



A game on partial orderings[☆]

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Abstract

We study the determinacy of the game $G_\kappa(A)$ introduced in Fuchino, Koppelberg and Shelah (to appear) for uncountable regular κ and several classes of partial orderings A . Among trees or Boolean algebras, we can always find an A such that $G_\kappa(A)$ is undetermined. For the class of linear orders, the existence of such A depends on the size of $\kappa^{<\kappa}$. In particular we obtain a characterization of $\kappa^{<\kappa} = \kappa$ in terms of determinacy of the game $G_\kappa(L)$ for linear orders L .

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We consider in this paper the question whether for every partially ordered set (A, \leq) , the game $G_\kappa(A)$ described below is determined, i.e., whether one of the players has a winning strategy. Here and in the following, except for the motivation given below, κ is always a regular uncountable cardinal. More precisely we study the question for trees, Boolean algebras and linear orderings. In fact there are trees, respectively Boolean algebras, A of size κ^+ for which $G_\kappa(A)$ is not determined (Propositions 6 and 11); for linear orders, the situation is more complex: if $\kappa^{<\kappa} = \kappa$, then for every linear order L , $G_\kappa(L)$ is determined (Proposition 2); otherwise there is a linear order L of size κ^+ such that $G_\kappa(L)$ is not determined (Proposition 8).

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The motivation for this question comes from the paper [1] which in turn was motivated by [2]. A Boolean algebra A is said to have the Freese–Nation property if there exists a function f which assigns to every $a \in A$ a finite subset $f(a)$ of A such that if $a, b \in A$ satisfy $a \leq b$, then $a \leq x \leq b$ holds for some $x \in f(a) \cap f(b)$. This property is closely related to projectivity; in fact, every projective Boolean algebra has the Freese–Nation property (but not conversely). Heindorf proved that the Freese–Nation property is equivalent to open-generatedness, a notion originally introduced in topology by Ščepin. In [1], it is generalized to from ω to regular cardinals κ and from Boolean algebras to arbitrary partial orderings. This generalization is called κ -Freese–Nation property and the following equivalence was proved: a partial ordering A has the κ -Freese–Nation property iff there is a closed unbounded subset C of $[A]^\kappa$ such that $C \leq_\kappa A$ holds (see below for the definition) for all $C \in \mathbb{C}$ iff in the game $G_\kappa(A)$, Player II has a winning strategy. In fact, in all examples considered in [1], either I or II has a winning strategy.

Let us define the game $G_\kappa(A)$ and some relevant notions for a partial ordering A . $X \subseteq A$ is said to be cofinal (coinitial) in A if, for every $a \in A$, there is some $x \in X$ such that $a \leq x$ ($a \geq x$). $\text{cf } A$ ($\text{ci } A$ respectively) is the smallest cardinality of a cofinal (coinitial respectively) subset of A .

For $R \subseteq A$ and $a \in A$, we write $R \uparrow a$ for the set $\{x \in R: a \leq x\}$ and $R \downarrow a$ for $\{x \in R: x \leq a\}$. The *type* of a over R is the pair

$$\text{tp}(a, R) = (\text{cf } R \downarrow a, \text{ci } R \uparrow a).$$

$R \subseteq A$ is said to be a κ -subset or a κ -substructure of A , written $R \leq_\kappa A$, if for all $a \in A$, the sets $R \downarrow a$ and $R \uparrow a$ have cofinality respectively coinitiality less than κ .

The game $G_\kappa(A)$ is played on A as follows. Players I and II alternately choose terms of an increasing chain of subsets x_α and y_α of A for $\alpha < \kappa$ (i.e., I chooses x_0 , II chooses y_0 , I chooses x_1 , II chooses y_1 , etc.) such that x_α and y_α have size less than κ , $x_\alpha \subseteq y_\alpha$ and $\bigcup_{\nu < \alpha} y_\nu \subseteq x_\alpha$. In the end of a play, II wins iff the result $R = \bigcup_{\alpha < \kappa} x_\alpha = \bigcup_{\alpha < \kappa} y_\alpha$ of the play is a κ -subset of A ; I wins otherwise.

Note that in this game, Player II has a winning strategy for any partial ordering A of size at most κ : she plays so that every element of A is gradually captured in one of the y_α 's.

The main body of the paper is organized as follows. In 5 we define a tree $T = T(S)$ depending on a subset S of $\lambda = \kappa^+$. If neither S nor $\lambda \setminus S$ are in the ideal I_λ defined in 3 then T is not determined (Proposition 6). From T we define a linear order L_T in 7. and a Boolean algebra B_T in 10. such that $G_\kappa(L_T)$ and $G_\kappa(B_T)$ are not determined (Propositions 8 and 11). The construction of L_T requires the extra assumption $\kappa^{<\kappa} > \kappa$ — cf. Proposition 2.

Let us start with an easy example.

Example 1. If κ^+ (or $(\kappa^+)^*$, the reverse order type of κ^+) embeds into A , then Player I has a winning strategy in $G_\kappa(A)$: assume, for simplicity, that $\kappa^+ \subseteq A$. We define a partial function f from A into κ^+ by letting $f(a)$ for $a \in A$ be the least $\alpha \in \kappa^+$ such

that $a \leq \alpha$, if such an α exists. Clearly f is order preserving and satisfies $f(a) = a$ for $a \in \kappa^+$. Player I wins by assuring that the result R of a play satisfies

- (a) $R \cap \kappa^+$ has cofinality κ ,
- (b) if $a \in R$ and $f(a)$ exists, then $f(a) \in R$.

The following proposition shows that the assumption $\kappa^{<\kappa} > \kappa$ in 6 and 7 cannot be dispensed with.

Proposition 2. *Assume that $\kappa^{<\kappa} = \kappa$. If $(L, <_L)$ is a linear order of cardinality $> \kappa$, then Player I has a winning strategy in $G_\kappa(L)$. Hence the game $G_\kappa(L)$ is determined for any linear order L whenever $\kappa^{<\kappa} = \kappa$.*

Proof. Let χ be sufficiently large. $\mathcal{H}(\chi)$ denotes the set of all sets which are hereditarily of size less than χ . We show:

Claim. *Suppose M is an elementary submodel of $(\mathcal{H}(\chi), \in)$ such that $(L, <_L) \in M$ and ${}^{<\kappa}M \subseteq M$. Then for any $d \in L \setminus M$, either $L \cap M \downarrow d$ has cofinality $\geq \kappa$ or $L \cap M \uparrow d$ has coinitality $\geq \kappa$.*

Proof of Claim. Otherwise, some $d \in L \setminus M$ fills a gap (X, Y) in $L \cap M$ such that $|X|, |Y| < \kappa$ and (X, Y) is unfilled inside $L \cap M$. But $(X, Y) \in M$ by ${}^{>\kappa}M \subseteq M$ and $M \prec \mathcal{H}(\chi)$, a contradiction. \square

Now Player I wins in $G_\kappa(L)$ by choosing an increasing sequence M_α , $\alpha < \kappa$, of elementary submodels of $\mathcal{H}(\chi)$ along with his moves x_α , $\alpha < \kappa$, such that $(L, <_L) \in M_0$, $x_\alpha \subseteq M_\alpha$, ${}^{<\kappa}M_\alpha \subseteq M_\alpha$, $|M_\alpha| = \kappa$ and $\bigcup_{\alpha < \kappa} x_\alpha = M \cap L$ where $M = \bigcup_{\alpha < \kappa} M_\alpha$. Such a choice is possible because of our assumption $\kappa^{<\kappa} = \kappa$. The result of the game $L \cap M$ is not a κ -subset of L , by Claim above. \square

Construction 3 (of the ideal I_λ). For the rest of the paper, fix $\lambda = \kappa^+$ (where κ was a regular uncountable cardinal). Let us first recall the definition and some properties of the ideal I_λ on λ introduced by Shelah, see, e.g., [4, Chapter VIII]. Fix a sufficiently large cardinal $\chi > \lambda$; we work in the structure $(\mathcal{H}(\chi), \in, <^*)$ where $<^*$ is some fixed well-ordering of $\mathcal{H}(\chi)$. For $x \in \mathcal{H}(\chi)$ and $\gamma < \lambda$, call $(M_i)_{i < \kappa}$ an x -approximation of γ if:

- (1) $M_i \prec (\mathcal{H}(\chi), \in, <^*)$, $|M_i| < \kappa$,
- (2) $x, \lambda \in M_0$,
- (3) $(M_i)_{i < \kappa}$ is a continuously increasing chain,
- (4) $(M_i)_{i \leq j} \in M_{j+1}$ for all $j < \kappa$,
- (5) $M = \bigcup_{i < \kappa} M_i$ satisfies $M \cap \lambda = \gamma$.

For $x \in \mathcal{H}(\chi)$, put $C_x = \{\gamma \in \lambda: \text{there is an } x\text{-approximation of } \gamma\}$ and define I_λ by

$$I_\lambda = \{A \subseteq \lambda: A \cap C_x = \emptyset, \text{ for some } x \in \mathcal{H}(\chi)\}.$$

Then I_λ is a λ -complete proper ideal containing all singletons and

$$N = \{\gamma \in \lambda: \text{cf } \gamma = \kappa\} \in I_\lambda^*$$

(i.e., $\lambda \setminus N \in I_\lambda$). By Ulam's Theorem (cf. [3, 27.8]), every $A \subseteq \lambda$ not in I_λ can be represented as the disjoint union $A = A_1 \cup A_2$ where $A_1, A_2 \notin I_\lambda$.

Game 4 ($G_\kappa(T)$ for a tree T). Assume that $(T, <_T)$ is a tree of height $\kappa + 1$. We call $Y \subseteq T$ a *subtree of T* if for all $y \in Y$: if $x <_T y$ then $x \in Y$. Y is *closed in T* if the following holds: if $x \in T$ is in the κ th level and all predecessors of x are in Y , then $x \in Y$.

In $G_\kappa(T)$ each of the players can ensure that the result Y of a play will be a subtree of T . In this case Player II wins (i.e., $Y \leq_\kappa T$) iff Y is closed in T .

Construction 5 (of the tree $T = T(S)$). Recall that $\lambda = \kappa^+$ and $N = \{\gamma \in \lambda: \text{cf } \gamma = \kappa\}$. For each subset S of N , we construct a tree $T = T(S)$; in fact, we shall show that if $T = T(S)$ where $S \subseteq N$ and $S, N \setminus S \notin I_\lambda$, then none of the players has a winning strategy.

Assume $S \subseteq N$. For each $\gamma \in S$, fix a function $f_\gamma: \kappa \rightarrow \gamma$ such that range f_γ is cofinal in γ . Let

$$T = T(S) = \{f_\gamma \upharpoonright \alpha: \gamma \in S, \alpha \leq \kappa\},$$

a tree under set-theoretic inclusion. Clearly T has height $\kappa + 1$ if S is nonempty, $\{f_\gamma: \gamma \in S\}$ is the κ th level of T , and $|T| = \lambda$ if $|S| = \lambda$.

Proposition 6. Let $T = T(S)$ for $S \subseteq N$.

(a) If $S \notin I_\lambda$, then Player II has no winning strategy in $G_\kappa(T)$.

(b) If $N \setminus S \notin I_\lambda$, then Player I has no winning strategy in $G_\kappa(T)$.

Thus if both S and $N \setminus S$ are not in I_λ , then the game $G_\kappa(T)$ is undetermined.

Proof. (a) Suppose that σ is a strategy for Player II; we show that it is not a winning strategy. Let $x = (\sigma, (f_\gamma)_{\gamma \in S})$. Since $S \notin I_\lambda$, there is a $\delta \in S \cap C_x$; let $(M_i)_{i < \kappa}$ be an x -approximation of δ . In a game in which Player II plays according to σ , Player I can ensure that the result $Y \subseteq T$ of the play will be the subtree

$$Y = \{f_\gamma \upharpoonright \alpha: \gamma \in S \cap \delta, \alpha \leq \kappa\}.$$

More precisely, in the i th move, Player I may take a subset x_i of $T \cap M_{i+1}$ so that all elements of Y are gradually captured. Furthermore, using the well-ordering $<^*$, Player I can ensure that each of his moves x_i is definable so that $(x_j, y_k)_{j \leq i, k < i}$ and hence also the next move $\sigma((x_j, y_k)_{j \leq i, k < i})$ by Player II will be an element of M_{i+1} .

Now $\delta \in S$ and thus f_δ witnesses that Y is not closed in T , i.e., Player I wins.

The proof of (b) is similar to (a). If Player I plays according to a strategy τ , Player II can assure that the result $Y \subseteq T$ has the form $Y = \{f_\gamma \upharpoonright \alpha: \gamma \in S \cap \delta, \alpha \leq \kappa\}$ for some $\delta \in N \setminus S$. Thus Y is closed in T and Player II wins. \square

Construction 7 (of the linear order L_T). Assume that $\kappa^{<\kappa} > \kappa$; let $(T, <_T)$ be any tree of height $\kappa + 1$ and size $\lambda = \kappa^+$. We shall construct a linear order $L = L_T$ of size λ . Moreover, we shall define for each $Y \subseteq T$ a subset L_Y of L such that $|L_Y| = |Y|$ holds

for infinite Y and in the game $G_\kappa(L)$, each player can ensure that the result R has the form L_Y for Y a subtree of T .

Let us first note that there exists a linear order I of size λ without any sequences (i.e., increasing or decreasing sequences) of type κ . This holds because our assumption $\kappa^{<\kappa} \geq \kappa^+ = \lambda$ implies that $\lambda \leq 2^\mu$, for some $\mu < \kappa$, and the lexicographic ordering on ${}^\mu 2$ has no sequence of type μ^+ (cf. [3, 29.4]), hence no sequence of type κ . It follows that, letting I be any subordering of ${}^\mu 2$ of cardinality λ , every subset of I has cofinality and coinitality less than κ .

The following notation concerning the tree $(T, <_T)$ will be used in the rest of 7 and in 8: for $\alpha \leq \kappa$, $\text{lev}_\alpha T$ is the α th level of T . For $t \in T$, $\text{pred } t$ is the set of predecessors of t in T and $\text{ht } t$ is the height of t . For $\alpha \leq \text{ht } t$, $\text{pr}_\alpha t$, the projection of t to level α , is the unique predecessor of t in the α th level. Call $x, y \in T$ equivalent and write $x \sim y$ if $\text{pred } x = \text{pred } y$ and let \bar{x} be the equivalence class of x . For each equivalence class \bar{x} , since $|\bar{x}| \leq \lambda$, we can fix a linear order $\leq_{\bar{x}}$ on \bar{x} without any sequences of type κ .

The linear order we construct is a sort of squashing of T with respect to $\leq_{\bar{x}}$, $x \in T$: we put $L = \{a_t, b_t : t \in T\}$ where the elements $a_t, b_t, t \in T$, are all pairwise distinct. The linear order $<_L$ on L is defined as follows: we will have $a_t <_L b_t$ for all $t \in T$. Now assume $x, y \in T$. If $x <_T y$, then we put $a_x <_L a_y <_L b_y <_L b_x$. If x and y are incomparable in T , let $\alpha \leq \kappa$ be minimal such that $\text{pr}_\alpha x \neq \text{pr}_\alpha y$; thus $\text{pr}_\alpha x \sim \text{pr}_\alpha y$. Then if $\text{pr}_\alpha x <_{\overline{\text{pr}_\alpha x}} \text{pr}_\alpha y$, we let $a_x <_L b_x <_L a_y <_L b_y$. Finally, for $Y \subseteq T$ let $L_Y = \{a_t, b_t : t \in Y\}$.

Proposition 8. *If Y is a subtree of T , then $L_Y \leq_\kappa L_T$ iff Y is closed in T . In particular, if $G_\kappa(T)$ is undetermined, then so is $G_\kappa(L_T)$.*

From Propositions 2, 6 and 8 (plus the observation in 7 that $\kappa^{<\kappa} > \kappa$ implies the existence of a linear order of size λ without sequences of type κ), we obtain the following equivalences to the condition $\kappa^{<\kappa} = \kappa$.

Corollary 9. *Let κ be a regular uncountable cardinal.*

(a) *If $\kappa^{<\kappa} > \kappa$, then there is a linear order L of cardinality $\lambda = \kappa^+$ such that $G_\kappa(L)$ is undetermined.*

(b) *The following are equivalent:*

- (1) $\kappa^{<\kappa} = \kappa$;
- (2) *in every linear order of cardinality $> \kappa$, there is an increasing or a decreasing sequence of order type κ ;*
- (3) $G_\kappa(L)$ *is determined for every linear order L .*

Let us explain how the second assertion of Proposition 8 follows from the first one: each of the players in $G_\kappa(L_T)$ (say II, playing against some strategy τ of Player I) can ensure that the result of the play is $R = L_Y$, for some subtree Y of T . Playing simultaneously on T as in the proof of Proposition 6, II can ensure that Y is closed in T . Thus $R = L_Y$ is a κ -substructure of L_T and she wins.

Proof of Proposition 8. Suppose first that Y is not closed. Pick some t in the highest level K of T such that $t \notin Y$ but $\text{pred } t \subseteq Y$. Then $\{a_y: y \in \text{pred } t\}$ is an increasing sequence of type κ , and it is a cofinal subset of $L_Y \downarrow l$ where $l = a_t$. Thus L_Y is not a κ -subset of L .

Now assume that Y is closed in T . Fix $l \in L \setminus L_Y$. We have to analyze the cofinality of $L_Y \downarrow l$ and the coinitality of $L_Y \uparrow l$; by symmetry, we will consider only $\text{cf}(L_Y \downarrow l)$. Now let $l = a_t$ or $l = b_t$ for some $t \in T \setminus Y$; since Y is a subtree of T , a_t and b_t realize the same cut in L_Y . Thus we assume that $l = a_t$.

We may also assume that $\text{ht } t < \kappa$ and $\text{pred } t \subseteq Y$. For this, consider the least element t^* of $\text{pred } t \setminus Y$. Now $\text{ht } t^* < \kappa$ since Y is a closed subtree of T ; moreover, a_t and a_{t^*} realize the same cut in L_Y . Thus we consider t^* instead of t .

To prove $\text{cf}(L_Y \downarrow l) < \kappa$, consider the following subsets of L respectively Y : let

$$N = \{a_x: x <_T t\};$$

thus N is a subset of $L_Y \downarrow l$ of size less than κ . Next, put $\gamma = \text{ht } t$ and

$$Y' = \{z \in Y: z \in \text{lev}_\gamma T, z \sim t, z <_{\bar{t}} t\}.$$

Y' is included in the \sim -equivalence class of t , thus it has a cofinal subset Y'' of size less than κ . We put

$$N' = \{b_z: z \in Y''\},$$

again a subset of $L_Y \downarrow l$ of size less than κ .

We prove that $N \cup N'$ is cofinal in $L_Y \downarrow l$. For, let $x \in L_Y$ and $x <_L l$, say $x \in \{a_y, b_y\}$ where $y \in Y$. Consider the relative position of t and y in T . It is impossible that $t <_T y$, since Y is a subtree of T and $t \notin Y$.

If $y <_T t$, then $a_y <_L a_t = l <_L b_t <_L b_y$ holds, hence $x = a_y \in N$. Otherwise, let α be minimal such that $\text{pr}_\alpha y \neq \text{pr}_\alpha t$; thus $\alpha \leq \gamma$.

If $\alpha < \gamma$, then let $z = \text{pr}_\alpha t$; it follows that $x \leq_L b_y <_L a_z \in N$. Otherwise $\alpha = \gamma$, $\text{pr}_\alpha y \sim t$ and hence $\text{pr}_\alpha y \in Y'$. Take $z \in Y''$ such that $\text{pr}_\alpha y \leq_{\bar{t}} z$; then $x \leq b_y \leq b_z \in N'$. \square

Construction 10 (of the Boolean algebra B_T). Let $(T, <_T)$ be any tree of height $\kappa + 1$ and size λ . We shall construct a Boolean algebra B_T of size λ . Moreover, we shall define for $Y \subseteq T$ a subalgebra B_Y of B_T such that $|B_Y| = |Y|$ holds for infinite Y . In the game $G_\kappa(B_T)$, each player can ensure that the result R has the form B_Y for Y a subtree of T .

In fact, we define B_T to be the Boolean algebra generated by a set $\{x_t: t \in T\}$ freely except that $s \leq_T t$ implies $x_s \leq x_t$. More precisely, let $\text{Fr}(x_t: t \in T)$ be the free Boolean algebra over $\{x_t: t \in T\}$, let B_T be the quotient algebra $\text{Fr}(x_t: t \in T)/K$ where K is the ideal of $\text{Fr}(x_t: t \in T)$ generated by $\{x_s \cdot \neg x_t: s \leq_T t\}$ and let $\pi: \text{Fr}(x_t: t \in T) \rightarrow B_T$ be the canonical homomorphism. We write $x_t (\in B_T)$ for $\pi(x_t)$, since π is one-one on the generators x_t (see the proof of 11 below). For $Y \subseteq T$, we define B_Y to be the subalgebra of B_T generated by $\{x_t: t \in Y\}$.

Proposition 11. *If Y is a subtree of T , then $B_Y \leq_\kappa B_T$ iff Y is closed in T . In particular, if $G_\kappa(T)$ is undetermined, then so is $G_\kappa(B_T)$.*

Proof. Similarly to the remark after Corollary 9 it is easy to see that the second assertion follows from the first. For the first assertion we start with a normal form lemma on the generators of B_T .

Step 1. Let $w \subseteq T$ be finite and assume $f: w \rightarrow 2$. Then the elementary product

$$q_f = \prod_{f(t)=1} x_t \cdot \prod_{f(t)=0} -x_t$$

is nonzero in B_T iff f is monotone, i.e., $s \leq_T t$ in w implies $f(s) \leq f(t)$. This follows immediately from the definition of the ideal K of $\text{Fr}(x_t: t \in T)$ in 9.

Step 2. If $Y \subseteq T$ is not closed, then B_Y is not a κ -subalgebra of B_T .

To see this, fix an element t in the highest (i.e., κ th) level of T such that $t \notin Y$ but all predecessors of t in T are in Y and consider the ideal $I = B_Y \downarrow x_t$ of B_Y . The set $J = \{x_s: s <_T t\}$ is a chain of order type κ included in I ; we show that J generates I as an ideal. Thus suppose $x \in I$ with the aim of finding some $s <_T t$ such that $x \leq x_s$. We may assume that x is a nonzero elementary product q_f where $f: w \rightarrow 2$. By $q_f \leq x_t$ and Step 1, it follows that f is monotone but $f \cup \{(t, 0)\}$ is not. Hence there is some $s \in w$ such that $s <_T t$ and $f(s) = 1$; thus $x = q_f \leq x_s$.

Step 3. The following remark simplifies Step 4: assume B is a Boolean algebra, A a subalgebra and M, N are finite subsets of B such that for all $m \in M$ and $n \in N$, there is an element α of A separating m and n , i.e., we have $m \leq \alpha$ and $n \leq -\alpha$ or $n \leq \alpha$ and $m \leq -\alpha$. Then there is an $a \in A$ separating $\sum M$ and $\sum N$: simply let

$$a = \prod_{n \in N} \sum_{m \in M} a_{mn}$$

where $a_{mn} \in A$ is such that $m \leq a_{mn}$ and $n \leq -a_{mn}$.

Step 4. If Y is a closed subtree of T , then $B_Y \leq_\kappa B_T$.

For the proof, fix an element b of B_T and consider the ideal

$$I = \{x \in B_Y: x \cdot b = 0\}$$

of B_Y . We shall find $Z \subseteq T$ such that $|Z| < \kappa$ and each element of I is separated from b by an element of B_Z ; since $|B_Z| < \kappa$, this shows that I is generated by less than κ elements.

Fix a finite subset of T generating b , say

$$b \in \langle x_{s_1}, \dots, x_{s_n}, x_{t_1}, \dots, x_{t_m} \rangle,$$

where every s_i is in Y and every t_j is in $T \setminus Y$. We put

$$Z = \{s_1, \dots, s_n\} \cup \bigcup \{\text{pred } t_j \cap Y: 1 \leq j \leq m\},$$

where, for $t \in T$, $\text{pred } t$ is the set of predecessors of t in the tree $(T, <_T)$. Z has size less than κ since Y is closed and a subtree of T .

Now let $x \in I$ with the aim of finding an element of B_Z which separates x and b . By Step 3, we may assume that both b and x are elementary products over the generators of B_T , say

$$\begin{aligned} b &= q_h, & h: \{s_1, \dots, s_n, t_1, \dots, t_m\} &\rightarrow 2, \\ x &= q_f, & f: w &\rightarrow 2, \quad w \subseteq Y, \end{aligned}$$

where h and f are monotone. Define

$$h' = h \upharpoonright \{s_1, \dots, s_n\}, \quad f' = f \upharpoonright (w \cap Z);$$

we show that either $q_{h'}$ or $q_{f'}$ separate x and b .

Case 1. $f \cup h'$ is not a function or not monotone. Then $b \leq q_{h'}$ and $x \cdot q_{h'} = 0$.

Note that if Case 1 does not hold, then also $f \cup h$ is a function: otherwise, let $r \in w \cap \{s_1, \dots, s_n, t_1, \dots, t_m\}$ be such that $f(r) \neq h(r)$. Then $r \in Y$ and thus $r = s_i$ for some i , hence $r \in \text{dom } f \cap \text{dom } h'$. Note also that, since $x \cdot b = 0$, $f \cup h$ cannot be monotone. Hence the remaining case is the following.

Case 2. $f \cup h'$ is a monotone function and $f \cup h$ is a function but not monotone. In this case, there are $r, u \in T$ such that $r <_T u$ and $f(r) = 1$, $h(u) = 0$. For otherwise, we have $r <_T u$ satisfying $h(r) = 1$, $f(u) = 0$. It follows that $u \in w \subseteq Y$, $r \in Y$ since Y is a subtree of T , and $r \in \text{dom } h'$, contradicting the fact that $f \cup h'$ is monotone.

Now $r \in w \subseteq Y$ and $u \in \{s_1, \dots, s_n, t_1, \dots, t_m\}$. In fact, $u = t_j$ for some j , since $u = s_i$ would imply that $u \in \text{dom } h'$, but $f \cup h'$ was monotone. But then $r \in \text{pred } t_j \cap Y \subseteq Z$, $r \in \text{dom } f'$, and $f' \cup h$ is not monotone. Thus $b \cdot q_{f'} = q_h \cdot q_{f'} = 0$ and $x = q_f \leq q_{f'}$ show that $q_{f'}$ separates x and b . \square

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