

ON THE p -RANK OF $\text{Ext}(A, B)$ FOR
COUNTABLE ABELIAN GROUPS A AND B^*

BY

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ABSTRACT

In this note we show that the p -rank of $\text{Ext}(A, B)$ for countable torsion-free abelian groups A and B is either countable or the size of the continuum.

1. Introduction

The structure of $\text{Ext}(A, B)$ for torsion-free abelian groups A has received much attention in the literature. In particular, in the case of $B = \mathbb{Z}$ complete characterizations are available in various models of ZFC (see [EkMe, Sections on the structure of Ext], [ShSt1], and [ShSt2] for references). It is easy to see that

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$\text{Ext}(A, B)$ is always a divisible group for any torsion-free group A . Hence it is of the form

$$\text{Ext}(A, B) = \bigoplus_{p \in \Pi} \mathbb{Z}(p^\infty)^{(\nu_p)} \oplus \mathbb{Q}^{(\nu_0)}$$

for some cardinals ν_p, ν_0 ($p \in \Pi$) which are uniquely determined. Here $\mathbb{Z}(p^\infty)$ is the p -Prüfer group and \mathbb{Q} is the group of rational numbers. Thus, the obvious question that arises is which sequences $(\nu_0, \nu_p : p \in \Pi)$ can appear as the cardinal invariants of $\text{Ext}(A, B)$ for some (which) torsion-free abelian group A and arbitrary abelian group B ? As mentioned above for $B = \mathbb{Z}$, the answer is pretty much known but for general B there is little known so far. Some results were obtained in [Fr] and [FrSt] for countable abelian groups A and B . However, one essential question was left open, namely if the situation is similar to the case $B = \mathbb{Z}$ when it comes to p -ranks. It was conjectured that the p -rank of $\text{Ext}(A, B)$ can only be either countable or the size of the continuum whenever A and B are countable. Here we prove that the conjecture is true.

Our notation is standard and we write maps from the left. If H is a pure subgroup of the abelian group G , then we will write $H \subseteq_* G$. The set of natural primes is denoted by Π . For further details on abelian groups and set-theoretic methods we refer to [Fu] and [EkMe].

2. Proof of the conjecture on the p -rank of Ext

It is well-known that for torsion-free abelian groups A the group of extensions $\text{Ext}(A, B)$ is divisible for any abelian group B . Hence it is of the form

$$\text{Ext}(A, B) = \bigoplus_{p \in \Pi} \mathbb{Z}(p^\infty)^{(\nu_p)} \oplus \mathbb{Q}^{(\nu_0)}$$

for some cardinals ν_p, ν_0 ($p \in \Pi$) which are uniquely determined.

The invariant $r_p(\text{Ext}(A, B)) := \nu_p$ is called the p -**rank** of $\text{Ext}(A, B)$ while $r_0(\text{Ext}(A, B)) := \nu_0$ is called the **torsion-free rank** of $\text{Ext}(A, B)$. The following was shown in [FrSt] and gives an almost complete description of the structure of $\text{Ext}(A, B)$ for countable torsion-free A and B . Recall that the **nucleus** G_0 of a torsion-free abelian group G is the largest subring R of \mathbb{Q} such that G is a module over R .

PROPOSITION 2.1: *Let A and B be torsion-free groups with A countable and $|B| < 2^{\aleph_0}$. Then either*

- (i) $r_0(\mathbf{Ext}(A, B)) = 0$ and $A \otimes B_0$ is a free B_0 -module or
- (ii) $r_0(\mathbf{Ext}(A, B)) = 2^{\aleph_0}$.

Recall that for a torsion-free abelian group G the p -rank $r_p(G)$ is defined to be the $\mathbb{Z}/p\mathbb{Z}$ -dimension of the vectorspace G/pG .

PROPOSITION 2.2: *The following holds true:*

- (i) *If A and B are countable torsion-free abelian groups and A has finite rank, then $r_p(\mathbf{Ext}(A, B)) \leq \aleph_0$. Moreover, all cardinals $\leq \aleph_0$ can appear as the p -rank of some $\mathbf{Ext}(A, B)$.*
- (ii) *If A and B are countable torsion-free abelian groups with $\mathbf{Hom}(A, B) = 0$, then*

$$r_p(\mathbf{Ext}(A, B)) = \begin{cases} 0 & \text{if } r_p(A) = 0 \text{ or } r_p(B) = 0, \\ r_p(A) \cdot r_p(B) & \text{if } 0 < r_p(A), r_p(B) < \aleph_0, \\ \aleph_0 & \text{if } 0 < r_p(A) < \aleph_0 \text{ and } r_p(B) = \aleph_0, \\ 2^{\aleph_0} & \text{if } r_p(A) = \aleph_0 \text{ and } 0 < r_p(B) \leq \aleph_0. \end{cases}$$

Recall the following way of calculating the p -rank of $\mathbf{Ext}(A, B)$ for torsion-free groups A and B (see [EkMe, p. 389]).

LEMMA 2.3: *For a torsion-free abelian group A and an arbitrary abelian group B , let φ^p be the map that sends $\psi \in \mathbf{Hom}(A, B)$ to $\pi \circ \psi$ with $\pi : B \rightarrow B/pB$ the canonical epimorphism. Then*

$$r_p(\mathbf{Ext}(A, B)) = \dim_{\mathbb{Z}/p\mathbb{Z}}(\mathbf{Hom}(A, B/pB)/\varphi^p(\mathbf{Hom}(A, B))).$$

We now need some preparation from descriptive set-theory in order to prove our main result. Recall that a subset of a topological space is called **perfect** if it is closed and contains no isolated points. Moreover, a subset X of a Polish space V is **analytic** if there is a Polish space Y and a Borel (or closed) set $B \subseteq V \times Y$ such that X is the projection of B ; that is,

$$X = \{v \in V \mid (\exists y \in Y) \langle v, y \rangle \in B\}.$$

PROPOSITION 2.4: *If E is an analytic equivalence relation on $\Gamma = \{X : X \subseteq \omega\}$ that satisfies*

- (†) *if $X, Y \subseteq \omega, n \notin Y, X = Y \cup \{n\}$ then X and Y are not E -equivalent,*

then there is a perfect subset T of Γ of pairwise nonequivalent $X \subseteq \omega$.

Proof. See [Sh, Lemma 13]. ■

We are now in the position to prove our main result.

THEOREM 2.5: *Let A be a countable torsion-free abelian group and B an arbitrary countable abelian group. Then either*

- $r_p(\text{Ext}(A, B)) \leq \aleph_0$ or
- $r_p(\text{Ext}(A, B)) = 2^{\aleph_0}$.

Proof. The proof uses descriptive set-theory, relies on [Sh, Lemma 13] and is inspired by [HaSh, Lemma 2.2]. By the above Lemma 2.3 the p -rank of $\text{Ext}(A, B)$ is the dimension κ of the $\mathbb{Z}/p\mathbb{Z}$ vector space $L := \text{Hom}(A, B/pB)/\varphi^p(\text{Hom}(A, B))$ where φ^p is the natural map. We choose a basis $\{[\varphi_\alpha] \mid \alpha < \kappa\}$ of L with $\varphi_\alpha \in \text{Hom}(A, B/pB)$ and assume that $\aleph_0 < \kappa$. Note that $[\varphi_\alpha] \neq 0$ means that $\varphi_\alpha : A \rightarrow B/pB$ has no lifting to an element $\psi \in \text{Hom}(A, B)$ such that $\varphi_\alpha = \pi \circ \psi$.

Now let $A = \bigcup_{i < \omega} A_i$ with $\text{rk}(A_i)$ finite. By the pigeonhole principle there are $\alpha_1 \neq \beta_1$ such that $\varphi_{\alpha_1} \upharpoonright A_1 = \varphi_{\beta_1} \upharpoonright A_1$. We define $\psi_1 := \varphi_{\alpha_1} - \varphi_{\beta_1}$ and obtain $A_1 \subseteq \text{Ker}(\psi_1)$. Obviously, ψ_1 has no lifting because $\{[\varphi_\alpha] \mid \alpha < \kappa\}$ is a basis of L , hence $[\psi_1] \neq 0$. Since $\alpha_1 \neq \beta_1$ there exists $x_1 \in A$ satisfying $\varphi_{\alpha_1}(x_1) \neq \varphi_{\beta_1}(x_1)$. Let n be minimal with $x_1 \in A_n \setminus A_{n-1}$. Because A_n has finite rank we similarly get $\alpha_2 \neq \beta_2$ such that $\varphi_{\alpha_2} \upharpoonright A_n = \varphi_{\beta_2} \upharpoonright A_n$ and define $\psi_2 := \varphi_{\alpha_2} - \varphi_{\beta_2}$ with $A_n \subseteq \text{Ker}(\psi_2)$. Clearly we have $\psi_1 \neq \psi_2$ since $\psi_1(x_1) \neq 0 = \psi_2(x_1)$ and $[\psi_2] \neq 0$. Continuing this construction we get \aleph_0 pairwise different $\psi_n \in \text{Hom}(A, B/pB)$ with $[\psi_n] \neq 0$.

Now let $\eta \in {}^\omega 2$, which means that η is a countable $\{0, 1\}$ -sequence and choose $\psi_\eta := \sum_{n \in \text{supp}(\eta)} \psi_n$ where $\text{supp}(\eta) := \{n \in \omega \mid \eta(n) = 1\}$. This is well-defined since, for any $a \in A$, the sum $\psi_\eta(a)$ consists of only finitely many summands because there is $n \in \mathbb{N}$ such that $a \in A_n$ and A_n is in the kernel of ψ_m for sufficiently large m . Note that also $\psi_\eta \neq \psi_{\eta'}$ for all $\eta \neq \eta'$. It remains to prove that the size of $\{[\psi_\eta] : \eta \in {}^\omega 2\}$ is 2^{\aleph_0} which then implies that $\kappa = 2^{\aleph_0}$.

We now define an equivalence relation on $\Gamma = \{X \subseteq \omega\}$ in the following way: $X \sim X'$ if and only if $[\psi_{\eta_X}] = [\psi_{\eta_{X'}}]$, where $\eta_X \in {}^\omega 2$ is the characteristic function of X . We claim that

(‡) for $X, X' \subseteq \omega$ and $n \notin X$ such that $X' = X \cup \{n\}$ we have $X \not\sim X'$.

But this is obvious since in this case $\psi_{\eta_X} - \psi_{\eta_{X'}} = -\psi_n$ and $[-\psi_n] \neq 0$, so $[\psi_{\eta_X}] \neq [\psi_{\eta_{X'}}]$. We claim that the above equivalence relation is analytic. It

then follows from Proposition 2.4 that there is a perfect subset of Γ of pairwise nonequivalent $X \subseteq \omega$ and hence there are 2^{\aleph_0} distinct equivalence classes for \sim . Thus the p -rank of $\mathbf{Ext}(A, B)$ must be the size of the continuum as claimed.

In order to see that \sim is analytic, recall that both the cantor space ${}^\omega 2$ (and hence Γ) and the Baire space ${}^\omega \omega$ are Polish spaces with the natural topologies. For \sim to be analytic, we therefore have to show that

$$\Delta = \{(X, X') : X, X' \in \Gamma \text{ and } X \sim X'\}$$

is an analytic subset of the product space $\Gamma \times \Gamma$. We enumerate $A = \{a_n : n \in \omega\}$ and $B = \{b_n : n \in \omega\}$ and let $C_1 \subseteq {}^\omega \omega$ be the set of all $\rho \in {}^\omega \omega$ such that ρ induces a homomorphism $f_\rho \in \text{Hom}(A, B/pB)$, i.e., the map $f_\rho : A \rightarrow B/pB$ sending a_n onto $b_{\rho(n)} + pB$ is a homomorphism. C_2 is defined similarly replacing $\text{Hom}(A, B/pB)$ by $\text{Hom}(A, B)$, i.e., $\nu \in C_2$ if the map $g_\nu : A \rightarrow B$ with $a_n \mapsto b_{\nu(n)}$ is a homomorphism. Clearly C_1 and C_2 are closed subsets of ${}^\omega \omega$. Put

$$P = \{(\rho_1, \rho_2, \nu) : \rho_1, \rho_2 \in C_1; \nu \in C_2 \text{ and } f_{\rho_1} - f_{\rho_2} = \varphi^p(g_\nu)\}.$$

Then P is a closed subset of ${}^\omega \omega \times {}^\omega \omega \times {}^\omega \omega$. Now the construction above gives a continuous function Θ from ${}^\omega 2$ to ${}^\omega \omega$ by sending η to $\rho = \Theta(\eta)$, where ρ is induced by the homomorphism ψ_η . Then $X \sim X'$ if and only if there is $\nu \in C_2$ such that $(\Theta(\eta_X), \Theta(\eta_{X'}), \nu) \in P$ and thus \sim is an analytic equivalence relation. ■

References

- [EkMe] P. C. Eklof and A. Mekler, *Almost Free Modules, Set-Theoretic Methods (revised edition)*, North-Holland Mathematical Library, Vol. 65, North-Holland, Amsterdam, 2002.
- [Fr] S. Friedenberg, *Extensions of Abelian Groups*, Südwestdeutscher Verlag für Hochschulschrift, 2010.
- [FrSt] S. Friedenberg and L. Strüngmann, *The structure of $\text{Ext}(A, B)$ for countable abelian groups A and B* , Archiv der mathematik, 2013, to appear.
- [Fu] L. Fuchs, *Infinite Abelian Groups, Vols. I and II*, Academic Press, New York, 1970 and 1973.
- [HaSh] L. Harrington and S. Shelah, *Counting equivalence classes for $co-\kappa$ -Souslin relation*, in *Logic Colloquium '80 (Prague 1980)*, Studies in Logic and the Foundations of Mathematics, Vol. 108, North-Holland, Amsterdam, 1982, pp. 147–152.

- [Sh] S. Shelah, *Can the fundamental (homotopy) group of a space be the rationals?*, Proceedings of the American Mathematical Society **103** (1988), 627–632.
- [ShSt1] S. Shelah and L. Strüngmann, *A characterization of $\text{Ext}(G, \mathbb{Z})$ assuming $(V=L)$* , Fundamenta Mathematicae **193** (2007), 141–151.
- [ShSt2] S. Shelah and L. Strüngmann, *On the p -rank of $\text{Ext}(G, \mathbb{Z})$ in certain models of ZFC*, Algebra and Logic **46** (2007), 200–215.