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REMARK TO "LOCAL DEFINABILITY THEORY" OF REYES *

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Abstract: Here we correct and improve a theorem of G.E.Reyes (in this journal) which generalizes a result of Chang and Makkai, on weak definability.

In Reyes [6], Theorem 3.2.1, p. 132 the following error occurs: in (i), and (ii) 2^μ should be replaced by 2^κ, and the sequence of models $(\mathfrak{B}_\xi, P_\alpha^{(\xi)}, f_{\alpha,\beta}^{(\xi)})_{\alpha,\beta \in \xi_2}$ is defined only for $\xi < \kappa$, for if $\kappa \leq \alpha < \mu$ may be $2^{|\alpha|} > \mu$, and so he gets a model of cardinality $> \mu$.

We shall show that a stronger theorem follows, and that this theorem is the best possible.

Let L be a (first-order) language, $L(P)$ – a language obtained from L by adding a new predicate P. T will be a fixed theory in L_1, $L(P) \subset L_1$. Let $|L_1|$ be the number of formulas of L_1. We say an $L(P)$-model is a model of T if it is a reduct of a model of T. T, L, $L(P)$, L_1 will be fixed. Let

Definition 1. (1) Df(λ) is the first cardinal μ such that for every L-model $\mathfrak{B}$ of cardinality λ

$$|\{P \quad (\mathfrak{B}, P) \text{ is a model of } T\}| < \mu \,.$$

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(2) $\mathrm{Df}_1(\lambda)$ is the first cardinal μ such that for every $L(P)$-model $(\mathfrak{B}, P)$ of T of cardinality λ

$$|\{P' : (\mathfrak{B}, P') \cong (\mathfrak{B}, P)\}| < \mu .$$

Remark. Clearly for every λ, $\mathrm{Df}(\lambda) \geq \mathrm{Df}_1(\lambda)$; and we can restrict the definition to $\lambda \geq |L_1|$ (and we shall assume it implicitly).

Definition 2. $\mathrm{Ded}(\lambda)$ is the first cardinal μ, such that there is no ordered set J_1 with a dense subset J, $|J_1| = \mu$, $|J| = \lambda$. (Where $|A|$ is the cardinality of A.)

Definition 3. $\mathrm{Ded}_\kappa(\lambda)$ is the first cardinal μ such that there is no ordered set J_1 with a dense subset J, $|J_1| \geq \mu$, $|J| \leq \lambda$ which satisfy: for every $s \in J_1$, $s \notin J$ there are in $J s^k$ $k < \kappa_1$, s_k $s < \kappa_2$, such that:

(1) $k < \ell \Rightarrow s^k < s^\ell < s < s_\ell < s_k$

(2) for every $t \in J$, $t \neq s$

either for some k, $t < s^k$; or for some ℓ, $s_\ell < t$

(3) $\kappa_1 + \kappa_2 = \kappa$.

Remark. By Theorem 1 we can replace (3) by $\kappa_1 = \kappa_2 = \kappa$, as the number of $s \in J_1^{\cdot}$, for which $\kappa_1 \neq \kappa_2$ is $\leq \lambda$ (as s is the last or first element in every high enough member of the branch A_s defined in the proof of th. 1.) Clearly $\kappa \leq \lambda$, otherwise $\mathrm{Ded}_\kappa(\lambda) = 0$.

Definition 4. $\mathrm{Ded}^*(\lambda) = \sum_{\kappa \leq \lambda} \mathrm{Ded}_\kappa(\lambda)$

Clearly $\lambda^+ < \mathrm{Ded}^*(\lambda) \leq \mathrm{Ded}(\lambda) \leq (2^\lambda)^+$. (Let $\kappa = \inf\{\kappa : 2^\kappa > \lambda\}$, and J_1 be the set of sequences of ones and zeroes of length κ, ordered lexicographically. Then clearly $\lambda < 2^\kappa < \mathrm{Ded}^*(\lambda)$.) If $\mu = \mathrm{Ded}^*(\lambda) < \mathrm{Ded}(\lambda)$ then $\mu^+ = \mathrm{Ded}(\lambda)$, and the cofinality of μ, $\mathrm{cf}(\mu)$ is $\leq |\alpha|$, where $\lambda = \aleph_\alpha$. It is known that ZFC + $[\mathrm{Ded}(\aleph_1) < (2^{\aleph_1})^+]$ is consistent. See Baumgartner [1, 2], Mitchell [5].

Theorem 1. *The following conditions are equivalent (in Reyes [6], their negations appear)*

(i) *there is* $\lambda \geq |L_1|$ *such that* $\mathrm{Df}(\lambda) > \lambda^+$

(ii) *for every* $\lambda \geq |L_1|$, $\mathrm{Df}_1(\lambda) \geq \mathrm{Ded}^*(\lambda)$ (*hence* $\mathrm{Df}(\lambda) > \lambda^+$)

(iii) *there are no formulas* $\theta_i(\bar{x}, \bar{y})$ $i = 1, \ldots, n$ *such that*

$$T \vdash \bigvee_{i=1}^{n} (\exists \bar{y})(\forall \bar{x})[P(\bar{x}) \equiv \theta_i(\bar{x}, \bar{y})] .$$

Remark. If in L the equality sign appears, and every model of T has at least two elements, then we can replace (iii) by

(iii)* there is no $\theta(\bar{x}, \bar{y})$ such that $T \vdash (\exists \bar{y})(\forall \bar{x})(P(\bar{x}) \equiv \theta(\bar{x}, \bar{y}))$.

For clearly (iii) implies (iii)*, and if (iii) does not hold, then

$$\theta(\bar{x}, \bar{y}, z_1, \ldots, z_{2n}) = \bigwedge_{i=1}^{n} [z_{2i-1} = z_{2i} \rightarrow \theta_i(\bar{x}, \bar{y})]$$

show that (iii)* does not hold.

Remark. Makkai [4] proved if (iii), $\lambda^+ = 2^\lambda > |L_1|$ then $\mathrm{Df}_1(\lambda^+) = (2^{\lambda^+})^+$. Chang [3] proved this and, in addition, that (iii), $\mu < \lambda \Rightarrow 2^\mu < \lambda$ and $\lambda > |L_1|$ implies $\mathrm{Df}_1(\lambda) = (2^\lambda)^+$. Reyes [5] proved, in fact, that if $\lambda = \sum_{\kappa < |L_1|} 2^\kappa$, and (iii) then $\mathrm{Df}_1(\lambda) > 2^{|L_1|}$. Of course it is trivial that (ii) $\rightarrow$ (i) $\rightarrow$ (iii).

Theorem 1 cannot be improved as shown by

Theorem 2. (1) *There exist a language L, and a finite theory T in L(P) such that for T; for every* λ, $\mathrm{Df}(\lambda) = \mathrm{Ded}(\lambda)$, $\mathrm{Df}_1(\lambda) = \mathrm{Ded}^*(\lambda)$.

(2) *There exist languages* $L \subset L(P) \subset L_1$ *and a finite theory T in* L_1, *such that for T; for every* λ, $\mathrm{Df}(\lambda) = \mathrm{Df}_1(\lambda) = \mathrm{Ded}^*(\lambda)$.

Proof of Theorem 2. We shall only give T for 2.1. The construction of the other example is similar; and the proofs depend on the remark to Definition 3, and the definition itself.

Let L contain the equality sign and the predicate $x < y$; and P be a one place predicate. Now T will be the theory of order with that axiom that P is a head. That is

$$T = \{(\forall xyz)(x < y \wedge y < x \rightarrow x < z),$$

$$(\forall xy)(x < y \vee y < x \vee x = y), (\forall x)(\neg x < x),$$

$$(\forall xy)[x < y \wedge P(y) \rightarrow P(x)]\}.$$

Remark. So in the case $L_1 = L(P)$, and when $(\exists\lambda)[\mathrm{Ded}^*(\lambda) < \mathrm{Ded}(\lambda)]$, and (iii), we do not know whether $(\forall\lambda)[\mathrm{Df}(\lambda) = \mathrm{Ded}(\lambda)]$ can be proved. Naturally arise the conjecture:

Conjecture. If for at least one λ, $\mathrm{Df}(\lambda) > \mathrm{Ded}(\lambda)$ then for every μ, $\mathrm{Df}_1(\mu) = (2^\mu)^+$.

As we have already mentioned, by Mitchell [5], $\mathrm{Ded}(\aleph_1) < (2^{\aleph_1})^+$ is consistent with ZFC, hence the conjecture is not meaningless. There is a corresponding syntactical condition; which implies that for every μ, $\mathrm{Df}_1(\mu) = (2^\mu)^+$. But the condition is not elegant, and there is no proof of the other part. A similar weaker theorem is Shelah [7] Theorem 4.3.

Proof of Theorem 1. As has been mentioned, (ii) $\rightarrow$ (i) $\rightarrow$ (iii) is trivial. So we should prove only (iii) $\rightarrow$ (ii). Hence suppose (iii) holds. So let $\lambda \geq |L_1|$, $\mu < \mathrm{Ded}^*(\lambda)$. We should prove only that $\mathrm{Df}_1(\lambda) > \mu$. So we should prove only that for some model $(\mathfrak{B}, P)$ of T

$$|\{P' : (\mathfrak{B}, P') \cong (\mathfrak{B}, P)\}| \geq \mu .$$

Clearly without loss of generality we can assume T is complete. For simplicity we assume $\mathrm{cf}(\mu) > \lambda$. (See remark at the end.)

The pair $\langle I, < \rangle$ is a tree if $<$ is a well-ordering of I, which can be a partial order. For any $s \in I$, the level of s, $\ell(s)$, is the order type of $\{t \in I : t < s\}$ which is an ordinal. Let $I^\alpha = \{s \in I : \ell(s) = \alpha\}$. A branch B of I is a maximal totally ordered subset of I; its level, $\ell(B)$, is its order type, and $\mathrm{Br}_\alpha(I) = \{B : B \text{ a branch}, \ell(B) = \alpha\}$.

Now we shall prove that there is a tree $\langle I, <\rangle$ and ordinal $\alpha_0 \leq \lambda$ such that:

(A) $|I| \leq \lambda$, and $|I^0| = 1$,

(B) for every $s \in I$, $\ell(s) < \alpha_0$,

(C) for every $s \in I^\beta$, $|\{t \in I^{\beta+1} : s < t\}| \leq 2$,

(D) for every $s \in I^\beta$ except one $|\{t \in I^{\beta+1} : s < t\}| = 1$,

(E) if $\{t \in I : t < s_1\} = \{t \in I : t < s_2\}$ and $\ell(s_1)$ is a limit ordinal, then $s_1 = s_2$.

(F) $|\mathrm{Br}_{\alpha_0}(I)| \geq \mu$.

It suffices to find a tree satisfying A, B, C, E, F, as from it we can easily build a tree satisfying all the properties. This is the bisection tree.

By definition there is an ordered set J_1, $|J_1| = \mu$, with a dense subset J, $|J| = \lambda$; and their order is $<$. We can assume J, J_1 are dense. Let $J_1 = \{a_k : k < \lambda\}$.

Let us define by induction on $\alpha < \lambda$ a family K_α of subsets of J_1, such that for each $A \in K_\alpha$; $a, b \in A$, $a < c < b \Rightarrow c \in A$.

(1) Let $K_0 = \{J_1\}$.

(2) Suppose K_α is defined. For every $A \in K_\alpha$, $|A| > 1$, we define a_A as the first $a_k \in A \cap J$ that is not the first or last in $A \cap J$ (that is, with the smallest index k). We define

$$F^1(A) = \{a \in J_1 : a \in A, a < a_A\},$$

$$F^2(A) = \{a \in J_1 : a \in A, a_A < a\},$$

and

$$K_{\alpha+1} = \{F^1(A) : A \in K_\alpha, |A| > 1\} \cup \{F^2(A) : A \in K_\alpha, |A| > 1\}.$$

(3) Suppose K_α is defined for every $\alpha < \delta$, where δ is a limit ordinal. Then

$$K_\delta = \{\bigcap_{\alpha<\delta} A_\alpha : A_\alpha \in K_\alpha, \alpha < \beta \Rightarrow A_\beta \subset A_\alpha, |\bigcap_{\alpha<\delta} A_\alpha| > 1\}.$$

Now on $K = \bigcup_{\alpha<\lambda} K_\alpha$ we define an order $<$: $A < B$ iff $B \subset A$. Clearly $\langle K, < \rangle$ is a tree, and $A \in K_\alpha$ iff it is in the α-th level. It is also clear that the tree satisfies conditions A, C, E. Clearly, if $s \in J_1$, $s \notin J$ then $A_s = \{A \in K : s \in A\}$ is a branch of the tree, and every branch of the tree is of level $\leq \lambda$; hence $\mu \leq |J_1| \leq |\bigcup_{\alpha\leq\lambda} \mathrm{Br}_\alpha(K)| = \sum_{\alpha\leq\lambda} |\mathrm{Br}_\alpha(K)|$. As $\mathrm{cf}(\mu) > \lambda$, for some $\alpha_0 \leq \lambda$ $|\mathrm{Br}_{\alpha_0}(K)| \geq \mu$. Now for $I = \bigcup_{\alpha<\alpha_0} K_\alpha$, $\langle I, < \rangle$ is the required tree.

Now after we have the tree $\langle I, < \rangle$, we shall describe shortly the construction, which is like Reyes's construction. For simplicity we assume $L(P) = L_1$. We shall define by induction on α the following: a model $\mathfrak{B}_\alpha$, relations P_s for $s \in I^\alpha$, and isomorphisms $f_{s,t}$ for $s, t \in I^{\alpha}$, such that:

(1) If $s \in I^0$, then $(\mathfrak{B}_0, P_s)$ is any model of T, of cardinality λ.

(2) If $s < t$, $t \in I^{\ell(s)+1}$ then $(\mathfrak{B}_{\ell(s)}, P_s)$ is an elementary submodel of $(\mathfrak{B}_{\ell(t)}, P_t)$, and their cardinalities are λ.

(3) If $t_1, t_2 \in I^{\alpha+1}$, $s \in I^\alpha$, $s < t_1$, $s < t_2$, $t_1 \neq t_2$ then $P_{t_1} \neq P_{t_2}$.

(4) If $\alpha = \ell(s)$ is a limit ordinal, then $(\mathfrak{B}_\alpha, P_s)$ is the union of $\{(\mathfrak{B}_\beta, P_t) : \ell(t) = \beta < \alpha, t < s\}$.

(5) If $s, t \in I^\alpha$, then $f_{s,t}$ is an automorphism between $(\mathfrak{B}_\alpha, P_s)$ and $(\mathfrak{B}_\alpha, P_t)$. (If $s = t$, $f_{s,t}$ is the identity.)

(6) If $s, t \in I^\alpha$, $s_1, t_1 \in I^{\alpha+1}$, $s < s_1$, $t < t_1$, then the reduction of f_{s_1,t_1} to $\mathfrak{B}_\alpha$ is $f_{s,t}$.

(7) If $s, t \in I^\delta$, δ a limit ordinal, then $f_{s,t}$ is the union of $\{f_{s_\alpha,t_\alpha} : \alpha < \delta, s_\alpha < s, t_\alpha < t; s_\alpha, t_\alpha \in I^\alpha\}$.

The definition is straightforward, with the use of the Robinson Theorem in the case $\alpha + 1$. (Only here (iii) is used.)

Now if $\mathfrak{B}$ is the union of $\{\mathfrak{B}_\alpha : \alpha < \alpha_0\}$ and for any $B \in \mathrm{Br}_{\alpha_0}(I)$ we define $P_B = \bigcup_{t\in B} P_t$, then the cardinality of $\mathfrak{B}$ is λ, and for any $B_1 \in \mathrm{Br}_{\alpha_0}(I)$

$$\{P' : (\mathfrak{B}, P') \cong (\mathfrak{B}, P_{B_1})\} \supset \{P_B : B \in \mathrm{Br}_{\alpha_0}(I)\} ,$$

hence

$$|\{P' : (\mathfrak{B}, P') \cong (\mathfrak{B}, P_{B_1})\}| \geq \mu .$$

So the theorem is proved.

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Remark. If $\mathrm{cf}(\mu) \leq \lambda$, then we will have $\leq \lambda$ trees $\{\langle I_k, < \rangle : k < k_0 \leq \lambda\}$, each of them satisfying (A)–(E) with α_k instead of α_0, and $\mathrm{cf}(\alpha_k) = \mathrm{cf}(\alpha_0)$; and $\sum_{k<k_0} |\mathrm{Br}_{\alpha_k}(I_k)| = \mu$. Then we do a similar construction using all the trees together. (We use that $\mu < \mathrm{Ded}^*(\lambda)$, to insure $\mathrm{cf}(\alpha_k) = \mathrm{cf}(\alpha_0)$.) (Here is the only place where $\mu < \mathrm{Ded}^*(\lambda)$ and not $\mu < \mathrm{Ded}(\lambda)$ is used.)

Added in proof, 8 December 1970

1) Baumgartner tells me that for very λ, $\mathrm{Ded}(\lambda^+) = \mathrm{Ded}^*(\lambda^+)$, and the consistency of ZFC implies the consistency of ZFC + $[\mathrm{Ded}^* \aleph_\alpha < \mathrm{Ded}\, \aleph_\alpha]$ for limit cardinal $\aleph_\alpha < 2^{\aleph_0}$. The proof is by the construction of Easton [8] for singular $\aleph_\alpha$, and by Baumgartner [2] for regular $\aleph_\alpha$. So by 2.1 it is possible that $\mathrm{Df}(\lambda) \neq \mathrm{Df}_1(\lambda)$ for some λ.

2) Theorem 2.2. can be improved to $T \subset L(P)$, $L_1 = L(P)$.

Conjecture. If for one λ $\mathrm{Df}_1(\lambda) > \mathrm{Ded}^*(\lambda)$ then for every μ $\mathrm{Df}_1(\mu) = (2^\mu)^+$.

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