Non standard uniserial module over a uniserial domain exists

Our aim is to prove:

Theorem: (ZFC) There exist a non standard uniserial modules over some uniserial domain (see 12).

The paper is self contained. It uses forcing - this can be eliminated easily but for me this has no point. Our example is in $\aleph_1$ - we can replace it by any regular $\kappa > \aleph_0$. The problem appears in the version of a book of Fuchs and Salce on modules over uniserial domains in existence in April 1984.. An answer in the other direction would have simplified the subject, and I think, make unnecessary several proofs and distinctions.

I thank Silvana Bazzoni, Elizabetta Martinez and Claudia Mettel for going our of their way to tell me the problem during a dinner at the conference in Udine, to Fuch's for mentioning it and to Salce for impressing upon me the importance of solving it.

Subsequently Fuchs continues this work, investigating for which uniserial R there are such modules.

0. Definition and Notation: 1) Let R denote a uniserial domain, i.e., no zero divisors and $Id(R) = \{I: I$ an ideal of $R\}$ is linearly ordered by inclusion. Let $Q = Q_R$ be the field quotient. Let a, b, c, r, s denote member of R, x, y, z denote members of an R-module, M, N denote R-modules. Let $a \mid b$ mean a divides b.

2) An R-module is called standard if it is a homomorphic image of an R-submodule of Q (which is trivially an R-module) and $M \neq 0$.

3) An R-module is uniserial if its family of submodules is linearly ordered. (So we are assuming R itself is uniserial.)

0A Remark: Any standard R-module is uniserial.

This is well known.

1. Fact: Let M be a uniserial R-module; if $x \in M$, $ax \neq 0$ then for every $b \in R,(b \neq 0)$: $bx = 0$ if $(b/a)(ax) \neq 0$ and a divides b in R.

Proof: If in R $a \mid b$ let $b = ca$ so $bx = 0 \Longleftrightarrow cax = 0 \Longleftrightarrow (b/a)(ax) = 0$. So it suffices to prove $a \mid b$ assuming $bx = 0$, but if a does not divide b, b divides a so $a = db$, so $ax = dbx = d = 0$ contradicting, an assumption.

2. Definition : 1) We call $\langle a_{i,j} : i < j < \delta \rangle$ an I-representation of M (for M a uniserial module over a uniserial domain R) *if:*

(i) I is an ideal of $R, I \neq R$.

(ii) $a_{i,j} \in R$, $a_{i,j} \neq 0$.

(iii) for $\alpha < \beta < \gamma < \delta$, $a_{\alpha,\gamma} - a_{\alpha,\beta} a_{\beta,\gamma} \in a_{0,\gamma} I$.

(iv) there are $x_i \in M (i < \delta)$ such that M is generated by $\{x_i : i < \delta\}$, and:

$$I = \{r \in R : rx_0 = 0\}, \ \ a_{ij} x_j = x_i$$

2) We call $\langle a_{i,j} : i < j < \delta \rangle$ an I- representation for R) if (i),(ii),(ii) above holds.

3. Claim: Every uniserial R-module M has an I-representation (for some ideal I of R).

Proof : Easy. Choose by induction on i, $x_i \in M (\neq 0)$ x_i not in the submodule generated by $\{x_j : j < i\}$. Say δ is the first for which x_δ is not defined. Clearly δ exists and is $< \|M\|^+$. For $i < j$, as $x_j \notin Rx_i$, by uniseriality $x_i \in Rx_j$ so for some $a_{i,j} \in R$, $x_i = a_{i,j} x_j$. Now for $\alpha < \beta < \gamma < \delta$, $a_{\alpha,\gamma} x_\gamma = x_\alpha = a_{\alpha,\beta} x_\beta = a_{\alpha,\beta}(a_{\beta,\gamma} x_\gamma)$. So $(a_{\alpha,\gamma} - a_{\alpha,\beta} a_{\beta,0}) x_\gamma = 0$. As $a_{0,\gamma} x_\gamma = x_0 \neq 0$, we finish by Fact 1.

Remark: Clearly $\delta > 0$ for $M \neq 0$, and if δ is a successor ordinal then M is standard.

4. Claim: 1) If $\langle a_{i,j} : i < j < \delta \rangle$ is an I- representation for R then some R-module M is I-represented by $\langle a_{i,j} : i < j < \delta \rangle$.

2) Moreover M is unique up to isomorphism and is uniserial

Proof: Let M be an R-module generated freely by $\{x_i : i < \delta\}$ except the relations:

(a) $rx_0 = 0$ (for $r \in I$)

(b) $x_i - a_{i,j}x_j = 0$ for $i < j < \delta$.

2) The uniqueness is trivial, so we shall prove that M constructed in (1) is uniserial. It is easy to see that (by the relations (b)).

(*) for every $y \in M$ for some $i < \delta$, $r \in R$: $y = rx_i$.

Now suppose K is a submodule of $M, K \neq M$, and we shall prove that for some $\xi < \delta$, $K \subseteq Rx_\xi$. This suffices [if K_1, K_2 are submodules of M, if $K_1 = M$ or $K_2 = M$ they are comparable so we finish; if $K_1, K_2 \neq M$ there are $\xi_1, \xi_2 < \delta$ such that $K_1 \subseteq Rx_{\chi_1}, K_2 \subseteq Rx_{\xi_2}$; let $\xi = Max\{\xi_1, \xi_2\}$, so K_1, K_2 are R- submodules of Rx_ξ, which is uniserial by 0A, hence $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$].

As $K \neq M$ for some ξ $x_\xi \notin K$. Assume $K \not\subseteq Rx_\xi$, so for some $y \in K$, $y \notin Rx_\xi$. By (*) above for some $\zeta < \delta$ and $r \in R$, $y = rx_\zeta$. Now $\xi < \zeta$ [otherwise $y = rx_\zeta \in Rx_\zeta \subseteq Rx_\xi$ contradiction to the choice of y]; As $y \neq 0, r \neq 0$, and $a_{\xi,\zeta} \neq 0$, in R r divides $a_{\xi,\zeta}$ or $a_{\xi,\zeta}$ divides r (or both).

If $a_{\xi,\zeta}$ divides r, then

$$y = rx_\zeta = (r / a_{\xi,\zeta})(a_{\xi,\zeta}x_\zeta) = (r / a_{\xi,\zeta})x_\xi \in Rx_\xi$$

contradiction to the choice of y.

If r divides $a_{\xi,\zeta}$ then

$$x_\xi = a_{\xi,\zeta} x_\zeta = (a_{\xi,\zeta}/r) r x_\zeta \in R(r x_\zeta) = Ry \subseteq K$$

contradiction to the choice of ξ.

So $K \subseteq Rx_\xi$. We previously show that this (i.e. for every R-submodule K of M, $K \subseteq Rx_\xi$ for some ξ) suffice.

5. Lemma : A uniserial R-module with I-representation $\langle a_{i,j} : i < j < \delta \rangle$ is standard *iff* for some $c_i \in R (i < \delta)$ for every $i < j < \delta$:

(i) $\dfrac{c_i^{-1}}{a_{0,i}} - \dfrac{c_j^{-1}}{a_{0,j}/a_{i,j}} \in I$

(ii) $c_i^{-1} \in R$, i.e., each c_i is a unit.

5A. Remark: We can replace is (i),(iii), c_i^{-1} by c_i,c_j^{-1} by c_j.

Proof : First suppose that there are such $c_i (i < \delta)$. Let $J_i = (1/a_{0,i})R \subseteq Q$ and define a function from J_i into M by

$$f_i((1/a_{0,i})r) = r c_i x_i \quad \text{for } r \in R$$

Clearly f_i is a homomorphism from one R-module to another.

It is onto Rx_i as c_i is invertible in R.

We shall prove that

(*) for $i < j < \delta$, $f_i \subseteq f_j$.

This suffice as then $\bigcup_{i<\delta} f_i$ is a homomorphism from $\bigcup_{i<\delta} J_i$ onto M. For proving (*) it suffices to prove:

(**) $f_i(1/a_{0,i}) = f_j(1/a_{0,i})$

First $1/a_{0,i} \in \text{Dom}(f_j)$, [this is equivalent to $1/a_{0,i} \in R(1/a_{0,j})$ which is equivalent to $a_{0,j} \in Ra_{0,i}$, if this fails then by the uniseriality of R, for some $s \in R$ which is not a unit, $a_{0,i} = s\ a_{0,j}$ so

$$a_{0,j}\ (1 - s a_{i,j}) = a_{0,j} - a_{0,i} a_{i,j} \in a_{0,j} I$$

as R has no zero divisors, $1-sa_{i,j} \in I$; as s is not a unit sR is a proper ideal, but $1 = sa_{i,j} + (1-sa_{i,j}) \in sR + I$, but $sR \subseteq I$ or $I \subseteq sR$, so necessarily $I \subseteq sR, 1 \in I$ but then $x_0 = 0$ contradiction]. Second, we can confirm (**) remember we have shown above $a_{0,j} \in R\, a_{0,i}$ hence $a_{0,j}/a_{0,i} \in R$):

$$f_j(1/a_{0,i}) = f_j((a_{0,j}/a_{0,i})(1/a_{0,j})) = (a_{0,j}/a_{0,i})\, c_j x_j$$
$$f_i(1/a_{0,i}) = c_i x_i = c_i a_{i,j} x_j$$

So it is enough to show that

$$\left(\frac{a_{0,j}}{a_{0,i}} c_j - a_{i,j} c_i\right) x_j = 0$$

equivalently (see Fact 1):

$$\frac{a_{0,j}}{a_{0,i}} c_j - a_{i,j} c_i \in a_{0,j} I$$

equivalently

$$\frac{c_j}{a_{0,i}} - \frac{c_i}{a_{0,j}/a_{i,j}} \in I$$

Multiplying by $c_j^{-1} c_i^{-1}$ we get (i) of the hypothesis , i.e., the demand holds (Note that for a unit c, $cI = I$).

We have proved the "if" part of Lemma 5.

For the only "if" part suppose J is an R-submodule of Q, $f: J \to M$ an onto homomorphism. W.l.o.g. $f(1) = x_0$ so $R \subseteq \mathrm{Dom}\ f$, $1 \notin \mathit{Ker}\ f = I$. For every i, let $x_i = f(y_i)$. $y_i \in J$. If $y_i \in R(1/a_{0,i})$ let for some $r \in R$, $y_i = r/a_{0,i}$, then $a_{0,i} y_i = r$ hence $f(r) = f(a_{0,i} y_i) = a_{0,i} f(y_i) = a_{0,i} x_i = x_0 = f(1)$, so $f(1-r) = 0$ hence $1-r \in I$, hence $r^{-1} \in R$ [otherwise $Rr \not\subseteq R$, so $Rr \cup R(1-r)$ is a proper ideal contradiction]. So $[y_i \in R(1/a_{0,i}) \Longrightarrow 1/a_{0,i} \in Ry_i]$. As $y_i, 1/a_{0,i} \in Q$, Q a uniserial R-module this implies $1/a_{0,i} \in Ry_i$, so for some $c_i \in R$, $1/a_{0,i} = c_i y_i$. As $y_i \in J$ clearly $1/a_{0,i} \in J$. Now

$$x_0 = f(1) = f(a_{0,i} \cdot (1/a_{0,i})) = a_{0,i} f(1/a_{0,i}) = a_{0,i} c_i x_i = c_i x_0$$
$$= a_{0,i} f(c_i y_i) = a_{0,i} c_i f(y_i)$$

so $(1-c_i)x_0 = 0$ hence $1-c_i \in I$, so as in an argument above c_i is a unit except when $I=R$ which is excluded.

So $1/a_{0,i} = c_i y_i$, $c_i \in R$ a unit. By (iii) of Definition 2 with G,i,j here standing for α,β,γ there, $1 - \frac{a_{0,i}a_{i,j}}{a_{0,j}} \in I$ so (when $I \neq R$) $a' \stackrel{def}{=} \frac{a_{0,i}}{a_{0,j}} a_{i,j}$ is a unit of R, as $a_{i,j} \in R$ this implies $\frac{a_{0,j}}{a_{0,i}} = a_{i,j}/a' \in R$. Now

$$0 = f(0) = f(1/a_{0,i} - 1/a_{0,i}) = f(1/a_{0,i}) - f((a_{0,j}/a_{0,i}) \cdot 1/a_{0,j}) =$$
$$f(c_i y_i) - (a_{0,j}/a_{0,i}) f(c_j y_j) =$$
$$c_i x_i - (a_{0,j}/a_{0,i}) c_j x_j = c_i a_{i,j} x_j - (a_{0,j}/a_{0,i}) c_j x_j =$$
$$(c_i a_{i,j} - (a_{0,j}/a_{0,i}) c_j) x_j$$

hence $[c_i a_{i,j} - [(a_{0,j}/a_{0,i})c_j]/a_{0,j} \in I$ and we can finish.

For a while we make

6. Assumption: M is a non-standard model of $Th(\mathbb{Z})$ of power $\aleph_1$ not $\aleph_1$-like, $M = \bigcup_{i<\omega_1} M_i$, $M_i \prec M, M_i$ increasing continuous, each M_i countable, $p \in M$ a prime $R = R^p_M$ is $\{a/b\,; a,b \in M,\ M \models "p$ does not divide $b"\}$./

Let $Q \supseteq R$ be the field of quotients of R.

Easily R is a uniserial domain. Let b be a member of M. let $\langle d(\alpha) : \alpha < \omega_1 \rangle$ be a sequence of members of M increasing, $d(\alpha) < b$, $b, p \in M_0$, $d(\alpha) \in M_{\alpha+1}$. Let Q_i be the field of quotients of M_i, $R_i = R \cap Q_i$.

Clearly we can find M as above, and then $b, d(\alpha)$.

7. Definition : Let $I = \{c \in R : p^b \mid c\}$, it is an ideal.

We define a set P; its members have the form:

$$\langle a_{i,j} : i < j, i \in u, j \in u \rangle$$

such that

(i) u a finite subset of $\omega_1, 0 \in u$.

(ii) for $\alpha < \beta < \gamma$ all in u,

$$\left(\frac{a_{\alpha,\gamma}-a_{\alpha,\beta}a_{\beta,\gamma}}{a_{0,\gamma}}\right) \in I.$$

(iii) $a_{\alpha,\beta}$ is divisible by $p^{d(\beta)-d(\alpha)}$ but not by $p^{d(\beta)-d(\alpha)+1}$ in R (exponentiation in M).

(iv) $a_{i,j} \in R_{j+1}$

[we write $a_{ij} = a^r_{ij}$, $u = u^r$ where $r = \langle a_{i,j} : i < j,\ i \in u,\ j \in u \rangle$].

We stipulate $a_{i,i} = 1$. The order of P is natural.

8. Fact: If $r = \langle a^r_{ij} : i < j \in u^r \rangle \in P$, $\xi < \omega_1$ *then* there is $q, r < q \in P, \xi \in u^q$.

Proof : If $\xi \in u^r$ let $q = p$, otherwise suppose $i_1 < \cdots < i_\ell < \xi < i_{\ell+1} < \cdots < i_m$, $u^r = \{i_1, \ldots, i_m\}$, (remember $i_0 = 0$) and let $a_{i,j} = a^r_{i,j}$.

We now define q:

$$u^q = u^r \cup \{\xi\}$$

$$a^q_{i,j} = \begin{cases} a_{i,j} & \text{if } i < j, i \in u^r,\ j \in u^r \\ a_{i,i_\ell} p^{d(\xi)-d(i_\ell)} & \text{if } i \in \{i_1, \ldots, i_\ell\},\ j = \xi \\ \dfrac{a_{i_{\ell+1},j}\, a_{i_\ell, i_{\ell+1}}}{p^{d(\xi)-d(i_\ell)}} & \text{if } j \in \{i_{\ell+1}, \ldots, i_n\}\ \ i = \xi \end{cases}$$

We shall now check that $q \in P$.

Properties (i), (iii) and (iv) of Definition 7 are easy, so let us check (ii)).

So let $\alpha < \beta < \gamma$ be in u^r.

Case A: $\alpha = \xi$.

$$\frac{a^q_{\alpha,\gamma} - a^q_{\alpha,\beta}\ \ a^q_{\beta,\gamma}}{a^q_{0,\gamma}} = \text{(by the third case in the definition of } a^q_{i,j}).$$

$$\frac{a_{i_{\ell+1},\gamma}a_{i_\ell,i_{\ell+1}}p^{-(d(\alpha)-d(i_\ell))}-a_{i_{\ell+1},\beta}a_{i_\ell,i_{\ell+1}}p^{-(d(\alpha)-d(i_\ell))}a_{\beta,\gamma}}{a_{0,\gamma}} =$$

$$\frac{a_{i_\ell,i_{\ell+1}}}{p^{d(\alpha)-d(i_\ell)}}\ \frac{a_{i_{\ell+1},\gamma}-a_{i_{\ell+1},\beta}a_{\beta,\gamma}}{a_{0,\gamma}} \in I$$

Because the left term is in R (by (iii) of Definition 7 for p) and the right term is in I (by (ii) of Definition 7 for p).

Case B: $\beta = \xi$.

$$\frac{a^q_{\alpha,\gamma} - a^q_{\alpha,\beta}a^q_{\beta,\gamma}}{a^q_{0,\gamma}} =$$

$$\frac{a_{\alpha,\gamma} - (a_{\alpha,i_\ell}p^{d(\beta)-d(i_\ell)})(a_{i_{\ell+1},\gamma}a_{i_\ell,i_{\ell+1}}p^{-(d(\beta)-d(i_\ell))})}{a_{0,\gamma}} =$$

$$\frac{a_{\alpha,\gamma}-a_{\alpha,i_\ell}a_{i_\ell,i_{\ell+1}}a_{i_{\ell+1},\gamma}}{a_{0,\gamma}} =$$

$$\frac{a_{\alpha,\gamma}}{a_{0,\gamma}} - a_{\alpha,i_\ell}\frac{a_{i_\ell,i_{\ell+1}}a_{i_{\ell+1},\gamma}}{a_{0,\gamma}} =$$

$$\frac{a_{\alpha,\gamma}}{a_{0,\gamma}} - a_{\alpha,i_\ell}\frac{a_{i_\ell,\gamma}}{a_{0,\gamma}} + a_{\alpha,i_\ell}\left(\frac{a_{i_\ell,\gamma}}{a_{0,\gamma}} - \frac{a_{i_\ell,i_{\ell+1}}a_{i_{\ell+1},\gamma}}{a_{0,\gamma}}\right)$$

$$= \frac{a_{\alpha,\gamma}-a_{\alpha,i_\ell}a_{i_\ell,\gamma}}{a_{0,\gamma}} + a_{\alpha,i_\ell}\frac{a_{i_\ell,\gamma}-a_{i_\ell,i_{\ell+1}}a_{i_{\ell+1},\gamma}}{a_{0,\gamma}} \in I$$

as the first term is in I (by (ii) of Definition 7 for p) and the second term is in I as a members of I times $a_{\alpha,i_\ell} \in R$ so as I is an ideal it belongs to I.

Case C: $\gamma = \xi$

$$\frac{a^q_{\alpha,\gamma}-a^q_{\alpha,\beta}a^q_{\beta,\gamma}}{a^q_{0,\gamma}} =$$

$$= \frac{a_{\alpha,i_\ell}p^{(d(\gamma)-d(i_\ell)}-a_{\alpha,\beta}a_{\beta,i_\ell}p^{d(\gamma)-d(i_\ell)}}{a_{0,i_\ell}p^{d(\gamma)-d(i_\ell)}}$$

$$= \frac{a_{\alpha,i_\ell} - a_{\alpha,\beta}a_{\beta,i_\ell}}{a_{0,i_\ell}} \in I$$

Case D: $\alpha,\beta,\gamma \neq \xi$.

Trivial.

So we have proved $q \in P$. Easily $p \le q, \xi \in u^q$, so we finish.

9. Main Fact: Suppose $u_0 < u_1 < u_2$ (all finite subsets of ω_1, not empty for simplicity, $u < v$ means $\forall \alpha \in u \ \ \forall \beta \in v \ \ \alpha < \beta$) non empty, and

$$r^\ell \in P \quad \text{for} \quad \ell = 0,1,2,$$
$$u^{r^0} = u_0 \,, \; u^{r^1} = u_0 \cup u_1, \;\; u^{r^1} = u_0 \cup u_2$$
$$r^0 \le r^1, \; r^0 \ge r^1$$

Let $\xi_\ell = Min \ u_\ell$ for $\ell = 1,2$, and $c_1, c_2 \in R$ are units of R.

Then we can find $r \in P$, $r^1 \le r$, $r^2 \le r$, such that

$$\frac{c_1}{a^r_{0,\xi_1}} - \frac{c_2}{a^r_{0,\xi_2} / a^r_{\xi_1,\xi_2}} \notin I$$

Let $\zeta_\ell = Max \ u_\ell$.

10. Subfact: We can find an element a of R such that

(α) $p^{d(\xi_2)-d(\xi_1)}$ divides a but $p^{d(\xi_2)-d(\xi_1)+1}$ does not divides a (in R).

(β) $\dfrac{a^{r^2}_{\zeta_0,\xi_2} - a^{r_1}_{\zeta_0,\zeta_1} a}{a^{r^2}_{0,\xi_2}} \in I$

(γ) $\dfrac{c_1}{a^{r^1}_{0,\xi_1}} - \dfrac{c_2}{a^{r^2}_{0,\xi_2} / (a^{r^1}_{\xi_1,\zeta_1} a)} \notin I$

(δ) $a \in M_{\xi_2+1}$

Proof: We shall choose some $t \in I \cap M_{\xi_2+1}$ and let

$$a = \frac{a^{r^2}_{\zeta_0,\xi_2} - a^{r^2}_{0,\xi_2} t}{a^{r_1}_{\zeta_0,\zeta_1}}$$

Now $t \in I$ guarantees (β) (just substitute and compute, and you shall get t) and $t \in M_{\xi_2+1}$ guarantee (δ) (as $\zeta_1, \zeta_0 \le \xi_2$ and use (iv) from 7). Also (α) is immediate: $a^{r^2}_{0,\xi_2}$ is divisible by $p^{d(\xi_2)}$ hence $a^{r_2}_{0,\xi_2} t$ is divisible by

$p^{d(\xi_2)-d(\zeta_0)+1}$, but $a^{r^2}_{\zeta_0,\xi_2}$ is not; so $a^{r^2}_{\zeta_0,\xi_2} - a^{r^2}_{0,\xi_2}t$ is divisible by $p^{d(\xi_2)-d(\zeta_0)}$ but not by $p^{d(\xi_2)-d(\zeta_0)+1}$. Using (iii) of Definition 7 on $a^{r_1}_{\zeta_0,\zeta_1}$ we finish.

We are left with (γ), it means now

$$\frac{c_1}{a^{r^1}_{0,\xi_1}} - \frac{c_2}{a^{r^2}_{0,\xi_2}} \cdot a^{r^1}_{\xi_1,\zeta_1}\left(\frac{a^{r^2}_{\zeta_0,\xi_2} - a^{r^2}_{0,\xi_2}t}{a^{r_1}_{\zeta_0,\zeta_1}}\right) \notin I$$

this is equivalent to:

$$(*)\ \frac{c_1}{a^{r^1}_{0,\xi_1}} - \frac{c_2 a^{r^1}_{\xi_1,\zeta_1} a^{r^2}_{\zeta_0,\xi_2}}{a^{r^2}_{0,\xi_2} a^{r^1}_{\zeta_0,\zeta_1}} + \frac{c_2 a^{r^1}_{\xi_1,\zeta_1} a^{r^2}_{0,\xi_2} t}{a^{r^2}_{0,\xi_2} a^{r^1}_{\zeta_0,\zeta_1}} \notin I$$

If for $t = 0$ $(*)$ holds, we finish, so we can assume

$s \overset{\text{def}}{=} \dfrac{c_1}{a_{0,\xi_1}} - \dfrac{c_1 a^{r^1}_{\xi_1,\zeta_1} a^{r_2}_{\zeta_0,\xi_2}}{a^{r_1}_{0,\xi_2} a^{r^1}_{\zeta_0,\zeta_1}} \in I$, so $(*)$ is then equivalent to

$$(*)'\ \frac{c_2 a^{r^1}_{\xi_1,\zeta_1} a^{r^2}_{0,\xi_2}}{a^{r^2}_{0,\xi_2} a^{r^1}_{\zeta_0,\zeta_1}} t \notin I \qquad \text{i.e.,}\ \frac{c_2 a^{r^1}_{\xi_1,\zeta_1}}{a^{r^1}_{\zeta_0,\zeta_1}}\ t \notin I$$

By applying (iii) of Definition 7 to all $a_{i,j}$'s appearing in $(*)'$ and remembering that for a unit c of R $cI = I$ and $c \in R$ is a unit iff p does not divide c for R, $(*)'$ is equivalent to

$$(*)''\ t \in I \text{ but } \frac{p^{d(\zeta_1)-d(\xi_1)} p^{d(\xi_2)}}{p^{d(\xi_2)} p^{d(\zeta_1)-d(\zeta_0)}} t \notin I$$

which means $t \in I$ but $t / p^{d(\xi_1)-d(\zeta_0)} \notin I$, which is easily accomplished by choosing $t = p^b \in M_0$.

Now we define r:

$$u^r = u^{r^1} \cup u^{r^2} = u_0 \cup u_1 \cup u_2$$

$$a^r_{ij} = \begin{cases} a^{r^1}_{i,j} & \text{if } i,j \in u^{r^1_1} \quad (a) \\ a^{r^2}_{i,j} & \text{if } i,j \in u^{r^2_2} \quad (b) \\ a_{i,\xi_1}\, a\, a_{\xi_2,j} & \text{if } i = u_1,\ j \in u_2 \quad (c) \end{cases}$$

(remember $a_{\xi_2,\xi_2} = 1$)

Again condition (i) + (iii) + (iv) are easy. Let us try (ii).

So $\alpha < \beta < \gamma$.

Case A: $\alpha \in u_0, \beta \in u_1, \gamma \in u_2$.

$$\frac{a^r_{\alpha,\gamma} - a^r_{\alpha,\beta} a^r_{\beta,\gamma}}{a_{0,\gamma}} = \frac{a^{r^2}_{\alpha,\gamma} - a^{r^1}_{\alpha,\beta} a^{r^1}_{\beta,\xi_1} a\ a^{r^2}_{\xi_2,\gamma}}{a_{0,\gamma}} \equiv mod\ I$$

$$\frac{a^{r^2}_{\alpha,\xi_2} a^{r^2}_{\xi_2,\gamma} - a^{r^1}_{\alpha,\beta} a^{r^1}_{\beta,\xi_1} a\ a^{r^2}_{\xi_2,\gamma}}{a^{r^2}_{0,\gamma}} \equiv$$

$$a^{r^2}_{\xi_2,\gamma} \frac{a^{r^2}_{\alpha,\xi_2} - a^{r^1}_{\alpha,\beta} a^{r^1}_{\beta,\xi_1} a}{a^{r^2}_{0,\gamma}} \equiv$$

$$\equiv \frac{a^{r^2}_{\xi_2,\gamma} a^{r^2}_{0,\xi_2}}{a^{r^2}_{0,\gamma}} \quad \frac{a^{r^2}_{\alpha,\xi_2} - a^{r^1}_{\alpha,\beta}\ a^{r^1}_{\beta,\xi_1} a}{a^{r^2}_{0,\xi_2}}$$

Now $\dfrac{a^{r^2}_{\xi_2,\gamma} a^{r^2}_{0,\xi_2}}{a^{r^2}_{0,\gamma}}$ is a unit, so we can forget it

$$\frac{a^{r^2}_{\alpha,\xi_2} - a^{r^1}_{\alpha,\beta} a^{r^1}_{\beta,\xi_1} a}{a^{r^2}_{0,\xi_2}} \equiv \frac{a^{r^0}_{\alpha,\zeta_0} a^{r^2}_{\zeta_0,\xi_2} - a^{r^1}_{\alpha,\beta} a^{r^1}_{\beta,\zeta_1} a}{a^{r^2}_{0,\xi_2}}$$

Now $(a^{r^1}_{\alpha,\beta} a^{r^1}_{\beta,\zeta_1}) \dfrac{a}{a^{r^2}_{0,\xi_2}} \equiv (a^{r^0}_{\alpha,\zeta_0} a^{r^1}_{\zeta_0,\zeta_1}) \dfrac{a}{a^{r^2}_{0,\xi_2}}\ mod\ I$ holds

[as $\dfrac{a^{r^1}_{\alpha,\beta} a^{r^1}_{\beta,\zeta_1} - a^{r^1}_{\alpha,\zeta_0} a^{r^1}_{\zeta_0,\zeta_1}}{a_{0,\zeta_1}} \equiv 0\ mod\ \dfrac{a_{0,\xi_2}}{a_{0,\zeta_1}, a} I$ holds, which hold by using twice (ii) of Definition 7, and computing power of p in the left side].

So

$$\frac{a^{r^0}_{\alpha,\zeta_0}a^{r^2}_{\zeta_0,\xi_2}-a^{r^1}_{\alpha,\beta}a^{r^1}_{\beta,\zeta_1}a}{a^{r}_{0,\xi_2}} = \frac{a^{r^0}_{\alpha,\zeta_0}a^{r^2}_{\zeta_0,\xi_2}-a^{r^1}_{\alpha,\zeta_0}a^{r^1}_{\zeta_0,\zeta_1}a}{a^{r}_{0,\xi_2}}$$

$$= a^{r^0}_{\alpha,\zeta_0}\frac{a^{r^2}_{\zeta_0,\xi_2}-a^{r^1}_{\zeta_0,\zeta_1}a}{a_{0,\xi_2}} \in I$$

the "$\in$" holds by (β) above. So we finish Case A.

Case B: $\alpha,\beta \in u_1$, $\gamma \in u_2$

$$\frac{a^{r}_{\alpha,\gamma}-a^{r}_{\alpha,\beta}a^{r}_{\beta,\gamma}}{a^{r}_{0,\gamma}} = \frac{a^{r^1}_{\alpha,\zeta_1}a\ a^{r^2}_{\xi_2,\gamma}-a^{r^1}_{\alpha,\beta}a^{r^1}_{\beta,\zeta_1}a\ a^{r^2}_{\xi_2,\gamma}}{a^{r^2}_{0,\gamma}} =$$

$$a^{r^2}_{\xi_2,\gamma}\ a\ [\frac{a^{r^1}_{\alpha,\zeta_1}-a^{r^1}_{\alpha,\beta}a^{r^1}_{\beta,\zeta_1}}{a_{0,\gamma}}]$$

by computing power of p this term belongs to I iff

$$\frac{a^{r^1}_{\alpha,\zeta_1}-a^{r^1}_{\alpha,\beta}a^{r^1}_{\beta,\zeta_1}}{a^{r^1}_{0,\zeta_1}} \in I$$

which holds.

Case C: $\alpha \in u_1$ $\beta,\gamma \in u_2$.

$$\frac{a^{r}_{\alpha,\gamma}-a^{r}_{\alpha,\beta}a^{r}_{\beta,\gamma}}{a^{r}_{0,\gamma}} = \frac{a^{r^1}_{\alpha,\zeta_1}a\ a^{r^2}_{\xi_2,\gamma}-a^{r^1}_{\alpha,\zeta_1}a\ a^{r^2}_{\xi_2,\beta}a^{r^1}_{\beta,\gamma}}{a_{0,\gamma}}$$

$$= a^{r^1}_{\alpha,\zeta_1}a[\frac{a^{r^2}_{\xi_2,\gamma}-a^{r^2}_{\xi_2,\beta}a^{r^2}_{\beta,\gamma}}{a_{0,\gamma}}] \in I$$

Case D: $\{\alpha,\beta,\gamma\} \subseteq u_0 \cup u_1$ or $\{\alpha,\beta,\gamma\} \subseteq u_0 \cup u_2$.

Trivial.

11. Conclusion: If $G \subseteq P$ is generic over V then in the new universal $V[G]$ over R there is a non standard uniserial R-module.

Proof : We can deal with I-representation. Let for $i < j < \omega_1$ $a_{i,j}$ be $a^{r}_{i,j}$ when $r \in G, \{i,j\} \subseteq u^r$, this is well defined as:

(A) $a_{i,j}$ has at most one value as G is directed.

(B) $a_{i,j}$ has at least one value [as by Fact B the sets $\{r \in P: i \in u^r\}$, $\{r \in P: j \in u^r\}$ are dense subsets of P, hence their intersection is. As G is generic, G is not disjoint to this intersection.] Now easily $\langle a_{i,j}: i < j < \omega_1 \rangle$ is an I-representation (over R). Why it represents a non standard uniserial module? Otherwise (letting $\underset{\sim}{a}_{i,j}$ be the name for $a_{i,j}$ defines above) there are P-name $\underset{\sim}{c}_i$ and $r \in P$ such that

(C) $r \Vdash_P$" $\underset{\sim}{c}_i$ is a unit of P, and $\dfrac{\underset{\sim}{c}_i}{\underset{\sim}{a}_{0,i}} - \dfrac{\underset{\sim}{c}_j}{\underset{\sim}{a}_{0,j} / \underset{\sim}{a}_{i,j}} \in I$ for every $i < j < \omega_1$".

As R consists of members of V, there are for $i < \omega_1$, $r_i \in P$, $r \leq r_i$ and $c_i^1 \in P$ $r_i \Vdash_P \underset{\sim}{c}_i = c_i^1$. Now using Fodor Lemma and Fact 9 we get a contradiction.

Originally we have then replaced forcing by $\diamondsuit_{\aleph_1}$, but it is better to have:

12. Theorem : (ZFC): There is a uniserial non standard module over some uniserial domain.

Proof : If we look carefully at the proof of this we can see that we have proved (and we shall prove):

(a) in $V[G]$, for every limit ordinal $\delta < \omega_1$ and unit $c \in R$, for every large enough $i < \delta$. $\dfrac{c}{a_{0,\delta} / a_{i,\delta}}$ is not I-equivalent to any member of R_δ.

13. Observation: If $\dfrac{c}{a_{0,\delta} / a_{i,\delta}} + I \not\subseteq \{x+I: x \in M_\delta\}$ and $i < j < \delta$ then $\dfrac{c}{a_{0,\delta} / a_{j,\delta}} + I \not\subseteq \{x+I: x \in M_\delta\}$.

Proof : Suppose $\dfrac{c}{a_{0,\delta} / a_{j,\delta}} = x+t$, $t \in I$, $x \in M_\delta$. Then

$$\frac{c}{a_{0,\delta} / a_{i,\delta}} = c\frac{a_{i,\delta}}{a_{0,\delta}} \equiv c\frac{a_{i,j} a_{j,\delta}}{a_{0,\delta}} = \quad mod\ I$$

$$a_{i,j}\left(\frac{c}{a_{0,\delta} / a_{j,\delta}}\right) = a_{i,j}(x+t) = a_{i,j}x + a_{i,j}t$$

Now $a_{i,j}x \in M_\delta$ (as $a_{i,j} \in M_{j+1} \subseteq M_\delta$, $x \in R_\delta$), and $a_{i,j}\, t \in I$ (as $a_{i,j} \in R$, $t \in I$).

Proof of (a): Suppose $r \in P$,

$r \Vdash_P$" $\delta < \omega_1$ is a limit ordinal, c a unit of R and δ, c, contradict (a)".

By Fact 8 w.l.o.g. $\delta \in u^r$. Now let $u_0 = u^r \cap \delta$, $u_2 = u^r - \delta$ $r^0 = r \restriction u_0$, $r^2 = r$, and find u_1, r_1 so that the assumptions of 9 holds ($u_2 \neq \phi$ as $\delta \in u_2$, $u_0 \neq \phi$ as $0 \in u_0$). Let $c_i = c$. We repeat the proof of 9 but in (γ) of 10 replace c_2 by c and $\notin I$ by $\notin I + M_{\xi_2}$, and drop $\frac{c_1}{a_{0,\xi_1}}$ i.e. we use

$$(\gamma)' \qquad \frac{c}{a^{r^2}_{0,\xi_2} / (a^{r^1}_{\xi_1,\zeta_1} a)} \notin I + M_\delta.$$

As we demand $a \in M_{\xi_2+1}$, and can assume M_{ξ_2+1} is quite large compared to M_{ξ_2} (though countable) there is no problem. [Let $e_i \in R$ $(i < \omega_1)$ be distinct units, $e_i - e_j$ not divisible by p then for $i \neq j$: $\frac{c\, a^{r_1}_{\xi_1,\zeta_1}}{a^{r_1}_{\zeta_0,\zeta_1}} (p^b e_i) - \frac{c\, a^{r_1}_{\xi_1,\zeta_1}}{a^{r_1}_{\zeta_0,\zeta_1}} (p^b e_j) \notin I$; as M_δ is countable, for some i $\frac{c\, a^{r_1}_{\xi_1,\zeta_1}}{a^{r_1}_{\zeta_0,\zeta_1}} (p^b e_i) \notin I + M_\delta$. For being able to repeat the argument in M_{ξ_2+1} it is enough that in M_{ξ_2+1} there is a "finite" set to which every $x \in M_{\xi_2}$ "belongs", which is easy. Alternatively change the forcing as to allow us to choose $a \in M$, so that the forcing fail the $\aleph_1$-c.c. but is still proper see [Sh 2], Ch. III.] So we find r^1.

$$r \leq r^1 \in P, \quad \frac{c}{a^{r^1}_{0,\delta} / a^{r^1}_{i,\delta}} \notin I + M_\delta$$

Contradiction, so (a) holds. Note also

14. Observation: If M_α: $(\alpha < \omega_1)$, $b, d(\alpha)(\alpha < \omega_1)$ are as in 6, $a_{i,j}$ satisfies (a) above, then $\langle a_{i,j} : i < j < \omega_1 \rangle$ is an I- representation of a non standard uniserial module.

Proof: Suppose $\langle c_i : i < \omega_1 \rangle$ exemplify the contrary. For a closed unbounded subset C of ω_1 for every $\delta \in C$

$$i < \delta \Longrightarrow c_i \in M_\delta$$

So $\frac{c_i}{a_{0,i}} \in M_\delta$ for $i < \delta$, hence $\frac{c_i}{a_{0,\delta} / a_{i,\delta}} + I \in \{x + I : x \in M_\delta\}$. Contradicting (a). So 14 holds.

Now the statement: there are $M_i (i < \omega_1)$ $b, d(\alpha)$ as in 6 and $a_{i,j}$ satisfying (a), can be expressed by a countable theory T in $L(aa)$ (note that we do not mind to replace ω_1 by a linear order K of power $\aleph_1$ such that $K = \bigcup_{i<\omega_1} K_i$, K_i increasing continuous each K_i countable $(\forall x \in K_i)(\forall y \in K_{i+1} - K_i)$ $(x < y)$ and K_i has a least upper bound). $L(aa)$ was introduced in Shelah [Sh 1], and thoroughly investigated in Barwise Kaufman and Makkai [BKM]. By the completeness theorem for $L(aa)$ (see [BKM]) the answer to "does T has a model" is absolute. As it has a model in $V[G]$ it has one in V.

15 Remark: We can replace $\aleph_1$ by any uncountable regular uncountable κ. Let $H(\aleph_2)$ be the family of sets of hereditary power $< \aleph_2$, and $\mathfrak{E}$ be $(H(\aleph_1), \in)$ expanded by (individual constants for) M, R, Q, I, $\langle M_i : i < \omega_1 \rangle$, $\langle d(i) : i < \omega_1 \rangle$, b and $\langle a_{i,j} : i < j < \omega_1 \rangle$. Now we can define by induction on $\alpha < \kappa^2$ $\mathfrak{E}_\alpha$ such that:

1) $\mathfrak{E}_\alpha$ is a model of power κ elementarily equivalent to $\mathfrak{E}$.

2) $\mathfrak{E}_\alpha$ $(\alpha < \kappa^2)$ is a continuous elementarily chain.

3) For every α there is $y_\alpha \in \mathfrak{E}_{\alpha+1}$ such that:

(a) $\mathfrak{E}_{\alpha+1} \models$ "y_α is a countable set".

(b) for every $x \in \mathfrak{E}_\alpha$, $\mathfrak{E}_{\alpha+1} \models$ "$x \in y_\alpha$".

(c) if α has cofinality κ and $\alpha < \beta \leq \kappa^2$ then $\mathfrak{E}_\beta \models$ "$x \in y_\alpha$", implies $x \in \mathfrak{E}_\alpha$.

Let $\mathbb{B}^* = \bigcup_{\alpha<\kappa^2} \mathbb{B}_\alpha$, $z_\alpha \in \mathbb{B}_{\alpha+1}$ be such that $\mathbb{B}_{\alpha+1} \models$"$z_\alpha$ is $\sup(y_\alpha \cap \omega_1)$".

There is no problem to do this (e.g. use saturated models, possible as we can construct the models say in L), see Mekler and Shelah [M Sh]. Now use M,R,I $\langle a_{i,j}: \mathbb{B}^* \models$ "$i < j < \omega_1^{\mathbb{B}^*}$"$\rangle$ or equivalently $\langle a_\beta: \alpha < \beta < \kappa \rangle$ with $M_\alpha = M^{\mathbb{B}_\alpha}$. Note that we could replace κ^2 by $\kappa\mu$ if $cf\ \mu \geq \aleph_0$.

References

[CK] C. C. Chang and H. J. Keisler, Model Theory, North Holland Publ. Co. 1973.

[F] L. Fuchs, Infinite Abelian groups I,II. Academic Press, N.Y. 1970,1973.

[J] T. Jech. Set Theory. Academic Press.

[BKM]
K. J. Barwise, M. Kaufman, M. Makkai. Stationary Logic, Annals of Math. Logic, 13 (1978), 171-229.

[MSh]
A. Mekler and S. Shelah, Stationary Logic and its Friends, Proc. of the Jerusalem Model Theory Year 1984, *Notre Dame J. of Formal Logic*, to appear.

[Sh1]
S. Shelah, Generalized quantifiers and compact logic, Trans. of A.M.S. 204 (1975), 342-364.

[Sh2]
S. Shelah, Proper Forcing, Springer Lecture Notes, 940 (1982).