks over M for some l. By symmetry, suppose the former. Clearly by V 3.1 $\{\bar{a}, \bar{c}\}$ is not independent over M, a contradiction.

(3) Let $A \subseteq M$, M $\boldsymbol{F}^a_\kappa$-saturated, $\mathrm{tp}_*(M, A \cup \bar{a})$ does not fork over A. By V

3.9 there are $n, \bar{d}_0, \cdots, \bar{d}_{n-1}$ such that: $\bar{d}_l$ realizes the stationarization of r over M, $n = w_r(\bar{a}, M) = w_r(\bar{a}, A)$, $\{\bar{d}_0, \cdots, \bar{d}_{n-1}\}$ is independent over (M, A) and each $\bar{d}_l$ realizes r, and $q_l = \text{tp}(\bar{d}_l, M \cup \bar{a})$ forks over M, hence q_l forks over A, hence for some $\bar{b}_l$, $\text{tp}(\bar{d}_l, A \cup \bar{a} \cup \bar{b}_l)$ forks over $A, \bar{b}_l \in M$.

The only property missing is $\text{tp}(\bar{d}_l, A \cup \bar{a})$ forks over M. If this fails we get a contradiction to part (2) (with A, $\bar{a}$, $\bar{b}_l$, $\bar{d}_l$ here standing for $\bar{A}$, $\bar{a}$, $\bar{b}$, $\bar{c}$ there).

(4) Left to the reader.

(5) Let $\{d_0, \cdots, \bar{d}_{n-1}\}$ be as in part (3) of the lemma (replacing A by B),

$$\boldsymbol{I}_l = \{\bar{d} \in r(\mathfrak{C}) : \text{tp}(\bar{d}, B \cup \bar{d}_l) \text{ forks over } B\},$$

and φ_l, $\bar{b}_l$ be such that $\bar{b}_l \in B$, $\models \varphi_l[\bar{d}_l, \bar{a}, \bar{b}_l]$ and $\varphi_l(\bar{x}, \bar{a}, \bar{b}_l)$ fork over B. By II 2.2(8) there are $\bar{c}_l \in A \cup \bigcup \boldsymbol{I}_l$, and ψ_l such that for every $\bar{d} \in \boldsymbol{I}_l$, $\varphi[\bar{d}, \bar{a}, \bar{b}_l]$ iff $\psi_l[\bar{d}, \bar{c}_l, \bar{b}_l]$, and w.l.o.g. (by increasing $\bar{b}_l, \bar{c}_l$) $\bar{c}_l$ is a concatanation of sequences from $\boldsymbol{I}_l$. Let $e_l \in \mathfrak{C}^{eq}$ be defined by $\bar{c}_l / E_l$, $E_l(\bar{y}, \bar{z}) \Rightarrow (\forall \bar{x})[\psi_l(\bar{x}, \bar{y}, \bar{b}_l) = \psi_l(\bar{x}, \bar{z}, \bar{b}_l)]$.

By the last phrase in part (3), for every automorphism F of $\mathfrak{C}$ which is the identity over $B \cup \bar{a}$, F maps $\boldsymbol{I}_l$ into some $\boldsymbol{I}_m$. Hence e_l can have at most n possible images (varying F). Hence e_l is algebraic over $B \cup \bar{a}$.

Let, for $i < |T|^+$, f_i be an elementary mapping with domain $C = B \cup \bar{a} \cup \bigcup_l (\bar{d}_l \cup \{e_l\} \cup \bar{c}_l)$, $f_i \restriction (A \cup \bar{a}) =$ the identity, $\text{stp}_*(f_i(C), \bigcup_{j \neq i} f_j(C))$ does not fork over $A \cup \bar{a}$, and extend $\text{stp}_*(C, A \cup \bar{a})$, and f_0 is the identity.

By V 1.11, $\text{tp}(f_i(\bar{d}_m), B \cup f_i(B) \cup \bigcup_l d_l)$ forks over A, hence by part (2) for some l $\text{tp}(f_i(\bar{d}_m), B \cup f_i(B) \cup \bar{d}_l)$, and by V 3.1 this l is unique, so by the indiscernibility, necessarily $l = m$.

As r is regular, for $l > 0$, $\text{tp}(\bar{d}_l, B \cup \bigcup_{m \neq l} I_m)$ does not fork over B, and easily $\text{tp}(\bar{d}_l, B \cup \bigcup_{m \neq l} (\bar{d}_m{}^\frown \bar{c}_m{}^\frown \langle e_m \rangle))$ does not fork over B, hence $\text{tp}(\bar{d}_l, B \cup \bigcup_{m \neq l} (\bar{d}_m{}^\frown \bar{c}_m{}^\frown \langle e_m \rangle) \cup \bigcup_{i > 0} f_i(B \cup \bar{d}_0{}^\frown \bar{c}_0{}^\frown \langle e_0 \rangle))$ does not fork over B. Hence we can find an elementary mapping g,

$$\text{Dom}(g) = \bar{a} \cup \bigcup_i f_i\left(B \cup \bigcup_m \bar{d}_m{}^\frown \bar{c}_m{}^\frown \langle e_m \rangle\right), g \restriction \bigcup_i f_i(B \cup \bar{d}_0{}^\frown \bar{c}_0{}^\frown \langle e_0 \rangle)$$

is the identity and $\text{stp}_*(\text{Rang}(g), \bigcup_i f_i(B \cup \bar{a}_0{}^\frown \bar{c}_0{}^\frown \langle e_0 \rangle))$ extends $\text{stp}_*(\text{Dom}(g), \bigcup_i f_i(B \cup \bar{d}_0{}^\frown \bar{c}_0{}^\frown \langle e_0 \rangle))$ and does not fork over $\bigcup_i f_i(B \cup \bar{d}_0{}^\frown \bar{c}_0{}^\frown \langle e_0 \rangle)$ and $\{d_m : m < n\} \cup \{g(d_m) : 0 < m < n\}$ is independent over B.

As we noted above e_0 is algebraic over $B \cup \bar{a}$, hence $e_0 = g(e_0)$ is algebraic over $g(B \cup \bar{a}) = B \cup g(\bar{a})$.

Checking closed, we see that for any automorphism F of $\mathfrak{C}^{eq}$ which is the identity over $B \cup \bar{a} \cup g(\bar{a})$, $F(e_0) = e_0$. So e_0 is definable over $B \cup \bar{a} \cup g(\bar{a})$, say by $\psi(x, \bar{a}, g(\bar{a}), \bar{b}^*)$ $(\bar{b}^* \in B)$. Let

$$E(\bar{y}_0{}^\frown\bar{z}_0, \bar{y}_1{}^\frown\bar{z}_1) \stackrel{\text{def}}{=} \text{“}(\forall x)[\psi(x,\bar{y}_0,\bar{z}_0,f_i(\bar{b}^*)) \equiv \psi(x,\bar{y}_1,\bar{z}_1,f_i(\bar{b}^*))]$$
$$\text{holds for infinitely many } i\text{'s”}.$$

By III 2.5, 1.7, E is a formula which is almost over A, and let $e^* = \bar{a}{}^\frown g(\bar{a})/E$.

Now exactly as in the proof of V 4.11, $e^* \notin A$, tp(e^*, A) is r-semi-simple not orthogonal to r, and $e^* \in \text{acl}(\bigcup f_i(B \cup \{e_0\})$, but here any two of $\{f_i(e_0): i\}$ depend on $\bigcup_i f_i(B)$, hence $w_r(e^*, A) = 1$.

The only point left is "$e^* \in \text{acl}(A \cup \bar{a})$", but for any i we know that (k large enough) $e^* \in \text{acl}(\bigcup_{j=i}^{i+k} f_j(B \cup \{e\}) \subseteq \text{acl}(A \cup \bar{a} \cup \bigcup_{j=i}^{i+k} f_j(B))$ (because $e_0 \in \text{acl}(B \cup \bar{a})$; see above). As this is true for every j and $\{f_j(B): j\}$ is independent over $A \cup \bar{a}$, clearly $e^* \in \text{acl}(A \cup \bar{a})$. The rest is obvious.

(6) Repeat the proof of (5).

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