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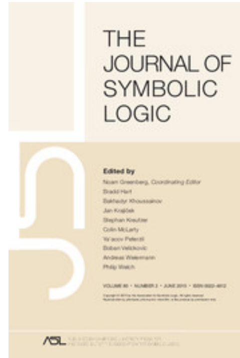

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SUBGROUPS OF SMALL INDEX IN INFINITE SYMMETRIC GROUPS. II

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§1. Introduction. Throughout this paper κ denotes an infinite cardinal, $S = \mathrm{Sym}(\kappa)$ and G is a subgroup of S. We shall be seeking the subgroups G with $[S\colon G] < 2^\kappa$. In [2], the following result was proved.

THEOREM 1. *If* $[S\colon G] \le \kappa$ *then there exists a subset* Δ *of* κ *such that* $|\Delta| < \kappa$ *and* $S_{(\Delta)} \le G$.

Here $S_{(\Delta)} = \mathrm{Sym}(\kappa \backslash \Delta)$ is the pointwise stabilizer of Δ in S.

However, the converse of Theorem 1 is not true. For if $\mathrm{cf}(\kappa) \le |\Delta| < \kappa$, then $[S\colon S_{(\Delta)}] \ge \kappa^{\mathrm{cf}(\kappa)} > \kappa$. This suggests that a substantially sharpened version of Theorem 1 may be true.

Question 1 [2]. Is it provable in ZFC, or even in ZFC with GCH, that if $[S\colon G] \le \kappa$ then there is a subset Δ of κ such that $|\Delta| < \mathrm{cf}(\kappa)$ and $S_{(\Delta)} \le G$?

At least two of the authors of [2] made a serious attempt to answer the above question positively. In §3, we shall see that they were essentially trying to prove that measurable cardinals do not exist.

The following result, due independently to Semmes [5] and Neumann [2], suggests a second way in which Theorem 1 might be improved.

THEOREM 2. *If* $\kappa = \aleph_0$ *and* $[S\colon G] < 2^{\aleph_0}$ *then there is a finite subset* Δ *of* κ *such that* $S_{(\Delta)} \le G$.

Question 2 [2]. Is it provable in ZFC that if $[S\colon G] < 2^\kappa$ then there is a subset Δ of κ such that $|\Delta| < \kappa$ and $S_{(\Delta)} \le G$?

This question will be answered negatively in §4.

§2. Stabilizers of filters. If $\mathscr{F}$ is a filter on the infinite cardinal κ, then the *depth* $d(\mathscr{F}) = \min\{|\Delta| \mid \Delta \in \mathscr{F}\}$. If $d(\mathscr{F}) = \kappa$, then the filter is said to be *uniform.* If $g \in S$, then $g[\mathscr{F}] = \{g[\Delta] \mid \Delta \in \mathscr{F}\}$. Define

$$S_{\{\mathscr{F}\}} = \{g \in S \mid g[\mathscr{F}] = \mathscr{F}\}, \qquad S_{(\mathscr{F})} = \{g \in S \mid \mathrm{fix}(g) \in \mathscr{F}\},$$

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where $\mathrm{fix}(g) = \{\alpha \in \kappa \mid g(\alpha) = \alpha\}$. Then it is easily checked that $S_{(\mathscr{F})}$ is a normal subgroup of $S_{\{\mathscr{F}\}}$. (See [2].) The following result was proved in [2]. A similar theorem was obtained earlier by Semmes [6].

THEOREM 3. *Suppose that* $G \leq S$ *and* $[S: G] < 2^\kappa$. *Define* $\mathscr{F} = \{\Gamma \subseteq \kappa \mid$ *there exists* $\Delta \subseteq \Gamma$ *such that* $|\kappa \setminus \Delta| = \kappa$ *and* $S_{(\Delta)} \leq G\}$. *Then*:

(i) $\mathscr{F}$ *is a filter on* κ;

(ii) *there exists* $\Gamma \in \mathscr{F}$ *such that* $|\Gamma| = |\kappa \setminus \Gamma| = \kappa$; *and*

(iii) $S_{(\mathscr{F})} \leq G \leq S_{\{\mathscr{F}\}}$.

Thus the groups G such that $S_{(\mathscr{F})} \leq G \leq S_{\{\mathscr{F}\}}$ for some filter $\mathscr{F}$ are the only candidates for subgroups of "small" index in S.

If $d(\mathscr{F}) = \mu$, then clearly $S_{(\Delta)} \leq G$ for some set $\Delta \subseteq \kappa$ of cardinality μ. Conversely, suppose that $S_{(\Delta)} \leq G$. Choose $\Gamma \in \mathscr{F}$ such that $|\Gamma| = |\kappa \setminus \Gamma| = \kappa$. There exists $g \in G$ such that $g[\Gamma \setminus \Delta] \subseteq \kappa \setminus \Gamma$. Thus $\Delta \cap \Gamma = g[\Gamma] \cap \Gamma \in \mathscr{F}$, and so $\mu \leq |\Delta \cap \Gamma| \leq |\Delta|$. Hence $d(\mathscr{F}) = \min\{|\Delta| \mid S_{(\Delta)} \leq G\}$. So both of the questions raised in the Introduction reduce to the question of computing $[S: S_{\{\mathscr{F}\}}]$ for a filter $\mathscr{F}$ of depth μ.

DEFINITION. Let θ and λ be cardinals and let $\mathscr{F}$ be a filter on λ. Then

$$\theta^\lambda/\mathscr{F} = |\{f/\approx_{\mathscr{F}} \mid f \in {}^\lambda\theta\}|$$

where $f \approx_{\mathscr{F}} h$ iff $\{\alpha \in \lambda \mid f(\alpha) = h(\alpha)\} \in \mathscr{F}$.

THEOREM 4. *Let* $\mathscr{F}$ *be a filter on* κ *which contains a set* Γ *with* $|\Gamma| = |\kappa \setminus \Gamma| = \kappa$. *Then* $[S: S_{\{\mathscr{F}\}}] = |\{g[\mathscr{F}] \mid g \in S\}| = \kappa^\kappa/\mathscr{F}$.

PROOF. As the first equality is obvious, we need only prove the second one. First we show that $|\{g[\mathscr{F}] \mid g \in S\}| \leq \kappa^\kappa/\mathscr{F}$. Suppose that $f, g \in S$ with $f \approx_{\mathscr{F}} g$. Then $\{\alpha \in \kappa \mid f(\alpha) = g(\alpha)\} \in \mathscr{F}$, and so $\mathrm{fix}(f^{-1}g) \in \mathscr{F}$. Thus $f^{-1}g \in S_{(\mathscr{F})} \leq S_{\{\mathscr{F}\}}$, and $fS_{\{\mathscr{F}\}} = gS_{\{\mathscr{F}\}}$. Hence $f[\mathscr{F}] = g[\mathscr{F}]$.

Now we prove the reverse inequality. Let $\{f_i \mid i < \kappa^\kappa/\mathscr{F}\}$ be a set of pairwise inequivalent elements of ${}^\kappa\kappa$. Choose $\Gamma \in \mathscr{F}$ with $|\Gamma| = |\kappa \setminus \Gamma| = \kappa$, and let $\{x^\alpha_\beta \mid \alpha, \beta < \kappa\}$ be a set of distinct elements of Γ. For each $i < \kappa^\kappa/\mathscr{F}$, there exists a permutation $g_i \in S$ such that

(a) if $\alpha \in \Gamma$, then $g_i(\alpha) = x^\alpha_{f_i(\alpha)}$, and

(b) $g_i \restriction \kappa \setminus \Gamma$ is a bijection from $\kappa \setminus \Gamma$ to $\kappa \setminus \{x_{f_i(\alpha)} \mid \alpha \in \Gamma\}$.

We claim that if $i \neq j$, then $g_i[\mathscr{F}] \neq g_j[\mathscr{F}]$. If not, then $\Delta = g_i[\Gamma] \cap g_j[\Gamma] \in g_i[\mathscr{F}]$. But, by construction, we have that

$$\Delta = \{x^\alpha_{f_i(\alpha)} \mid \alpha \in \Gamma, f_i(\alpha) = f_j(\alpha)\},$$

and so

$$g_i^{-1}[\Delta] = \{\alpha \in \Gamma \mid f_i(\alpha) = f_j(\alpha)\} \notin \mathscr{F}.$$

This contradiction establishes the claim. ∎

COROLLARY 1. *With the hypothesis of Theorem* 4, *suppose that* $\mathscr{U}$ *is an ultrafilter which extends* $\mathscr{F}$. *Then* $[S: S_{\{\mathscr{U}\}}] \leq [S: S_{\{\mathscr{F}\}}]$.

PROOF. Clearly $\kappa^\kappa/\mathscr{U} \leq \kappa^\kappa/\mathscr{F}$. ∎

§3. Singular cardinals and indecomposable ultrafilters. Let $\mathscr{F}$ be a filter on the infinite cardinal λ and let α be a cardinal. $\mathscr{F}$ is *α-decomposable* if there are sets Y_ξ, $\xi < \alpha$, such that

(i) $\lambda = \bigcup_{\xi < \alpha} Y_\xi$ is a partition of λ, and
(ii) for all $T \subseteq \alpha$, if $|T| < \alpha$ then $\bigcup_{\xi \in T} Y_\xi \notin \mathscr{F}$.

$\mathscr{F}$ is *α-indecomposable* if $\mathscr{F}$ is not α-decomposable.

LEMMA 1. *Let κ be a strong limit cardinal. Let $\operatorname{cf}(\kappa) < \lambda < \kappa$ and let $\mathscr{F}$ be a filter on λ.*

(i) *If $\mathscr{F}$ is $\operatorname{cf}(\kappa)$-indecomposable, then $\kappa^\lambda/\mathscr{F} = \kappa$.*

(ii) *If $\mathscr{F}$ is $\operatorname{cf}(\kappa)$-decomposable, then $\kappa^\lambda/\mathscr{F} = 2^\kappa$.*

PROOF. (i) Let $\kappa = \bigcup_{i < \operatorname{cf}(\kappa)} \theta_i$, where $\theta_i < \theta_j < \kappa$ for $i < j < \operatorname{cf}(\kappa)$. Let $f \in {}^\lambda\kappa$. Since $\mathscr{F}$ is $\operatorname{cf}(\kappa)$-indecomposable, there exists $i < \operatorname{cf}(\kappa)$ such that $f^{-1}[\theta_i] \in \mathscr{F}$. Hence there exists $h \in {}^\lambda\theta_i$ such that $h \approx_{\mathscr{F}} f$. As κ is a strong limit cardinal, this implies that $\kappa^\lambda/\mathscr{F} = \kappa$.

(ii) Let $\kappa = \bigcup_{\xi < \kappa} X_\xi$ be a partition of κ, where each X_ξ has cardinality λ. Since κ is a strong limit cardinal, $\kappa^{\operatorname{cf}(\kappa)} = 2^\kappa$ and $\operatorname{cf}(\kappa)$ is the least cardinal θ such that $\kappa^\theta > \kappa$. By 1.1.3 of [7], there exists a family $\mathscr{A}$ of 2^κ almost disjoint subsets of κ, each of cardinality $\operatorname{cf}(\kappa)$. Let $\lambda = \bigcup_{\alpha < \operatorname{cf}(\kappa)} Y_\alpha$ be a partition of λ such that whenever $T \subseteq \operatorname{cf}(\kappa)$ has cardinality less than $\operatorname{cf}(\kappa)$, then $\bigcup_{\alpha \in T} Y_\alpha \notin \mathscr{F}$. For each $A = \{\xi_\alpha^A \mid \alpha < \operatorname{cf}(\kappa)\} \in \mathscr{A}$, there exists $f_A \in {}^\lambda\kappa$ such that $f_A[Y_\alpha] \subseteq X_{\xi_\alpha^A}$ for all $\alpha < \operatorname{cf}(\kappa)$. Let $B \neq A$ be another element of $\mathscr{A}$. If $T = \{\alpha < \operatorname{cf}(\kappa) \mid \xi_\alpha^A = \xi_\alpha^B\}$, then $|T| < \operatorname{cf}(\kappa)$ and $\{\beta < \lambda \mid f_A(\beta) = f_B(\beta)\} \subseteq \bigcup_{\alpha \in T} Y_\alpha$. Hence $f_A \not\approx_{\mathscr{F}} f_B$, and the result is proved. ■

THEOREM 5. *If κ is a strong limit cardinal, then the following are equivalent.*

(i) *If $[S: G] < 2^\kappa$ then there is a subset Δ of κ such that $|\Delta| < \operatorname{cf}(\kappa)$ and $S_{(\Delta)} \leq G$.*

(ii) *If $\operatorname{cf}(\kappa) < \lambda < \kappa$ and $\mathscr{F}$ is a uniform ultrafilter on λ, then $\mathscr{F}$ is $\operatorname{cf}(\kappa)$-decomposable.*

PROOF. (i) $\Rightarrow$ (ii). Suppose that $\operatorname{cf}(\kappa) < \lambda < \kappa$ and that $\mathscr{F}$ is a uniform $\operatorname{cf}(\kappa)$-indecomposable ultrafilter on λ. Let $\mathscr{U} = \{\Gamma \subseteq \kappa \mid \Delta \subseteq \Gamma$ for some $\Delta \in \mathscr{F}\}$. If $f, g \in {}^\kappa\kappa$, then $f \approx_{\mathscr{U}} g$ iff $f \restriction \lambda \approx_{\mathscr{F}} g \restriction \lambda$. Thus, by Theorem 4 and Lemma 1, $[S: S_{\{\mathscr{U}\}}] = \kappa^\kappa/\mathscr{U} = \kappa^\lambda/\mathscr{F} = \kappa$. Since $\min\{|\Delta| \mid S_{(\Delta)} \leq S_{\{\mathscr{U}\}}\} = d(\mathscr{U}) = \lambda$, condition (i) fails.

(ii) $\Rightarrow$ (i). Suppose that $G \leq S$ witnesses the failure of condition (i). By Theorem 3, there exists a filter $\mathscr{G}$ on κ such that $S_{(\mathscr{G})} \leq G \leq S_{\{\mathscr{G}\}}$. Let $\lambda = d(\mathscr{G})$. Then $\operatorname{cf}(\kappa) \leq \lambda \leq \kappa$. Extend $\mathscr{G}$ to an ultrafilter $\mathscr{U}$ such that $d(\mathscr{U}) = \lambda$. By Corollary 1, $[S: S_{\{\mathscr{U}\}}] < 2^\kappa$.

Since κ is a strong limit cardinal, there exists a family of 2^κ almost disjoint subsets of κ, each of cardinality κ. By Observation 2 in [2], $\lambda \neq \kappa$. For the same reason, $\lambda \neq \operatorname{cf}(\kappa)$.

There exists $g \in S$ such that $\lambda \in g[\mathscr{U}]$. Since $S_{\{g[\mathscr{U}]\}} = gS_{\{\mathscr{U}\}}g^{-1}$, we have that $[S: S_{\{g[\mathscr{U}]\}}] = [S: S_{\{\mathscr{U}\}}] < 2^\kappa$. So we may suppose that $\lambda \in \mathscr{U}$. Then $\mathscr{F} = \{\Delta \in \mathscr{U} \mid \Delta \subseteq \lambda\}$ is a uniform ultrafilter on λ. Arguing as in the first half of the proof, $[S: S_{\{\mathscr{U}\}}] = \kappa^\kappa/\mathscr{U} = \kappa^\lambda/\mathscr{F}$. By Lemma 1, $\mathscr{F}$ is $\operatorname{cf}(\kappa)$-indecomposable, and so condition (ii) fails. ■

COROLLARY 2. *Assume GCH and that there is no inner model of ZFC which contains a measurable cardinal. Then for every infinite cardinal κ, $[S: G] < 2^\kappa$ if and only if there is a subset Δ of κ such that $|\Delta| < \operatorname{cf}(\kappa)$ and $S_{(\Delta)} \leq G$.*

PROOF. With these set-theoretic assumptions, Donder [3] has proved that if $\mathscr{F}$ is a uniform ultrafilter on an infinite cardinal λ, then $\mathscr{F}$ is α-decomposable for all regular cardinals $\omega \leq \alpha \leq \lambda$. So the result follows from Theorems 1 and 5. ■

By Theorem 5, the conclusion is false if a measurable cardinal exists.

§4. Consistency results for regular cardinals.

THEOREM 6. *Suppose that* $\lambda < \kappa < \mu$ *and that the following conditions are satisfied.*

(i) $\lambda^{<\lambda} = \lambda$ *and* $\lambda < \mathrm{cf}(\mu) < \kappa$.

(ii) μ *is a strong limit cardinal.*

(iii) *There exists a uniform* $\mathrm{cf}(\mu)$*-indecomposable filter* $\mathscr{D}$ *on* κ.

Let $\mathbf{P} = \{p \mid |p| < \lambda, p$ *is a partial function from* μ *to* $2\}$. *Then* $\mathbf{P}$ *preserves cardinals and cofinalities, and if* H *is* $\mathbf{P}$*-generic over* V *then the following are true in* $V[H]$.

(a) $2^\lambda = \mu$ *and* $2^\kappa = 2^\mu$.

(b) $|\{g[\mathscr{D}] \mid g \in \mathrm{Sym}(\kappa)\}| \le \mu$, *where* $\mathscr{D}$ *is identified with the filter which it generates in* $V[H]$.

PROOF. We only prove (b). The other assertions follow easily from Kunen [4]. If σ is a $\mathbf{P}$-name, then σ_H denotes the corresponding element of $V[H]$. For each $\gamma < \mu$, let $\mathbf{P} \restriction \gamma = \{p \in \mathbf{P} \mid \mathrm{dom}\, p \subseteq \gamma\}$.

Let $f \in V[H]$ be a permutation of κ and φ be a $\mathbf{P}$-name of f. There exists $p \in H$ such that $p \Vdash \varphi$ is a permutation of κ. For each $\alpha < \kappa$, let

$$F(\alpha) = \{\beta \in \kappa \mid (\exists q \le p)(q \Vdash \varphi(\alpha) = \beta)\}.$$

Since $\mathbf{P}$ satisfies the λ^+-c.c.c., $|F(\alpha)| \le \lambda$. For each $\beta \in F(\alpha)$, let $A_{\alpha,\beta}$ be a maximal antichain of $\{q \in \mathbf{P} \mid q \le p, q \Vdash \varphi(\alpha) = \beta\}$. Then $|A_{\alpha,\beta}| \le \lambda$, and so there exists $\gamma_\alpha < \mu$ such that $A_{\alpha,\beta} \subseteq \mathbf{P} \restriction \gamma_\alpha$ for all $\beta \in F(\alpha)$.

Let $\mu = \bigcup_{i<\mathrm{cf}(\mu)} \theta_i$, where $\theta_i < \theta_j < \kappa$ for $i < j < \mathrm{cf}(\mu)$. Define $\pi\colon \kappa \to \mathrm{cf}(\mu)$ by $\pi(\alpha) = \min\{i \mid \gamma_\alpha \in \theta_i\}$. Since $\mathscr{D}$ is $\mathrm{cf}(\mu)$-indecomposable, there exists $i < \mathrm{cf}(\mu)$ such that $\Delta = \pi^{-1}[\theta_i] \in \mathscr{D}$. Thus $A_{\alpha,\beta} \subseteq \mathbf{P} \restriction \theta_i$ for all $\alpha \in \Delta$ and $\beta \in F(\alpha)$. Let

$$\tau = \bigcup\{\{\langle\alpha,\beta\rangle\} \times A_{\alpha,\beta} \mid \alpha \in \Delta, \beta \in F(\alpha)\}.$$

Then $p \Vdash \tau = \varphi \restriction \Delta$.

Suppose that $g \in V[H]$ is a permutation of κ such that $g \restriction \Delta = f \restriction \Delta$. Then $g^{-1}f \in S_{(\mathscr{D})} \le S_{\{\mathscr{D}\}}$ and so $g[\mathscr{D}] = f[\mathscr{D}]$. Thus $f[\mathscr{D}]$ is determined by $f \restriction \Delta = \tau_H$. For each $i < \mathrm{cf}(\mu)$, there are less than μ possibilities for the $\mathbf{P} \restriction \theta_i$-name τ. Hence there are at most μ possibilities for τ, and (b) is proved. ■

COROLLARY 3. *Suppose that* κ *is a measurable cardinal. Then there exists a forcing extension in which* κ *is regular and*

($*$) *there exists a subgroup* $G \le S = \mathrm{Sym}(\kappa)$ *such that* $[S\colon G] < 2^\kappa$ *and* $S_{(\Delta)} \not\le G$ *for all* $\Delta \subseteq \kappa$ *of cardinality less than* κ. ■

COROLLARY 4. *Suppose that there exists a supercompact cardinal. Then there exists a forcing extension in which* ($*$) *holds for* $\kappa = \aleph_{\omega+1}$.

PROOF. By [1], there exists a forcing extension $V' \supseteq V$ in which

(i) GCH holds, and

(ii) there is a uniform ultrafilter on $\aleph_{\omega+1}$ which is λ-indecomposable for all $\aleph_0 < \lambda < \aleph_\omega$.

The result now follows from Theorem 6. ■

Question. Is ($*$) consistent for $\kappa = \aleph_1$?

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