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SPECIALIZING ARONSZAJN TREES AND PRESERVING SOME WEAK DIAMONDS

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Abstract. We show that $\diamondsuit(\mathbb{R}, \mathcal{N}, \in)$ together with CH and "all Aronszajn trees are special" is consistent relative to ZFC. The weak diamond for the covering relation of Lebesgue null sets was the only weak diamond in the Cichoń diagramme for relations whose consistency together with "all Aronszajn trees are special" was not yet settled. Our forcing proof gives also new proofs to the known consistencies of several other weak diamonds stemming from the Cichoń diagramme together with "all Aronszajn trees are special" and CH. The main part of our work is an application [15, Chapter V, §§ 1–7] for a special completeness system, such that we have a genericity game. Thus we show new preservation properties of the known forcings.

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1. Introduction

Let A and B be sets of reals and let $E \subseteq A \times B$. Here we work only with Borel sets A and B and absolute E, so that there are no difficulties in the interpretation of the notions in various ZFC models. The set A carries the topology inherited from the reals and 2^α carries the product topology. A function $F\colon 2^{<\omega_1} \longrightarrow A$ is called Borel function if each part $F \restriction 2^\alpha$, $\alpha < \omega_1$, is a Borel function. The complexity of the set of $\aleph_1$ parts can be high.

Definition 1.1 (Definition 4.4. of [14]). Let $\diamondsuit(A, B, E)$ be the following statement: For every Borel map $F\colon 2^{<\omega_1} \longrightarrow A$ there is some $g\colon \omega_1 \longrightarrow B$ such that for every $f\colon \omega_1 \longrightarrow 2$ the set

$$\{\alpha \in \omega_1 \,:\, F(f \restriction \alpha) E g(\alpha)\}$$

is stationary. Commonly, if E is not the equality $\diamondsuit(A, B, E)$ is called a weak diamond.

The original diamond, $\diamondsuit_{\omega_1}$, is $\diamondsuit(A, B, E)$ with $A = B = 2^{<\omega_1}$ (so here we do not have subsets of the reals), E being equality, in the special case of F being the identity function. Jensen [9] showed that $\diamondsuit_{\omega_1}$ holds in L. Devlin and Shelah [7] showed that in the case $|B| = 2$ some diamond principles follow from $2^{\aleph_0} < 2^{\aleph_1}$.

In the mentioned work Jensen also showed that $\diamondsuit_{\omega_1}$ implies the existence of a Souslin tree. Since then it has been interesting to investigate which weakenings of $\diamondsuit_{\omega_1}$ still imply the existence of a Souslin tree. Moore, Hrušák and Džamonja [14] introduce and investigate numerous versions of weak diamonds. Let $\mathrm{Unif}(\mathcal{M})$ denote the relation (F_σ meager sets, $\omega^\omega, \not\ni$), and let $\mathrm{Unif}(\mathcal{N})$ denote the relation (G_δ null sets, $\omega^\omega, \not\ni$). They show that $\diamondsuit(\mathrm{Unif}(\mathcal{M}))$ implies the existence of a Souslin tree, and from work by Hirschorn [8] they derive that $\diamondsuit(\mathrm{Unif}(\mathcal{N}))$ does not imply the existence of a Souslin tree. Another model (with larger continuum) is given by Laver [11]. Since the Borel Galois-Tukey connections (see Vojtáš [16]) in the Cichoń diagramme can be translated into implications of the corresponding weak diamonds [14, Proposition 4.9], there is a Cichoń's diagramme of weak diamonds. So all its entries above $\diamondsuit(\mathrm{Unif}(\mathcal{M}))$ imply the existence of a Souslin tree, see Figure 1.

Also $\diamondsuit(\omega^\omega, \omega^\omega, \leq^*)$ together with "all Aronszajn trees are special" is consistent relative to ZFC according to [12]. In this model, the continuum is $\aleph_2$.

So, before this work, there was one question regarding the existence of Souslin trees and the weak diamonds in Cichoń's diagramme left open: Does the weak diamond for the covering relation ($\mathbb{R}, F_\sigma$ null sets$, \in$) imply that there is a Souslin tree? The answer is negative:

$$\begin{tikzcd}
\diamondsuit(\mathrm{Cov}(\mathcal{N})) \arrow[dd] & \boxed{\diamondsuit(\mathrm{Unif}(\mathcal{M}))} \arrow[l] \arrow[d] & \boxed{\diamondsuit(\mathrm{Cof}(\mathcal{M}))} \arrow[l] \arrow[d] & \boxed{\diamondsuit(\mathrm{Cof}(\mathcal{N}))} \arrow[l] \arrow[dd] \\
 & \diamondsuit(\omega^\omega, \omega^\omega, \not\geq^*) \arrow[d] & \diamondsuit(\omega^\omega, \omega^\omega, \leq^*) \arrow[l] \arrow[d] & \\
\diamondsuit(\mathrm{Add}(\mathcal{N})) & \diamondsuit(\mathrm{Add}(\mathcal{M})) \arrow[l] & \diamondsuit(\mathrm{Cov}(\mathcal{M})) \arrow[l] & \diamondsuit(\mathrm{Unif}(\mathcal{N})) \arrow[l]
\end{tikzcd}$$

FIGURE 1. The framed weak diamonds imply the existence of a Souslin tree. The arrows indicate implications.

Theorem 1.2. *$\diamondsuit(\mathbb{R}, F_\sigma$ null sets$, \in)$ together with CH and with "all Aronszajn trees are special" is consistent relative to ZFC.*

Now we give an outline: An essential tool in the analysis of proper forcings are countable elementary substructures: We let $\chi > 2^{\aleph_2}$ (this is the concrete interpretation of the phrase "sufficiently large" in our context, and sometimes smaller lower bounds suffice, but let us be definite) be regular and denote by $H(\chi)$ the set of all sets of hereditary cardinality $< \chi$. Let $<^*_\chi$ be a fixed well-ordering of $H(\chi)$ such that $x \in y$ implies $x <^*_\chi y$. We work with countable elementary substructures $M \prec (H(\chi), \in)$, and when we want to perform constructions along a well-order we take $M \prec (H(\chi), \in, <^*_\chi)$. There are at most $2^{\aleph_0}$ isomorphism types of transitive collapses $(N, \in, (<^*_\chi)^N)$ of $(M, \in, <^*_\chi)$. By our proviso on $<^*_\chi$, the relation $(<^*_\chi)^N$ is still a well-order. In general we let the letter N (also with subscripts) stand for transitive models (Mostowski collapses of the M's), and let M stand for a countable elementary submodel.

We shall define a game played in countable parts of the iterated proper forcings from [15, Chapter V, Section 5]. The countable elementary submodel M, P, $p \in P \cap M$, $\underset{\sim}{f}$, $\dots$ are parameters. The number of rounds is $\alpha = \mathrm{otp}(M \cap \gamma)$, where γ is the iteration length. The generic player gives a real ν_ε and the antigeneric player gives a real η_ε dominating it in round $\varepsilon < \alpha$. The strategy of the game depends only on the isomorphism type of the Mostowski collapse of the given countable elementary submodel (M, P_γ, p), P_γ an iteration of length γ. In the central Theorem 3.4, we prove the existence of a Borel functions $\mathbf{B}_\alpha \colon (\omega^\omega)^\alpha \times \mathcal{P}(\omega) \longrightarrow \mathcal{P}(\omega)$ for $\alpha < \omega_1$, such that $\mathbf{B}_\alpha$ has the play and the isomorphism type of the collapse as arguments and then yields as value a bounded (M, P_γ)-generic filter iff the generic player wins. An (M, P_γ)-generic filter G is called *bounded* if there is a $q \in P_\gamma$ such that $G = \{p \in M \cap P_\gamma \,:\, p \leq q\}$. We will prove that there is a winning strategy for the generic player and let the antigeneric player play in such a way that the generic real or a Borel function applied to the generic real will be contained in the sets of branches of a meagre measure zero tree. Then from $\diamondsuit_{\omega_1}$ in $\mathbf{V}$, which shows that all the Mostowski collapses N and all used (finitely many) predicates on them are guessed stationarily often in ω_1, we will derive that the extension preserves certain weak diamonds. Juhász' question [13], whether $\clubsuit$ (a definition can be found, e.g., in [15, Chapter I, Definition 7.1]) implies the existence of a Souslin tree, remains open. It cannot be attacked by forcings adding no reals since in the presence of CH, $\clubsuit$ implies $\diamondsuit$.

2. Proper forcings adding no new reals

We first recall the definition of the forcings "specializing an Aronszajn tree without adding reals" from [2] and [15, Chapter V, Section 6]. It is known that these forcings are α-proper for all $\alpha < \omega_1$ and are $\mathbb{D}$-complete for a simple $\aleph_1$-completeness system $\mathbb{D}$, which guarantees that their countable support iterations do not add reals [15, Theorem V.7.1]). Abraham gives a nice didactic exposition of the method of $\mathbb{D}$-completeness systems in [1, Section 5]. Here, we will take a simple $\aleph_1$-completeness system $\mathbb{D}$ similar to the one from Abraham and Shelah's work [2].

Jensen (see [6]) showed that the property of not adding reals is in general not preserved in countable support iterations of proper forcings at limit steps of cofinality ω. So some stronger requirement has to be imposed on the iterands. The method of completeness systems that has been developed by Shelah [15, Chapter V] is appropriate for our aim.

Recall, a *specialization* of an Aronszajn tree $\mathbf{T} = (\omega_1, <_\mathbf{T})$ is a function $f\colon \omega_1 \longrightarrow \mathbb{Q}$ such that for any $s, t \in \omega_1$, $s <_\mathbf{T} t \rightarrow f(s) < f(t)$. We call such a function monotone. Now we work with monotone functions f, that specialize only a part of $\mathbf{T}$, namely the union of countably many of its levels, so that the indices of the levels form a closed set C. We call such a pair (f, C) an *approximation*. For $\alpha < \omega_1$ let T_α denote the α-th level of $\mathbf{T}$. For $x \in T_\alpha$ and $\beta < \alpha$ we let $x \lceil \beta$ be the $y \in T_\beta$ such that $y <_\mathbf{T} x$. For making the notation easier, we consider only Aronszajn trees $\mathbf{T}$ whose α-th level, T_α, is $[\omega\alpha, \omega(\alpha+1))$. This is no loss of generality since specializing all these Aronszajn trees suffices.

For any closed C of ω_1, every monotone $f\colon \bigcup_{\alpha \in C} T_\alpha \longrightarrow \mathbb{Q}$ can be extended to a total specialization (see, e.g., [8, Lemma 3.7]), and hence working with approximations on a closed set of levels is the same as working with all levels. We follow the exposition in [2], where the promises (see Definition 2.3) are not only finite parts of the Aronszajn trees as in the book [15], but they are functions from these finite parts into $\mathbb{Q}$. We follow the book [15] in that we use club sets of levels on which the approximations will be defined and not just initial segments $\bigcup_{\beta\leq\alpha} T_\alpha$ as in [2].

We follow the Israeli convention that the stronger forcing condition is the larger one. We assume that each poset P has a weakest element and denote it by 0_P.

Definition 2.1 (A modification of [2, Definition 4.1]).

(1) An *approximation* is a pair (f, C) such that there is a countable ordinal α and $C \subseteq \alpha + 1$, C is closed and $\alpha \in C$, $f \colon \bigcup_{i\in C} T_i \longrightarrow \mathbb{Q}$ is a partial specialization function. The ordinal α is called last(f). We say "(f_2, C_2) extends (f_1, C_1)" and write $(f_1, C_1) \leq (f_2, C_2)$ iff $f_1 \subseteq f_2$ and $C_1 \subseteq C_2$ and $(C_2 \setminus C_1) \cap (\bigcup C_1) = \emptyset$.

(2) We say H is a *requirement* of height $\gamma < \omega_1$ iff for some $n = n(H) < \omega$, H is a countable set of functions of the form $h \colon \mathrm{dom}(h) \longrightarrow \mathbb{Q}$ with $\mathrm{dom}(h) \in [T_\gamma]^n$.

(3) We say that a finite function $h \colon T_\alpha \longrightarrow \mathbb{Q}$ *bounds* an approximation f with last$(f) = \alpha$ iff $\forall x \in \mathrm{dom}(h)$, $f(x) < h(x)$. More generally, if $\beta \geq \alpha = \mathrm{last}(f)$, then $h \colon T_\beta \longrightarrow \mathbb{Q}$ bounds f iff $\forall x \in \mathrm{dom}(h)(f(x\lceil\alpha) < h(x))$.

(4) An approximation f with last$(f) = \alpha$ is said to *fulfil* the requirement H of height $\gamma \geq \alpha$ iff for every $t \in [T_\alpha]^{<\omega}$ there is some $h \in H$ which bounds f and such that $\{x\lceil\alpha : x \in \mathrm{dom}(h)\}$ is disjoint from t.

If f fulfils the requirement H, then any approximation f' with the same last level that is dominated everywhere by f fulfils the requirement as well. Note that according to Definition 2.1 (4) only infinite requirements H can be fulfilled. For $\gamma = \alpha$ the necessary property is equivalent to having an infinite set of pairwise disjoint $\mathrm{dom}(h)$, $h \in H$ and is equivalent to a property we call dispersedness:

Definition 2.2. $H \subseteq \mathbb{Q}^{[T_\gamma]^n}$ is called *dispersed* iff for each $t \in [T_\gamma]^{<\omega}$, there is some $h \in H$ such that $t \cap \mathrm{dom}(h) = \emptyset$.

A forcing condition will be an approximation together with a **T**-promise. The promises function as side-conditions and ensure that the forcing and also all of its countable support iterations (see Theorem 2.20) do not add new reals.

In order to describe how elements of $\Gamma(\gamma)$ are seen at lower levels in the tree, we extend our $\lceil$-notation: Let $\alpha < \gamma$. For $h \colon T_\gamma \longrightarrow \mathbb{Q}$ we let $\mathrm{dom}(h\lceil\alpha) \subseteq T_\alpha$ and $h\lceil\alpha(x) = \min\{h(y) : y\lceil\alpha = x, y \in \mathrm{dom}(h)\}$. For a requirement H of height γ and $\alpha < \gamma$ we set $H\lceil\alpha = \{h\lceil\alpha : h \in H\}$.

Definition 2.3 (See [2, Definition 4.1 (4)]). Γ is a $\mathbf{T}$-promise iff $\mathrm{dom}(\Gamma)$ is club in ω_1 and $\Gamma = \langle \Gamma(\gamma) : \gamma \in \mathrm{dom}(\Gamma)\rangle$ has the following properties:

(a) For each $\gamma \in \mathrm{dom}(\Gamma)$, $\Gamma(\gamma)$ is a countable set of requirements of height γ.
(b) $(\forall \gamma \in \mathrm{dom}(\Gamma))(\forall H \in \Gamma(\gamma))$ H is dispersed.
(c) $(\forall \alpha < \gamma \in \mathrm{dom}(\Gamma))(\Gamma(\alpha) \supseteq \{H\lceil\alpha : H \in \Gamma(\gamma)\})$. This condition implies that $\{H\lceil\alpha : (\exists\gamma > \alpha)(H \in \Gamma(\gamma))\}$ is countable.

Definition 2.4 ([2, Definition 4.1 (5)]). We say that an approximation (f, C) *fulfils* the promise Γ iff $\mathrm{last}(f) \in \mathrm{dom}(\Gamma)$ and f fulfils each requirement H in $\Gamma(\mathrm{last}(f))$.

Finally we can describe the iterands of our iteration of length ω_2. $Q_{\mathbf{T}}$ is called $\mathcal{S}(\mathbf{T})$ in [2]. We do not know whether it is equivalent to the forcing notion Q^{NNR} or $\mathrm{NNR}(\mathbf{T})$ from [15, V, 6.3]. NNR means “no new reals”.

Definition 2.5 ([2, 4.2]). $Q_{\mathbf{T}}$ is the set of (f, C, Γ) such that (f, C) is an approximation, and Γ is a promise and (f, C) fulfils Γ. The partial order is defined as $(f_0, C_0, \Gamma_0) \le (f_1, C_1, \Gamma_1)$ iff

(1) f_1 extends f_0,
(2) C_1 is an end-extension of C_0 and $C_1 \setminus C_0 \subseteq \mathrm{dom}(\Gamma_0)$, and
(3) $(\forall \gamma \in \mathrm{dom}(\Gamma_0 \setminus \mathrm{last}(f_1))(\gamma \in \mathrm{dom}(\Gamma_1)$ and $\Gamma_0(\gamma) \subseteq \Gamma_1(\gamma))$.

If $p = (f, C, \Gamma)$, we write $f = f^p$, $C = C^p$ and $\Gamma = \Gamma^p$, and we write $\mathrm{last}(p) = \mathrm{last}(f^p) = \max(C^p)$.

Do not confound the countable, closed C's that are the second coordinate of the approximations with the true clubs $\mathrm{dom}(\Gamma)$ in ω_1 that are the domains of the promises Γ: the first ones are approximations to the latter ones as in the forcing adding a club through a stationary set by countable approximations [4]. However, we take club sets $\mathrm{dom}(\Gamma)$ and not co-stationary sets as there, as we want to work with proper forcings.

Now we want to extend a given condition to a stronger condition of a given height, and we want to show that the set of promises can be enlarged.

Lemma 2.6 ([2, Lemma 4.3], The extension lemma). *Let $\mu < \omega_1$. If $p \in Q_{\mathbf{T}}$ and if $\mathrm{last}(p) < \mu \in \mathrm{dom}(\Gamma^p)$, then there is some $q \ge p$ such that $\Gamma^q = \Gamma^p$ and $\mathrm{last}(q) = \mu$. Moreover, if $h \colon T_\mu \longrightarrow \mathbb{Q}$ is finite and bounds f^p, then q can be chosen such that h bounds f^q.*

Proof. The proof is done by induction on μ.

First case: $\mu = \mu_0 + 1$ is a successor. We may assume that $\mathrm{last}(p) = \mu_0$ and we have to extend f^p onto T_{μ_0+1}, fulfilling all the countably many requirements in $\Gamma^p(\mu)$. We know that every requirement $H\lceil\mu_0$ for $H \in \Gamma^p(\mu)$ is fulfilled by f^p. So $H\lceil\mu_0$ contains infinitely many functions h that bound f. We have countably many H, and we enumerate them as H_0, H_1, ... There are enough points in $T_{\mu_0+1} \setminus T_{\mu_0}$ such that in each H_i there will be some h_i such that $\mathrm{dom}(h_i) \cap \bigcup\{\mathrm{dom}(h_j) : j < i\} = \emptyset$.

Since it will be used in the limit step, we now prove the "moreover"-clause. If h bounds p as in the Lemma, we first choose any extension p_1 of p with $\mu = \mathrm{last}(p_1)$ and then we correct p_1 as follows to obtain q: There is some $d > 0$ such that $\forall x \in \mathrm{dom}(h)$, $h(x) > f^p(x\lceil\mu_0) + d$. Now we take $\delta\colon \mathbb{Q}^+ \longrightarrow (0, d)$ order-preserving and such that $\delta(x) < x$ for all $x \in \mathbb{Q}^+$. Now we set $f^q(x) = f^p(x\lceil\mu_0) + \delta(f^{p_1}(x) - f^p(x\lceil\mu_0))$. Hence h bounds q.

Second case: μ is a limit of $\mathrm{dom}(\Gamma^p)$. We pick an increasing sequence of ordinals μ_i, $i < \omega$, converging to μ. We define an increasing sequence $p_i \in Q_{\mathbf{T}}$, $i \in \omega$, beginning with $p_0 = p$ and finite $h_i, g_i\colon T_\mu \longrightarrow \mathbb{Q}$ which bound p_i and whose union of domains will be T_μ. The passage from μ_i to μ_{i+1} uses the inductive assumption for μ_{i+1} of the stronger claim in the "moreover" clause. The h_i and g_i ensure that f^q is bounded on each branch in $T_{<\mu}$ and that f^q on the level T_μ fulfils all the promises in $\Gamma(\mu)$. Then we can define $q = (f, C, \Gamma)$ by $C = C^p \cup \{\mu\}$ and $\Gamma = \Gamma^p$. We let $f' = \bigcup\{f^{p_i} : i < \omega\} \cup \{(x, \limsup_{i\longrightarrow\omega} f^{p_i}(x\lceil\mu_i)) : x \in T_\mu\}$. The values on level μ might be irrational. We correct them to slightly larger values in $\mathbb{Q}$ that are small as to fulfil all the promises in $\Gamma^q(\mu)$ and let the resulting function be f^q. Such a choice is possible since all (ω,ω)-gaps in $\mathbb{R}$ are filled with sequences with values in $\mathbb{Q}$.

To carry out the step from i to $i+1$, let $\Gamma^p(\mu) = \{H_i : i < \omega\}$. At step i, we choose $h_i \in H_i$ such that $\mathrm{dom}(h_i) \cap \bigcup\{\mathrm{dom}(h_j) : j < i\} = \emptyset$ and we choose $g_i \in \{g\colon [T_\mu]^{n(H_i)} \longrightarrow \mathbb{Q} : g(x) = f^p(x\lceil\mathrm{last}(p)) + 1/2^i\}$ and fulfil both. In addition we take care that $\bigcup\{\mathrm{dom}(h_i) \cup \mathrm{dom}(g_i) : i < \omega\} = T_\mu$. Then we choose ℓ_i so high that $\mathrm{dom}(h_i\lceil\mu_{\ell_i}) \cap \bigcup\{\mathrm{dom}(h_j\lceil\mu_{\ell_i}) : j < i\} = \emptyset$. By the induction hypothesis of the statement together with the "moreover"-clause we have some $\varepsilon_i > 0$ and p_i such that for all $j \leq i$, $\forall x \in \mathrm{dom}(h_j)$ $f^{p_i}(x\lceil\mu_{\ell_i}) < h_j\lceil\mu_{\ell_i}(x\lceil\mu_{\ell_i}) - \varepsilon_i$ and $\mathrm{last}(p_i) = \mu_{\ell_i}$, and the same can be arranged for the g_j. Since $h_j \in H_j$ is taken care of at each step $i \geq j$, in the end also $f(x) < h_j(x)$ for all $x \in \mathrm{dom}(h_j)$. □

Definition 2.7. Let p be a condition of height μ and let Ψ be a promise. We say that p *includes* Ψ iff $\mathrm{dom}(\Psi) \subseteq \mathrm{dom}(\Gamma^p)$ and for all $\gamma \in \mathrm{dom}(\Psi)$, $\Psi(\gamma) \subseteq \Gamma^p(\gamma)$.

If p includes Ψ, then p fulfils Ψ. There is a sufficient condition for the existence of an extension q of p such that q includes Ψ:

Lemma 2.8 (Modification [2, Lemma 4.4.], Addition of promises). *Let $p \in Q_{\mathbf{T}}$ and $\mu = \mathrm{last}(p)$. Let Ψ be a promise with $\mu < \beta = \min(\mathrm{dom}(\Psi))$ and $\mathrm{dom}(\Psi) \subseteq \mathrm{dom}(\Gamma^p)$. Suppose that for some finite $g\colon T_\mu \longrightarrow \mathbb{Q}$ called a basis for Ψ, g bounds f^p and*

$$(\forall \gamma \in \mathrm{dom}(\Psi))(\forall H \in \Psi(\gamma))(\forall h \in H)(h\lceil\mu = g).$$

Then there is an extension q of p in $Q_{\mathbf{T}}$ that includes Ψ.

Proof. Since g is finite, there is some rational $d > 0$ such that $(\forall x \in \mathrm{dom}(g))(g(x) > f^p(x) + d)$. Now every $H \in \Psi(\beta)$ is a dispersed collection of functions h with $h\lceil\mu \geq g$. Let p_1 be any extension of p of height β. For $\gamma \geq \beta$ we set $\Gamma^q(\gamma) = \Psi(\gamma) \cup \Gamma^p(\gamma)$, and $\gamma \in [\mu, \beta)$ we set $\Gamma^q(\gamma) = \{H\lceil\gamma : H \in \Psi(\beta)\} \cup \Gamma^p(\gamma)$. The desired extension of p is obtained by correcting f^{p_1} to get f^q that fulfils $\Psi(\beta) \cup \Gamma(\beta)$ as in the "moreover"-part of the previous lemma. $\square$

In the following lemma $\chi > 2^{\aleph_1}$ is sufficiently large.

Lemma 2.9 ([2], [15, Fact V.6.7]). *Let $\mathbf{T}$ be an Aronszajn tree. Let $M \prec (H(\chi), \in)$ be a countable elementary substructure with a sufficiently large regular χ, $Q_{\mathbf{T}} \in M$, $p \in Q_{\mathbf{T}} \cap M$, $\mu = \omega_1 \cap M$ and $h\colon T_\mu \longrightarrow \mathbb{Q}$ be a finite function which bounds f^p. Let $D \in M$, $D \subseteq Q_{\mathbf{T}}$ be dense open. Then there is an $q \geq p$, $q \in D \cap M$, that h bounds q.*

Proof. We assume that the contrary is the case. Let $\mathbf{T}$, M, p, h be a counterexample. Let $\mu_0 = \mathrm{last}(p) < \mu$ and let $\{x_0, \dots, x_{n-1}\} = \mathrm{dom}(h) \in [T_\mu]^n$. Let $v_i = x_i\lceil\mu_0$. We assume that $v_i \neq v_j$ for $i \neq j$ otherwise we extend p upwards with Lemma 2.6 to get some $p' \geq p$ with $\mathrm{last}(p') < \mu$ and $x_i\lceil\mathrm{last}(p') \neq x_j\lceil\mathrm{last}(p')$.

Put $g_0 = h\lceil\mu_0$. Then $g_0 \in M$, as it is finite. We say that that a finite partial $g\colon T_\gamma \longrightarrow \mathbb{Q}$ is *bad* iff $\mu_0 \leq \gamma$ and $g\lceil\mu_0 = g_0$ and, whenever $q \in D$ extends p and $\gamma \geq \mathrm{last}(q)$, g does not bound q. So g is bad iff it has the similar behaviour as $h\lceil\mu_0$. For every $\gamma \in [\mu_0, \mu)$, $h\lceil\gamma$ is bad. So in M and hence in $H(\chi)$ there are uncountably many bad g's. We set

$$B = \{\mathrm{dom}(g) : g \text{ is bad}\}.$$

Then B is an uncountable and closed downwards in $<_{\mathbf{T}}$ (above μ_0) subset of $\bigcup_{\mu_0 \leq \gamma < \omega_1} [T_\gamma]^n$. As $\mathbf{T}$ is an Aronszajn tree, [6, Lemma VI.7] implies that there is some $\beta \geq \mu_0$ and some $B^0 \subseteq B$ such that:

(1) For $\beta \leq \gamma_0 < \gamma_1$, $B^0 \cap T_{\gamma_0} = (B^0 \cap T_{\gamma_1})\lceil\gamma_0$
(2) $B^0 \cap T_\beta$ is dispersed.

Here we take $X \lceil \gamma = \{x \lceil \gamma : x \in X\}$ for $X \subseteq \mathbf{T}$. We may find B^0 in M, since only parameters in M were mentioned in its definition. For $\beta \leq \gamma < \omega_1$ let $\Psi(\gamma) = \{H_\gamma\}$ with $H_\gamma = \{g : g \text{ is bad and } \operatorname{dom}(g) \subseteq B^0 \cap T_\gamma\}^M$. By Lemma 2.8, read in M, there is an extension q of p in M of height β which includes Ψ, i.e., $H_\gamma \in \Gamma^q(\gamma)$.

Now let $r \in D$ be any condition extending q. Let $\gamma = \operatorname{last}(r)$. Since r fulfils Γ, for some g in H_γ, g bounds r. But this contradicts the fact that g is bad. $\square$

Lemma 2.9 will be used in the induction in Claim 2.16 to get point (5).

Definition 2.10. Now we assume $\mathbf{V} \models CH + \diamondsuit_{\omega_1} + 2^{\aleph_1} = \aleph_2$ and let $P_{\omega_2} = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration with $\underset{\sim}{Q}_\alpha = Q_{\underset{\sim}{\mathbf{T}}_\alpha}$ being as above for some Aronszajn tree $\mathbf{T}_\alpha \in \mathbf{V}[G_\alpha]$, where the filter G_α is P_α-generic over $\mathbf{V}$, such that $\Vdash_{P_\alpha}$ "$\underset{\sim}{\mathbf{T}}_\alpha$ is an Aronszajn tree and for $\gamma < \omega_1$ its γ-th level is $[\omega\gamma, \omega\gamma{+}\omega)$". The book-keeping shall be arranged so that every P_{ω_2}-name for an Aronszajn tree is used in some iterand.

Why does every Aronszajn tree in $\mathbf{V}^{P_{\omega_2}}$ have a P_α-name for some $\alpha < \omega_2$? We have $|Q_\mathbf{T}| = \aleph_2$, so that we cannot work with the $\aleph_2$-chain condition for each iterand. Now [15, Chapter VIII, Section 2] helps: Basically by Lemma 2.8, each $Q_\mathbf{T}$ has the $\aleph_2$ p.i.c. (proper isomorphism condition), see [15, Chapter VIII, Definition 2.1], and hence by [15, Chapter VIII, Lemma 2.4], P_{ω_2} has the $\aleph_2$-c.c, *if* $\mathbf{V}_0$ fulfils the CH.

Since P_{ω_2} has the $\aleph_2$-c.c., by a lemma similar to the one of [5, 5.10], now for subsets of ω_1 instead of real numbers, every subset of ω_1 in a countable support iteration of proper forcings with the $\aleph_2$-c.c. at each initial segment has a name at some stage of cofinality ω_1. So we an carry out the desired book-keeping.

In the remainder of this section, we shall prove that P_{ω_2} does not add new reals. Towards this aim, we first recall some general theory for $< \omega_1$-proper forcings P adding no reals. Then we shall show that our specific forcing and a suitable completeness system $\mathbb{D}(M, P, p)$ exhibit these properties. Note that adding no reals and adding no new ω-sequence of ordinals is the same for proper forcings. In the application, P is of the form $Q_\mathbf{T}$ or is some countable support iteration of $Q_\mathbf{T}$'s.

Recall, $p \in P$ is (M, P)-generic if for every P-generic filter G over V with $p \in G$, $p \Vdash M[\underset{\sim}{G}] \cap On = M \cap On$. Now in the context of proper forcings that do not add reals we find completely (M, P)-generic conditions.

Definition 2.11. A condition p is *completely* (M, P)*-generic* if $G = \{q \in P \cap M : q \leq p\}$ is an (M, P)-generic filter. G is called bounded.

Indeed, P is proper and does not add reals iff for every $M \prec (H(\chi), \in)$, for every $p \in M \cap P$ there is a completely (M,P)-generic $q \geq p$. Given a name $\underset{\sim}{f}$ for a real, consider the dense sets $D_n = \{p : (\exists m \in \omega)(p \Vdash \underset{\sim}{f}(n) = m)\}$. Completeness systems that are closed under finite intersections — we shall have countably closed ones — help to find completely generic conditions in a first order definable way and allow to prove that no new reals sneak in at the limit steps. Only the case of cofinality ω is hard, since every real in a countable support iteration of proper forcings appears for the first time at some stage of at most countable cofinality [1, Corollary 2.9 (1)]. An important point is that some parameters of the members of the completeness system, that are subsets of M, here called x, need to be guessed. Since intersections over countable parts of the completeness system are not empty, the guessing can be performed in M', when $M' \prec (H(\chi), \in, <^*_\chi)$ and $M \in M'$. One not so aesthetic feature stays: There is neither a completeness system for the two-step iteration nor for the limit forcing, we only know that no reals are added. From the proof we get a description of the bounded generic filters and of the generic filters for some towers of elementary submodels that appear as helpers in the proofs.

Definition 2.12 ([15, V, 5.5])**.**

(1) We call $\mathbb{D}$ a *completeness system* if for some μ, $\mathbb{D}$ is a function defined on the set of triples $\langle M, P, p\rangle$, $p \in M \cap P$, $P \in M$, $M \prec (H(\mu), \in)$, M countable, such that $\mathbb{D}(M,P,p)$ is a family of non-empty subsets of

$$\begin{aligned}\mathrm{Gen}(M,P,p) =\{G : \; & G \subseteq M \cap P, G \text{ is directed and } p \in G \\ & \text{and } G \cap \mathcal{I} \neq \emptyset \\ & \text{for every dense subset } \mathcal{I} \text{ of } P \text{ which belongs to } M\}.\end{aligned}$$

(2) We call $\mathbb{D}$ a λ-*completeness system* if each family $\mathbb{D}(M,P,p)$ has the property that the intersection of any i elements is non-empty for $i < 1+\lambda$ (so for $\lambda \geq \aleph_0$, $\mathbb{D}(M,P,p)$ generates a filter). $\aleph_1$-completeness systems are also called countably closed completeness systems.

(3) We say $\mathbb{D}$ is on μ if $M \prec (H(\mu), \in)$. We do not always distinguish strictly between $\mathbb{D}$ and its definition.

The notion of forcing $Q_{\mathbf{T}}$ has size $2^{\aleph_1}$, and the set of all approximations has size $\aleph_1^{\aleph_0}$. So for a countable $M \prec (H(\chi), \in, <^*_\chi)$, we never have $P \subseteq M$. If $\mathbf{T} \in M$, we can read the definition of $P = Q_{\mathbf{T}}$ in M and get P^M. Since $\mathbf{T}$ is definable from $Q_{\mathbf{T}}$ ($x \not<_{\mathbf{T}} y$ iff there is an approximation with $f(x) = f(y)$), $Q_{\mathbf{T}} \in M$ implies $\mathbf{T} \in M$. If $\chi > 2^{\aleph_1}$ is regular, then $P^M = P \cap M$. In our description via first order formulae, P, x, and G are predicates on M.

Definition 2.13. Suppose that $\mathbb{D}$ is a completeness system on χ. We say P is $\mathbb{D}$-*complete*, if for every countable $M \prec (H(\chi), \in)$ with $P \in M$, $\mathbb{D} \in M$, $p \in P \cap M$, the following set contains as a subset a member of $\mathbb{D}(M, P, p)$:

$$\mathrm{Gen}^+(M, P, p) = \{G \in \mathrm{Gen}(M, P, p) : \text{ there is an upper bound for } G \text{ in } P\}.$$

Definition 2.14 ([15, V, 5.5])**.**

(1) A completeness system $\mathbb{D}$ is called *simple* if there is a first order formula ψ such that

$$\mathbb{D}(M, P, p) = \{A_x : x \text{ is a finitary relation on } M, \text{i.e., } x \subseteq M^k \text{ for some } k \in \omega\},$$

where

$$A_x = \{G \in \mathrm{Gen}(M, P, p) : (M \cup \mathcal{P}(M), \in, p, M, P) \models \psi(x, G)\}.$$

(2) A completeness system $\mathbb{D}$ is called almost simple over $\mathbf{V}_0$ ($\mathbf{V}_0$ a class, usually a subuniverse) if there is a first order formula ψ such that

$$\mathbb{D}(M, P, p) = \{A_{x,z} : x \text{ is a finitary relation on } M, \text{i.e., } x \subseteq M^k \text{ for some } k \in \omega, z \in \mathbf{V}_0\},$$

where

$$A_{x,z} = \{G \in \mathrm{Gen}(M, P, p) : (\mathbf{V}_0 \cup M \cup \mathcal{P}(M), \in^{\mathbf{V}_0}, \in^{M \cup P \cup \mathcal{P}(M)}, p, M, \mathbf{V}_0, P) \models \psi(x, z, G)\},$$

where $\in^A = \{(x, y) \in A \times A : x \in y\}$.

(3) If in (2) we omit z, we call $\mathbb{D}$ simple over $\mathbf{V}_0$.

We shall give an example ψ and a simple $\aleph_1$-completeness system $\mathbb{D}$ on any regular $\chi > 2^{\aleph_2}$, so that $Q_{\mathbf{T}}$ is $\mathbb{D}$-complete. From now on we use the requirement from Definition 2.10 that the α-th level of $\mathbf{T} = (\omega_1, <_{\mathbf{T}})$ is $[\omega\alpha, \omega(\alpha + 1))$. Let $\chi > 2^{\aleph_2}$ be a regular cardinal. If we have a countable $M \prec (H(\chi), \in)$, then $M \cap \mathbf{T} = T_{<\mu}$ for $\mu = M \cap \omega_1$. We take an increasing sequence $\bar{\beta} = \langle \beta_n : n \in \omega \rangle$ that is cofinal in μ. Now we take for $x_1 \subseteq M$ a code of the branches through $T_{<\mu}$, for example $x_1 \colon T_{<\mu} \longrightarrow \omega$, x_1 is eventually constant on each branch. We also code in x_1 the branches through $T_{<\mu}$ that have $<_{\mathbf{T}}$ successors in T_μ. Indeed the other branches are unimportant. If we want to find an (M, P)-generic condition with last level T_μ we have to take care that the approximations to the specialization function do not diverge on any branch that is continued in T_μ. Since we are looking for a condition $q \geq p$ and $p \in M$, we also code into another component $x_2 \subseteq M$ the set $\bigcup_{\gamma \geq \mu} \Gamma^q(\gamma) \lceil \mu$ of promises for each $q \in M \cap P$. The codes $x = (x_1, x_2, \bar{\beta})$ are in general not in M, but they are predicates

$\subseteq M^k$. The point is that countably many A_x from Definition 2.14 (the ψ appearing in A_x will be given in Lemma 2.15) have a non-empty intersection. This works also for countably many guesses for codes, which is crucial in the proofs of Theorems 2.20 and 3.4.

The technique of the following lemma comes from [2]. Actually a sketch of the elements of the $\aleph_1$-completeness system is also given in the end of the proof of [15, Chapter V, Theorem 6.1] on page 236. We conceive $x = (x_1, x_2, \bar\beta)$ as one relation in M.

Lemma 2.15. *$Q_{\mathbf{T}}$ is $\mathbb{D}$-complete for the simple $\aleph_1$-completeness system $\mathbb{D}$ given by $\psi(x, G) = \psi_0(x) \wedge \psi_1(x, G)$, with*

$$\psi_0(x) \equiv x = (x_1, x_2, \bar\beta) \wedge \bar\beta = \langle \beta_n : n \in \omega\rangle \text{ increasing}$$
$$\wedge M \cap \omega_1 = \bigcup \{\beta_n : n < \omega\}$$

and

$$\psi_1(x, G) \equiv (\forall \varepsilon > 0)(\exists m < \omega)(\forall n_1 < n_2 \in [m, \omega))(\forall t \in T_\mu)(\forall y_1, y_2 <_{\mathbf{T}} t)$$
$$\Big((y_1 \in T_{\beta_{n_1}} \wedge y_2 \in T_{\beta_{n_2}} \wedge y_1 <_{\mathbf{T}} y_2 \rightarrow \underset{\sim}{f}[G](y_2) < \underset{\sim}{f}[G](y_1) + \frac{\varepsilon}{2^{n_2}}\Big)$$
$$\wedge \text{ “}G \text{ is a filter”}$$
$$\wedge\, p \in G \wedge \forall D \in M((D \subseteq P \wedge D \text{ dense in } P) \rightarrow D \cap G \neq \emptyset)$$
$$\wedge\, (\forall H \in x_2)(\forall n)(\forall t \in [T_{\beta_n}]^{<\omega})(\exists h \in H)$$
$$(\operatorname{dom} h\lceil\beta_n \cap t = \emptyset \wedge \underset{\sim}{f}[G] \restriction T_{\beta_n} \text{ fulfils } h\lceil\beta_n).$$

Here M, P, x and G appear in the formulas as (names for) predicates and p is a constant. To ease readability, we write T_μ instead of x_1 (though T_μ is not a subset of M) and $\bigcup_{\gamma\geq\mu}\Gamma^p(\gamma)\lceil\mu$ instead of x_2.

Proof. First we proof the following claim:

Claim 2.16. *Let $\mu = M \cap \omega_1 = \sup\langle\beta_n : n < \omega\rangle$ and let the β_n be increasing. If*

$$\Big(M \cup \mathcal{P}(M), \in^{M\cup\mathcal{P}(M)}, p, M, Q_{\mathbf{T}}\Big) \models \psi_0(x),$$

then there is $G \subseteq Q_{\mathbf{T}}$, $G \in G(M, Q_{\mathbf{T}}, p) \cap A_x$ such that

$$\Big(M \cup \mathcal{P}(M), \in^{M\cup\mathcal{P}(M)}, p, M, Q_{\mathbf{T}}\Big) \models \psi(x, G).$$

Proof. Let $\{I_n : n \in \omega\}$ be an enumeration of all open dense subsets of $Q_{\mathbf{T}}$ that are in M. Let $\{t_n : n \in \omega\}$ enumerate T_μ: Now we choose by induction on $n < \omega$, p_n such that

(1) $p_0 = p$,

(2) $p_{n+1} \geq p_n \in M$,
(3) $\mathrm{last}(p_{n+1}) \geq \beta_{n+1}$,
(4) $p_{n+1} \in I_n$,
(5) $(\forall t \in \{t_k : k \leq n\})(\forall y <_{\mathbf{T}} t)$
$$\left(y \in T_{\beta_{n+1}} \rightarrow f^{p_{n+1}}(y) < f^{p_n}(y\lceil\beta_n) + \frac{1}{2^{n+1+n}}\right).$$

Then $G = \{r : (\exists n \in \omega)(r \leq p_n)\} \in \mathrm{Gen}(M, Q_{\mathbf{T}}, p) \cap A_x$.

Why is this choice possible? For Properties (4) and (5) we use Lemma 2.9 for h with

$$\begin{aligned} \mathrm{dom}(h) =& \{t_k\lceil\beta_{n+1} : k \leq n\}, \\ h(y) =& f^{p_n}(y\lceil\beta_n) + \frac{1}{2^{n+1+n}}, \end{aligned}$$

which is a finite function that bounds p_n and we find some p_{n+1} of length β_{n+1}. $\square$

Claim 2.17. *If $(M \cup \mathcal{P}(M), \in, p, M, Q_{\mathbf{T}}) \models \psi(x, G)$ for some x, then G has an upper bound in $Q_{\mathbf{T}}$.*

Proof. Again let $\{I_n : n \in \omega\}$ be an enumeration of all open dense subsets of $Q_{\mathbf{T}}$ that are in M. Let x be as in $\psi(x, G)$. Let $G \supseteq \{q_n : n \in \omega\}$, $q_n \in M \cap I_n$, $\mathrm{last}(q_n) = \beta_n$ such that the β_n and the q_n are increasing. We set $\mu = M \cap \omega_1 = \bigcup \beta_n$, f^q as in the proof of Lemma 2.6 a slightly larger rational variant of $\bigcup f^{q_n} \cup \{(z, \sup\{f^{q_n}(z\lceil\beta_n) : n \in \omega\}) : z \in T_\mu\}$, $C^q = \bigcup_{n\in\omega} C^{q_n} \cup \{\mu\}$, which is closed since for each n, $C^{q_{n+1}}$ is an end extension of C^{q_n}, $\mathrm{dom}(\Gamma^q) = (\bigcup_{n\in\omega} \mathrm{dom}\,\Gamma^{q_n} \cap [\mu, \omega_1)) \cup \{\mu\}$, and for $\mu' > \mu$, $\Gamma^q(\mu') = \bigcup_{n\in\omega} \Gamma^{q_n}(\mu')$ and $\Gamma^q(\mu) = \bigcup_{\mu'\geq\mu} \bigcup_{n\in\omega} \Gamma^{q_n}(\mu')\lceil\mu$.

We claim that q is an upper bound of G: First we check that $q \in Q_{\mathbf{T}}$. Note that if ν dominates all $h_{\bar{\beta},z}$, $z \in T_\mu$, then for every $z \in T_\mu$ the limit $f^q(z)$ exists, because if $h_{z,\bar{\beta}} \leq^* \nu$, then for almost all n, $z\lceil\beta_n = \omega\beta_n + h_{z,\bar{\beta}}(n)$ and $h_{z,\bar{\beta}}(n) \leq \nu(n)$. So we have that (f^q, C^q) is an approximation. Now let $H \in \Gamma^q(\mu)$ be a $\mathbf{T}$-promise. For some $\mu' \geq \mu$, $k \in \omega$, $H \in \Gamma^{q_k}(\mu')\lceil\mu$. Then, since q_k fulfils the promise, also q fulfils the promise. $\square$

Proof of Lemma 2.15 continued. We showed that $A_x \subseteq G^+(N, Q_{\mathbf{T}}, p)$. So we have that $Q_{\mathbf{T}}$ is $\mathbb{D}$-complete. It remains to show that $\mathbb{D}$ is countably closed, i.e., that given x_ℓ with $\psi(x_\ell, G)$, $\ell < \omega$, the intersection $\bigcap_{\ell\in\omega} A_{x_\ell}$ is not empty. But this is now easy: Let $x_\ell = (x_{1,\ell}, x_{2,\ell}, \bar{\beta}_\ell)$. $x_{1,\ell}$, coding the cofinal branches in $T_{<\mu}$, and $x_{2,\ell}$, coding the promise $\Gamma(\mu)$, are defined from $\mathbf{T}$ and p and do depend on ℓ at most in the way the coding is chosen, not in the content they code.

There is only some little twist because the $\bar{\beta}_\ell = \langle \beta_{\ell,u} : u < \omega \rangle$ are not the same. We choose $\beta = \langle \beta_m : m < \omega \rangle$ such that $\beta_0 = 0$, $(\forall \ell \leq m)(\exists u < \omega)(\beta_{\ell,u} \in [\beta_m, \beta_{m+1}))$. Then we let $x_1 = x_{1,0}$, $x_2 = x_{2,0}$ and $x = (x_1, x_2, \bar{\beta})$. Then $A_x \subseteq A_{x_\ell}$, $\ell < \omega$. □

Definition 2.18. We call P *α-proper* if the following holds: Let M_i, $i < \alpha$, be countable elementary submodels of $(H(\chi), \in)$. Let $P \in M_0$ and let $\langle M_i : i < \alpha \rangle$ be an increasing sequence such that $\langle M_j : j \leq i \rangle \in M_{i+1}$ and for limit ordinals j, $M_j = \bigcup_{i<j} M_i$. Then for every $p \in P \cap M_0$ there is some $q \geq p$ that is (M_i, P)-generic for all $i < \alpha$. Such a sequence $\langle M_i : i < \alpha \rangle$ is called a *tower* of models and α is the height or the length of the tower.

Lemma 2.19. *$Q_{\mathbf{T}}$ is α-proper for all $\alpha < \omega_1$.*

Proof. The upper bound from Claim 2.17 gives a completely $(M, Q_{\mathbf{T}})$-generic $q \geq p$. Given a tower of height α, we can repeat the construction α steps, using a "diagonalised" version of Claim 2.16 for countably many M and countably many enumerations of dense sets simultaneously, so that in the end we get some q that is $(M_i, Q_{\mathbf{T}})$-generic for all $i < \alpha$. □

Now we can cite Theorem V.7.1 (2) of [15] for $\aleph_1$-complete systems. A very clear proof, even in a more general context when "almost simple over $\mathbf{V}_0$" is replaced by "in $\mathbf{V}_0$", is given in [1, Theorem 5.17].

Theorem 2.20. *Let $P_\gamma = \langle P_j, \underset{\sim}{Q}_i : j \leq \gamma, i < \gamma \rangle$ be a countable support iteration. If each $\underset{\sim}{Q}_i$ is β-proper for every $\beta < \omega_1$ and $\mathbb{D}_i$-complete for some almost simple $\aleph_1$-completeness system $\mathbb{D}_i$ over $\mathbf{V}_0$ (not over the current stage of the iteration), then P_γ does not add reals.*

So we know that P_{ω_2} from Definition 2.10 exists and specializes all Aronszajn trees and does not add reals. The remaining task is to obtain the weak diamond $\diamondsuit(\mathbb{R}, \mathcal{N}, \in)$ in $\mathbf{V}^{P_{\omega_2}}$.

3. Games for the generic filters over countable models

In this section we show that certain weak diamonds hold when forcing with a countable support iteration of $Q_{\mathbf{T}}$'s (of arbitrary iteration length γ) over a ground model fulfilling $\diamondsuit_{\omega_1}$. In order to specialize all Aronszajn trees, we start with a ground model of CH and $2^{\aleph_1} = \aleph_2$ and perform an iteration of length $\gamma = \omega_2$ with a suitable book-keeping.

For the weak diamonds, we rework the facts used in the proof of Lemma 2.15 to give some stronger, descriptive statement about $G \cap M$.

The basic idea is: The parameters x_1 and x_2 of the A_x in the completeness system $\mathbb{D}(M,P,p)$ from Lemma 2.16 can be coded into functions $\nu\colon \omega \longrightarrow \omega$ in a way that each $\eta \geq^* \nu$ also serves as a code for a parameter. The proof of Theorem 2.20, which works with guessing parameters, will be translated into a game whose innings give $\leq^*$-sufficiently large codes of parameters. Let γ be the iteration length. The result, stated in Theorem 3.4, is that bounded (M_0, P_γ)-generic filters containing p_0 can be computed in a Borel manner from the isomorphism type of (M_0, P_γ, p_0) and a game played according to a strategy. The length of the game is $\alpha = \mathrm{otp}(M_0 \cap \gamma)$.

In the following $\chi > 2^{\aleph_2}$ suffices. Let $<^*_\chi$ be a fixed well-ordering of $H(\chi)$ such that $x \in y$ implies $x <^*_\chi y$. Assume that $M \prec (H(\chi), \in, <^*_\chi)$ is a countable model and $\mathbf{T}$, $Q_{\mathbf{T}} \in M$. From now on we shall use the well-order $<^*_\chi$. In the following, let M always be a model of this kind. We reserve the letter N for transitive collapses of the M's. Fix a bijective pairing function $e\colon \omega \times \omega \longrightarrow \omega$ that is so low in complexity such that it is an element of every M.

Now we want to get rid of the two parameters x_1 and x_2 that depend on p, T_μ and $\bigcup_{\mu' \geq \mu} \Gamma^p(\mu') \lceil \mu$ and are relations over M but not elements in M. The trick is to find a real ν coding them (after a transitive collapse) and code in such a way that every $\eta \geq^* \nu$ codes even better. Coding means we want to imitate Lemma 2.16 now with η taking the role of x_1 and of x_2. The parameter $\bar\beta$ can stand as it is, since it depends only on the transitive collapse of M and not on P and p.

We translate the task of x_1:

Definition 3.1. Let $\mathbf{T}$ be an Aronszajn tree with levels $T_\alpha = [\omega\alpha, \omega(\alpha+1))$. Let μ be a limit ordinal in ω_1. Given $\bar\beta$ converging to μ, we can write cofinally many nodes of a branch b of $T_{<\mu}$ into a function $h_{b,\bar\beta}\colon \omega \longrightarrow \omega$, such that for all n,

$$b \cap T_{\beta_n} = \{\omega\beta_n + h_{b,\bar\beta}(n)\}$$

and we can describe each node $t = \omega\mu + k \in T_\mu$, by $h_{t,\bar\beta}\colon \omega \longrightarrow \omega$, such that for all n,

$$t\lceil\beta_n = \omega\beta_n + h_{t,\bar\beta}(n).$$

If $t = \omega\beta_n + k \in T_{\beta_n}$, then we define $h_{t,\bar\beta}\colon n+1 \longrightarrow \omega$, such that for all $m \leq n$,

$$t\lceil\beta_m = \omega\beta_m + h_{t,\bar\beta}(m).$$

Now we translate the task of x_2:

Definition 3.2. Let $\mu = M \cap \omega_1$. Given $\bar\beta$ converging to μ, and $p \in M \cap Q_{\mathbf{T}}$ with $\mathrm{last}(p) = \beta_0$, let $\Gamma^p(\mu) = \{H_n : n \in \omega\}$, and let $h_{n,m} \in H_n$ be such that p fulfils $h_{n,m}$, and such that $\{\mathrm{dom}(h_{n,m}) : m < \omega\}$ is dispersed and pairwise disjoint. We define $h_{p,H_n}\colon T_\mu \longrightarrow \omega$, such that for all $x \in T_\mu$, for all m

$$h_{n,m}(x) - 2^{-h_{p,H_n}(x)} > f^p(x\lceil \mathrm{last}(p)).$$

That is, the growth of $f^q \supseteq f^p$ along the branch leading to $x \in T_\mu$ and a promise $H_n \in \Gamma^p(\mu)$ shall be bounded, only the small increase $2^{-h_{p,H_n}(x)}$ above f^p is allowed. We code the level $T_\mu \subseteq M$ in a predicate on M and we code the promise $\Gamma^p(\mu)$ into the natural numbers via a bijection $l\colon \omega \longrightarrow T_\mu$. Then $h_{p,H_n} \circ l\colon \omega \longrightarrow \omega$ is a function we want to eventually dominate with a good parameter ν. The parameter does not know the actual functions h_{p,H_n}. That aim ist: if a parameter ν dominates all the $h_{t,\bar\beta}$, $t \in T_\mu$, and all the h_{p,H_n}, $n \in \omega$. then we can choose the conditions in an (M,P)-generic filter only with the knowledge of η for any $\eta \geq^*$ the parameter ν and without T_μ (or x_1) and $\Gamma^p(\mu)$ (or x_2). To make the induction in the next lemma going, the parameter need also to be larger than the codes of the $\Gamma^{p_n}(\mu)$ for $n \in \omega$. So we code all $h_{q,H}$ for $H \in \Gamma^q(\mu)$, $q \in M \cap P$, into x_2.

Lemma 3.3. *Let $p \in Q_{\mathbf{T}} \cap M$. Let $\mu = M \cap \omega_1 = \sup\langle \beta_n : n < \omega\rangle$, $\beta_{n+1} > \beta_n$. Let $c\colon \omega \longrightarrow M$ be a bijection with $c(0) = Q_{\mathbf{T}}$, $c(1) = p$, $c(2n+2) = \beta_n$, and let*

$$\begin{aligned} U &= U(M, Q_{\mathbf{T}}, p) \\ &= \{2e(n_1,n_2) : c(n_1) \in c(n_2)\} \cup \{2e(n_1,n_2)+1 : c(n_1) <^*_\chi c(n_2)\}. \end{aligned}$$

We let $\Gamma^p(\mu) = \{H_n : n \in \omega\}$ and we let the functions $h_{y,\bar\beta}$ and h_{p,H_n} be defined as in Definitions 3.1 and 3.2.

There is a Borel function $\mathbf{B}_1\colon \omega^\omega \times \mathcal{P}(\omega) \longrightarrow \mathcal{P}(\omega)$, such that for every $\eta \in \omega^\omega$, if

$$(\forall y \in T_\mu)(h_{y,\bar\beta} \leq^* \eta), \tag{3.1}$$

and

$$(\forall n)(h_{p,H_n}(l(\cdot)) \leq^* \eta) \tag{3.2}$$

for

$$G = \{c(n) : n \in \mathbf{B}_1(\eta, U)\}$$

the following holds: G is $(M, Q_{\mathbf{T}})$-generic and $p \in G$ and there is an upper bound r of G as in Claim 2.17.

Remark. r is an upper bound of G iff we have for every $Q_{\mathbf{T}}$-generic filter $G^{\mathbf{V}}$ over $\mathbf{V}$ with $r \in G^{\mathbf{V}}$ and name $\underset{\sim}{G}^{\mathbf{V}}$ that

$$r \Vdash_{Q_{\mathbf{T}}} \underset{\sim}{G}^{\mathbf{V}} \cap M = \{c(n) : n \in \mathbf{B}_1(\eta, U)\}.$$

Proof. We verify that each step in the proof of Lemma 2.15 is Borel-computable from (η, U). Let $M \prec (H(\chi), \in, <^*_\chi)$ be countable. Then we take an enumeration $\langle I_n : n \in \omega\rangle$ of all dense subsets of $Q_{\mathbf{T}}$ that are in M, ordered according to $<^*_\chi$.

Now, we compute from η and U by induction on $n < \omega$, p_n such that

(1) $p_0 = p$, $\mathrm{last}(p) = \beta_0$

(2) p_{n+1} is the $<^*_\chi$-least element of M such that

  (2a) $p_{n+1} \geq p_n$,

  (2b) $\mathrm{last}(p_{n+1}) \geq \beta_{n+1}$,

  (2c) $p_{n+1} \in I_n$,

  (2d) $(\forall x \in T_{\beta_{n+1}})$

$$\left(h_{x,\bar\beta}(n+1) \leq \eta(n+1) \rightarrow f^{p_{n+1}}(x) < f^{p_n}(x\lceil\beta_n) + \frac{1}{2^{n+1+n+\eta(l(x))}}\right).$$

For finding such an p_{n+1} we use the Lemma 2.9 for the finitely many initial segments of branches $y \restriction (\beta_{n+1}+1)$ with $y(\beta_{n+1}) \leq \eta(n+1)$ and with the following bound h:

$$\mathrm{dom}(h) = \{x \in T_{\beta_{n+1}} : h_{x,\bar\beta}(n+1) \leq \eta(n+1)\},$$
$$h(x) = f^{p_n}(x\lceil\beta_n) + \frac{1}{2^{n+1+n+\eta(l(x))}}.$$

If Equations (3.1) and (3.2) hold, then η is sufficiently large to take care of all branches of $T_{<\mu}$ that lead to points $x \in T_\mu$. Set $\mathbf{B}_1(\eta, U) = \{q \in N \cap Q_{\mathbf{T}} : (\exists n) q \leq p_n\}$.

Then $\mathbf{B}_1(\eta, U) \in \mathrm{Gen}^+(N, Q_{\mathbf{T}}, p) \cap A_x$ and there is an upper bound of $\mathbf{B}_1(\eta, U)$ as in Claim 2.17. □

Strictly speaking we must write $U = U(M, P, p, \bar\beta)$, since by the boundedness theorem (see, e.g., [10, Theorem 31.1]) a cofinal sequence $\bar\beta$ cannot be computed in a Borel manner from $(M, \in, <^*_\chi)$, and for each n, β_n is coded by the stipulation $c(2n+2) = \beta_n$. The arguments (M, P, p) of U will change during the iteration, and one of the main tasks is to show that all the changes are Borel computable, see for example Equation (3.8). Fortunately, since in proper forcing P the ordinary height of N and $N[\mathcal{G}]$ (we use the letters N and $\mathcal{G}$ for the objects after the transitive collapse) are the same for all (M, P)-generic filters G, $\bar\beta$ will not change and it does not hide features of the proof if we do not write it during the proof of the iteration theorem. However, $\bar\beta$ needs to be guessed as one component in Lemma 3.11 and will be written there. Since our notation is already heavily burdened, we write only $U(M, P, p)$ until the end of the proof of Theorem 3.4.

Since each η dominating all $h_{t,\bar\beta}$, $t \in T_\mu$, and dominating h_{p,H_n}, $n \in \omega$, gives an $(M, Q_{\mathbf{T}})$-generic G, the generic player can play ν fulfilling all theses largeness requirements and thereafter any $\eta \geq^* \nu$ can be used as an argument of $\mathbf{B}$. We use this option to build a game between two players, and to establish properties that say: The $\leq^*$-larger the argument η in the Borel function $\mathbf{B}_1$ is, the better it aims at the envisaged weak diamond. See also the remark [15, V, Remark 5.4 (2)] about the influence of the guessed parameters on the generic filter. The knowledge that the $\leq^*$-larger parameter can be inserted in the Borel function $\mathbf{B}_1$ will help us later to see that in the iteration every name of a real (called $\mathbf{B}'$ in Lemma 3.10 as it is another Borel function) can be forced into a slalom from the ground model (called $\mathcal{C}$ there) that is meagre and of Lebesgue measure zero.

The following theorem is an iterated version of Lemma 3.3. It is related to Theorem 2.20, however now we want to compute bounded (M, P_γ)-generic filters (that witness that no reals are added) as Borel functions of certain arguments. As in Theorem 2.20 we use $< \omega_1$-properness and a tower $\langle M_i : i \leq \alpha \rangle$ with $\alpha = \mathrm{otp}(M \cap \gamma) < \omega_1$, $\gamma =$ iteration length, of elementary submodels in order to prove facts about $M = M_0$ and P_γ. The tower appears only in the proof, not in the statement of the theorem. The following theorem would work for arbitrary iteration length, but we use it only for length ω_2 and notate it only for this length.

Theorem 3.4. *Let $P_{\omega_2} = \langle P_\alpha, \underset{\sim}{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a countable support iteration of iterands of the form $Q_{\mathbf{T}}$. If χ is sufficiently large and regular and if $M \prec (H(\chi), \in, <^*_\chi)$ is countable and*

(a) $P_\gamma \in M$, $\gamma \leq \omega_2$,
(b) $p \in P_\gamma \cap M$,
(c) $\alpha = \mathrm{otp}(M \cap \gamma)$,
(d) *Let $\bar{\beta}$ be cofinal in $M \cap \omega_1$. Let $c \colon \omega \longrightarrow M$ be a bijection with $c(0) = P_\gamma$, $c(1) = p$, $c(2n+2) = \beta_n$, and let*

$$\begin{aligned} U &= U(M, P_\gamma, p) \\ &= \{2e(n_1, n_2) : c(n_1) \in c(n_2)\} \cup \{2e(n_1, n_2) + 1 : c(n_1) <^*_\chi c(n_2)\}. \end{aligned}$$

Then there is a Borel function $\mathbf{B} = \mathbf{B}_\alpha \colon (\omega^\omega)^\alpha \times \mathcal{P}(\omega) \longrightarrow \mathcal{P}(\omega)$, such that in the following game $\partial_{(M, P_\gamma, p)}$ the generic player has a winning strategy σ, which depends only on the isomorphism type of $(M, \in, <^*_\chi, P_\gamma, p, \bar{\beta})$:

(α) *a play lasts α moves,*
(β) *in the ε-th move the generic player chooses some real ν_ε and the anti-generic player chooses some $\eta_\varepsilon \in \omega^\omega$, such that $\eta_\varepsilon \geq^* \nu_\varepsilon$,*
(γ) *in the end the generic player wins iff the following is true:*

$G_\gamma = \{c(n) : n \in \mathbf{B}_\alpha(\langle \eta_\varepsilon : \varepsilon < \alpha \rangle, U)\}$ *is (M, P_γ)-generic and*
$p \in G_\gamma$ *and*
$(\exists q \in P_\gamma)(p \leq q$ *and* q *bounds* $G_\gamma)$.

Proof. We follow Abraham's exposition in [1, Theorem 5.17]. This theorem works only inductively: For Q_α in $\mathbf{V}^{P_\alpha}$ to be $\mathbb{D}$–complete with respect to a system that lies in $\mathbf{V}$ we need that P_α does not add new countable sets of ordinals. So every countable transitive set in $\mathbf{V}^{P_\alpha}$ is in $\mathbf{V}$.

To prove the theorem we shall first define for every countable $M \prec (H(\chi), \in, <^*_\chi)$ with $P_\gamma \in M$, $p \in P_\gamma \cap M$, with $\alpha = \text{otp}(M \cap \gamma)$, an (M, P_γ)-generic filter $G_\gamma = \mathbf{B}_\alpha(\langle \eta_i : i < \alpha \rangle, U)$; and then we shall prove that G_γ is bounded in P_γ by a completely (M, P_γ)-generic condition. The bounding condition is not computed in a Borel manner. Its existence is sufficient, and its existence is proved along the iteration.

Remark. The bounding condition also appears in an argument about the truth in forcing extensions at the very end of our Lemma 3.11.

The definition of G_γ is by induction and we shall define for every $\gamma_0 < \gamma$ and G_{γ_0} that is (M, P_{γ_0})-generic and every $p \in P_\gamma \cap M$ with $p \restriction \gamma_0 \in G_0$ a filter G_γ that extends G_{γ_0} and contains p. Once the induction is performed, we shall set $\gamma_0 = 0$, $G_0 = \{0_{P_0}\}$. There will be two main cases in this definition: γ successor and γ limit, and likewise there will be two cases in the proofs that G_γ is bounded. We start with the preparations for the successor case. When looking at complexity, we regard G_0 as a parameter.

Two step iteration

Let P be a poset and let $\underset{\sim}{Q} \in \mathbf{V}^P$ be a name forced by 0_P to be a poset. Let χ be sufficiently large and regular (as said, $\chi = (2^{\aleph_2})^+$ is always sufficiently large) and $M_0 \prec (H(\chi), \in, <^*_\chi)$ be a countable elementary submodel such that P, $\underset{\sim}{Q} \in M_0$. Henceforth we write just $H(\chi)$ instead of $(H(\chi), \in, <^*_\chi)$. We want to find a criterion for when a condition $(q_0, q_1) \in P * \underset{\sim}{Q}$ is completely $(M_0, P * \underset{\sim}{Q})$-generic. Let $\pi \colon M_0 \longrightarrow N_0$ be a transitive collapsing map. Suppose that $q_0 \in P$ is completely generic over (M_0, P) and let $G_0 \subseteq P \cap M_0$ be the (M_0, P)-generic filter induced by q_0. Then $\mathcal{G}_0 = \pi'' G_0$ is an $(N_0, \pi(P))$-generic filter and we can form the transitive extension $N^*_0 = N_0[\mathcal{G}_0]$. $\pi(\underset{\sim}{Q})$ is a name in N_0, and its interpretation $Q^*_0 = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$ is a poset in N^*_0.

Let $\underset{\sim}{G} \in \mathbf{V}^P$ be the canonical name of the P-generic filter over $\mathbf{V}$. If F is a $(\mathbf{V}, P)$ generic filter containing q_0 then $M_0[F] \prec H(\chi)[F]$ can be formed and the collapsing map π on M_0 can be extended to collapse $M_0[F]$ onto N^*_0. Let $\underset{\sim}{\pi}$ be the name of the extended collapse. Then $q_0 \Vdash_P \underset{\sim}{\pi} \colon M_0[\underset{\sim}{G}] \longrightarrow N^*_0$. We phrase now the desired criterion and we shall use the direction from right to left later.

Lemma 3.5. *Using the above notation,* (q_0, q_1) *is completely generic over* $(M_0, P * \underset{\sim}{Q})$*, iff*

1. q_0 *is completely* (M_0, P)*-generic, and*
2. *for some* $\mathcal{G}_1 \subseteq Q_0^*$ *that is* (N_0^*, Q_0^*)*-generic* $q_0 \Vdash$ "$\underset{\sim}{\pi}^{-1\prime\prime}\mathcal{G}_1$ *is bounded by* q_1".

In this case the filter induced by (q_0, q_1) *over* $M_0 \cap P * \underset{\sim}{Q}$ *is* $\pi^{-1\prime\prime}\mathcal{G}_0 * \mathcal{G}_1$.

Given a countable $M_0 \prec H(\chi)$ such that the two step iteration $P * \underset{\sim}{Q}$ is in M_0, our aim is to extent each (M_0, P)-generic filter G_0 to an $(M_0, P * \underset{\sim}{Q})$-generic filter. This definition depends not only on M_0 but also on another countable elementary submodel $M_1 \prec H(\chi)$ such that $M_0 \in M_1$ and $G_0 \in M_1$. In addition we fix a $p_0 \in P * \underset{\sim}{Q}$ which we want to include in the extended filter. All of this leads us to a five place function $\mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$ that we define now.

Definition 3.6. Let P be a poset that adds no new countable sets of ordinals and suppose that $\underset{\sim}{Q}$, $\underset{\sim}{\mathbb{D}} \in \mathbf{V}^P$ are such that

$$\Vdash_P \underset{\sim}{\mathbb{D}} \in \mathbf{V} \text{ is an } \aleph_1\text{-completeness system and}$$
$$\underset{\sim}{Q} \text{ is } \mathbb{D}\text{-complete with respect to } \underset{\sim}{\mathbb{D}}.$$

Let χ be sufficiently large and $M_0 \prec M_1 \prec (H(\chi), \in, <^*_\chi)$ be countable elementary submodels with $M_0 \in M_1$ and P, $\underset{\sim}{Q}$, $\underset{\sim}{\mathbb{D}} \in M_0$. Let $G_0 \subseteq M_0 \cap P$ be (M_0, P)-generic and suppose that $G_0 \in M_1$. Let $p_0 \in P * \underset{\sim}{Q} \cap M_0$ be given $p_0 = (a, \underset{\sim}{b})$ with $a \in G_0$. Then we define

$$G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0),$$

an $(M_0, P * \underset{\sim}{Q})$-generic filter containing p_0 (dominating G_0) by the following procedure:

Let $\pi \colon M_1 \longrightarrow N_1$ with $\pi(M_0) = N_0$ be the transitive collapse and $\mathcal{G}_0 = \pi''G_0$. Form $N_0^* = N_0[\mathcal{G}_0]$. Observe that $N_0^* \in N_1$. Let $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$, and let $\mathbb{D}_0 = \pi(\underset{\sim}{\mathbb{D}})[\mathcal{G}_0]$. Then $\mathbb{D}_0 \in N_0$, because it is forced to be in the ground model. So $\mathbb{D}_0 = \pi(\mathbb{D})$ where $\mathbb{D} \in M_0$ is a countably closed completeness system. Thus $\mathbb{D}_0(N_0^*, Q_0^*, b^*)$ is defined in N_1, where $b^* = \pi(\underset{\sim}{b})[\mathcal{G}_0]$ is a condition in Q_0^*. Since $N_1 \cap \mathbb{D}_0(N_0^*, Q_0^*, b^*)$ is countable,

$$\text{there is some } \mathcal{G}_1 \in \bigcap(N_1 \cap \mathbb{D}_0(N_0^*, Q_0^*, b^*)). \tag{3.3}$$

$\mathcal{G}_1$ is (N_0^*, Q_0^*)-generic and $b^* \in \mathcal{G}_1$. Form $\mathcal{G}_0 * \mathcal{G}_1 = \mathcal{G}$, an $(N_0, \pi(P * \underset{\sim}{Q}))$-generic filter. Then $\pi(p_0) \in \mathcal{G}$. Finally we define

$$G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0) = \pi^{-1\prime\prime}\mathcal{G}. \tag{3.4}$$

Now observe that if η fulfils Equations (3.1) and (3.2) for (N_0^*, Q_0^*, b^*) instead of $(M, Q_{\mathbf{T}}, p)$, then the existence of Equation (3.3) is given by

$$\pi^{-1\prime\prime}\mathcal{G}_1 = \mathbf{B}_1(\eta, U(M_0[G_0], \underset{\sim}{Q}_0[G_0], \underset{\sim}{b}[G_0]))$$

and hence is Borel computable from η and the code U of the intermediate model (N_0^*, Q_0^*, b^*).

In fact, we want to define a formula ψ so that

$$H(\chi) \models \psi(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$$

iff Equation (3.4) holds. That is, we want to define $\mathbb{E}$ in $H(\chi)$. We cannot take the above definition verbally, because it relies on the assumption that M_0 and M_1 are elementary substructures of $H(\chi)$, something which is not expressible in $H(\chi)$. Whenever the definition above relies on some fact that happens not to hold we let $\mathcal{G}$ have an arbitrary value. For example if N_0^* is not in N_1 or if $N_1 \cap \mathbb{D}_0(N_0^*, Q_0^*, b^*)$ is empty, then we let $\mathcal{G}$ be some arbitrary fixed N_0-generic filter. The Borel computation does not invoke N_1, since $\pi^{-1}{}''\mathcal{G}_1 = \mathbf{B}_1(\eta, U(M_0[G_0], \underset{\sim}{Q}_0[G_0], \underset{\sim}{b}[G_0]))$. Here, G_0 is a parameter and will be set $\{0_{P_0}\}$ later, so that in the end (that means in Lemma 3.11) only the possible isomorphism types of $(M_0, \in\restriction M_0, <^*_\chi\restriction M_0, P_\gamma, p, \bar{\beta})$ need to be guessed stationarily often alongside with names for the F and f from the statement of the weak diamond.

The following lemma shows the second part of the argument: We want to show the G given in Equation (3.4) is bounded. The lemma analyses the iteration of two posets when the second is $\mathbb{D}$-complete.

Lemma 3.7. *The One Step Extension Lemma.*
Let P be poset and suppose that $\underset{\sim}{Q}$, $\underset{\sim}{\mathbb{D}} \in \mathbf{V}^P$ are such that

$$\Vdash_P \underset{\sim}{\mathbb{D}} \in \mathbf{V} \text{ is an } \aleph_1\text{-completeness system and}$$
$$\underset{\sim}{Q} \text{ is } \mathbb{D}\text{-complete with respect to } \underset{\sim}{\mathbb{D}}.$$

Let χ be sufficiently large and $M_0 \prec M_1 \prec H_\chi$ be countable elementary submodels with $M_0 \in M_1$ and P, $\underset{\sim}{Q}$, $\underset{\sim}{\mathbb{D}} \in M_0$. Suppose that $q_0 \in P$ is (M_1, P)-generic as well as completely (M_0, P)-generic, and let $G_0 \subseteq M_0 \cap P$ be the M_0 filter over $M_0 \cap P$ induced by q_0. Let $p_0 \in P * \underset{\sim}{Q}$, $p_0 \in M_0$ be given, so that $p_0 = (a, \underset{\sim}{b})$ and $a \in G_0$. Then there is $q_1 \in \mathbf{V}^{\tilde{P}}$ such that (q_0, q_1) is completely generic over $(M_0, P * Q)$ and $p_0 \leq (q_0, q_1)$, in fact (q_0, q_1) bounds $G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0) = \tilde{G}_0 * \mathbf{B}_1(\eta, U(N_0^*, Q_0^*, \pi(\underset{\sim}{b})))$.

Proof. This is literally [1, The Gambit Lemma]. For completeness' sake we repeat Abraham's proof here. Notice that $G_0 \in M_1$ by the following argument: Let R be the collection of all conditions $r \in P$ that are completely generic over M_0. Then $R \in M_1$ and $q_0 \in R \cap M_1$. Since q_0 is (M_1, P)-generic, it follows that it is compatible with some $r \in R \cap M_1$. But any two compatible conditions in R induce the same filter, and hence G_0 is the filter induced by r.

Let $\pi\colon M_1 \longrightarrow N_1$, $\pi(M_0) = N_0$, be the transitive collapse and $\mathcal{G}_0 = \pi'' G_0$. We recall the definition of $\mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$. Form $N_0^* = N_0[\mathcal{G}_0]$ and let $Q_0^* = \pi(\underset{\sim}{Q})[\mathcal{G}_0]$, and let $\mathbb{D}_0 = \pi(\underset{\sim}{\mathbb{D}})[\mathcal{G}_0]$. Then $\mathbb{D}_0 \in N_0$ because it is forced to be in the ground model. So $\mathbb{D}_0 = \pi(\mathbb{D})$ where $\mathbb{D} \in M_0$ is a countably closed completeness system. Thus $\mathbb{D}_0(N_0^*, Q_0^*, b^*)$ is defined in N_1, where $b^* = \pi(\underset{\sim}{b})[\mathcal{G}_0]$ is a condition in Q_0^*. Since $N_1 \cap \mathbb{D}_0(N_0^*, Q_0^*, b^*)$ is countable, there is some $\mathcal{G}_1 \in \bigcap(N_1 \cap \mathbb{D}_0(N_0^*, Q_0^*, b^*))$. $\mathcal{G}_1$ is (N_0^*, Q_0^*)-generic and $b^* \in \mathcal{G}_1$. Form $\mathcal{G}_0 * \mathcal{G}_1 = \mathcal{G}$, an $(N_0, \pi(P * \underset{\sim}{Q}))$-generic filter. Then $\pi(p_0) \in \mathcal{G}$. We defined $G = \mathbb{E}(M_0, M_1, P * \underset{\sim}{Q}, G_0, p_0)$ as $\pi^{-1\prime\prime}\mathcal{G}$.

Let $\underset{\sim}{G} \in \mathbf{V}^P$ be the canonical name of the generic filter over P. Then q_0 forces that π can be extended to a collapse $\underset{\sim}{\pi}$ which is onto N_0^*, that is

$$q_0 \Vdash_P \underset{\sim}{\pi}\colon M_0[\underset{\sim}{G}] \longrightarrow N_0^*.$$

The conclusion of our lemma follows if we show that

$$q_0 \Vdash_P \underset{\sim}{\pi}^{-1\prime\prime}\mathcal{G}_1 \text{ is bounded in } \underset{\sim}{Q}. \tag{3.5}$$

In this case, if we define $q_1 \in \mathbf{V}^P$ so that $q_0 \Vdash_P q_1$ bounds $\underset{\sim}{\pi}^{-1\prime\prime}\mathcal{G}_1$, then the previous lemma implies that the $(M_0, P * \underset{\sim}{Q})$-generic filter induced by (q_0, q_1) is $\pi^{-1\prime\prime}\mathcal{G}_0 * \mathcal{G}_1$.

So let F be $(\mathbf{V}, P)$-generic with $q_0 \in F$. $\underset{\sim}{\pi}[F]$ collapses $M_0[F]$ onto N_0^* and there is a set $X \in \mathbb{D}_0(N_0^*, Q_0^*, b^*)$, so that if $\mathcal{H} \in X$ is any filter then $\pi^{-1\prime\prime}\mathcal{H}$ is bounded in $\underset{\sim}{Q}[F]$. As $N_1[F] \prec H_\chi[F]$, we can have $X \in N_1[F]$. But since $\mathbb{D}_0$ is in the ground model, $X \in N_1$. Thus $\mathcal{G}_1 \in X$, where $\mathcal{G}_1$ is the filter defined above. This proves Equation (3.5). $\square$

The iteration theorem.

Let P_γ be a countable support iteration of length γ obtained by choosing iterands $Q_\alpha \in \mathbf{V}^{P_\alpha}$ as in the theorem. That is, each Q_α is $\mathbb{D}$-complete in $\mathbf{V}^{P_\alpha}$ for some $\aleph_1$-completeness system taken from $\mathbf{V}$. Let χ be a sufficiently large regular cardinal. To prove the theorem we first describe a machinery for obtaining generic filters over countable submodels of $H(\chi)$. We define a function $\mathbb{E}$ that takes five arguments, $\mathbb{E}(M_0, \bar{M} \restriction [1, \alpha), P_\gamma, G_0, p_0)$ of the following types.

1. $M_0 \prec H_\chi$ is countable, $P_\gamma \in M_0$, so $\gamma \in M_0$. Moreover, $p_0 \in M_0 \cap P_\gamma$.
2. For some $\gamma_0 \in M_0 \cap \gamma$, G_0 is an (M_0, P_{γ_0})-generic filter and such that $p_0 \restriction \gamma_0 \in G_0$. We assume that $G_0 \in M_1$.
3. The order type of $M_0 \cap [\gamma_0, \gamma)$ is α.
4. $\bar{M} = \langle M_\xi : 0 \le \xi \le \alpha \rangle$ is an $\alpha + 1$-tower of countable elementary submodels of $H(\chi)$ and $M_0 = M$. Note that only $M_0 = M$ appears in the statement of the theorem. The rest $\langle M_\xi : 1 \le \xi \le \alpha \rangle$ of the tower is a technical means for the proof.

The value returned, $G_\gamma = \mathbb{E}(M_0, \bar{M} \restriction [1,\alpha), P_\gamma, G_0, p_0)$ is an (M_0, P_γ)-generic filter that extends G_0 and contains p_0. Formally, in saying that G_γ extends G_0, we mean that the restriction projection takes G_γ onto G_0. The definition of $\mathbb{E}(M_0, \bar{M} \restriction [1,\alpha), P_\gamma, G_0, p_0)$ is by induction on $\alpha < \omega_1$.

Assume that $\alpha = \alpha' + 1$ is a successor ordinal. Then $\gamma = \gamma' + 1$ is also a successor. Assume first that $\gamma_0 = \gamma'$. Then $\alpha = 1$ and we have only two structures: M_0 and M_1. Since P_γ is isomorphic to $P_{\gamma_0} * Q_{\gamma_0}$ we can define G_γ by Equation (3.4). So, if η fulfils Equations (3.1) and (3.2) for $(M_0[G_0], \underset{\sim}{Q}_0[G_0], \underset{\sim}{b}[G_0])$ in the role of of $(M, Q_{\mathbf{T}}, p)$, then

$$\begin{aligned} G_\gamma =& \mathbb{E}(M_0, M_1, P_{\gamma_0} * Q_{\gamma_0}, G_0, p_0) \\ =& G_0 * \mathbf{B}_1(\eta_0, U(M_0[G_0], \underset{\sim}{Q}_0[G_0], \underset{\sim}{b}[G_0])). \end{aligned}$$

Assume next that $\gamma_0 < \gamma'$. Then by induction hypothesis, if all η_i, $i < \alpha'$, are sufficiently large, then

$$\begin{aligned} G_{\gamma'} &= \mathbb{E}(M_0, \langle M_\xi : 1 \le \xi \le \alpha' \rangle, P_{\gamma'}, G_0, p_0 \restriction \gamma') \\ &= G_0 * \mathbf{B}_{\alpha'}(\langle \eta_i : 0 \le i < \alpha' \rangle, U(M_0[G_0], P_{[\underset{\sim}{\gamma_0}, \gamma')}[G_0], p_0 \restriction [\underset{\sim}{\gamma_0}, \gamma')[G_0])) \end{aligned} \tag{3.6}$$

is defined and is an $(M_0, P_{\gamma'})$-generic filter that extends G_0 and contains $p_0 \restriction \gamma'$. Moreover by elementarity, $G_{\gamma'} \in M_\alpha$. When we finish this definition it will be evident that it continues for every $\alpha < \omega_1$ since $M_\alpha \prec H(\chi)$ and the parameters are all in M_α. This brings us to the previous case and we choose $\eta_{\alpha'}$ such that it fulfils Equations (3.1) and (3.2) in which $(M, Q_{\mathbf{T}}, p)$ is replaced by

$$(M_0[G_{\gamma'}], \underset{\sim}{Q}_\gamma[G_{\gamma'}], p_0(\underset{\sim}{\gamma'})[G_{\gamma'}]).$$

Now from Equation (3.6) we define temporarily

$$U' = U(M_0[G_0], P_{[\underset{\sim}{\gamma_0}, \gamma')}[G_0], p_0 \restriction [\underset{\sim}{\gamma_0}, \gamma')[G_0]]). \tag{3.7}$$

Then

$$\begin{aligned} G_\gamma =& \mathbb{E}(M_0, M_\alpha, P_{\gamma'} * Q_{\gamma'}, G_{\gamma'}, p_0) \\ =& G_0 * \mathbf{B}_1(\eta_{\alpha'}, U(M_0[G_0 * \mathbf{B}_{\alpha'}(\langle \eta_i : i < \alpha' \rangle, U')], \\ & \qquad \underset{\sim}{Q}_\gamma[G_0 * \mathbf{B}_{\alpha'}(\langle \eta_i : i < \alpha' \rangle, U')], \\ & \qquad p_0(\underset{\sim}{\gamma'})[G_0 * \mathbf{B}_{\alpha'}(\langle \eta_i : i < \alpha' \rangle, U')])) \\ =:& G_0 * \mathbf{B}_\alpha(\langle \eta_i : i < \alpha \rangle, U(M_0[G_0], P_{[\underset{\sim}{\gamma_0}, \gamma)}[G_0], \underset{\sim}{p_0}[G_0])) \end{aligned} \tag{3.8}$$

and the middle U' is defined above in Equation (3.7). This justifies that the Borel functions given by induction hypothesis can be composed to one Borel function of the required arguments.

Now it is also clear how to define the *strategy* $\sigma(\langle \nu_i, \eta_i : i < \alpha' \rangle)$: The generic player plays $\nu_{\alpha'}$ so that it fulfils Equations (3.1) and (3.2), where $(M, Q_{\mathbf{T}}, p)$ is replaced by $(M_0[G_{\gamma'}], \underset{\sim}{Q}_\gamma[G_{\gamma'}], p_0(\underset{\sim}{\gamma'})[G_{\gamma'}])$ with $G_{\gamma'}$ as in Equation (3.6).

Now assume that α is a limit ordinal and let $\langle \alpha_n : n \in \omega \rangle$ be an increasing cofinal sequence with $\alpha_0 = 0$. Let $\gamma_n \in M_0$ be such that $\alpha_n = \mathrm{otp}(M_0 \cap [\gamma_0, \gamma_n))$. Let $\langle I_n : n \in \omega \rangle$ be an enumeration of all dense subsets of P_γ that are in M_0 in such a way that I_n is the $<^*_\chi$-least dense subset of P_γ that is not among $\{I_m : m < n\}$.

We define

$$\begin{aligned} G_\gamma =& \mathbb{E}(M_0, \bar{M} \restriction [1, \alpha), P_\gamma, G_0, p_0) \\ =& G_0 * \mathbf{B}_\alpha(\langle \eta_i : i < \alpha \rangle, U(M_0[G_0], \underset{\sim}{P}_{[\gamma_0, \gamma)}[G_0], p_0 \restriction \underset{\sim}{[\gamma_0}, \gamma)[G_0])) \end{aligned}$$

as follows. We define by induction on $n \in \omega$ a condition $p_n \in P_\gamma \cap M_0$ and an (M_0, P_{γ_n})-generic filter $G_n \in M_{\alpha_{n+1}}$ such that

1. G_0 and p_0 are given. $p_n \restriction \gamma_n \in G_n$.
2. $p_n \leq p_{n+1}$ and $p_{n+1} \in I_n$.

Suppose that G_n and p_n are defined. First we can find $p_{n+1} \in I_n \cap M_0$ such that $p_{n+1} \restriction \gamma_n \in G_n$ (for an existence proof see [1, Lemma 1.2]) and we take the $<^*_\chi$-least in M_0 so that it is Borel computed. Now define

$$\begin{aligned} G_{n+1} =& \mathbb{E}(M_0, \langle M_\xi : \alpha_n + 1 \leq \xi \leq \alpha_{n+1} \rangle, P_{\gamma_{n+1}}, G_n, p_{n+1} \restriction \gamma_{n+1}) \\ =& G_0 * \mathbf{B}_{\alpha_{n+1} - \alpha_n}(\langle \eta_i : i \in [\alpha_n, \alpha_{n+1}) \rangle, U^*). \end{aligned}$$

Here we have

$$\begin{aligned} U^* =& U(M_0[G_0 * \mathbf{B}_{\alpha_n}(\langle \eta_i : i < \alpha_n \rangle, U'')], \\ & \underset{\sim}{P}_{[\gamma_n, \gamma_{n+1})}[G_0 * \mathbf{B}_{\alpha_n}(\langle \eta_i : i < \alpha_n \rangle, U'')], \\ & p_{n+1} \restriction [\underset{\sim}{\gamma_n}, \gamma_{n+1})[G_0 * \mathbf{B}_{\alpha_n}(\langle \eta_i : i < \alpha_n \rangle, U'')]) \text{ and} \\ U'' =& U(M_0[G_0], \underset{\sim}{P}_{[\gamma_0, \gamma_n)}[G_0], p_{n+1} \restriction \underset{\sim}{[}\gamma_0, \gamma_n)[G_0]). \end{aligned}$$

Finally let

$$G_\gamma = \text{ the generic filter generated in } M_0 \text{ by } \{p_n : n \in \omega\}.$$

From the above induction on $n < \omega$ and from the induction hypothesis it is clear that there is a Borel function $\mathbf{B}_\alpha$ such that

$$G_\gamma = G_0 * \mathbf{B}_\alpha(\langle \eta_i : i < \alpha \rangle, U(M_0[G_0], \underset{\sim}{P}_{[\gamma_0, \gamma)}[G_0], p_0 \restriction \underset{\sim}{[\gamma_0}, \gamma)[G_0])). \quad (3.9)$$

This ends the definition of $\mathbb{E}(M_0, \bar{M} \restriction [1, \alpha), P_\gamma, G_0, p_0)$ and of $\mathbf{B}_\alpha$.

The *strategy* σ for the generic player is defined by the prescription, that in the limit game of length α he plays according to the strategies for the initial segments of the game. (This justifies that σ_α is just named σ, for all lengths α.) This is a winning strategy, as the Borel function was just derived. It gives a generic filter. We still have to show that the given generic filter is bounded.

Now the missing part is to show that "all the generic filters are bounded" is preserved in the limit steps of the iteration. Again there is nothing new to our work and we repeat Abraham's proof to [1, The Extension Lemma].

Lemma 3.8. *Let $\langle P_\alpha, \underset{\sim}{Q}_\beta : \beta < \gamma, \alpha \le \gamma \rangle$ be a countable support iteration of forcing posets such that each iterand Q_α satisfies the following in $\mathbf{V}^{P_\alpha}$:*

1. *Q_α is δ-proper for every countable δ.*
2. *Q_α is $\mathbb{D}$-complete with respect to some countably closed completeness system in the ground model that has the property that all $\eta \ge^* \nu$ serve as parameters.*

Suppose that $M_0 \prec H(\chi)$ is countable, $P_\gamma \in M_0$ and $p_0 \in P_\gamma \cap M_0$. For any $\gamma_0 \in \gamma \cap M_0$ with $\alpha = \operatorname{otp}(M_0 \cap [\gamma_0, \gamma))$ and $\bar{M} = \langle M_\xi : \xi \le \alpha \rangle$ is a tower of countable elementary substructures starting with the given M_0, then the following holds:

For every $q_0 \in P_{\gamma_0}$ that is completely (M_0, P_{γ_0})-generic as well as $(\bar{M}, P_{\gamma_0})$-generic, if $p_0 \restriction \gamma_0 < q_0$, then there is some $q \in P_\gamma$ such that $q_0 = q \restriction \gamma_0$ and $p_0 < q$ and q is completely (M_0, P_γ)-generic. In fact, the filter induced by q is $\mathbb{E}(M_0, \langle M_\xi : 1 \le \xi \le \alpha \rangle, P_\gamma, G_0, p_0)$ where $G_0 \subseteq P_{\gamma_0} \cap M_0$ is the filter induced by q_0.

Proof. Let $G_0 \subseteq P_{\gamma_0} \cap M_0$ be the M_0-generic filter induced by q_0. Observe that $G_0 \in M_1$ follows from the assumption that q_0 is also M_1-generic. We shall prove by induction on $\alpha = \operatorname{otp}(M_0 \cap [\gamma_0, \gamma))$ that q can be found that bounds $G_\gamma = \mathbb{E}(M_0, \langle M_\xi : 1 \le \xi \le \alpha \rangle, P_\gamma, G_0, p_0)$.

Suppose first that $\alpha = \alpha' + 1$ and consequently $\gamma = \gamma' + 1$ are successor ordinals. Define in M_α, $X \subseteq P_{\gamma_0}$ as maximal antichain of conditions r so that

1. r bounds G_0,
2. r in $\langle M_\xi : 1 \le \xi \le \alpha' \rangle$-generic.

Then $X \in M_\alpha$ is predense above q_0. By our inductive assumption, every $r_0 \in X$ has a prolongation $r_1 \in P_{\gamma'}$ that bounds $G_{\gamma'} = \mathbb{E}(M_0, \langle M_\xi : 1 \le \xi \le \alpha' \rangle, G_0, p_0 \restriction \gamma')$. Since all the parameters are in M_α, we get that $G_{\gamma'} \in M_\alpha$. Since $M_\alpha \prec H(\chi)$ we can choose $r_1 \in M_\alpha$ whenever $r_0 \in X \cap M_\alpha$. This defines a name $\underset{\sim}{r}_1 \in \mathbf{V}^{P_{\gamma_0}}$, forced by q_0 to be in $M_\alpha \cap P_{\gamma'}$. Namely, if G is any $(\mathbf{V}, P_{\gamma_0})$-generic filter containing q_0, then $X \cap G$ contain a unique condition r_0, and we let $\underset{\sim}{r}_1[G] = r_1$. By the Properness Extension Lemma [1, Lemma 2.8] we can find $q_1 \in P_{\gamma'}$, $q_1 \restriction \gamma_0 = q_0$, q_1 is $(M_\alpha, P_{\gamma'})$-generic, and $q_1 \Vdash_{P_{\gamma'}}$ "$\underset{\sim}{r}_1$ is in the generic filter $\underset{\sim}{G}_{\gamma'}$". It follows that q_1 bounds $G_{\gamma'}$. We find $q_2 \in P_\gamma$, such that $q_2 \restriction \gamma' = q_1$ and q_2 bounds G_γ. In order to define $q_2(\gamma)$ we use the Two Step Lemma and Equation (3.5).

Now assume that α is a limit ordinal. We follow the definition of G_γ see Equation (3.9). Recall that we had an ω-sequence $\langle \alpha_n : n \in \omega \rangle$ cofinal in α and we defined γ_n cofinal in γ as the resulting sequence $\alpha_n = \text{otp}(M_0 \cap [\gamma_0, \gamma_n))$. We defined by induction $p_n \in P_\gamma \cap M_0$ and filters $G_n \subseteq P_{\gamma_n}$, $G_n \in M_{\alpha_n+1}$ and defined G_γ as the filter generated by the p_n's. We shall define now $q_n \in P_{\gamma_n}$ by induction on n so that the following hold

1. q_n bounds G_n,
2. $p_n \restriction \gamma_n \leq q_n$,
3. $q_n = q_{n+1} \restriction \gamma_n$,
4. q_n is $\langle M_\xi : \alpha_n + 1 \leq \xi \leq \alpha \rangle$-generic over P_{γ_n}.

Thus q_n gains in length and looses in status as an M_ξ-generic condition for $0 < \xi \leq \alpha_n$. But q_n is completely (M_0, P_{γ_n})-generic for all n. Finally $q = \bigcup q_n$ is not M_ξ-generic for any $\xi > 0$. However, q is completely (M_0, P_γ)-generic.

Suppose that q_n is defined. Let X be in $M_{\alpha_{n+1}+1}$ be a maximal antichain in P_{γ_n} of conditions r that induce G_n and are $\langle M_\xi : \alpha_n + 1 \leq \xi \leq \alpha_{n+1} \rangle$-generic over P_{γ_n}. Observer that X is predense above q_n. For each $r_0 \in X$, define by the induction assumption $r_1 \in P_{\gamma_{n+1}}$ such that r_1 bounds G_{n+1}, $p_{n+1} \restriction \gamma_{n+1} < r_1$ and $r_1 \restriction \gamma_n = r_0$. If $r_0 \in X \cap M_{\alpha_{n+1}+1}$, then r_1 is taken from $M_{\alpha_{n+1}+1}$. Now view $\{r_1 : r_0 \in X\}$ as a name $\underset{\sim}{r}$ for a condition forced by q_n to lie in $M_{\alpha_{n+1}+1}$. By the α-Extension Lemma [1, Lemma 5.6], define q_{n+1} that satisfies items 2 to 4 from the above list and such that $q_{n+1} \Vdash_{P_{\gamma_{n+1}}} \underset{\sim}{r} \in G_{\underset{\sim}{n}+1}$. Then q_{n+1} bounds G_{n+1} and is a required. $\square$

End of proof of Theorem 3.4. Now that the induction is performed, we set $\gamma_0 = 0$, $G_0 = \{0_{P_0}\}$, $p_0 = p \in P_\gamma$ from the statement of Theorem 3.4. Then $N_0^* = N_0 = \pi(M_0)$, $\pi(P_{[\underset{\sim}{\gamma_0},\gamma)})[\mathcal{G}_0] = \pi(P_\gamma)$ and $\pi(p_0)_{[\underset{\sim}{\gamma_0},\gamma)}[\mathcal{G}_0] = \pi(p)$ and the $\mathbf{B}_\alpha$'s second argument is just the isomorphism type of $(M_0, \in, <^*_\chi, P_\gamma, p, \bar{\beta})$. $\square_{3.4}$

The role of the antigeneric player in the game $\partial(M, P_\gamma, p)$ is now turned to good use:

Definition 3.9. Let $f, g \in \mathbf{V} \cap \omega^\omega$. A notion of forcing P^* has the (f,g)-*bounding property* when for every P^*-name $\underset{\sim}{u}$ for a function from ω to ω the following holds: If $p \Vdash_{P^*} \underset{\sim}{u} \leq^* g$, then there are $q \geq p$ and an f-slalom $\langle S_n : n < \omega \rangle$ in the ground model such that $q \Vdash_{P^*} (\forall n)(\underset{\sim}{u}(n) \in S_n)$. $\langle S_n : n < \omega \rangle$ is an *f-slalom* if for every n, $S_n \subseteq \omega$ and $|S_n| \leq f(n)$.

Lemma 3.10. *Suppose that*

(α) $\gamma < \omega_1$, *and*

(β) $\mathbf{B}'$ *is a Borel function from* $(\omega^\omega)^\gamma$ *to* 2^ω,

(γ) *$r\colon \omega \longrightarrow \omega$, r diverging to infinity, and* $\lim \dfrac{r(n)}{2^n} = 0$.

Then we can find some $\mathcal{C} = \mathcal{C}_{\mathbf{B}'}$ such that

(a) *$\mathcal{C}$ is a closed subset of 2^ω,*
(b) *$(\forall n)|\{\eta \restriction n : \eta \in \mathcal{C}\}| \le r(n)$, so if $\mathcal{C} = \lim(T) = \{f \in 2^\omega : \forall n f \restriction n \in T\}$, then $T \subseteq 2^{<\omega}$ is a tree with n-th level counting less than or equal to $r(n)$,*
(c) *in the following game $\partial_{(\gamma,\mathbf{B}')}$ between two players, IN and OUT, the player IN has a winning strategy, the play lasts γ moves and in the ε-th move OUT chooses $\nu_\varepsilon \in \omega^\omega$ and then IN chooses $\eta_\varepsilon \ge^* \nu_\varepsilon$. In the end IN wins iff $\mathbf{B}'(\langle \eta_\varepsilon : \varepsilon < \gamma \rangle) \in \mathcal{C}$.*

Proof. Assume that $P^*_\gamma = \langle P^*_\xi, \underset{\sim}{Q}^*_\zeta : \xi \le \gamma, \zeta < \gamma \rangle$ is a c.s. iteration of Laver forcing and assume that $p \in P^*_\gamma$ and $\langle \underset{\sim}{\rho}_\xi : \xi < \gamma \rangle$ is a sequence of names for the P^*_ξ-generics. Clearly $p \Vdash_{P^*_\gamma} \mathbf{B}'(\langle \underset{\sim}{\rho}_\varepsilon : \varepsilon < \gamma \rangle) \in 2^\omega$.

The (f,g)-bounding property is preserved in countable support iterations [3, p. 340]. The Laver forcing and any forcing not adding reals at all have the (f,g)-bounding property. Hence there are $p^* \in P^*_\gamma$ and $\mathcal{C}$ as in (a) and (b) above such that

$$p^* \Vdash_{P^*_\gamma} \mathbf{B}'(\langle \underset{\sim}{\rho}_\varepsilon : \varepsilon < \gamma \rangle) \in \mathcal{C}.$$

Now we show that player IN can play in a way that imitates the Laver-generic reals over a countable elementary submodel, so that actually everything is in the ground model.

Let $M^* \prec (H(\chi), \in)$ be countable such $\mathbf{B}', \mathcal{C} \in M^*$. (So M^* is not the M from the next proof, but rather contains a non-trivial part of the power-set of that M.) Now we prove by induction on $j \le \gamma$ for all $i < j$

$\boxtimes_{i,j}$ Assume that $P^*_j \in M^*$ and $G_i \subseteq P^*_i \cap M^*$ is generic over M^*, and p^* is such that $p^* \in P^*_j \cap M^*$ and $p^* \restriction i \in G_i$. Then in the following game $\partial^*_{(i,j,G_i,p^*)}$ player II has a winning strategy $\sigma_{(i,j,G_i,p^*)}$. There are $j - i$ moves indexed by $\varepsilon \in [i,j)$, and in the ε-th move $(p_\varepsilon, \nu_\varepsilon, \eta_\varepsilon)$ are chosen such that player I chooses $p_\varepsilon \in P_\varepsilon / G_i$, $p_\varepsilon \ge p^* \restriction \varepsilon$, and $\nu_\varepsilon \in \omega^\omega$ and player II chooses $\eta_\varepsilon \ge^* \nu_\varepsilon$.

First case: there is a (P^*_ε, M^*)-generic $G_\varepsilon \subseteq P^*_\varepsilon \cap M^*$, such that $p^*(\varepsilon) \in G_\varepsilon$ and $G_\varepsilon \supset G_i$ and $(\forall \xi \in [i,\varepsilon)) \underset{\sim}{\rho}_\xi[G_\varepsilon] = \eta_\xi$ and $M^*[G_\varepsilon \cap P^*_\xi] \models p_\xi \ge p^*(\xi)$. In this case player I chooses $p_\varepsilon \in G_\varepsilon$ forcing this and so that $M^*[G_\varepsilon] \models p^*(\varepsilon) \le_{P^*_\varepsilon} p_\varepsilon$. Then player I chooses ν_ε dominating $M^*[G_\varepsilon]$ and the second player chooses $\eta_\varepsilon \ge^* \nu_\varepsilon$.

Second case: There is no such G_ε. Then player I won the play.

We prove by induction on j that player II wins the game $\partial^*_{(i,j,G_i,p^*)}$: Case 1: $j = 0$. Nothing to do. Case 2: $j = j^* + 1$. For $\varepsilon \in [i,j)$ we use the strategy for $\partial^*_{(i,j,G_i,p^*)}$, and for $\varepsilon = j$ we make the following move: We show that there is a generic G^{j^*} of ${Q^*}_{j^*}^{M^*[G_{j^*}]}$ to which $p^*(j^*)$ belongs and such that $\underset{\sim}{\rho}_{j^*}[G^{j^*}] \geq^* \nu_{j^*}$. Then the move $\underset{\sim}{\rho}_{j^*}[G^{j^*}]$ dominates $\omega^\omega \cap M^*[G_{j^*}]$ and also player I's move ν_{j^*}.

First take $q \geq p^*(j^*)$ such that q is $(M^*[G_{j^*}], {Q^*}_{j^*}^{M^*[G_{j^*}]})$-generic. $q \in \mathbf{V}$ is a Laver condition. Now we take a stronger condition q' by letting $\operatorname{trunk}(q) = \operatorname{trunk}(q')$ and for every $s \in q'$ of length n,

$$\operatorname{succ}(q', s) = \{n \in \operatorname{succ}(q, s) \,:\, n \geq \nu_{j^*}(n)\}.$$

Now let $G^{j^*} = \{r \in M^*[G_{j^*}] \,:\, q' \geq r\}$. Since q' is a $(M^*[G_{j^*}], {Q^*}_{j^*}^{M^*[G_{j^*}]})$-generic condition, G^{j^*} is a $(M^*[G_{j^*}], {Q^*}_{j^*}^{M^*[G_{j^*}]})$-generic filter. The generic real is $\underset{\sim}{\rho}_{j^*}[G^{j^*}] = \bigcup\{\operatorname{trunk}(p) \,:\, p \in G^{j^*}\}$. Then $q' \Vdash \underset{\sim}{\rho}_{j^*} \geq^* \nu_{j^*}$. Now player II takes $\eta_{j^*} = \underset{\sim}{\rho}_{j^*}[G^{j^*}]$. We set $G_j = G_{j^*} * G^{j^*}$. Case 3: j is a limit. Like the proof of the preservation of properness.

Why does $\boxtimes_{i,j}$ suffice? Use $i = 0$, $j = \gamma$, $\mathbf{B}' \in M^*$. Take $P^*_\gamma \in M^*$, $p^* \in P^*_\gamma \cap M^*$. Let $\sigma(0, \gamma, \{\emptyset\}, p^*)$ be a winning strategy for player II in the game $\partial^*_{(0,\gamma,\{\emptyset\},p^*)}$. During the play of $\partial_{(\gamma,\mathbf{B}')}$ let ν_ε be chosen in stage $\varepsilon < \gamma$. The player IN simulates on the side a play of $\partial^*_{(0,\gamma,\{\emptyset\},p^*)}$: As a move of I he assumes the ν_ε chosen by OUT in the play of $\partial_{(\gamma,\mathbf{B}')}$ and p_ε, $p_\varepsilon \restriction \delta = p_\delta$ for $\delta < \varepsilon$, the p_δ gotten from earlier simulations. Then player IN uses $\sigma(0, \gamma, \{\emptyset\}, p^*)$ for player II, applied to $(p_\varepsilon, \nu_\varepsilon)$, to compute an η_ε, which he presents in this move in $\partial_{(\gamma,\mathbf{B}')}$. So p_ε forces that there is a Laver generic $\rho_\varepsilon[G^\varepsilon] =: \eta_\varepsilon$ over $M^*[G_\varepsilon]$ and that $\eta_\varepsilon \geq^* \nu_\varepsilon$. The requirement $\eta_\varepsilon \geq^* \nu_\varepsilon$ is fulfilled.

Suppose that they have played. So we have $\langle \nu_\varepsilon, \eta_\varepsilon \,:\, \varepsilon < \gamma\rangle$ and there is $p = \bigcup_{\varepsilon<\gamma} p_\varepsilon \geq p^*$, and for $\varepsilon < \gamma$ there is the name for the Q^*_ε-generic real, namely $\underset{\sim}{\rho}_\varepsilon \in M^*$, such that for all $\varepsilon < \gamma$, $p \Vdash_{P^*_\gamma} \underset{\sim}{\rho}_\varepsilon = \check{\eta}_\varepsilon$. So as $p \Vdash_{P^*_\gamma}$ "$\mathbf{B}'(\langle \underset{\sim}{\rho}_\varepsilon \,:\, \varepsilon < \gamma\rangle) \in \mathcal{C}$", we have $\mathbf{B}'(\langle \eta_\varepsilon \,:\, \varepsilon < \gamma\rangle) \in \underset{\sim}{\mathcal{C}}$. $\square$

Let $S \subseteq \omega_1$ be stationary and $\langle A_\delta \,:\, \delta \in S\rangle$ exemplify $\diamondsuit_S$. For example we can take the most frequent $S = \{\alpha < \omega_1 \,:\, \alpha \text{ limit ordinal}\}$, which gives $\diamondsuit_{\omega_1}$.

Lemma 3.11. *Let $r\colon \omega \longrightarrow \omega$ such that $\lim \dfrac{r(n)}{2^n} = 0$. Assume that $\mathbf{V} \models \diamondsuit_S$. Then*

$$\Vdash_{P_{\omega_2}} \diamondsuit$$
$$(2^\omega, \{\lim(T) \,:\, T \subseteq \mathbb{R} \textit{ perfect } \wedge (\forall n)|\{\eta \restriction n \,:\, \eta \in \lim(T)\}| \leq r(n)\}, \in).$$

Proof. Let G be P_{ω_2}-generic over $\mathbf{V}$. We use the $\diamondsuit_S$-sequence $\langle A_\delta : \delta \in S\rangle$ in the following manner: By easy integration and coding we have $\langle (N^\delta, \bar\beta^\delta, \underset{\sim}{f}^\delta, \underset{\sim}{F}^\delta, \underset{\sim}{C}^\delta, P^\delta_{\omega_2}, p^\delta, <^\delta) : \delta \in S\rangle$ such that

(a) N^δ is a transitive collapse of a countable $M \prec H(\chi, \in, <^*_\chi)$, $<^\delta$ is a well-ordering of N^δ, U^δ codes the isomorphism type of $(N^\delta, P^\delta_{\omega_2}, p^\delta, \bar\beta^\delta)$.

(b) $N^\delta \models P^\delta_{\omega_2} = \langle P^\delta_\alpha, \underset{\sim}{Q}^\delta_\beta : \alpha \le \omega_2^{N^\delta}, \beta < \omega_2^{N^\delta}\rangle$ is as in Definition 2.10.

(c) $N^\delta \models (p^\delta \in P^\delta_{\omega_2}, \underset{\sim}{f}^\delta$ is a $P^\delta_{\omega_2}$-name of a member of ${}^{\omega_1}2$ $\underset{\sim}{F}^\delta \colon 2^{<\omega_1} \longrightarrow 2^\omega)$.

(d) If $p \in P_{\omega_2}$,

$$p \Vdash_{P_{\omega_2}} \underset{\sim}{f} \in 2^{\omega_1} \wedge \underset{\sim}{F} \colon 2^{<\omega_1} \longrightarrow 2^\omega \text{ is Borel}, \underset{\sim}{C} \subseteq \omega_1 \text{ is club},$$

and p, P_{ω_2}, $\underset{\sim}{F}$, $\underset{\sim}{f}$, $\underset{\sim}{C} \in H(\chi)$, then

$$\begin{aligned} S(p, \underset{\sim}{F}, \underset{\sim}{f}) := \{\delta \in S : & \text{there is a countable } M \prec (H(\chi), \in, <^*_\chi) \\ & \text{such that } \underset{\sim}{f}, \underset{\sim}{F}, \underset{\sim}{C}, P_{\omega_2}, p \in M \\ & \text{and there is an isomorphism } h^\delta \text{ from } N^\delta \text{ onto } M \\ & \text{mapping } P^\delta_{\omega_2} \text{ to } P_{\omega_2}, \underset{\sim}{f}^\delta \text{ to } \underset{\sim}{f}, \\ & \underset{\sim}{F}^\delta \text{ to } \underset{\sim}{F}, \underset{\sim}{C}^\delta \text{ to } \underset{\sim}{C}, p^\delta \text{ to } p, <^\delta \text{ to } <^*_\chi \restriction M\} \end{aligned}$$

is a stationary subset of ω_1.

(e) Choose $\langle \mathbf{B}_{\gamma(\delta)} : \delta \in S\rangle$ such that $\gamma(\delta) = \mathrm{otp}(N^\delta \cap \omega_2)$ and

$$\begin{aligned} \mathbf{B}_{\gamma(\delta)} \colon (\omega^\omega)^{\gamma(\delta)} \times \mathcal{P}(\omega) &\longrightarrow \mathrm{Gen}^+(P^\delta_{\omega_2}) \\ &= \{G \subseteq P^\delta_{\omega_2} \cap N^\delta : G \text{ is } P^\delta_{\omega_2}\text{-generic over } N^\delta \text{ and bounded}\} \end{aligned}$$

be as in Theorem 3.4 with $U^\delta = U(N^\delta, P^\delta_{\omega_2}, p^\delta, \bar\beta^\delta)$.

We do not require uniformity, $\langle \nu_\varepsilon, \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ is indeed $\langle \nu^\delta_\varepsilon, \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ since we have the dependence on the δ in the definition of $\mathbf{B}_{\gamma(\delta)}$. We assume that $N^\delta \cap \omega_1 = \delta$. Since this holds on a club set of $\delta \in \omega_1$, this is no restriction.

Now assume the $p \in G$ and $\underset{\sim}{F}$, $\underset{\sim}{f}$, $\underset{\sim}{C}$ are as in (d).

We define a function $\mathbf{B}'_{\delta, U_\delta}$ with domain $(\omega^\omega)^{\gamma(\delta)}$.

$$\mathbf{B}'_{\delta,U^\delta}(\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle) = \begin{cases} \underset{\sim}{F}^\delta(\underset{\sim}{f}^\delta \restriction \delta)[\mathbf{B}_{\gamma(\delta)}(\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle, U^\delta)], \\ \qquad\qquad\qquad\qquad \text{if the argument is good;} \\ \langle 0, 0, \dots, \rangle \in 2^\omega, \qquad\qquad \text{otherwise.} \end{cases}$$

Here, we call $\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ a good argument if there is a play $\langle \nu_\varepsilon, \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ in the game $\partial_{(N^\delta, P^\delta, p^\delta)}$ from Theorem 3.4 in which the generic player plays according his winning strategy and the antigeneric player plays according to the rules. Goodness is a Borel predicate because the ν_ε are irrelevant, just check whether the η_ε are large enough for Equations (3.1) and (3.2) in the respective iteration step. So $\mathbf{B}'_{\delta,U^\delta}(\langle \eta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ is a Borel function. Now we choose a "very good" argument $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ that player IN plays with his strategy in $\partial_{(\gamma(\delta),\mathbf{B}'_{\delta,U_\delta})}$ from Lemma 3.10 applied to $\mathbf{B}'_{\delta,U_\delta}$ and the $(r, 2^n)$ bounding property, answering to a good argument $\langle \nu^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ played by player OUT.

Now we derive a guessing function g. We consider for every $\delta \in S$ a very good argument $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$. We assume that G is P_{ω_2}-generic over V and that $p \in G$. Then we also have by the rules of the game $\partial_{(N^\delta,P^\delta,p^\delta)}$ that

$$\mathbf{B}_{\gamma(\delta)}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle, U^\delta) \text{ has an upper bound } q^\delta.$$

Lemma 3.10 gives a closed set $\mathcal{C}_{\mathbf{B}'_{\delta,U^\delta}}$ with small levels as in 3.10 (b), such that for $\delta \in S$, and we have

$$\mathbf{B}'_{\delta,U_\delta}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle) \in \mathcal{C}_{\mathbf{B}'_{\delta,U^\delta}}. \tag{3.10}$$

Note that $\mathcal{C}_{\mathbf{B}'_{\delta,U^\delta}}$ does not depend on $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$. So (3.10) also holds for $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ that are the answers of player IN in the game $\partial_{(\gamma(\delta),\mathbf{B}'_{\delta,U^\delta})}$ to any good sequence $\langle \nu^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ given by the generic player that is so fast growing ν^δ_ε that $\mathbf{B}_{\delta,U_\delta}(\langle \nu^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle)$ computes a bounded generic filter over M as in Theorem 3.4. This is important, since the isomorphism h^δ does not preserve the knowledge (that is which branches are continued and what are the values of the promises in these continuations) about the level δ for the Aronszajn trees involved in $P \cap M$.

We set

$$\mathcal{C}_{\mathbf{B}'_{\delta,U^\delta}} =: g(\delta).$$

Both sides are conceived as Borel codes for closed sets. Since $\omega \subseteq M$ and $\omega \subseteq N^\delta$ we have that $h^\delta(\mathcal{C}_{\mathbf{B}'_{\delta,U^\delta}}) = \mathcal{C}_{\mathbf{B}'_{\delta,U^\delta}}$. We show that g is a diamond function.

Since P_{ω_2} is proper, $S(p, \underset{\sim}{f}, \underset{\sim}{F})$ is also stationary in $\mathbf{V}[G]$. Now we take a very good sequence $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle$ that is suitable so that $\mathbf{B}_{\delta,U_\delta}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle)$ computes a bounded (M, P)-generic filter for M that witnesses that $\delta \in S$. So now we take the game $\partial_{(M,P,p)}$ for the choice of the $\langle \nu^\delta_\eta : \eta < \gamma_\delta\rangle$ and then again we take the winning strategy in the game $\partial_{(\gamma(\delta),\mathbf{B}'_{\delta,U_\delta})}$, which is unchanged by the collapse, for choosing $\langle \eta^\delta_\varepsilon : \varepsilon < \gamma_\delta\rangle$. We take q to be a bound of $\mathbf{B}_{\delta,U_\delta}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle)$. Now we have that $q \geq p$ and

$$q \Vdash \text{“}\mathbf{B}_{\gamma(\delta)}(\langle \eta^\delta_\varepsilon : \varepsilon < \gamma(\delta)\rangle, U^\delta) \text{ is } (M, P)\text{-generic and bounded by } q\text{”}.$$

Now for $\delta \in S(p, \underset{\sim}{f}, \underset{\sim}{F})$ we have by the isomorphism property of h^δ and by (3.10),

$$q \Vdash h^{\delta\prime\prime}\underset{\sim}{F}^\delta(\underset{\sim}{f}^\delta \restriction \delta) = \underset{\sim}{F}(\underset{\sim}{f} \restriction \delta) \ \wedge \ \underset{\sim}{F}(\underset{\sim}{f} \restriction \delta) \in g(\delta) \ \wedge \ \delta \in \underset{\sim}{C}.$$

So we have that p forces that $\{\alpha \in S \ : \ F(f \restriction \delta) \in g(\delta)\}$ contains a stationary subset of $S(p, \underset{\sim}{f}, \underset{\sim}{F})$. Note that the stationary subset depends on F (and f of course), but the guessing function g does not. So actually we proved a diamond of the kind:

There is some $g \colon \omega_1 \longrightarrow B$ such that for every Borel map $F \colon 2^{<\omega_1} \longrightarrow A$ and for every $f \colon \omega_1 \longrightarrow 2$ the set

$$\{\alpha \in \omega_1 \ : \ F(f \restriction \alpha) E g(\alpha)\}$$

is stationary. $\square$

Corollary 3.12. $\Vdash_{P_{\omega_2}} \diamondsuit(\mathbb{R}, \mathcal{N}, \in)$.

Proof. $\mathrm{Leb}(g(\delta)) = 0$ for the functions $g \colon \omega_1 \longrightarrow \{\text{closed subsets of } 2^\omega\}$ from the previous lemma. Thus, for every Borel $F \colon 2^{<\omega_1} \longrightarrow 2^\omega$, the function $g \colon \omega_1 \longrightarrow \mathcal{N}$ is a guessing sequence showing $\Vdash_{P_{\omega_2}} \diamondsuit(\mathbb{R}, \mathcal{N}, \in)$, and we finish the proof of Theorem 1.2. $\square_{1.2}$

Since $\mathcal{C}$ from Lemma 3.10 is also meagre, the same proof also yields

Corollary 3.13. $\Vdash_{P_{\omega_2}} \diamondsuit(\mathbb{R}, \mathcal{M}, \in)$.

If $S \subseteq \omega_1$ is stationary and we start with $\diamondsuit_S$ in the ground model, then we get the respective weak diamonds on S. We conclude with an open question: The forcing from Definition 2.10 could easily be mixed with proper $\aleph_2$-p.i.c. iterands, for example iterands with $|Q_\alpha| \leq \aleph_1$ (by [15, Lemma VIII 2.5] this is sufficient for the $\aleph_2$-p.i.c.) that add reals. Still we specialize all Aronszajn trees in the new mixed iteration. However, the parallel of our main technique for the weak diamonds, that is Theorems 3.4 and 3.11, does not work anymore, since the completeness systems are no longer in the ground model. So there is the question:

Question 3.14. *Is $2^{\aleph_0} = \aleph_2$ and $\diamondsuit(\mathrm{Cov}(\mathcal{N}))$ and "all Aronszajn trees are special" consistent relative to ZFC?*

References

[1] Abraham, U., *Proper forcing*, in "Handbook of Set Theory", M. Foreman, A. Kanamori and M. Magidor, Eds., Kluwer, (to appear).

[2] Abraham, U., Shelah, S., *A Δ^2_2 well-order of the reals and incompactness of $L(Q^{MM})$*, Ann. Pure Appl. Logic **59** (1993), 1–32.

[3] Bartoszyński, T., Judah, H., *Set Theory, On the Structure of the Real Line*, A K Peters, Wellesley, Massachusetts, 1995.

[4] Baumgartner, J., Harrington, L., Kleinberg, E., *Adding a closed unbounded set*, J. Symbolic Logic, **41** (1976), 481–482.

[5] Blass, A., Shelah, S., *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed*, Ann. Pure Appl. Logic, **33** (1987), 213–243.

[6] Devlin, K., Johnsbråten, H., *The Souslin Problem*, Lecture Notes in Math. **405**, Springer-Verlag, Berlin-New York, 1974.

[7] Devlin, K. J., Shelah, S., *A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel J. Math. **29** (1978), 239–247.

[8] Hirschorn, J., *Random trees under CH*, Israel J. Math. **157** (2007), 123–154.

[9] Jensen, R. B., *The fine structure of the constructible hiercharchy*, Ann. Math. Logic **4** (1972), 229–308.

[10] Kechris, A., *Classical Descriptive Set Theory*, Grad. Texts in Maths. **156**, Springer-Verlag, New York, 1995.

[11] Laver, R., *Random reals and Souslin trees*, Proc. Amer. Math. Soc. **100**(3) (1987), 531–534.

[12] Mildenberger, H., *Creatures on ω_1 and weak diamonds*, J. Symbolic Logic **74** (2009), 1–16.

[13] Miller, A., *Arnie Miller's problem list*, in "Set theory of the reals" (Ramat Gan, 1991), *Israel Math. Conf. Proc.* **6** (1993), Bar-Ilan Univ., Ramat Gan, 645–654.

[14] Moore, J. T., Hrušák, M., Džamonja, M., *Parametrized $\diamondsuit$-principles*, Trans. Amer. Math. Soc. **356** (2004), 2281–2306.

[15] Shelah, S., *Proper and Improper Forcing*, 2nd edition, Springer-Velag, Berlin, 1998.

[16] Vojtáš, P., *Generalized Galois-Tukey connections between explicit relations on classical objects of real analysis*, in "Set theory of the reals" (Ramat Gan, 1991), *Israel Math. Conf. Proc.* **6** (1993), Bar-Ilan Univ., Ramat Gan, 619–643.

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