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ON TWO PROBLEMS OF ERDŐS AND HECHLER: NEW METHODS IN SINGULAR MADNESS

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Abstract. For an infinite cardinal μ, $\mathrm{MAD}(\mu)$ denotes the set of all cardinalities of *nontrivial maximal almost disjoint families* over μ.

Erdős and Hechler proved in 1973 the consistency of $\mu \in \mathrm{MAD}(\mu)$ for a singular cardinal μ and asked if it was ever possible for a singular μ that $\mu \notin \mathrm{MAD}(\mu)$, and also whether $2^{\mathrm{cf}\,\mu} < \mu \Longrightarrow \mu \in \mathrm{MAD}(\mu)$ for every singular cardinal μ.

We introduce a new method for controlling $\mathrm{MAD}(\mu)$ for a singular μ and, among other new results about the structure of $\mathrm{MAD}(\mu)$ for singular μ, settle both problems affirmatively.

1. Introduction

1.1. **Background.** Let μ be an infinite cardinal. A family of sets $\mathcal{A}$ is *μ-almost disjoint* (*μ-ad* for short) if $|A| = \mu = |\bigcup \mathcal{A}|$ for every $A \in \mathcal{A}$ and $|A \cap B| < \mu$ for every distinct $A, B \in \mathcal{A}$. $\mathcal{A}$ is *maximal μ-almost disjoint* (*μ-mad*) if there is no $C \subseteq \bigcup \mathcal{A}$ such that $\mathcal{A} \cup \{C\}$ is μ-almost disjoint; in this case we also say that $\mathcal{A}$ is *mad in μ*. It is clear that every μ-almost disjoint family consisting of fewer than $\mathrm{cf}\,\mu$ sets is mad in μ; such a family will be called *trivial*. We denote by $\mathrm{MAD}(\mu)$ the set of all cardinalities of nontrivial mad families in μ. A standard diagonalization argument shows that $\mathrm{cf}\,\mu \notin \mathrm{MAD}(\mu)$. Therefore, $\mathrm{MAD}(\mu)$ is contained in the interval of cardinals $[\mathrm{cf}(\mu)^+, 2^\mu]$.

W. W. Comfort asked (see [7]) under what conditions it follows that $\mu \in \mathrm{MAD}(\mu)$ for a singular cardinal μ. P. Erdős and S. Hechler [7] proved that $\mu \in \mathrm{MAD}(\mu)$ if $\lambda^{\mathrm{cf}\,\mu} < \mu$ for every $\lambda < \mu$. Thus, if $2^{\aleph_0} < \aleph_\omega$, then the interval $[2^{\aleph_0}, \aleph_\omega]$ of cardinals is contained in $\mathrm{MAD}(\aleph_\omega)$.

Erdős and Hechler asked in [7] whether it is consistent that $\mu \notin \mathrm{MAD}(\mu)$ for some singular cardinal μ and, more concretely, whether Martin's axiom together with $2^{\aleph_0} > \aleph_\omega$ implies that $\aleph_\omega \notin \mathrm{MAD}(\aleph_\omega)$. They also asked whether $2^{\mathrm{cf}\,\mu} < \mu$ implies $\mu \in \mathrm{MAD}(\mu)$ for singular cardinals μ other than $\aleph_\omega$.

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Both problems are settled affirmatively by the general results below on $\mathrm{MAD}(\mu)$ for a singular μ.

1.2. **Notation.** Let $\mathfrak{a}_\mu = \min \mathrm{MAD}(\mu)$, and let $\mathfrak{a} = \mathfrak{a}_{\aleph_0}$. For a singular μ it follows that $\mathrm{MAD}(\mathrm{cf}\,\mu) \subseteq \mathrm{MAD}(\mu)$; therefore $\mathfrak{a}_\mu \leqslant \mathfrak{a}_{\mathrm{cf}\,\mu}$.

A crucial role in the results is played by two *bounding numbers*: $\mathfrak{b}_\mu$ and $\mathfrak{b}_{\mathrm{cf}\,\mu}$.

For every quasi-ordering $(P, \leqslant)$ with no maximum, the *bounding number* $\mathfrak{b}(P, \leqslant)$ is the least cardinality of a subset of P with no upper bound. For a regular cardinal κ, let $\mathfrak{b}_\kappa$ denote the bounding number of $(\kappa^\kappa, \leqslant^*)$, where $f \leqslant^* g$ means that $|\{i < \kappa : f(i) > g(i)\}| < \kappa$; let $\mathfrak{b} = \mathfrak{b}_{\aleph_0}$. It is well known that $\kappa < \mathfrak{b}_\kappa \leqslant \mathfrak{a}_\kappa$ for a regular cardinal κ (for $\kappa = \aleph_0$ see [6]; the general case is similar) and that under Martin's axiom, $\mathfrak{b} = 2^{\aleph_0}$.

Suppose that μ is a singular cardinal of cofinality κ and that $\langle \mu_i : i < \kappa \rangle$ is a strictly increasing sequence of regular cardinals with supremum μ. Standard diagonalization shows that $\mathfrak{b}(\prod_{i<\kappa} \mu_i, \leqslant^*) > \mu$. Denote by $\mathfrak{b}_\mu$ the *supremum of* $\mathfrak{b}(\prod \mu_i, \leqslant^*)$ *over all strictly increasing sequences of regular cardinals* $\langle \mu_i : i < \kappa \rangle$ *with supremum* μ.

Each of the following three relations is consistent with ZFC: $\mathfrak{b} < \mathfrak{b}_{\aleph_\omega}$, $\mathfrak{b} = \mathfrak{b}_{\aleph_\omega}$ and $\mathfrak{b} > \mathfrak{b}_{\aleph_\omega}$.

1.3. **The results.** We prove that for every singular cardinal μ:

(1) $\mathfrak{a}_\mu \geqslant \min\{\mathfrak{b}_\mu, \mathfrak{b}_{\mathrm{cf}\,\mu}\}$;
(2) $\mathfrak{a}_\mu \leqslant \lambda < \mathfrak{b}_\mu \Longrightarrow \lambda \in \mathrm{MAD}(\mu)$.

Thus, if $\mathfrak{b}_{\mathrm{cf}\,\mu} > \mu$, it follows from (1) that $\mathfrak{a}_\mu > \mu$, and hence $\mu \notin \mathrm{MAD}(\mu)$; and if $\mathfrak{a}_{\mathrm{cf}\,\mu} < \mu$, it follows from (2) that $\mu \in \mathrm{MAD}(\mu)$. In particular,

(b) $MA + 2^{\aleph_0} > \aleph_\omega \Longrightarrow \aleph_\omega \notin \mathrm{MAD}(\aleph_\omega)$,
(a) $2^{\mathrm{cf}\,\mu} < \mu \Longrightarrow \mu \in \mathrm{MAD}(\mu)$ for every singular μ,

which, respectively, settle in the affirmative both problems of Erdős and Hechler from [7].

If one assumes the consistency of large cardinals, $\mathfrak{b}_{\aleph_\omega}$ can be shifted up arbitrarily high below $\aleph_{\omega_1}$. Following this with a ccc forcing for controlling $\mathfrak{b}$ proves the following:

(3) for every regular $\lambda \in (\aleph_\omega, \aleph_{\omega_1})$ and regular uncountable $\theta \leqslant \lambda^+$, it is consistent that
$$\mathrm{MAD}(\aleph_\omega) = [\theta, \lambda^+].$$

So, e.g., the following are consistent:

- $\mathrm{MAD}(\aleph_\omega) = \{\aleph_1, \aleph_2, \ldots, \aleph_{\omega+\beta+2} = 2^{\aleph_\omega}\}$ for an arbitrary $\beta < \omega_1$,
- $\mathrm{MAD}(\aleph_\omega) = \{\aleph_{\omega+\beta+2}\}$ for an arbitrary $\beta < \omega_1$,
- $\mathrm{MAD}(\aleph_\omega) = [\aleph_{\omega+\alpha+1}, \aleph_{\omega+\beta+2}]$ for arbitrary $\alpha \leqslant \beta < \omega_1$,

and so on.

We refer the reader to the comprehensive list of references in D. Monk's recent paper [12], in which maximal almost disjoint families are viewed as partitions of unity in the Boolean algebra $\mathcal{P}(\mu)/_{[\mu]^{<\mu}}$.

1.4. **Preliminary facts.** We will use the following facts from [7]:

(1) $\mathrm{MAD}(\mathrm{cf}\,\mu) \subseteq \mathrm{MAD}(\mu)$, and
(2) $\mathrm{MAD}(\mu)$ is closed under singular suprema.

The latter fact is stated in [7] in a less general form. So we give a proof here.

Lemma 1.1. *Assume that* $\lambda = \sup_{i<\theta} \lambda_i$, *where* $\{\lambda_i : i < \theta\} \subseteq \mathrm{MAD}(\mu)$ *and* $\theta < \lambda$. *Then* $\lambda \in \mathrm{MAD}(\mu)$.

Proof. We may assume that $\theta \leqslant \lambda_0$. Let $\mathcal{A}$ be a mad family in μ with $|\mathcal{A}| = \lambda_0$. Write $\mathcal{A} = \{A_i : i < \lambda_0\}$ and for each $i < \theta$ choose a mad family $\mathcal{B}_i$ with $\bigcup \mathcal{B}_i = A_i$ and $|\mathcal{B}_i| = \lambda_i$. Set

$$\mathcal{C} = \bigcup_{i<\theta} \mathcal{B}_i \cup \{A_j : \theta \leqslant j < \lambda_0\}.$$

Then $|\mathcal{C}| = \lambda$ and $\mathcal{C}$ is mad in μ. □

The following fact will also be used in some proofs.

Lemma 1.2. *Let* $\kappa = \mathrm{cf}\,\mu$, *and let* $\mathcal{A}$ *be a* μ*-almost disjoint family of size* κ. *Then there exists a mad family* $\mathcal{A}' \supseteq \mathcal{A}$ *such that* $|\mathcal{A}'| = \mathfrak{a}_\mu$ *and* $\bigcup \mathcal{A}' = \bigcup \mathcal{A}$.

Proof. Fix a μ-mad family $\mathcal{B}$ with $|\mathcal{B}| = \mathfrak{a}_\mu$. Choose $\mathcal{B}_0 = \{B_i : i < \kappa\} \subseteq \mathcal{B}$. Let $X = \bigcup_{i<\kappa} B_i$ and define

$$\mathcal{B}' = \{B \cap X : X \in \mathcal{B} \setminus \mathcal{B}_0 \ \&\ |X \cap B| = \mu\}.$$

Let $\langle A_i : i < \kappa \rangle$ be a one-to-one enumeration of $\mathcal{A}$. Define a bijection $f\colon \bigcup \mathcal{A} \to X$ so that $f[A_i \setminus \bigcup_{j<i} A_j] = B_i \setminus \bigcup_{j<i} B_j$. Finally, set $\mathcal{A}' = \mathcal{A} \cup \{f^{-1}[B] : B \in \mathcal{B}'\}$. Observe that $\mathcal{A}'$ is mad and $|\mathcal{A}'| = \mathfrak{a}_\mu$. □

2. Inequalities

From now on, μ will always denote a singular cardinal whose cofinality is denoted by κ.

2.1. Bounding numbers and madness in singular cardinals.

Theorem 2.1. *For every singular cardinal* μ,

$$\mathfrak{a}_\mu \geqslant \min\{\mathfrak{b}_\mu, \mathfrak{b}_{\mathrm{cf}\,\mu}\}. \tag{1}$$

Proof. Let $\kappa = \mathrm{cf}\,\mu$. Suppose to the contrary that $\mathfrak{a}_\mu < \min\{\mathfrak{b}_\mu, \mathfrak{b}_\kappa\}$, and fix a strictly increasing sequence of regular cardinals $\langle \mu_i : i < \kappa \rangle$ with supremum μ such that $\mathfrak{b}(\prod_{i<\kappa} \mu_i, \leqslant^*) > \mathfrak{a}_\mu$.

Let $\mathcal{A} = \big\{\{i\} \times \mu : i < \kappa\big\}$. By Lemma 1.2, there exists a family $\mathcal{B} \subseteq [\kappa \times \mu]^\mu$ such that $\mathcal{B} \cup \mathcal{A}$ is mad in μ, $\mathcal{B} \cap \mathcal{A} = \emptyset$ and $|\mathcal{B}| = \mathfrak{a}_\mu$.

For each $B \in \mathcal{B}$, define a function $f_B : \kappa \to \kappa$ by $f_B(i) = \min\{j < \kappa : |B \cap (\{i\} \times \mu)| < \mu_j\}$. This function is well defined, since $|B \cap (\{i\} \times \mu)| < \mu$ for each $i < \kappa$.

Since $|\mathcal{B}| = \mathfrak{a}_\mu < \mathfrak{b}_\kappa$, there exists a function $f : \kappa \to \kappa$ so that $f_B <^* f$ for all $B \in \mathcal{B}$. Without loss of generality, we may assume that f is strictly increasing.

For each $B \in \mathcal{B}$, for all but boundedly many $i < \kappa$, it follows that $\sup\{\alpha < \mu_{f(i)} : (i, \alpha) \in B\} < \mu_{f(i)}$. Let $g_B(i)$ be defined by

$$g_B(i) = \begin{cases} 0 & \text{if } \sup\{\alpha < \mu_{f(i)} : (i,\alpha) \in B\} = \mu_{f(i)}, \\ \sup\{\alpha < \mu_{f(i)} : (i,\alpha) \in B\} & \text{otherwise.} \end{cases}$$

For each $B \in \mathcal{B}$ the function g_B belongs to $\prod_{i<\kappa} \mu_{f(i)}$. Since

$$\mathfrak{b}(\prod_{i<\kappa} \mu_{f(i)}, \leqslant^*) \geqslant \mathfrak{b}(\prod_{i<\kappa} \mu_i) > \mathfrak{a}_\mu,$$

we can fix a function $g \in \prod_{i<\kappa} \mu_{f(i)}$ so that $g_B <^* g$ for all $B \in \mathcal{B}$.

Define

$$C = \bigcup_{i<\kappa} \{i\} \times [g(i), \mu_{f(i)}).$$

Clearly, $|C| = \mu$. For each $B \in \mathcal{B}$ there exists $j_B < \kappa$ such that $g_B(i) < g(i)$ for all $i > j_B$. This implies that $\{i\} \times [g(i), \mu_{f(i)})$ is disjoint from B for all $i > j_B$. Hence $|B \cap C| \leqslant \mu_{f(j_B)} < \mu$. Clearly, $|C \cap (\{i\} \times \mu)| \leqslant \mu_{f(i)} < \mu$ for all $i < \kappa$; so $\mathcal{A} \cup \mathcal{B} \cup \{C\}$ is μ-almost disjoint, contrary to the maximality of $\mathcal{A} \cup \mathcal{B}$. □

A positive answer to the first question of Comfort, Erdős and Hechler follows now as a corollary:

Corollary 2.2. *If Martin's Axiom holds and* $2^{\aleph_0} > \mu > \operatorname{cf} \mu = \aleph_0$, *then* $\mu \notin \mathrm{MAD}(\mu)$.

2.2. **Between $\mathfrak{a}_\mu$ and $\mathfrak{b}_\mu$.** In this section we shall show that $\mathrm{MAD}(\mu)$ contains the interval of cardinals $[\mathfrak{a}_\mu, \mathfrak{b}_\mu)$ and even $[\mathfrak{a}_\mu, \mathfrak{b}_\mu]$ in the case that $\mathfrak{b}_\mu$ is a successor of a regular cardinal.

Theorem 2.3. *For every singular cardinal μ and every cardinal λ,*

$$\mathfrak{a}_\mu \leqslant \lambda < \mathfrak{b}_\mu \Longrightarrow \lambda \in \mathrm{MAD}(\mu). \tag{2}$$

If $\mathfrak{b}_\mu$ is a successor of a regular cardinal, then $\mathfrak{a}_\mu \leqslant \mathfrak{b}_\mu \implies \mathfrak{b}_\mu \in \mathrm{MAD}(\mu)$.

To prove the theorem it suffices, by Lemma 1.1, to show that every regular $\lambda \in [\mathfrak{a}_\mu, \mathfrak{b}_\mu)$ belongs to $\mathrm{MAD}(\mu)$.

The proof of this will now be divided into two cases. First we prove that every regular $\mathfrak{a}_\mu < \lambda < \mu$ belongs to $\mathrm{MAD}(\mu)$. The proof in this case does not require any specialized techniques. Then we prove the same for regular $\mu < \lambda < \mathfrak{b}_\mu$ and for $\mathfrak{b}_\mu$ itself when it is the successor of a regular cardinal. In this case the proof requires some machinery from pcf theory.

Despite the technical differences between both proofs, they are similar, and could, in fact, be combined to a single proof. Both follow the same scheme of gluing together λ different μ-mad families, each of size $\mathfrak{a}_\mu$, to a single μ-mad family of size λ. In the case $\lambda < \mu$, a simple presentation of μ as a disjoint union of λ parts works; in the second part we need to rely on smooth pcf scales to get a presentation of μ as an *almost increasing* and *continuous* union of length λ of sets of size μ.

2.2.1. *The case $\lambda < \mu$.*

Lemma 2.4. *Suppose $\mu > \operatorname{cf} \mu = \kappa$. Then for every regular cardinal λ,*

$$\mathfrak{a}_\mu \leqslant \lambda < \mu \implies \lambda \in \mathrm{MAD}(\mu).$$

Proof. Suppose λ is regular and $\mathfrak{a}_\mu \leqslant \lambda < \mu$. Since $\mathfrak{a}_\mu > \kappa = \operatorname{cf} \mu$, $\lambda > \kappa$.

Fix a strictly increasing sequence of regular cardinals $\langle \mu_i \colon i < \kappa \rangle$ such that $\sup_{i<\kappa} \mu_i = \mu$ and $\lambda < \mu_0$. We will work in $\mu \times \lambda$ instead of μ. Let $S = \{\delta < \lambda \colon \operatorname{cf} \delta = \kappa\}$. For each $\delta \in S$ fix a strictly increasing, continuous sequence $D_\delta = \langle \gamma_i^\delta \colon i < \kappa \rangle$ with limit δ such that $\gamma_0^\delta = 0$. Define

$$F_j^\delta = \bigcup \{\mu \times \{\beta\} \colon \gamma_j^\delta \leqslant \beta < \gamma_{j+1}^\delta\}.$$

Thus $\mathcal{F}_\delta = \{F_j^\delta \colon j < \kappa\}$ is a disjoint family of sets, each set of size μ, which covers $\mu \times \delta$. Let $\mathcal{A}_\delta \subseteq [\mu \times \delta]^\mu$ be such that $\mathcal{A}_\delta \cup \mathcal{F}_\delta$ is mad in $\mu \times \delta$, $\mathcal{A}_\delta \cap \mathcal{F}_\delta = \emptyset$ and $|\mathcal{A}_\delta| = \mathfrak{a}_\mu$.

Define

$$\mathcal{B} = \{\mu \times \{\alpha\} \colon \alpha < \lambda\} \cup \bigcup_{\delta \in S} \mathcal{A}_\delta.$$

Then $|\mathcal{B}| = \lambda$ and $\mathcal{B} \subseteq [\mu \times \lambda]^\mu$. We will show that $\mathcal{B}$ is μ-mad.

First, observe that $\mathcal{B}$ is almost disjoint: clearly each element of $\mathcal{A}_\delta$ is almost disjoint from any set of the form $\mu \times \{\alpha\}$, because if $\alpha < \delta$, then $\mu \times \{\alpha\} \subseteq F^\delta_j$ for $j < \kappa$ such that $\gamma^\delta_j \leqslant \alpha < \gamma^\delta_{j+1}$. Finally, consider $A_i \in \mathcal{A}_{\delta_i}$, $i < 2$, with $\delta_0 < \delta_1$. Then $A_0 \subseteq \bigcup_{j<j_0} F^{\delta_1}_j$, where $j_0 < \kappa$ is such that $\delta_0 < \gamma^{\delta_1}_{j_0}$. Thus $|A_0 \cap A_1| < \mu$.

To see that $\mathcal{B}$ is mad, fix an arbitrary $Z \in [\mu \times \lambda]^\mu$. There exists a sequence $\langle \alpha_i \colon i < \kappa\rangle$ in λ such that

$$|Z \cap (\mu \times \{\alpha_i\})| \geqslant \mu_i.$$

If $|\{\alpha_i \colon i < \kappa\}| < \kappa$, then $|Z \cap (\mu \times \{\alpha\})| = \mu$ for some α. So suppose that $|Z \cap (\mu \times \{\alpha_i\})| < \mu$ for every $i < \kappa$. Taking a subsequence, we may assume that $\langle \alpha_i \colon i < \kappa\rangle$ is strictly increasing. Let δ be its supremum. By regularity of λ, $\delta \in S$ and therefore $Z \in [\mu \times \delta]^\mu$. Shrinking Z if necessary, assume that $Z \subseteq \bigcup_{i<\kappa} \mu \times \{\alpha_i\}$. Then $|Z \cap F^\delta_j| < \mu$ for every $j < \kappa$. Thus, $|Z \cap A| = \mu$ for some $A \in \mathcal{A}_\delta$. This completes the proof. □

Corollary 2.5. *Let $\mu > \operatorname{cf} \mu = \kappa$. If $\mathfrak{a}_\kappa \leqslant \mu$, then $[\mathfrak{a}_\kappa, \mu] \subseteq \mathrm{MAD}(\mu)$. In particular, if $2^\kappa < \mu$, then $\mu \in \mathrm{MAD}(\mu)$.*

Corollary 2.5 anwers affirmatively the second question of Erdős and Hechler in [7].

2.2.2. *The case $\lambda > \mu$.* A (μ, λ)-*scale*, for a regular cardinal $\lambda > \mu$, is a sequence $\overline{f} = \langle f_\alpha \colon \alpha < \lambda\rangle \subseteq \prod_{i<\kappa} \mu_i$ such that $\langle \mu_i \colon i < \kappa\rangle$ is a strictly increasing sequence of regular cardinals with limit μ, and so that $\alpha < \beta < \lambda \implies f_\alpha <^* f_\beta$ and for every $g \in \prod_{i<\kappa} \mu_i$ there is $\alpha < \lambda$ with $g <^* f_\alpha$. The relation $f <^* g$ means that the set $\{i < \kappa \colon f(i) \geqslant g(i)\}$ is bounded in κ. When μ is fixed, "(μ, λ)-scale" will be abbreviated by "λ-scale". A λ-scale $\overline{f}$ is *smooth* if for every $\delta < \lambda$ with $\operatorname{cf} \delta > \kappa$, the sequence $\overline{f} \restriction \delta = \langle f_\alpha \colon \alpha < \delta\rangle$ is cofinal in $(\prod_{i<\kappa} f_\delta(i), <^*)$. In this case we say that f_δ is an *exact upper bound* of $\overline{f} \restriction \delta$. We will denote by $[f, g)$ the set $\{(i, \alpha) \colon i < \kappa \wedge f(i) \leqslant \alpha < g(i)\}$.

The proof in the present case goes through two steps. First, it is shown that whenever a smooth (μ, λ)-scale exists and $\mathfrak{a}_\mu < \lambda$, it follows that $\lambda \in \mathrm{MAD}(\mu)$. Then it is shown that for every $\mu < \lambda < \mathfrak{b}_\mu$ there is a smooth (μ, λ)-scale and that in case $\mathfrak{b}_\mu$ is a successor of a regular cardinal there is also a smooth $(\mu, \mathfrak{b}_\mu)$-scale.

Lemma 2.6. *Assume $\lambda > \mu > \operatorname{cf} \mu = \kappa$ and there exists a smooth (μ, λ)-scale. If $\mathfrak{a}_\mu \leqslant \lambda$, then $\lambda \in \mathrm{MAD}(\mu)$.*

Proof. Suppose there exists a smooth λ-scale $\langle g_\xi \colon \xi < \lambda\rangle \subseteq \prod_{i<\kappa} \mu_i$. Let $S = \{\delta < \lambda \colon \operatorname{cf} \delta = \kappa\}$, and for each $\delta \in S$ fix a strictly increasing, continuous, sequence $\langle \gamma^\delta_i \colon i < \kappa\rangle$ with limit δ such that $\gamma^\delta_0 = 0$ and put $D_\delta = \{\gamma^\delta_i \colon i < \kappa\}$.

By induction on $\xi < \lambda$ we construct a smooth λ-scale $\overline{f} = \langle f_\xi \colon \xi < \lambda\rangle \subseteq \prod_{i<\kappa} \mu_i$ that satisfies the following two conditions:

(1) If $\delta < \lambda$ is a limit and $\operatorname{cf} \delta \leqslant \kappa$, then $f_\delta(i) = \sup_{\xi \in D_\delta} f_\xi(i)$.
(2) For each $\xi < \lambda$ the set $[f_\xi, f_{\xi+1}) = \{(i, \alpha) \colon f_\xi(i) \leqslant \alpha < f_{\xi+1}(i)\}$ has cardinality μ.

By induction on $\xi < \lambda$ we define an increasing and continuous sequence of ordinals $\zeta(\xi) < \lambda$ and a $<^*$-increasing sequence of functions $f_\xi \in \prod_{i<\kappa} \mu_i$ so that $f_\xi = g_{\zeta(\xi)}$ for all $\xi < \lambda$ *except* when ξ is limit of cofinality $\leqslant \kappa$. Then $\overline{f} := \langle f_\xi \colon \xi < \lambda \rangle$ will be a smooth λ-scale as required.

At a limit stage ξ of cofinality $\leqslant \kappa$, let $\zeta(\xi) = \bigcup_{\xi'<\xi} \zeta(\xi')$ and use condition (1) to define f_ξ; at successor $\xi+1$ choose $\zeta(\xi+1)$ so that $\max\{f_\xi, g_{\zeta(\xi)}\} <^* g_{\zeta(\xi+1)}$ and (2) holds, and let $f_{\xi+1} = g_{\zeta(\xi+1)}$. Suppose now that ξ is a limit of cofinality $> \kappa$. By the smoothness of $\overline{g}$, and since $\langle g_{\zeta(\xi')} \colon \xi' < \xi \rangle$ is $<^*$-increasing, after defining $\zeta(\xi) = \bigcup_{\xi'<\xi} \zeta(\xi')$ we get that $g_{\zeta(\xi)}$ is an exact upper bound of $\langle g_{\zeta(\xi')} \colon \xi' < \xi \rangle$. But then $g_{\zeta(\xi)}$ is also an exact upper bound of $\langle f_{\zeta(\xi')} \colon \xi' < \xi \rangle$, and we let $f_\xi = g_{\zeta(\xi)}$.

Let f_λ be defined on κ by $f_\lambda(i) = \mu_i$.

Claim 2.7. Suppose $\delta \leqslant \lambda$ and $A \subseteq [0, f_\delta)$ has cardinality μ. If $\operatorname{cf} \delta > \kappa$, there is some $\delta' < \delta$ so that $|A \cap [0, f_{\delta'})| = \mu$.

Proof. Find $g < f_\delta$ so that $\sum_{i<\kappa} |A \cap (i \times g(i))| = \mu$. By smoothness there exists some $\delta' < \delta$ so that $g <^* g_{\delta'}$. □

For every $\xi < \lambda$, let $A_\xi = [f_\xi, f_{\xi+1})$ and let $\mathcal{A} = \{A_\xi \colon \xi < \lambda\}$. Then $\mathcal{A} \subseteq \mathcal{P}\big([0, f_\lambda)\big)$ is μ-almost disjoint and $|\mathcal{A}| = \lambda$.

For each $\delta \in S$ and $i < \kappa$, let $F_i^\delta = [f_{\gamma_i^\delta}, f_{\gamma_{i+1}^\delta})$. Then $\mathcal{F}_\delta = \{F_i^\delta \colon i < \kappa\}$ is a μ-almost disjoint family whose union is, by condition (1) on $\overline{f}$, equal to $[0, f_\delta)$. Fix a μ-ad family $\mathcal{B}_\delta \subseteq \mathcal{P}([0, f_\delta))$ such that $|\mathcal{B}_\delta| = \mathfrak{a}_\mu$, $\mathcal{B}_\delta \cup \mathcal{F}_\delta$ is μ-mad and $\mathcal{B}_\delta \cap \mathcal{F}_\delta = \emptyset$ (by Lemma 1.2).

Claim 2.8. If $\delta \in S$ and $B \in \mathcal{B}_\delta$, then for all $i < \kappa$, it follows that $|B \cap [0, f_{\gamma_i^\delta})| < \mu$.

Proof. If not so, let $i_0 < \kappa$ be the largest value so that $|B \cap [0, f_{\gamma_{i_0}^\delta})| < \mu$; i_0 exists because D_δ is closed. Now $|B \cap F_{i_0}^\delta| = \mu$, a contradiction. □

Let $\mathcal{B} = \bigcup_{\delta \in S} \mathcal{B}_\delta$. Then $|\mathcal{B}| = \mathfrak{a}_\mu \cdot \lambda = \lambda$, and therefore $|\mathcal{A} \cup \mathcal{B}| = \lambda$. We will show now that $\mathcal{A} \cup \mathcal{B}$ is μ-mad.

Suppose that $A = A_\xi \in \mathcal{A}$ and $B \in \mathcal{B}_\delta$ for some $\delta \in S$. If $\xi \geqslant \delta$, then clearly $|A \cap B| < \mu$, and if $\xi < \delta$, there is some $i < \kappa$ so that $A_\xi \subseteq^* F_i^\delta$ and $|A \cap B| < \mu$ follows from Claim 2.8.

If $B_1 \in \mathcal{B}_{\delta_1}$ and $B_2 \in \mathcal{B}_{\delta_2}$ with $\delta_1 < \delta_2$ in S, then there is some $i < \kappa$ so that $f_{\delta_1} <^* f_{\gamma_i^{\delta_2}}$ and Claim 2.8 gives $|B_1 \cap B_2| < \mu$.

This establishes that $\mathcal{A} \cup \mathcal{B}$ is μ-mad. To verify maximality, let $Z \subseteq [0, f_\lambda)$ be arbitrary of size μ. By Claim 2.7 the first $\xi \leqslant \lambda$ for which $|Z \cap [0, f_\xi)| = \mu$ is either a successor or of cofinality $\leqslant \kappa$. Cofinality $< \kappa$ is ruled out by condition (1) on $\overline{f}$. The case ξ successor implies that $|Z \cap A_\xi| = \mu$. Finally, in the remaining case $\xi = \delta \in S$, there is some $B \in \mathcal{B}_\delta$ so that $|Z \cap B| = \mu$. □

Now the proof of Theorem 2.3 will be completed by the following lemma, whose proof is actually found implicitly in [15]. We shall sketch a proof here too.

Lemma 2.9. *Suppose μ is singular and $\mu < \lambda < \mathfrak{b}_\mu$, λ regular. Then there is a smooth (μ, λ)-scale. If $\mathfrak{b}_\mu$ is a successor of a regular cardinal, there is also a smooth $(\mu, \mathfrak{b}_\mu)$-scale.*

Proof. Since $\lambda < \mathfrak{b}_\mu$, there exists a product $\prod_{i<\kappa} \mu_i$, where $\kappa = \operatorname{cf} \mu$, so that $\mathfrak{b}(\prod_{i<\kappa} \mu_i, <^*) > \lambda$.

By Claim 1.3 in [15] there exists a λ-scale $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$ in some $\prod_{i<\kappa} \mu'_i$ such that for all regular $\theta \in (\kappa, \mu)$, every $\alpha < \lambda$ with $\operatorname{cf}\alpha = \theta$ satisfies that $\overline{f} \restriction \alpha$ is *flat*, that is, is equivalent modulo the bounded ideal on κ to a strictly increasing sequence of ordinal functions on κ.

By Lemma 15 in [10], *every* $\alpha < \lambda$ with $\operatorname{cf}\alpha > \kappa$ satisfies that $\overline{f} \restriction \alpha$ has an exact upper bound. Now it is clear how to replace $\overline{f}$ by a smooth λ-scale.

Suppose now that $\mathfrak{b}_\mu = \lambda^+$, $\lambda = \operatorname{cf}\lambda$. By [14], 4.1, the set $S^{\lambda^+}_{<\lambda} := \{\alpha : \alpha < \lambda^+ \wedge \operatorname{cf}\alpha < \lambda\}$ is a union of λ sets, each of which carries a square sequence. Therefore, $S^{\lambda^+}_{<\lambda} \in I[\lambda]$. By 2.5 in chapter 1 of [15], there exists a $(\mu, \mathfrak{b}_\mu)$-scale in which all points of cofinality $< \mu$ are flat and therefore a smooth $(\mu, \mathfrak{b}_\mu)$-scale. $\square$

In contrast to the case of singular μ, let us mention the following result of A. Blass [4], which generalizes Hechler's result [8]: it is consistent that $\operatorname{MAD}(\aleph_0) = C$, for any prescribed closed set of uncountable cardinals C that satisfies that $[\aleph_1, \aleph_1 + |C|] \subseteq C$ and $\lambda^+ \in C$ whenever $\lambda \in C$ has countable cofinality. For example, by Blass' or by Hechler's results there are universes of set theory in which $\operatorname{MAD}(\aleph_0) = \{\aleph_1, \aleph_{\omega+1}\}$. By Corollary 2.5, in any universe that satisfies this, it follows that $[\aleph_1, \aleph_{\omega+1}] \subseteq \operatorname{MAD}(\aleph_\omega)$.

Recently Brendle [5], using techniques from [16], proved the consistency of $\mathfrak{a} = \aleph_\omega$.

Problem 2.10. Is it consistent that $\mathfrak{a}_{\aleph_\omega} = \aleph_\omega$?

3. Consistency results on $\operatorname{MAD}(\aleph_\omega)$ from large cardinal axioms

The inequality (1) can be used to control $\operatorname{MAD}(\aleph_\omega)$ by first increasing $\mathfrak{b}_{\aleph_\omega}$ and then increasing $\mathfrak{b}$. PCF theory implies that whenever the SCH fails at a singular cardinal μ, it follows that $\mathfrak{b}_\mu > \mu^+$. On the other hand, $\mathfrak{b}_\mu$ cannot be changed by a ccc forcing.

Before we state the result, let us recall some pcf terminology:

$$\operatorname{pcf}\{\aleph_n : n < \omega\} = \Big\{\mathfrak{b}\big(\prod_n \aleph_n, \leqslant_I\big) : I \subseteq \mathcal{P}(\omega) \text{ is a proper ideal}\Big\}.$$

The relation $<_I$ is defined by $f <_I g \Leftrightarrow \{n : f(n) \geqslant g(n)\} \in I$.

$\operatorname{pcf}\{\aleph_n : n < \omega\}$ is an interval of regular cardinals and has a maximum. For every $\lambda \in \operatorname{pcf}\{\aleph_n : n < \omega\}$ there exists a *pcf generator* $B_\lambda \subseteq \omega$ so that the following holds: denote by $J_{<\lambda}$ the ideal that is generated by $\{B_\theta : \theta \in \operatorname{pcf}\{\aleph_n : n < \omega\} \wedge \theta < \lambda\}$; then

$$\lambda = \mathfrak{b}\big(\prod_n \aleph_n, \leqslant_{J_{<\lambda}}\big).$$

Finally, $(\aleph_\omega)^{\aleph_0} = \max\operatorname{pcf}\{\aleph_n : n < \omega\} \times 2^{\aleph_0}$. Therefore, if $\aleph_\omega$ is a strong limit, $2^{\aleph_\omega} = \max\operatorname{pcf}\{\aleph_n : n < \omega\}$.

Fact 3.1. For every $\beta < \omega_1$ it is consistent (from large cardinal axioms) that $2^{\aleph_\omega} = \mathfrak{b}_\mu = \aleph_{\omega+\beta+1}$.

Proof. Let V be any universe of set theory in which $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} = \max\operatorname{pcf}\{\aleph_n : n \in \omega\} = \aleph_{\omega+\beta+1}$ [13], [9].

In V, the ideal $J_{<\max\operatorname{pcf}\{\aleph_n : n<\omega\}}$ is proper and is generated by countably many sets. Therefore, by simple diagonalization there exists an infinite $B \subseteq \omega$ so that

$J_{<\max \mathrm{pcf}\{\aleph_n : n<\omega\}} \restriction B$ is contained in the ideal of finite subsets of B. Since $\mathfrak{b}(\prod_n \aleph_n, \leqslant_{J_{<\max \mathrm{pcf}\{\aleph_n:n<\omega\}}}) = \aleph_{\omega+\beta+1}$, it follows that $\mathfrak{b}(\prod_{n\in B} \aleph_n, \leqslant^*) = \aleph_{\omega+\beta+1}$; hence $\mathfrak{b}_{\aleph_\omega} = \aleph_{\omega+\beta+1}$. □

Theorem 3.2. *For every $\beta < \omega_1$ and $\alpha \leqslant \omega + \beta + 2$, it is consistent (from large cardinals) that $2^{\aleph_\omega} = \aleph_{\omega+\beta+2}$ and $\mathrm{MAD}(\aleph_\omega) = [\aleph_\alpha, \aleph_{\omega+\beta+2}]$.*

Proof. Start from a model V in which $2^{\aleph_0} = \aleph_1$, $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+\beta+2}$. Such a model exists by the previous Fact.

For every regular $\aleph_\omega < \lambda \leqslant \aleph_{\omega+\beta+2}$, there is a smooth λ-scale by Lemma 2.9. Consequently, there is also a smooth $\aleph_{\omega+\beta+2}$-scale.

Now apply Theorem 2.3 to finish the proof. □

By Theorem 5.4(b) in [3], after adding many Cohen subsets to ω_1, $\max \mathrm{MAD}(\aleph_\omega)$ does not increase by much. Therefore, it is consistent to have $\mathrm{MAD}(\aleph_\omega) = [\aleph_1, \aleph_{\omega+\beta+2}]$ as above, and to have $2^{\aleph_\omega}$ arbitrarily large.

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