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More on SOP_1 and SOP_2

Saharon Shelah[a,b,*], Alexander Usvyatsov[c,d]

[a] Mathematics Department, Hebrew University of Jerusalem, 91904 Givat Ram, Israel
[b] Department of Mathematics, Hill Center-Busch Campus Rutgers, The State University of New Jersey, 110 Frelinghuysen Rd Piscataway, NJ 08854-8019, USA
[c] UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555, USA
[d] Universidade de Lisboa, Centro de Matemática e Aplicações Fundamentais, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal

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Abstract

This paper continues the work in [S. Shelah, Towards classifying unstable theories, Annals of Pure and Applied Logic 80 (1996) 229–255] and [M. Džamonja, S. Shelah, On $\triangleleft^*$-maximality, Annals of Pure and Applied Logic 125 (2004) 119–158]. We present a rank function for $NSOP_1$ theories and give an example of a theory which is $NSOP_1$ but not simple. We also investigate the connection between maximality in the ordering $\triangleleft^*$ among complete first order theories and the (N)SOP_2 property. We prove that $\triangleleft^*$-maximality implies SOP_2 and obtain certain results in the other direction. The paper provides a step toward the classification of unstable theories without the strict order property.

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1. Introduction and preliminaries

We continue the work started by Mirna Džamonja and the first author in [4] and [1]. The main goal of this project is to throw more light on first order theories with the tree property (that is, non-simple) and without the strict order property (more specifically, without the SOP_3, see Definition 1.1). We pursue and finalize certain directions started in [1] and answer several questions asked there, providing a more general and complete picture.

The reader may be familiar with a former version of this paper that has been available as a preprint on Shelah's archive (under the number "ShUs:E32") and on Usvyatsov's webpage, and is referred to in the most recent version of [1].

Some connections between the work of Džamonja and Shelah and this article have been already explained in the introduction of [1]. In particular, our results provide a generalization of the main theorem of section 1 in [1], expand

* Corresponding author at: Mathematics Department, Hebrew University of Jerusalem, 91904 Givat Ram, Israel.
E-mail addresses: shelah@rci.rutgers.edu, Shelah@math.huji.ac.il (S. Shelah).

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the results of section 2 there and answer certain questions which were left open in section 3. One of the answers leads to a complete proof of a theorem which had been the original motivation of section 3 of [1], Corollary 3.15 here (see also Discussion 3.12 here, Theorem 0.5 in [1] and the discussion preceding it). We give more details below.

Before describing the background and the results obtained in this paper, let us recall the definitions of SOP_n hierarchy, starting with the more classical concepts introduced in [4].

Let T be a complete first order theory, $\mathfrak{C}$ — the monster model of T (a κ^* — saturated model for κ^* big enough).

Definition 1.1. (1) Let $n \geq 3$. We say $\varphi(\bar{x}, \bar{y})$ (with $\lg(x) = \lg(y)$) exemplifies the strong order property of order n (SOP_n) in T if it defines on $\mathfrak{C}$ a directed graph with infinite indiscernible chains and no cycles of length n.

(2) We say $\varphi(\bar{x}, \bar{y})$ (with $\lg(x) = \lg(y)$) exemplifies the *strict order property* in T if it defines on $\mathfrak{C}$ a partial order with infinite indiscernible chains.

Fact 1.2. *For a theory T, strict order property $\Longrightarrow \mathrm{SOP}_{n+1} \Longrightarrow \mathrm{SOP}_n$ for all $n \geq 3$.*

Proof. The first implication is trivial, for the other one see [4], Claim (2.6). □

We also remind the reader of the following equivalent definition of SOP_3:

Fact 1.3. *T has SOP_3 if and only if there is an indiscernible sequence $\langle \bar{a}_i : i < \omega \rangle$ and formulae $\varphi(\bar{x}, \bar{y})$, $\psi(\bar{x}, \bar{y})$ such that*

(a) *$\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$ is contradictory,*

(b) *for some sequence $\langle \bar{b}_j : j < \omega \rangle$ we have*

$$i \leq j \Longrightarrow \models \varphi[\bar{b}_j, \bar{a}_i] \text{ and } i > j \Longrightarrow \models \psi[\bar{b}_j, \bar{a}_i],$$

(c) *for $i < j$, the set $\{\varphi(\bar{x}, \bar{a}_j), \psi(\bar{x}, \bar{a}_i)\}$ is contradictory.*

Proof. Easy, or see [4], Claim (2.20). □

Remark 1.4. Note that if in the previous definition $\psi = \neg\varphi$, we get the strict order property.

Now we recall the definitions of SOP_1, SOP_2 and related properties:

Definition 1.5. (1) T has SOP_2 if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies this property in $\mathfrak{C}$, and this means:

There are $\bar{a}_\eta \in \mathfrak{C}$ for $\eta \in {}^{\omega>}2$ such that

(a) For every $\eta \in {}^{\omega}2$, the set $\{\varphi(\bar{x}, \bar{a}_{\eta\restriction\ell}) : l < \omega\}$ is consistent.

(b) If $\eta, \nu \in {}^{\omega>}2$ are incomparable, $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$ is inconsistent.

(2) T has SOP_1 if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies this in $\mathfrak{C}$, which means:

There are $\bar{a}_\eta \in \mathfrak{C}$, for $\eta \in {}^{\omega>}2$ such that:

(a) for $\rho \in {}^{\omega}2$ the set $\{\varphi(\bar{x}, \bar{a}_{\rho\restriction n}) : n < \omega\}$ is consistent.

(b) if $\nu \frown \langle 0 \rangle \trianglelefteq \eta \in {}^{\omega>}2$, <u>then</u> $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_{\nu\frown\langle 1 \rangle})\}$ is inconsistent.

(3) NSOP_2 and NSOP_1 are the negations of SOP_2 and SOP_1 respectively.

(4) T has SOP'_1 if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies this property in $\mathfrak{C}$, and this means:

there are $\langle \bar{a}_\eta : \eta \in {}^{\omega>}2 \rangle$ in $\mathfrak{C}_T$ such that

(a) $\{\varphi(\bar{x}, \bar{a}_{\eta\restriction n})^{\eta(n)} : n < \omega\}$ is consistent for every $\eta \in {}^{\omega}2$, where we use the notation

$$\varphi^l = \begin{cases} \varphi & \text{if } l = 1, \\ \neg\varphi & \text{if } l = 0 \end{cases}$$

for $l < 2$.

(b) If $\nu \frown \langle 0 \rangle \trianglelefteq \eta \in {}^{\omega>}2$, <u>then</u> $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$ is inconsistent.

(5) T has SOP''_2 if there is a formula $\varphi(\bar{x}, \bar{y})$ which exemplifies this property in $\mathfrak{C}$, and this means:

there are $n < \omega$ and a sequence

$$\left\langle \bar{a}_{\bar{\eta}} : \bar{\eta} = \langle \eta_0, \ldots \eta_{n-1} \rangle, \eta_0 \lhd \eta_1 \lhd \cdots \lhd \eta_{n-1} \in {}^{\lambda>}2 \text{ and } \lg(\eta_i) \text{ successor} \right\rangle$$

such that

(a) for each $\eta \in {}^{\lambda}2$, the set

$$\left\{ \begin{array}{c} \varphi(\bar{x}, \bar{a}_{\bar{\eta}}) : \bar{\eta} = \langle \eta \restriction (\alpha_0 + 1), \eta \restriction (\alpha_1 + 1), \ldots \eta \restriction (\alpha_{n-1} + 1) \rangle \\ \text{and } \alpha_0 < \alpha_1 < \ldots \alpha_{n-1} < \lambda \end{array} \right\}$$

is consistent

(b) for every large enough m, if h is a one-to-one function from ${}^{n\geq}m$ into ${}^{\lambda>}2$ preserving $\eta \lhd \nu$ and $\eta \perp \nu$ (incomparability) <u>then</u> $\{\varphi(\bar{x}, \bar{a}_{\bar{\nu}}) :$ for some $\eta \in {}^{n}m$ we have $\bar{\nu} = \langle h(\eta \restriction \ell) : \ell \leq n \rangle\}$ is inconsistent.

Fact 1.6. (1) *For a theory* T, $\mathrm{SOP}_3 \Longrightarrow \mathrm{SOP}_2 \Longrightarrow \mathrm{SOP}_1$
(2) T *has* SOP_1 *if and only if it has* SOP'_1.

Proof. See [1]. □

It is still not known whether the implications in 1.6(1) are strict, but for now we investigate each one of these order properties on its own.

In the second section we expand our knowledge on SOP_1. We present a rank function measuring type-definable "squares", i.e. pairs of types of the form $(p(\bar{x}), q(\bar{y}))$ and show the rank is finite for every such a pair if and only if T does not have SOP'_1 (if and only if T does not have SOP_1). In fact, if one calls a tree of parameters $\{\bar{a}_\eta : \eta \in {}^{\omega>}2\}$ showing that $\varphi(\bar{x}, \bar{y})$ exemplifies SOP'_1 in $\mathfrak{C}$ (as in the definition of SOP'_1) a $\varphi-\mathrm{SOP}'_1$ *tree*, the rank measures exactly the maximal depth of a tree like this that can be built in $\mathfrak{C}$. We also show a small application of the rank.

It is easy to see (see [1]) that if $\varphi(\bar{x}, \bar{y})$ exemplifies SOP_1 in $\mathfrak{C}$ then it also exemplifies the tree property, so T has $\mathrm{SOP}_1 \Longrightarrow T$ is not simple. We show that the implication is proper, i.e. find an example of a theory T which is not simple, but is NSOP_1. This theory which we call T^*_{feq}, was first defined in [3], and is used in [4] as an example of an NSOP_3 non-simple theory. Here we use a slightly different definition of the same theory, as given in [1].

Definition 1.7. (1) T_{feq} is the following theory in the language $\{Q, P, E, R, F\}$:

(a) Predicates P and Q are unary and disjoint, and $(\forall x)\,[P(x) \vee Q(x)]$,

(b) E is an equivalence relation on Q,

(c) R is a binary relation on $Q \times P$ such that

$$[x\, R\, z \;\&\; y\, R\, z \;\&\; x\, E\, y] \Longrightarrow x = y.$$

(so R picks for each $z \in Q$ (at most one) representative of any E-equivalence class).

(d) F is a (partial) binary function from $Q \times P$ to Q, which satisfies

$$F(x, z) \in Q \;\&\; (F(x, z))\; R\, z \;\&\; x\, E\, (F(x, z)).$$

(so for $x \in Q$ and $z \in P$, the function F picks the representative of the E-equivalence class of x which is in the relation R with z).

(2) T^*_{feq} is the model completion of T_{feq}.

If the reader thinks about the definition above, they will find out that T^*_{feq} is just the model completion of the theory of infinitely many (independent) parameterized equivalence relations. The reader can also compare between the definition of T^*_{feq} here and in [3]. As we have already mentioned, it was shown in [4] that this theory does not have SOP_3 (but is not simple). Here we prove an (a priori) stronger result: T^*_{feq} does not have SOP_1.

In the third section we deal with $\lhd^*_\lambda$-maximality (see the beginning of the section for definitions). For a theory T to be $\lhd^*_\lambda$-maximal means to be complicated. In a way, it means that it is hard to make its models λ-saturated.

The motivation for considering this property comes from Classification Theory and the search for "dividing lines". The authors believe that a "good" property of a theory T should have several characterizations of different types, both "internal" (something happens in the monster model of T, such as SOP_n) and "external" (how T compares to other theories, such as maximality in a certain order). Although all the approximations to the strict order property (including SOP_n and the strict order property itself) seem to be very natural syntactic internal definitions, no external property is known to characterize any of them. There are natural conjectures, though. The following question partly guides our current work:

Question 1.8. Does $\lhd^*_\lambda$-maximality characterize either SOP_3 or SOP_2, maybe both?

There are several indications that the answer should be positive. It had been already known before our work that $\triangleleft^*_\lambda$-maximality lies strictly "above" the tree property (non-simplicity): Džamonja and Shelah showed in [1] that $T^*_{\rm feq}$ (which is not simple) is not $\triangleleft^*_\lambda$-maximal. The question where exactly above non-simplicity this property lies is still open, but we narrow the possibilities down significantly. It follows from our results here that $\mathrm{SOP}_3 \Longrightarrow \triangleleft^*_\lambda$-maximality $\Longrightarrow \mathrm{SOP}_2$. We also obtain a local version of the reversed direction of the second implication.

Our analysis also provides an alternative proof of the fact that $T^*_{\rm feq}$ is not $\triangleleft^*_\lambda$-maximal, Theorem 1.17 in [1]: no NSOP_2 theory is $\triangleleft^*_\lambda$-maximal, and $T^*_{\rm feq}$ is NSOP_1, therefore NSOP_2. So by bringing the "internal" and the "external" dividing lines close together, we also give many examples of non-simple theories which are not $\triangleleft^*_\lambda$-maximal, $T^*_{\rm feq}$ being a particular case. See also Discussion 3.18.

Let us now give more details concerning some results in the paper and explain how exactly they fit in the general picture. In [4] it was stated that SOP_3 implies $\triangleleft^*_\lambda$-maximality, but the proof there is not complete: it is shown that every theory with SOP_3 is $\triangleleft^*_\lambda$-above $T^*_{\rm tr}$, the model completion of the theory of trees. The first theorem in section 3, Theorem 3.5, fills the missing part, showing explicitly that $T^*_{\rm tr}$ is $\triangleleft^*_\lambda$-maximal for every $\lambda > \aleph_0$. This also continues [2], chapter VI, where Keisler's order, a relative of $\triangleleft^*_\lambda$, is studied.

One of the reasons for giving an explicit proof for Theorem 3.5 here was to provide more tools for strengthening the result above to SOP_2 theories, i.e. showing that $\mathrm{SOP}_2 \Longrightarrow \triangleleft^*_\lambda$-maximality. A step in this direction is Theorem 3.11 where we show a "local" version: if a formula ϑ exemplifies SOP_2 in T, then the pair (T, ϑ) is $\triangleleft^*_\lambda$-above the pair $(T^*_{\rm tr}, y < x)$ for every regular $\lambda > |T|$ (again, see the beginning of section 3 for precise definitions). This result, although interesting on its own, is insufficient for "global" $\triangleleft^*_\lambda$-maximality of SOP_2 theories, as explained in Discussion 3.17. Nevertheless, combined with Theorem 3.5 and its proof, it gives more information on the behavior of SOP_2 theories and $\triangleleft^*_\lambda$-order altogether.

As for the other direction ($\triangleleft^*_\lambda$-maximality $\Longrightarrow SOP_2$), we provide a complete proof, based on several related results achieved by Džamonja and the first author, who showed in [1] that a property similar to $\triangleleft^*_\lambda$-maximality (which also follows from $\triangleleft^*_\lambda$-maximality for some λ under certain set theoretic conditions) implies SOP''_2. One of the questions left open in [1] is the connection between SOP''_2 and the SOP_n hierarchy. Of course, it would be natural to connect between SOP''_2 and SOP_2, and indeed we prove here that these two properties are equivalent for a theory T (not necessarily for a formula), Theorem 3.13.

So we can conclude $\mathrm{SOP}_3 \Longrightarrow \triangleleft^*_\lambda$-maximality $\Longrightarrow \mathrm{SOP}_2$, while very little is known at this point concerning implications in the other directions.

The following definitions and facts are going to be very useful.

In [1] two notions of "tree indiscernibility" were defined. We recall the definitions:

Definition 1.9. (1) Given an ordinal α and sequences $\bar\eta_l = \langle \eta^l_0, \eta^l_1, \dots, \eta^l_{n_l}\rangle$ for $l = 0, 1$ of members of ${}^{\alpha>}2$, we say that $\bar\eta_0 \approx_1 \bar\eta_1$ iff

(a) $n_0 = n_1$,

(b) the truth values of

$$\eta^l_{k_3} \trianglelefteq \eta^l_{k_1} \cap \eta^l_{k_2}, \quad \eta^l_{k_1} \cap \eta^l_{k_2} \triangleleft \eta^l_{k_3}, \quad (\eta^l_{k_1} \cap \eta^l_{k_2}) \frown \langle 0\rangle \trianglelefteq \eta^l_{k_3},$$

for $k_1, k_2, k_3 \le n_0$, do not depend on l.

(2) We say that the sequence $\langle \bar a_\eta : \eta \in {}^{\alpha>}2\rangle$ of $\mathfrak{C}$ (for an ordinal α) are *1-fully binary tree indiscernible (1-fbti)* iff whenever $\bar\eta_0 \approx_1 \bar\eta_1$ are sequences of elements of ${}^{\alpha>}2$, <u>then</u>

$$\bar a_{\bar\eta_0} =: \bar a_{\eta^0_0} \frown \cdots \frown \bar a_{\eta^0_{n_0}}$$

and the similarly defined $\bar a_{\bar\eta_1}$, realize the same type in $\mathfrak{C}$.

(3) We replace 1 by 2 in the above definition iff $(\eta^l_{k_1} \cap \eta^l_{k_2}) \frown \langle 0\rangle \trianglelefteq \eta^l_{k_3}$ is omitted from clause (b) above.

We will need the following fact proved in [1], (2.11):

Fact 1.10. *If $t \in \{1, 2\}$ and $\langle \bar b_\eta : \eta \in {}^{\omega>}2\rangle$ are given, and $\delta \ge \omega$, <u>then</u> we can find $\langle \bar a_\eta : \eta \in {}^{\delta>}2\rangle$ such that*

(a) *$\langle \bar a_\eta : \eta \in {}^{\delta>}2\rangle$ is t-fbti,*

(b) *if $\bar\eta = \langle \eta_m : m < n\rangle$, where each $\eta_m \in {}^{\delta>}2$ is given, and Δ is a finite set of formulae of T, <u>then</u> we can find $\nu_m \in {}^{\omega>}2\, (m < n)$ such that with $\bar\nu =: \langle \nu_m : m < n\rangle$, we have $\bar\nu \approx_t \bar\eta$ and the sequences $\bar a_{\bar\eta}$ and $\bar b_{\bar\nu}$, realize the same Δ-types.*

Convention 1.11. We work with a complete first order theory T, let $\mathfrak{C}$ be its "monster" model (saturated in some very big κ^*). Let $\mathcal{L} = \mathcal{L}(T)$ (the language of T). Every formula we mention is an $\mathcal{L}$-formula, maybe with parameters from $\mathfrak{C}$.

2. More on SOP_1

SOP_1 was introduced by Džamonja and Shelah as an intermediate property between simplicity and SOP_3. A natural question is: does the class $NSOP_1$ coincide with either simple theories or $NSOP_3$? Here we give a negative answer to the first question above. The answer for the second one is still not known.

Theorem 2.1. *T^*_{feq} does not have* SOP_1.

Proof. Suppose there exists $\varphi(\bar{x}, \bar{y})$ with $\ell g(\bar{x}) = n$, $\ell g(\bar{y}) = m$, and $\langle \bar{a}_\eta : \eta \in {}^{\omega>} 2\rangle$ in ${}^m\mathfrak{C}$ which exemplify SOP_1 in $\mathfrak{C}$ ($\mathfrak{C}$ is the monster model of T^*_{feq}). Without loss of generality, (by Fact 1.10) $\langle \bar{a}_\eta : \eta \in {}^{\omega>}2\rangle$ if 1-full tree indiscernible. Also, by elimination of quantifiers, we may assume that $\varphi(\bar{x}, \bar{y})$ is quantifier free. As the only function symbol in the language is F and $F^{\mathfrak{C}}$ has the property $F^{\mathfrak{C}}(F^{\mathfrak{C}}(x, z), y) = F^{\mathfrak{C}}(x, y)$ for all z, we will also assume wlog that $\bar{x}$ and $\bar{y}$ in $\varphi(\bar{x}, \bar{y})$ are closed under F and $\varphi(\bar{x}, \bar{y})$ gives the full diagram of $\bar{x} \frown \bar{y}$. We shall regard $\bar{x}$ as $\langle x^0, \ldots, x^{n-1}\rangle$, $\bar{y}$ as $\langle y^0, \ldots, y^{m-1}\rangle$, $\bar{a}_\eta$ as $\langle a^0_\eta, \ldots, a^{m-1}_\eta\rangle$.

By the definition of SOP_1, there exist $\bar{e} = \langle e^0, \ldots, e^{n-1}\rangle$, $\bar{d} = \langle d^0, \ldots, d^{n-1}\rangle$ in ${}^n\mathfrak{C}$ s.t.

$$\mathfrak{C} \models \varphi(\bar{e}, \bar{a}_{\langle\,\rangle}) \wedge \varphi(\bar{e}, \bar{a}_{\langle 0\rangle}) \wedge \varphi(\bar{e}, \bar{a}_{\langle 00\rangle})$$

and

$$\mathfrak{C} \models \varphi(\bar{d}, \bar{a}_{\langle\,\rangle}) \wedge \varphi(\bar{d}, \bar{a}_{\langle 1\rangle}).$$

Denote $\eta = \langle 00\rangle$. Let $B = \mathfrak{C} \restriction \bar{a}_\eta \frown \bar{a}_{\langle 1\rangle}$. By our assumptions, there exists a model N_0 whose universe is $\bar{x} \frown \bar{a}_\eta$, extending $\mathfrak{C} \restriction \bar{a}_\eta$, whose basic diagram is $\varphi(\bar{x}, \bar{a}_\eta)$. Similarly, there exists a model N_1 with universe $\bar{x} \frown \bar{a}_{\langle 1\rangle})$ and basic diagram $\varphi(\bar{x}, \bar{a}_{\langle 1\rangle})$. We shall amalgamate B, N_0 and N_1 into a model of T_{feq}, N. This will immediately give a contradiction: first, extend N to $N^* \models T^*_{\mathrm{feq}}$, then amalgamate N^* and $\mathfrak{C}$ over B into some $\mathfrak{C}^+ \models T^*_{\mathrm{feq}}$. By model completeness of T^*_{feq}, $\mathfrak{C} \prec \mathfrak{C}^+$, but $\mathfrak{C}^+ \models \exists\bar{x}(\varphi(\bar{x}, \bar{a}_\eta) \wedge \varphi(\bar{x}, \bar{a}_{\langle 1\rangle}))$, which is a contradiction to the definition of SOP_1.

It is left, therefore, to show that we can define on $|N_0| \cup |N_1|$ a structure which will be a model of T_{feq}, extending B.

We define N as follows:

$$|N| = |N_1| \cup |N_2|, \quad P^N = P^{N_1} \cup P^{N_2}, \qquad Q^N = Q^{N_1} \cup Q^{N_2}.$$

Note that the diagram of $\bar{x}$ in N_0 is the same as the diagram of $\bar{x}$ in N_1 (both implied by $\varphi(\bar{x}, \bar{y})$, and the diagrams of $\bar{a}_\eta, \bar{a}_{\langle 1\rangle}$ in N_i are the same as in $\mathfrak{C}$, hence the same as in B. Therefore, P^N and Q^N are well defined and give a partition of $|N|$. Also, so far N extends B (as a structure).

Considering E and R, we define

$$R^N = R^{N_1} \cup R^{N_2} \cup R^B$$
$$E^N = E^{N_1} \cup E^{N_2} \cup E^B.$$

Once we have proven the following claims, we will be able to define F^N in a natural way, and in fact will be done.

Claim 2.1.1. *E^N is an equivalence relation on Q^N, extending E^B.*

Claim 2.1.2. *R^N is a two-place relation on N, $R^N \subseteq P^N \times Q^N$, satisfying:*
for every $y \in P^N$ and every equivalence class C of E^N, there exists a unique $z \in C$ such that $(y, z) \in R^N$.

Proof of 2.1.1. The only non-obvious thing is transitivity. We check two main cases, all the rest are either similar or trivial.

(1) Assume $x^i E^N a^j_\eta$, $x^i E^N a^k_{\langle 1\rangle}$ for some i, j, k. We want to show $a^j_\eta E^N a^k_{\langle 1\rangle}$. It is enough to see $a^j_\eta E^{\mathfrak{C}} a^k_{\langle 1\rangle}$. We will write E instead of $E^{\mathfrak{C}}$.

$N \models x^i E a^j_\eta \Rightarrow N_0 \models x^i E a^j_\eta \Rightarrow \varphi(\bar{x}, \bar{y}) \vdash x^i E y^j$. Similarly, $\varphi(\bar{x}, \bar{y}) \vdash x^i E y^k$, and we get (by the choice of $\bar{e}, \bar{d} \in {}^n\mathfrak{C}$) $e^i E a^j_\eta,\ e^i E a^j_{\langle\rangle},\ e^i E a^k_\eta,\ e^i E a^k_{\langle\rangle}, d^i E a^j_{\langle 1\rangle},\ d^i E a^j_{\langle\rangle}\ , d^i E a^k_{\langle 1\rangle},\ d^i E a^k_{\langle\rangle}$. Now it is easy to see that all the above elements are E-equivalent in $\mathfrak{C}$, in particular a^j_η and $a^k_{\langle 1\rangle}$, as required.

(2) Assume $x^i E^N a^n_\eta,\ a^k_{\langle 1\rangle} E^N a^n_\eta$, and we show $x^i E^N a^k_{\langle 1\rangle}$, i.e. $\varphi(\bar{x}, \bar{y}) \vdash x^i E y^k$. As $\varphi(\bar{d}, \bar{a}_{\langle 1\rangle})$ holds in $\mathfrak{C}$ and $\varphi(\bar{x}, \bar{y})$ gives a full diagram, it will be enough to see $d^i E a^k_{\langle 1\rangle}$.

We know that $\varphi(\bar{x}, \bar{y}) \vdash x^i E y^j$ therefore $e^i E a^j_\eta,\ e^i E a^j_{\langle\rangle},\ d^i E a^j_{\langle 1\rangle},\ d^i E a^j_{\langle\rangle}$. In particular, $d^i E a^j_\eta$, but, by our assumption, $a^j_\eta E a^k_{\langle 1\rangle}$, so we are done. $\Box_1$

Proof of 2.1.2. Like in the previous lemma, the only non-trivial thing to prove is the last part, and we will deal with two main cases.

(1) $N \models (a^i_\eta R\, a^j_{\langle 1\rangle}) \wedge (a^i_\eta R x^k) \wedge (x^k E a^j_{\langle 1\rangle})$. We aim to show $N \models (x^k = a^j_{\langle 1\rangle})$. We know:

$(*)_1$ $\mathfrak{C} \models a^i_\eta R\, a^j_{\langle 1\rangle}$

$(*)_2$ $N_0 \models a^i_\eta R x^k$, therefore $\varphi(\bar{x}, \bar{y}) \vdash y^i R x^k$

$(*)_3$ $N_1 \models x^k E a^j_{\langle 1\rangle}$, therefore $\varphi(\bar{x}, \bar{y}) \vdash x^k E y^j$.

So we can conclude:

$(*)_2 \Rightarrow a^i_{\langle\rangle} R e^k,\quad a^i_{\langle\rangle} R d^k$

$(*)_3 \Rightarrow e^k E a^j_{\langle\rangle},\quad d^k E a^j_{\langle\rangle} \Rightarrow e^k E d^k$.

As the above two relations hold in $\mathfrak{C}$, which is a model of T_{feq}, we get $\mathfrak{C} \models e^k = d^k$. Denote $e^* = e^k = d^k$.

$(*)_1 \Rightarrow a^i_\eta R\, a^j_{\langle 1\rangle}$

$(*)_2 \Rightarrow a^i_\eta R e^*$

$(*)_1 \Rightarrow e^* E a^j_{\langle 1\rangle}$.

Together (once again, $\mathfrak{C} \models T_{\text{feq}}$) we get $e^* = a^j_{\langle 1\rangle}$, therefore $\varphi(\bar{x}, \bar{y}) \vdash x^k = y^j$, so $N_1 \models x^k = a^j_{\langle 1\rangle}$, and we are done.

(2) $N \models (x^i R\, a^j_{\langle 1\rangle}) \wedge (x^i R\, a^k_\eta) \wedge (a^k_\eta E a^j_{\langle 1\rangle})$ and we aim to show $N \models (a^k_\eta = a^j_{\langle 1\rangle})$.

We know:

$(*)_1$ $N_1 \models x^i R\, a^j_{\langle 1\rangle}$, so $\varphi(\bar{x}, \bar{y}) \vdash x^i R y^j$

$(*)_2$ $N_0 \models x^i R\, a^k_\eta$, so $\varphi(\bar{x}, \bar{y}) \vdash x^i R y^k$

$(*)_3$ $\mathfrak{C} \models a^k_\eta E a^j_{\langle 1\rangle}$.

Note that by indiscernibility of $\langle \bar{a}_r : r \in {}^{w>}2\rangle$ and $(*)_3$ we get $a^k_{\langle 0\rangle} E a^j_{\langle 1\rangle}$, therefore $a^k_{\langle 0\rangle} E a^k_\eta$. Now, by $(*)_2$, $e^i R\, a^k_\eta\ \ \&\ \ e^i R\, a^k_{\langle 0\rangle}$. Therefore, by $\mathfrak{C} \models T_{\text{feq}}$, $a^k_{\langle 0\rangle} = a^k_\eta$. Now by indiscernibility

$$a^k_{\langle 0\rangle} = a^k_{\langle\rangle},\quad a^k_{\langle 1\rangle} = a^k_{\langle\rangle}.$$

So we get that all of the above are equal (and in fact $a^k_{r_1} = a^k_{r_2}$ for all $r_1, r_2 \in {}^{w>}2$).

Now:

$(*)_1 \Rightarrow d^i R\, a^j_{\langle 1\rangle}$

$(*)_2 \Rightarrow d^i R\, a^k_{\langle 1\rangle} \Rightarrow d^i R\, a^k_\eta$ (as $a^k_{\langle 1\rangle} = a^k_\eta$)

$(*)_3 \Rightarrow a^k_\eta E a^j_{\langle 1\rangle}$.

By $\mathfrak{C} \models T_{\text{feq}}$, we conclude $a^k_\eta = a^k_{\langle 1\rangle}$, which finishes the proof of the lemma, and therefore the proof of the theorem. $\Box_2$

$\Box$

Our next goal is to show that there is a rank function closely related to being (N)SOP$_1$. Let $\varphi(\bar{x}, \bar{y})$ be a formula.

Definition 2.2. Given (partial) types $p(\bar{x}), q(\bar{y})$. By induction on $n < \omega$ we define when

$$\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p(\bar{x}), q(\bar{y})) \geq n :$$

If $\underline{n = 0}$, this happens if both $p(\bar{x}), q(\bar{y})$ are consistent.
For $\underline{n+1}$, the rank is $\geq n+1$ if for some $\bar{c} \models q(\bar{y})$, both

$$\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p(\bar{x}) \cup \{\varphi(\bar{x},\bar{c})\}, q(\bar{y})) \geq n$$

and

$$\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p(\bar{x}), q(\bar{y}) \cup \{\neg(\exists \bar{x})(\varphi(\bar{x},\bar{y}) \wedge \varphi(\bar{x},\bar{c}))\}) \geq n.$$

We say $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p(\bar{x}), q(\bar{y})) = \infty$ iff $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p(\bar{x}), q(\bar{y})) \geq n$ for all n.
We say the rank is -1 if it is not bigger than or equal to 0.

Remark 2.3. (1) (Definability) Given formulae θ_1, θ_2 and $n < \omega$, the statement $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(\theta_1(\bar{x};\bar{a}), \theta_2(\bar{x};\bar{b})) \geq n$ is a first order formula with parameters $\bar{a}, \bar{b}$.
(2) (Finite Character) If $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p(\bar{x}), q(\bar{y})) = n$, then for some finite $p_0(\bar{x}) \subseteq p(\bar{x})$ and $q_0(\bar{y}) \subseteq q(\bar{y})$ we have $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p_0(\bar{x}), q_0(\bar{y})) = n$.
(3) (Monotonicity) If $p' \vdash p''$ and $q' \vdash q''$, then $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p', q') \leq \mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p'', q'')$.
(4) We can continue to define when $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p(\bar{x}), q(\bar{y})) \geq \alpha$ for any ordinal α, but by the compactness theorem, part (1) (Definability) and part (2) (the Finite Character) it follows that $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p, q) \geq \alpha$ for some $\alpha \geq \omega$ iff $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p, q) \geq \omega$ iff $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p, q) = \infty$.
(5) If $p' \equiv p''$, and $q' \equiv q''$, then $\mathrm{rk}^1_{\varphi}(p', q') = \mathrm{rk}^1_{\varphi}(p'', q'')$.

We aim to show that $rk^1_{\varphi}(p(\bar{x}), q(\bar{y}))$is finite for every $p(\bar{x}), q(\bar{y})$(or, equivalently, $rk^1_{\varphi}(\bar{x} = \bar{x}, \bar{y} = \bar{y})$ is finite) if and only if $\varphi(\bar{x},\bar{y})$ does not exemplify SOP'_1 in T. For this purpose we shall need another definition and several easy claims.

Definition 2.4. Given (partial) types $p(\bar{x})$and $q(\bar{y})$, we say that $\{\bar{a}_\eta : \eta \in {}^{n\geq}2\}$ is a φ-SOP'_1 *tree* for $p(\bar{x})$ and $q(\bar{y})$ (of depth n) if

(a) $p(\bar{x}) \cup \{\varphi^{\eta(i)}(\bar{x}, \bar{a}_{\eta\restriction i}) : i < n\}$ is consistent for every $\eta \in^n 2$.
(b) $\bar{a}_\eta \models q(\bar{y})$ for all $\eta \in {}^{n\geq}2$.
(c) If η, ν are in ${}^{n\geq}2$ satisfying $\eta \frown \langle 0\rangle \trianglelefteq \nu$, then the set $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$ is inconsistent.

Proposition 2.5. *Suppose $\{\bar{a}_\eta : \eta \in {}^{n\geq}2\}$ is a φ-SOP'_1 tree for $p(\bar{x})$and $q(\bar{y})$ of depth n, and denote $A^0 = \{\bar{a}_\eta : \langle 0\rangle \trianglelefteq \eta\}$, $A^1 = \{\bar{a}_\eta : \langle 1\rangle \trianglelefteq \eta\}$. Then*

(1) *A^1 is a φ-SOP'_1 tree for $p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_{\langle\rangle})\}$ and $q(\bar{y})$*
(2) *A^0 is a φ-SOP'_1 tree for $p(\bar{x})$and $q(\bar{y}) \cup \{\neg(\exists \bar{x})(\varphi(\bar{x},\bar{y}) \wedge \varphi(\bar{x}, \bar{a}_{\langle\rangle}))\}$.*

Proof. The clauses (a) and (c) of the definition easily hold both for A^1 *and* A^0, so we should only check (b), which is also obvious for A^1. Therefore, we are left to show that for every $\eta \in A^0$, $\bar{a}_\eta \models \neg(\exists \bar{x})(\varphi(\bar{x},\bar{y}) \wedge \varphi(\bar{x}, \bar{a}_{\langle\rangle}))$, and this is clear by clause (c) of the definition ($\{\bar{a}_\eta : \eta \in {}^{n\geq}2\}$ is a φ-SOP'_1 tree, and $\langle\rangle \frown 0 \trianglelefteq \eta$). □

Now we show the connection between the rank and SOP'_1trees.

Proposition 2.6. *$rk^1_{\varphi}(p(\bar{x}), q(\bar{y})) \geq n \iff$ there exists a φ-SOP'_1 tree for $p(\bar{x})$ and $q(\bar{y})$ of depth n.*

Proof. Both directions are proved by induction on n. The case $n = 0$ is obvious. For $n = m+1$, the right-to-left direction follows immediately by the induction hypothesis and 2.5. So we will elaborate more only about the other direction, although it is also straightforward.

Suppose $n = m+1$ and $rk^1_{\varphi}(p(\bar{x}), q(\bar{y})) \geq n$. By the definition of the rank and the induction hypothesis, for some $\bar{c} \models q(\bar{y})$, there are

(1) a φ-SOP_1' tree $A^1 = \{\bar{a}_\eta{}^1 : \eta \in {}^{m\geq}2\}$ for $p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{c})\}$ and $q(\bar{y})$
(2) a φ-SOP_1' tree $A^0 = \{\bar{a}_\eta{}^0 : \eta \in {}^{m\geq}2\}$ for $p(\bar{x})$ and $q(\bar{y}) \cup \{\neg(\exists\bar{x})(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{c}))\}$

(both of depth m). We define a tree $\{\bar{a}_\eta : \eta \in {}^{n\geq}2\}$ by

$$\bar{a}_{\langle\rangle} = \bar{c}$$
$$\bar{a}_{\langle\ell\rangle\frown\eta} = \bar{a}_\eta{}^\ell \text{ for } \ell \in \{0, 1\}$$

which is as required, i.e. a φ-SOP_1' tree for $p(\bar{x})$ and $q(\bar{y})$, since:

(a) of the definition obviously holds by (1) above.
(b) holds as $\bar{c} \models q(\bar{y})$.
(c) obviously holds by (2) above. □

The following remark is obvious:

Remark 2.7. $\varphi(\bar{x}, \bar{y})$ exemplifies SOP_1' in $T \iff$ there exists a φ-SOP_1' tree for $\bar{x} = \bar{x}$ *and* $\bar{y} = \bar{y}$ of any depth.

So we can conclude the following

Theorem 2.8. *A formula $\varphi(\bar{x}, \bar{y})$ does not exemplify SOP_1' in $T \iff rk^1_\varphi(\bar{x} = \bar{x}, \bar{y} = \bar{y}) < \omega \iff rk^1_\varphi(p(\bar{x}), q(\bar{y})) < \omega$ for every two (partial) types $p(\bar{x})$ and $q(\bar{y})$. Moreover, $rk^1_\varphi(\bar{x} = \bar{x}, \bar{y} = \bar{y})$ is exactly the maximal depth of a φ-SOP_1' tree that can be built in $\mathfrak{C}$.*

Corollary 2.9. *T does not have $SOP_1 \iff T$ does not have $SOP_1' \iff rk^1_\varphi(\bar{x} = \bar{x}, \bar{y} = \bar{y})$ is finite for every formula $\varphi(\bar{x}, \bar{y})$.*

Now we show an application of the rank.

Theorem 2.10. *Suppose that T satisfies $NSOP_1$. Assume that*

(a) *$M_1 \prec M_2 \prec \mathfrak{C}$.*
(b) *p is a (not necessarily complete) type over M_2, containing the formula $\varphi(\bar{x}, \bar{b}^*)$ for some $\bar{b}^* \in M^2 \setminus M^1$.*

Then for some finite $q' \subseteq tp(\bar{b}^*/M_1)$ at least one of the following holds:

(i) *If $\bar{b} \in M_1$ realizes $q'(\bar{y})$ then $\varphi(\bar{x}, \bar{b}) \notin p$, or*
(ii) *If $\bar{b} \in M_1$ realizes $q'(\bar{y})$ then $\{\varphi(\bar{x}, \bar{b}), \varphi(\bar{x}, \bar{b}^*)\}$ is consistent.*

In fact, all we need to assume for this Claim is that $\varphi(\bar{x}, \bar{y})$ does not exemplify SOP_1' in $\mathfrak{C}$.

Proof. Denote $q = tp(b^*/M_1)$. As T is $NSOP_1$, we have that $rk^1_{\varphi(\bar{x},\bar{y})}(p \upharpoonright M_1, q) = n^* < \omega$ (certainly $n^* \geq 0$). By the finite character of the rank, we have that for some finite $p_0 \subseteq p \upharpoonright M_1$ and $q_0 \subseteq q$,

$$rk^1_{\varphi(\bar{x},\bar{y})}(p_0, q_0) = n^*.$$

Hence for no $\bar{c} \models q_0(\bar{y})$ do we have that both $rk^1_{\varphi(\bar{x},\bar{y})}(p_0 \cup \{\varphi(\bar{x}, \bar{c})\}, q_0) \geq n^*$ and $rk^1_{\varphi(\bar{x},\bar{y})}(p_0, q_0 \cup \{(\neg\exists\bar{x})[\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{c})]\}) \geq n^*$. In particular, this holds for $\bar{c} = \bar{b}^*$ (remember that $\bar{b}^* \models q$ and therefore certainly $\bar{b}^* \models q_0$). So

⊗ **2.10.1.**

$$\text{If } rk^1_{\varphi(\bar{x},\bar{y})}(p_0 \cup \{\varphi(\bar{x}, \bar{b}^*)\}, q_0) \geq n^*, \text{ then } rk^1_{\varphi(\bar{x},\bar{y})}(p_0, q_0 \cup \{(\neg\exists\bar{x})[\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{b}^*)]\}) < n^*.$$

By Remark 2.3(1), there is a finite $q' \subseteq q$ such that

⊗ **2.10.2.**

$$\bar{b} \text{ realizes } q' \implies rk^1_{\varphi(\bar{x},\bar{y})}(p_0 \cup \{\varphi(\bar{x}, \bar{b})\}, q_0) = rk^1_{\varphi(\bar{x},\bar{y})}(p_0 \cup \{\varphi(\bar{x}, \bar{b}^*)\}, q_0).$$

We aim to show that q' is as required.

Case 1. $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p_0 \cup \{\varphi(\bar{x},\bar{b^*})\}, q_0) = n < n^*$.
We note that possibility (i) holds.
Namely, suppose $\bar{b}$ realizes q', then $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p_0 \cup \{\varphi(\bar{x},\bar{b})\}, q_0) = n < n^*$, so if $\varphi(\bar{x},\bar{b}) \in p$, we obtain a contradiction with monotonicity of the rank.

Case 2. $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p_0 \cup \{\varphi(\bar{x},\bar{b^*})\}, q_0) = n^*$.
We shall show that (ii) holds.
Suppose otherwise, so let $\bar{b} \in M_1$ realize q' and $\{\varphi(\bar{x},\bar{b}), \varphi(\bar{x},\bar{b^*})\}$ is contradictory. By 2.10.2,

$$\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p_0 \cup \{\varphi(\bar{x},\bar{b})\}, q_0) = n^*$$

and by 2.10.1,

$$\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p_0, q_0 \cup \{(\neg\exists\bar{x})(\varphi(\bar{x},\bar{y}) \wedge \varphi(\bar{x},\bar{b})\}) < n^*.$$

We have that $(\neg\exists\bar{x})[\varphi(\bar{x},\bar{y}) \wedge \varphi(\bar{x},\bar{b}] \in q$, hence $q_0 \cup \{(\neg\exists\bar{x})[\varphi(\bar{x},\bar{y}) \wedge \varphi(\bar{x},\bar{b})]\} \subseteq q$, in contradiction with monotonicity and $\mathrm{rk}^1_{\varphi(\bar{x},\bar{y})}(p \restriction M_1, q) = n^*$. $\square$

3. More on SOP_2, SOP_3 and $\triangleleft^*_\lambda$-order

We try to find a connection between the syntactic properties SOP_2, SOP_3 and the semantic property of being $\triangleleft^*_\lambda$-maximal. Our guess is that $\triangleleft^*_\lambda$-maximality should be equivalent to one of the above order properties (maybe both), but all we prove here is $\mathrm{SOP}_3 \Longrightarrow \triangleleft^*_\lambda$-maximality $\Longrightarrow \mathrm{SOP}_2$. We also give a weaker "local" result in the other direction.

First we generalize the definitions from [1], of $\triangleleft^*_\lambda$-maximality, making them local as well as global.

Definition 3.1. (1) For given (complete first order theories) T_1, T_2 and cardinals $\lambda \geq \mu > \kappa$, $\mu \geq \theta > |T_1|+|T_2|+\aleph_0$
(a) $T_1 \triangleleft^*_{<\lambda,<\mu,\kappa,<\theta} T_2$ means that there is a (complete first order theory) T^* and interpretations $\bar{\varphi}_1, \bar{\varphi}_2$ of T_1, T_2 in T^* respectively, $|T^*| < \theta$ such that:
– $\boxtimes^{<\lambda,<\mu,\kappa}_{T_*,\bar{\varphi}_1,\bar{\varphi}_2}$ if M is a κ-saturated model of T^* and $M_\ell = M^{[\bar{\varphi}^\ell]}$ for $\ell = 1, 2$ and M_2 is λ-saturated (model of T_2), then M_1 is μ-saturated.
(b) $(T_1, \vartheta_1(\bar{x},\bar{y})) \triangleleft^*_{<\lambda,<\mu,<\kappa} (T_1, \vartheta_2(\bar{x},\bar{y}))$ means that $\vartheta_\ell(\bar{x},\bar{y}) \in L(\tau_{T_\ell})$ and that there is a T^* and interpretations $\bar{\varphi}_1, \bar{\varphi}_2$ of T_1, T_2 in T^* respectively, $|T^*| < \mu$ such that $\boxtimes^{<\lambda,<\mu,\kappa}_{T^*,\vartheta_1,\vartheta_2,\bar{\varphi}^1,\bar{\varphi}^2}$ if M is a κ-saturated model of T^* and $M_\ell = M^{[\bar{\varphi}^\ell]}$ for $\ell = 1, 2$ and M_2 is $(\lambda, \vartheta_1(\bar{x},\bar{y}))$-saturated (see 3), then M_i is (μ, ϑ_2)-saturated.
(2) Instead of "$< \lambda^+$" we may write "λ", and instead of "$< \mu^+$" we may write μ, instead of $< \theta^+$ we may write θ. If we omit μ we mean $\mu = \lambda$, and if we write $\kappa = 0$ then "κ-saturated" becomes the empty demand, if we omit θ we mean $|T_1| + |T_2| + \aleph_0$ and if we omit κ and θ then we mean that $\mu = \lambda$, $\theta = |T_1| + |T_2| + \aleph_0$.
(3) We say M is (λ, Δ)-saturated when: if $p \subseteq \{\vartheta(\bar{x};\bar{a}) : \vartheta(\bar{x};\bar{y}) \in \Delta, \bar{a} \in {}^{\ell g(\bar{y})}M\}$ is finitely satisfiable of cardinality $< \lambda$ then p is realized in M. If $\Delta = \{\vartheta(\bar{x},\bar{y})\}$ we may write $\vartheta(\bar{x},\bar{y})$ instead of Δ.
(4) If T_1, T_2 are not necessarily complete, then above T^* is not necessarily complete and we demand: if $M_1 \models T_1$, $M_2 \models T_2$ then there is $M \models T^*$ such that $M^{[\bar{\varphi}_\ell]} \models Th(M_\ell)$ for $\ell = 1, 2$.
(5) We say T is $\triangleleft^*_{\lambda,\kappa}$-maximal if $|T'| < \lambda \Rightarrow T' \triangleleft^*_{\lambda,\kappa} T$. We say $(T, \vartheta(\bar{x};\bar{y}))$ is $\triangleleft^*_{\lambda,\kappa}$-maximal if $|T'| < \lambda \& \vartheta'(\bar{x}';\bar{y}') \in L(\tau_{T'}) \Rightarrow (T', \vartheta'(\bar{x}';\bar{y}')) \triangleleft^*_{\lambda,\kappa} (T, \vartheta(\bar{x};\bar{y}))$.

Definition 3.2. (1) T_{tr} is the theory of trees (i.e. the vocabulary is $\{<\}$ and the axioms state that $<$ is a partial order and $\{y : y < x\}$ is a linear order for every x), so T_{tr} is not complete, and let $\vartheta_{\mathrm{tr}}(x, y) = (y < x)$.
(2) T^*_{tr} is the model completion of T_{tr}.
(3) T_{ord} is the theory of linear orders, T^*_{ord} is its model completion (i.e. the theory of dense linear order without endpoints).

We note the connection to previous works and obvious properties:

Proposition 3.3. (1) *$T_1 \triangleleft^*_{\lambda,\mu,0} T_2$ is $T_1 \triangleleft^*_{\lambda,\mu} T_2$ of [1].*
(2) *$T_1 \triangleleft^*_{\lambda,\lambda;<\kappa} T_2$ implies $T_1 \triangleleft^*_{\lambda,\kappa} T_2$ of [4].*

(3) $\lhd^*_{\lambda,\mu;\kappa,\theta}$ *has the obvious monotonicity properties: if* $T_1 \lhd^*_{<\lambda_1,<\mu_1';<\kappa_1,<\theta_1} T_2$ *and* $\lambda_2 \geq \lambda_1, \mu_2 \leq \mu_1, \kappa_2 \geq \kappa_1, \theta_2 \geq \theta_1$ *then* $T_1 \lhd^*_{<\lambda_2,<\mu_2;<\kappa_2,<\theta_2} T_2$.

(4) $T \lhd^*_{\lambda,\mu;\kappa,\theta} T$ *if* $|T| < \theta, \lambda \geq \mu > \kappa, \mu \geq \theta$.

(5) *If* μ *is a limit cardinal, then* $T_1 \lhd^*_{<\lambda,<\mu;<\kappa,<\theta} T_2$ *iff for every* $\mu_1 < \mu, \mu_1 \geq \kappa$ *we have*

$$T_1 \lhd^*_{<\lambda,<\mu_1;<\kappa,<\theta} T_2.$$

(6) *Similar results hold for* $(T_\ell, \vartheta_\ell(\bar{x};\bar{y}))$.

Proof. Easy. □

Proposition 3.4. (1) *Assume* $T_1 \lhd^*_{<\lambda,<\mu;<\kappa,<\theta} T_2$. *Then for any theory* T^*, *we can find* $T^{**} \supseteq T^*$ *complete* $|T^{**}| < (|T^*|^{|\tau(T_1)|+|\tau(T_2)|})^+ + \theta$ *such that: for any interpretations* $\bar{\varphi}_1, \bar{\varphi}_2$ *of* T_1, T_2 *in* T^{**} *respectively the Definition* 3.1*(1) of* $T_1 \lhd^*_{<\lambda,<\mu;<\kappa,<\theta} T_2$ *holds.*

(2) *Assume* $\tau(T_1), \tau(T_2)$ *are disjoint. Then* $T_1 \lhd^*_{<\lambda,<\mu;<\kappa,<\theta} T_2$ *if for any* $T \supseteq T_1 \cup T_2$ *there is* $T^* \supseteq T$ *as demanded in Definition* 3.1*(1) for the trivial interpretations* $M^{[\bar{\varphi}^\ell]}$ *is the* $\tau(T_\ell)$*-reduct.*

Proof. Easy, or see [1], Observation 1.4. □

Now we will show that T^*_{tr} is $\lhd^*_\lambda$-maximal for every λ big enough, and conclude that $\text{SOP}_3 \Longrightarrow \lhd^*_\lambda$-maximality. The last result has already appeared in [4], Theorem 2.9, but the proof is not complete — in fact, the proof shows the following theorem:

Theorem 3.5. *Any theory* T, $|T| < \lambda$, *with* SOP_3 *is* $\lhd^*_\lambda$*-above* T^*_{ord}.

Proof. See [4], (2.12). □

Here we prove explicitly that T^*_{tr}, and therefore T^*_{ord} are maximal.

Theorem 3.6. T^*_{tr} *is* $\lhd^*_\lambda$*-maximal for any* $\lambda > \aleph_0$*; the witness* T^* *does not depend on* λ.

Proof. Let T be any complete theory, $|T| < \lambda$ and M_1 a model of T.

Let $\Phi = \{\varphi(x,\bar{a}) : \varphi(x,\bar{y}) \in L(\tau_T), \bar{a} \in {}^{\ell g(\bar{y})}(M_1)\}$, so $|\Phi| = \|M_1\|$. So $M = ({}^{\omega>}\Phi, \lhd)$ is a model of T_{tr} and there is a model M_2 of T^*_{tr} of cardinality $\|M_1\|$ extending M such that every member of M_2 is below some member of M.

Let χ be large enough such that $M_1, M_2 \in \mathcal{H}(\chi)$ and we define $\mathcal{B}^*$ expanding $(\mathcal{H}(\chi), \in)$ by $P_1 = |M_1|$, $P_2 = |M_2|$, $P = |M|$, $Q_0 = \Phi$, $<_1 = <^{M_2}$, $< = <_1 \restriction P$, $m =$ a constant symbol for a set M_1, $R^{\mathcal{B}^*} = R^{M_1}$ for $R \in \tau_T$ (wlog $\tau(T)$ does not contain any other predicate mentioned here)

$$Q = \{(\langle \varphi_\ell(x,\bar{a}_\ell) : \ell < n\rangle : M_1 \models \exists x[\wedge \varphi_\ell(x,\bar{a}_\ell)]\}.$$

H is a partial unary function with domain Q and range P_1, $H(\langle \varphi_\ell(x,\bar{a}_\ell) : \ell < n\rangle)$ satisfies $\{\varphi_\ell(x,\bar{a}_\ell) : \ell < n\}$, i.e. $\mathcal{B}^*$ satisfies the formula "$m \models (\exists x) \bigwedge_{\ell<n} \varphi_\ell(x,\bar{a}_\ell)$".

Let $T^* = Th(\mathcal{B}^*)$, let $\bar{\varphi}_1$ be the trivial interpretation of T in T^* (the restriction + reduct) and $\bar{\varphi}_2 = \langle P_2(x), x_0 <_1 x_1\rangle$ is an interpretation of T^*_{tr}. So $T^*, \bar{\varphi}_1, \bar{\varphi}_2$ does not depend on λ.

Now we assume $\mathcal{B}$ is a model of T^*, $N_1 = \mathcal{B}^{[\bar{\varphi}_1]}$, $N_2 = \mathcal{B}^{[\bar{\varphi}_2]}$, $N_3 = (P^{\mathcal{B}}, <^{\mathcal{B}})$ and we aim to show that (i) implies (iii). We will first show that (i) ⇒ (ii) and use this fact in the proof.

(i) N_2 is λ-saturated

(ii) in N_3 every branch has cofinality $\geq \lambda$, equivalently: every increasing sequence of length $< \lambda$ has an upper bound

(iii) N_1 is λ-saturated.

Let us first show (i) ⇒ (ii). If $\langle a_i : i < \delta\rangle$ is $<^{N_3}$-increasing, $\delta < \lambda$ then it is $<^{N_2}$-increasing hence has a $<^{N_2}$-upper bound a but $(\forall x \in P_2)(\exists y)(x <_1 y \& P(y))$ belongs to T^* so there is b, $a <^{N_2} b \in P^N = N_3$ so b is as required.

So we can assume clause (i) and we shall prove (iii).

Before we proceed, let us note several easy but important properties of $\mathcal{B}$.

(a) We can talk inside $\mathcal{B}$ about a set being a model, (standard coding of) a formula, a proof, etc. In particular, we can speak about m (as a model) satisfying or not satisfying certain sentences. Also, given a formula with free variables we can speak about substitution of other variables or parameters into the formula. Given $s \in \mathcal{B}$ which is a formula with free variables $\bar{x}$, we will allow ourselves to write $s = s(\bar{x})$, and if $\mathcal{B}$ thinks that substitution of $\bar{a} \in P_1$ into s will turn it into a true sentence in m as a model, we will write $m \models s(\bar{a})$ or just $s(\bar{a})$.
(b) $\mathcal{B} \models \forall z Q_0(z) \iff$ "z is a formula with one free variable with parameters from P_1". Moreover, suppose $\varphi(x, \bar{a})$ is a formula in $L(\tau_T)$ s.t. $\bar{a} \in P_1^{\mathcal{B}}$. $\mathcal{B}^*$ and therefore $\mathcal{B}$ satisfy $(\forall \bar{y} \in P_1)(\exists! s \in Q_0)$ such that $(\forall x \in P_1)\varphi(x, \bar{y}) \iff$ "$m \models s(x, \bar{y})$". Let us denote by $\ulcorner\varphi(x, \bar{a})\urcorner$ this "canonical encoding" of $\varphi(x, \bar{a})$ in $Q_0^{\mathcal{B}}$.
(c) $\mathcal{B} \models \forall s P(s) \iff$ "s is a finite sequence of members of Q_0, i.e. $(\exists n \in \omega)(s : n \to Q_0)$".
(d) For simplicity of notation, given $s \in P^{\mathcal{B}}$, we will write "$z \in s$" instead of "$z \in Im(s^{\mathcal{B}})$".
(e) For $z \in P^{\mathcal{B}}$, $c \in P_1^{\mathcal{B}}$, we write $z(c)$ meaning $(\forall s \in z)s(c)$.
(f) For every $\varphi(x, \bar{a}) \in L(\tau_T)$ for $a \in P_1^{\mathcal{B}}$, there exists an element of $P^{\mathcal{B}}$ corresponding to the finite sequence $\langle\varphi(x, \bar{a})\rangle$. We denote this element by $\langle\ulcorner\varphi(x, \bar{a})\urcorner\rangle$. Moreover, $\mathcal{B} \models \exists x(P_1(x) \wedge \varphi(x, \bar{a})) \to Q(\langle\ulcorner\varphi(x, \bar{a})\urcorner\rangle)$.

Subclaim 3.6.1. (1) Suppose $\mathcal{B} \models Q(z)$. Then $\mathcal{B} \models \forall w(Q(w) \wedge z < w) \to z(H(w))$.
(2) Let $\varphi(x, \bar{a}) \in L(\tau_T)$ and suppose $\mathcal{B} \models \exists x P_1(x) \wedge \varphi(x, \bar{a})$. Then $\mathcal{B} \models \forall z(Q(z) \wedge \langle\ulcorner\varphi(x, \bar{a})\urcorner\rangle < z) \to \varphi(H(z), \bar{a})$.

Proof. (1) Trivial as $\mathcal{B}^*$ satisfies it.
(2) Let $z^* = \langle\ulcorner\varphi(x, \bar{a})\urcorner\rangle$. First, $Q(z^*)$ holds by (f) above. By (1), $z^*(H(z))$ holds for each $z \in Q^{\mathcal{B}}$, $z^* < z$. Now by (b) and (f), $\mathcal{B} \models \forall x P_1(x) \to (z^*(x) \iff \varphi(x, \bar{a}))$. As $\mathcal{B} \models Range(H) \subseteq P_1$, we are done. □

We now proceed with the proof (i) $\Longrightarrow$ (iii). So let p be a 1-type in N_1 of cardinality $< \lambda$, so let $p = \{\varphi_\beta(x, \bar{a}_\beta) : \beta < \alpha\}$ with $\alpha < \lambda$, $\bar{a}_\beta \in N_1 \forall \beta$. Without loss of generality p is closed under conjunction, i.e. for every $\varepsilon, \zeta < \alpha$ for some $\xi < \alpha$ we have $\varphi_\xi(x, \bar{a}_\xi) = \varphi_\varepsilon(x, \bar{a}_\varepsilon) \wedge \varphi_\zeta(x, \bar{a}_\zeta)$. We shall now choose by induction on $\beta \le \alpha$ an element b_β of N such that

(A) $b_\beta \in P^{\mathcal{B}} = N_3$ moreover $b_\beta \in Q^{\mathcal{B}}$ and $\gamma < \beta \Rightarrow b_\gamma <^{N_3} b_\beta$
(B) if $\gamma < \beta$ then $\mathcal{B} \models (\forall z)(Q(z) \wedge (b_\beta \le z) \to \varphi_\gamma(H(z), \bar{a}_\gamma))$
(C) if $\gamma < \alpha$ (but not necessarily $\gamma < \beta$) then $\mathcal{B} \models (\exists z)[Q(z) \wedge (b_\beta \le z) \wedge (\forall y)(Q(y) \wedge z \le y \to \varphi_\gamma(H(y), \bar{a}_\gamma))]$.

If we succeed then $H^{\mathcal{B}}(b_\alpha)$ is as required.

Case 1: $\beta = 0$.
Define $b_0 = \langle\rangle$ (the element of $P^{\mathcal{B}}$ corresponding to the empty sequence). Clearly $\mathcal{B} \models Q(b_0)$, i.e. the demand (A) holds. (B) holds trivially. Why does (C) hold? Let $\gamma < \alpha$. $\mathcal{B} \models \exists x \varphi_\gamma(x, \bar{a}_\gamma)$ therefore denoting $z^*_\gamma = \langle\ulcorner\varphi_\gamma(x, \bar{a}_\gamma)\urcorner\rangle$, we have $\mathcal{B} \models Q(z^*_\gamma) \wedge b_0 < z^*_\gamma$. Now we finish by part (2) of the subclaim.

Case 2: $\beta = \upsilon + 1$.
$\mathcal{B}$ satisfies the sentence saying that for every $\eta \in Q$ and $\bar{y} \in P_1$ there exists an element of P that we denote by $Conc_\upsilon(\eta, \bar{y})$ corresponding to $\eta\hat{\ }\langle\ulcorner\varphi_\upsilon(x, \bar{y})\urcorner\rangle$. We define $b_\beta = Conc_\upsilon(b_\upsilon, \bar{a}_\upsilon)$. Now we have to check (A)–(C).

(A) By the induction hypothesis, clause (C) holds for b_υ and υ (standing for b_β and γ there). Therefore $\mathcal{B} \models \exists z \in Q(b_\upsilon \le z) \wedge \varphi_\upsilon(H(z), \bar{a}_\upsilon))$. But $\mathcal{B}^*$ (and so $\mathcal{B}$) satisfies that $\forall \bar{y} \in P_1$ if there exists $z \in Q$ s.t. $\varphi_\upsilon(H(z), \bar{y})$ holds, then $Conc_\upsilon(z, \bar{y})$ is an element of Q (as in $\mathcal{B}^*$ the assumption means that there exists an element of m satisfying all the formulae in z plus $\varphi_\upsilon(x, \bar{y})$). So we get what is required.
(B) is clear as by the induction hypothesis, $\varphi_\zeta(H(z), \bar{a}_\zeta)$ holds for every $\zeta < \upsilon$, $b_\beta \le z$ (recall that $b_\upsilon \le b_\beta$). As for $\varphi_\upsilon(x, \bar{a}_\upsilon)$, $\mathcal{B}^*$ clearly satisfies that for every $z \in Q$, $\bar{y} \in P_1$, if $b = Conc_\upsilon(z, \bar{y})$ is in Q then $\varphi_\upsilon(H(z), \bar{y})$ holds $\forall z \in Q, b \le z$.
(C) Let $\zeta < \alpha$. As p is closed under conjunctions, for some ξ, $\varphi_\gamma(x, \bar{a}_\gamma) \wedge \varphi_\zeta(x, \bar{a}_\zeta) = \varphi_\xi(x, \bar{a}_\xi)$. Now we apply clause (C) holding for b_υ to $\gamma = \xi$ and get $z \in Q$, $b_\upsilon \le z$ with $H(z)$ satisfying both $\varphi_\upsilon(x, \bar{a}_\upsilon)$ and $\varphi_\zeta(x, \bar{a}_\zeta)$. Once again using the satisfaction by $\mathcal{B}$ of natural sentences, we show that $b = Conc_\zeta(b_\beta, \bar{a}_\zeta)$ is in Q, $b_\beta \le b$ and $\forall z \in Q$ which is above b, $\varphi_\zeta(x, \bar{a}_\zeta)$ holds, i.e. b is as required.

Case 3: $\beta = \delta$ limit.

By our present assumption, clause (i), and therefore clause (ii), hold. Hence there is $b \in P^{\mathcal{B}}$ which is an upper bound to $\{b_\gamma : \gamma < \beta\}$. Now $\mathcal{B}$ satisfies "for every element z of P there is a $y \le z$ which is in Q and $x \le z \& Q(x) \to x \le y$". Apply this to b for z and get b'_δ for y. So $b'_\delta \in Q$ and $\gamma < \delta \Rightarrow b_\gamma \le b'_\delta$, as required in clauses (A) + (B) but not necessarily (C).

Define for each $\zeta < \alpha$ a formula $\psi_\zeta(w, \bar{a}_\zeta) = (\exists z)(w \le z \wedge Q(z) \wedge (\forall y)(z \le y \wedge Q(y) \to \varphi_\zeta(H(y), \bar{a}_\zeta))$. Now we find c_ζ (for $\zeta < \alpha$) such that:

(a) $c_\zeta \in Q^{\mathcal{B}}, c_\zeta \le b$.
(b) $\psi_\zeta(c_\zeta, a_\zeta)$ holds.
(c) under (a) + (b), the element c_ζ is maximal.

Why do c_ζ exist? $\mathcal{B}$ satisfies "for every element s of P there is a $w \le s$ which satisfies $\psi_\zeta(w, \bar{a}_\zeta)$, is in Q and $(x \le s \wedge \psi_\zeta(x, \bar{a}_\zeta) \wedge Q(x)) \to (x \le w)$".

By the induction hypothesis we have:

$$\gamma < \delta, \zeta < \alpha \Rightarrow b_\gamma <^{N_3} c_\zeta.$$

Clearly it suffices to find b_δ satisfying $Q(b_\delta)$ and $b_\gamma <^{N_3} b_\delta <^{N_3} c_\zeta$ for $\gamma < \delta, \zeta < \alpha$. As $N_3 \restriction \{c : c \le b\}$ is linearly ordered, this follows from N_2 being λ-saturated. □

Proposition 3.7. (1) *For every T^*, there is $T^{**} \supseteq T^*$, $|T^{**}| = |T^*| + \aleph_0$ such that for every model $\mathcal{B}$ of T^{**} we have*
(a) *for any λ, the following are equivalent*
(α) *if $\bar{\varphi}_1$ is an interpretation of T^*_{tr} in $\mathcal{B}$ (possibly with parameters) then $\mathcal{B}^{[\bar{\varphi}_1]}$ is λ_{tr}-saturated*
(β) *if $\bar{\varphi}_2$ is an interpretation of T_{ord} in $\mathcal{B}$ (possibly with parameters) then $\mathcal{B}^{[\bar{\varphi}_2]}$ is λ-saturated*
(b) *for any λ, the following are equivalent*
(α) *if $\bar{\varphi}_1$ is an interpretation of T_{tr} in $\mathcal{B}$ (possibly with parameters) then in $\mathcal{B}^{[\bar{\varphi}_1]}$, every branch with no last element has cofinality $\ge \lambda$*
(β) *if $\bar{\varphi}^*_2$ is an interpretation of T_{ord} in $\mathcal{B}$ (possibly with parameters) then in $\mathcal{B}^{[\bar{\varphi}_2]}$ there is no Dedekind cut (I_1, I_2) with both cofinalities $< \lambda$ and at least one $\ge \aleph_0$.*

Proof. Easy. □

Corollary 3.8. (1) *T^*_{ord} is $\lhd^*_\lambda$-maximal.*
(2) *If $|T| < \lambda$ and T has SOP_3 then T is $\lhd^*_\lambda$-maximal.*

Proof. (1) Follows from 3.7
(2) By (1) and 3.5. □

Question 3.9. Is the other direction of 3.8 (2) true?

Remark 3.10. See Theorem 3.13 and Corollary 3.15 for a proof of a weaker version of the other direction: we get SOP_2 instead of SOP_3.

We would like to know whether it is possible to weaken the assumptions of Corollary 3.8(2) to SOP_2. The following theorem is a step in this direction, showing a local version. See also Discussion 3.17.

Theorem 3.11. *If T has SOP_2 as exemplified by $\vartheta(\bar{x}; \bar{y})$, then $(T^*_{\mathrm{tr}}, \vartheta_{\mathrm{tr}}(x; y)) \lhd^*_\lambda (T, \vartheta(\bar{x}; \bar{y}))$ for any $\lambda \ge |T| + \aleph_0$ regular.*

Proof. We can find a model M_1 of T^*_{tr} and model M_2 of T and $\bar{a}_b \in {}^{\ell g(\bar{y})}M_2$ for $b \in M_1$ such that:

(α) if $M_1 \models b_0 < \cdots < b_{n-1}$ then $\{\vartheta(\bar{x}, \bar{a}_{b_\ell}) : \ell < n\}$ is satisfiable in M_2
(β) if b_1, b_2 are incomparable in M_1 then

$$M_2 \models \neg(\exists \bar{x})(\vartheta(\bar{x}, \bar{a}_{b_1}) \& \vartheta(\bar{x}, \bar{a}_{b_2}))$$

(γ) for no $\bar{d} \in {}^{\ell g(\bar{x})}(M_2)$ is $\{b \in M_1 : M_2 \models \vartheta(\bar{d}, \bar{a}_b)\}$ unbounded in M_1 (note that by (β) it is always linearly ordered in M_1, therefore (γ) means that for each $\bar{d} \in {}^{\ell g(\bar{x})}(M_2)$, there exists an element of M_1 which is above every b satisfying $\vartheta(\bar{d}, \bar{a}_b)$).

[The construction of M_1 and M_2 is as follows: choose by induction on n, $(M_{1,n}, M_{2,n}, \langle \bar{a}_b : b \in M_{1,n}\rangle : n < \omega)$ such that:

(a) $M_{1,n}$ is a model of T^*_{tr}
(b) $M_{2,n}$ is a model of T
(c) $M_{1,n} \prec M_{1,n+1}$ moreover, every branch of $M_{1,n}$ has an upper bound in $M_{1,n+1}$
(d) $M_{2,n} \prec M_{2,n+1}$
(e) $\bar{a}_b \in {}^{\ell g(\bar{y})}(M_{2,n})$ for $b \in M_{1,n}$
(f) clauses (α), (β) hold
(g) if $b \in M_{1,n+1}$ and $[b' \in M_{1,n} \Rightarrow M_{1,n+1} \models \neg(b < b')]$ then $\vartheta(\bar{x}, \bar{a}_b)$ is not satisfied by any sequence from $M_{1,n}$.

There is no problem to carry the definition.

Now $M_1 = \bigcup_n M_{1,n}$, $M_2 = \bigcup_n M_{2,n}$ and $\langle \bar{a}_b : b \in M_1\rangle$ are as required above.]

Now let χ be such that $M_1, M_2 \in \mathcal{H}(\chi)$, wlog $\tau_T = \tau(M_2)$, $\{<\} = \tau(T_{\mathrm{tr}}) = \tau(M_1)$ and $\{\in\}$ are pairwise disjoint. Now we define a model $\mathcal{B}_0$.

Its universe is $\mathcal{H}(\chi)$ relation $\in$ (membership)
$P_1 = |M_1|$,
$P_2 = |M_2|$,
$R = R^{M_\ell}$ if $R \in \tau(M_\ell)$, $\ell \in \{1, 2\}$ F_ℓ (for $\ell < \ell g(\bar{y})$) a partial unary function such that: $b \in M_1 \Rightarrow \langle F_\ell(b) : \ell < \ell g(\bar{y})\rangle = \bar{a}_b$.

Let $T^* = Th(\mathcal{B}_0)$. For the obvious $\bar{\varphi}$ and $\bar{\psi}$, T^* is (T, T_{tr})-superior and $|T^*| = |T| + \aleph_0$. Assume $\lambda = \mathrm{cf}(\lambda) > |T^*|$.

So let $\mathcal{B}$ be a model of T^* such that $M'_2 = \mathcal{B}^{[\bar{\varphi}]}$, the model of T interpreted in it, is λ^+-saturated. It will be enough to prove that $M'_1 = \mathcal{B}^{[\bar{\psi}]}$ satisfies: for every branch of cofinality $\theta \le \lambda$ there exists an upper bound. So let $\{b_i : i < \theta\}$ be $<^{M_1}$-increasing let $\bar{c}_i = \langle F^{\mathcal{B}}_\ell(b_\ell) : \ell < \ell g(\bar{y})\rangle$. Hence for any $n < \omega, i_0 < \cdots < i_{n-1} < \theta$ we have $M'_2 \models (\exists \bar{x})[\bigwedge_{m<n} \vartheta(\bar{x}, \bar{c}_i)]$ because $\mathcal{B}_0 \models (\forall z_0, \ldots, z_{n-1})[\bigwedge_{k<n} P_1(z_k) \Rightarrow z_0 < z_1 < \cdots < z_{n-1} \rightarrow (\exists \bar{x}) \bigwedge_{m<n} \vartheta(\bar{x}, \langle F_\ell(z_m) : \ell < \ell g(\bar{y})\rangle)]$.

So $\{\vartheta(\bar{x}, \bar{c}_i) : i < \theta\}$ is finitely satisfiable in M'_2 hence some $\bar{d} \in {}^{\ell g(\bar{x})}(M'_2)$ realizes it. Now we claim that $\{b \in M'_1 : \mathcal{B} \models \vartheta(\bar{d}, \bar{a}_b)\}$ is bounded in M'_1: recall that by clause (γ) $\mathcal{B}_0$ satisfies: for every $\bar{x} \in {}^{\ell g(y)}P_2$ there exists $z \in P_1$ such that z is $<^{\mathcal{B}}$— above all the elements $w \in P_1$ satisfying $\vartheta(\bar{x}, \bar{a}_w)$. Therefore $\mathcal{B}$ satisfies this sentence, and applying it to $\bar{d} \in {}^{\ell g(\bar{x})}(M'_2)$, we get $b^* \in M'_1$ — the required bound. As for each $i < \theta$, $\vartheta(\bar{d}, \bar{a_{b_i}})$ holds, clearly $\mathcal{B} \models b_i < b^*$ for all i, and we are done. □

The next goal is to complete the proof started in [1] of the fact that $\lhd^*$-maximality implies SOP_2. In [1] a property "$\lhd^{**}_\lambda$ - maximality", which is closely related to "$\lhd^*_\lambda$ - maximality" was defined, and it was shown (Theorem 3.6 there) that every T which is $\lhd^{**}_\lambda$-maximal for some (every) big enough regular λ, has an order property similar to SOP_2, that we call SOP''_2 (see Definition 1.5). We will show that SOP''_2 is equivalent to SOP_2 (for a theory). This answers Question (3.8)(3) from the original version of [1] (version 1 on the arXiv).

Discussion 3.12. In particular, Theorem 3.13 will lead to the following conclusion: assuming that T is $\lhd^*_{\lambda^+}$-maximal for some regular λ satisfying $2^\lambda = \lambda^+$, we get by [1], Claim (3.2) that T is $\lhd^{**}_\lambda$-maximal, so it has SOP''_2, and therefore SOP_2. So we will obtain $\lhd^*$-maximality implies SOP_2, see Corollary 3.15.

Theorem 3.13. *Let T be a theory.*

(1) *Suppose $\vartheta(\bar{x}, \bar{y})$ exemplifies SOP_2 in T. Then $\vartheta(\bar{x}; \bar{y})$ exemplifies SOP''_2 in T as well.*
(2) *Suppose $\vartheta(\bar{x}, \bar{y})$ exemplifies SOP''_2 in T. Then for some k, $\vartheta^{<k>}(\bar{x}; \bar{y})$ exemplifies SOP_2 in T (where $\vartheta^{<k>}(\bar{x}; \bar{y}^{<k>}) = \bigwedge_{\ell<k} \vartheta(\bar{x}; \bar{y}_\ell)$).*

Proof. (1) is easy.

(2) Denote $\mathfrak{I}^n_\lambda = \{\bar\eta : \bar\eta = \langle \eta_\ell : \ell \le n\rangle, \eta_\ell \lhd \eta_{\ell+1};$ and $\eta_\ell \in {}^{\lambda>}2\}$. So assume $\vartheta(\bar x; \bar y)$ has SOP''_2 as exemplified by $n, \bar{\mathbf{a}} = \langle a_{\bar\eta} : \bar\eta \in \mathfrak{I}^n_\omega\rangle$. Without loss of generality $\langle \bar a_{\bar\eta} : \bar\eta \in \mathfrak{I}^n_\omega\rangle$ is tree indiscernible in the relevant sense: $\eta \frown \langle 0\rangle$, $\eta \frown \langle 1\rangle$ look the same over η (2-*fbti* from 1.9). We can assume this by 1.10 (for more details, see [1], Claim (2.14)).

For $\nu \in {}^{\omega\ge}2$ let $p_\nu = \{\vartheta(\bar x, \bar a_{\bar\eta}) : \bar\eta = \langle \eta_\ell : \ell < n\rangle, \eta_\ell < \eta_{\ell+1} \trianglelefteq \nu\}$ so

$\circledast_1$ p_η for $\eta \in {}^\omega 2$ is consistent (in $\mathfrak{C}_T$).

Let

$$\Xi = \big\{(h, \Upsilon) : h \text{ is a one-to-one mapping from } {}^{n\ge}m \text{ to } {}^{\omega>}2$$
$$\text{preserving } \lhd, \perp \text{ and } \Upsilon \subseteq {}^n m \text{ and there is}$$
$$\langle \nu^*_\eta : \eta \in \Upsilon\rangle, h(\eta) \lhd \nu^*_\eta \in {}^\omega 2 \text{ for } \eta \in {}^n m$$
$$\text{such that } \cup\{p_{\nu^*_\eta} : \eta \in \Upsilon\} \text{ is inconsistent}\big\}.$$

Now

$\circledast_2$ Ξ is non-empty.

[By the definition of SOP''_2, clause (b), choose $\Upsilon = {}^n m$].

Choose $(h^*, \Upsilon^*) \in \Xi$ with $|\Upsilon^*|$ of minimal cardinality and $\langle \nu^*_\eta : \eta \in \Upsilon^*\rangle$ as there. By $\circledast_1$ clearly $|\Upsilon^*| \ge 2$. So choose $\eta_0 \ne \eta_1$ from Υ^* with $\nu^* = \nu^*_{\eta_0} \cap \nu^*_{\eta_1}$ $(= h(\eta_0) \cap h(\eta_1))$ being of maximal length and let $k^* = \ell g(\nu^*)$. We can find $\ell^* < \omega$ sufficiently large such that $\cup\{p_{\nu^*_\eta \restriction \ell^*} : \eta \in \Upsilon^*\}$ is inconsistent. We choose by induction on $i < \omega$ for every $\rho \in {}^\ell 2$, a sequence $\nu_\rho \in {}^{\omega>}2$ by $\nu_{\langle\rangle} = \langle\rangle$, $\nu_{\rho\hat{}<j>} = \nu_\rho\hat{}(\nu^*_{\eta_j} \restriction \ell^*)$.

Lastly for $\rho \in {}^{\omega>}2 \notin \{<>\}$ let $\vartheta^*(\bar x, \bar b^*_\rho)$ be the conjunction of

$$\bigcup\{p_{\nu^*_\eta \restriction \ell^*} : \eta \in \Upsilon^* \backslash \{\eta_0, \eta_1\}\} \cup \{\vartheta(\bar x, \bar a_{\bar\eta}) : \bar\eta = \langle \eta_\ell : \ell \le n\rangle,$$
$$\eta_\ell \lhd \eta_{\ell+1} \trianglelefteq \nu_\rho \text{ and } (\forall \ell \le n)[\ell g(\eta_\ell) \notin [k^*, \ell g(\nu_\rho) - \ell^* + k^*)]$$
$$\text{(the last condition is empty if } \ell g(\rho) = 1)\big\}.$$

In other words, we are taking the "upper part" of ν_ρ that "looks like" η_0 or η_1 after they split.

Now if $\rho^* \in {}^\omega 2$ then $\{\vartheta^*(x, \bar b_\rho) : \rho \lhd \rho^*\}$ is consistent as all its members are conjunctions of formulae from

$$\cup\{p_{\nu^*_\eta} : \eta \in \Upsilon^* \backslash \{\eta^*_0, \eta^*_1\} \cup p_{\rho^*}$$

and this is consistent as otherwise $(h^* \restriction (\Upsilon^* \backslash \{\eta^*_0, \eta^*_1\}) \cup \{\langle \eta^*_0, \rho^* \restriction \ell^{**}\rangle\}, \Upsilon^* \backslash \{\eta^*_1\})$ belongs to Ξ for some ℓ^{**}, thus contradicting the choice of (h^*, Υ^*), i.e. with minimal $|\Upsilon^*|$.

Lastly if $\rho_0, \rho_1 \in {}^{\omega>}2$ are $\lhd$-incomparable then $\{\vartheta^*(\bar x; \bar b_{\rho_0}), \vartheta^*(\bar x; \bar b_{\rho_1})\}$ is inconsistent: we know that

$\otimes$ 3.13.1.

$$\bigcup\{p_{\nu^*_\eta \restriction \ell^*} : \eta \in \Upsilon^* \backslash \{\eta_0, \eta_1\}\} \cup \{\vartheta(\bar x, \bar a_{\bar\eta}) : \bar\eta = \langle \eta_\ell : \ell \le n\rangle,$$
$$\eta_\ell \lhd \eta_{\ell+1} \trianglelefteq \nu^*_{\eta_0} \restriction \ell^*\} \cup \{\vartheta(\bar x, \bar a_{\bar\eta}) : \bar\eta = \langle \eta_\ell : \ell \le n\rangle, \eta_\ell \lhd \eta_{\ell+1} \trianglelefteq \nu^*_{\eta_1} \restriction \ell^*\}$$

is inconsistent (by the choice of $(h^*, \Upsilon^*) \in \Xi$ and the choice of ℓ^*). Now, by the fact that $\nu^* = \nu^*_{\eta_0} \cap \nu^*_{\eta_1}$ was chosen to be maximal among other pairs in Υ^*, we see that if

$$\bar\eta_0 = \langle \eta^0_\ell : \ell \le n\rangle, \text{ where for each } \ell,\ \eta^0_\ell \lhd \eta^0_{\ell+1} \trianglelefteq \nu^*_{\eta^*_0} \restriction \ell^*$$

and

$$\bar\eta_1 = \langle \eta^1_\ell : \ell \le n\rangle, \text{ where for each } \ell,\ \eta^1_\ell \lhd \eta^1_{\ell+1} \trianglelefteq \nu^*_{\eta^*_1} \restriction \ell^*$$

while

$$\bar\eta_3 = \langle \eta^3_\ell : \ell \le n\rangle, \text{ where for each } \ell,\ \eta^3_\ell \lhd \nu^*_{\eta^*} \text{for some } \eta^* \in \Upsilon^* \setminus \{\eta^*_0, \eta^*_1\}$$

then

$\otimes$ **3.13.2.**

$$\bar{\eta}_1 \frown \bar{\eta}_2 \frown \bar{\eta}_3 \equiv \bar{\varsigma}_1 \frown \bar{\varsigma}_2 \frown \bar{\eta}_3$$

where $\bar{\varsigma}_j = \langle \varsigma^j_\ell : \ell \leq n \rangle$ and

$$\varsigma^j_\ell = \eta^j_\ell, \quad if \ \lg(\eta^j_\ell) \leq k^*$$
$$\varsigma^j_\ell = \nu_{\rho_j} \restriction [\lg(\nu_{\rho_j}) - (\ell^* - \lg(\eta^j_\ell))], \text{ otherwise.}$$

In simpler words: we replace every η^j_ℓ (an initial segment of $\nu_{\eta_j} \restriction \ell^*$) whose length is bigger than k^* (in particular, it is not below any element in the image of Υ^* other than ν_{η_j} itself) by an appropriate initial segment of ν_{ρ_j}, and get a similar sequence over the image of $\Upsilon^* \setminus \{\eta^*_0, \eta^*_1\}$.

Now, by indiscernibility of $\langle \bar{a}_{\bar{\eta}} \rangle$, the definition of $\vartheta^*(\bar{x}, \bar{b}^*_\rho)$, 3.13.1 and 3.13.2, we conclude $\{\vartheta^*(\bar{x}; \bar{b}_{\rho_0}), \vartheta^*(\bar{x}; \bar{b}_{\rho_1})\}$ is also inconsistent. $\square$

Let us summarize the main results of this section.

Definition 3.14. (1) We call a theory T $\lhd^*$*-maximal* if it is $\lhd^*_\lambda$-maximal for every regular $\lambda > |T| + \aleph_0$.
(2) We call a pair of theory and formula (T, ϑ) $\lhd^*$*-maximal* if it is $\lhd^*_\lambda$-maximal for every regular $\lambda > |T| + \aleph_0$.

Corollary 3.15. (1) *If T has* SOP_3 *then it is* $\lhd^*$*-maximal.*
(2) *If T is* $\lhd^*$*-maximal then it has* SOP_2.

Proof. (1) Corollary 3.8.
(2) By [1] Claim 3.2, [1] Theorem 3.6 and Theorem 3.13. $\square$

So we have shown $\mathrm{SOP}_3 \implies \lhd^*$-maximality $\implies SOP_2$, and for the second implication we also have a weak (local) "converse", Theorem 3.11. See Discussion 3.17.

Question 3.16. Is any of the two implications above reversible?

Discussion 3.17. Note that Theorem 3.11 is a step in (possibly) reversing the second implication above: we show that if T has SOP_2 exemplified by a formula ϑ, then the pair $(T^*_{\mathrm{tr}}, \vartheta_{\mathrm{tr}})$ is $\lhd^*$-below the pair (T, ϑ). By Theorem 3.6 (and quantifier elimination), in order to obtain $\mathrm{SOP}_2 \implies \lhd^*$-maximality it is enough to show that $(T^*_{\mathrm{tr}}, \{\vartheta_{\mathrm{tr}}, \neg\vartheta_{\mathrm{tr}}\})$ is $\lhd^*$-below (T, Δ) where Δ is some fragment of the language of T. This was our original motivation for proving Theorem 3.11, which is in a sense a "local" or "positive" version of what we are interested in, but right now it is unclear to us whether similar techniques will lead to the desired "global" result.

One should remark that Theorem 3.11 is not weaker than the global version since $\Delta = \{\vartheta\}$, so it is really localized to a single formula, with no use of negation (hence "positive"). Therefore, although it does not quite does what one would hope for, we find Theorem 3.11 interesting on its own.

Discussion 3.18. We would also like to point out that our analysis provides an alternative (in fact, in a sense a more conceptual) proof of Theorem 1.17 and Conclusion 1.18 in [1]. Theorem 3.5 here shows that T^*_{tr}, and therefore T^*_{ord} is maximal in $\lhd^*_\lambda$, and therefore is $\lhd^*_\lambda$-above T^*_{feq}. By Theorem 2.1, T^*_{feq} does not have SOP_1 (in particular, does not have SOP_2), and so by Corollary 3.15, cannot be $\lhd^*_\lambda$-maximal. So T^*_{feq} is *strictly below* T^*_{tr} (and T^*_{ord}) in $\lhd^*_\lambda$-ordering, which is precisely the statement of Conclusion 1.18 in [1]. There is no surprise here: in [1] it was shown that T^*_{feq} is on the "good" side of an "external" dividing line. Here we showed that it is also on the "good" side of an "internal syntactic" dividing line, and brought the "internal" and "external" lines close together. So our paper also provides a generalization of section 1 of [1].

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