Arch. Math. Logic (2000) 39: 213–218

Archive for Mathematical Logic

The PCF Trichotomy Theorem does not hold for short sequences

Menachem Kojman[1,2], **Saharon Shelah**[3]

[1] Department of Mathematics and Computer Science, Ben Gurion University of the Negev, Beer Sheva, Israel
[2] Department of Mathematical Sciences, Carnegie-Mellon University, Pittsburgh, PA, USA (e-mail: kojman@cs.bgu.ac.il)
[3] Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel (e-mail: shelah@math.huji.ac.il)

Received: 17 January 1998

Abstract. The PCF Trichotomy Theorem deals with sequences of ordinal functions on an infinite κ modulo some ideal I. If a $<_I$-increasing sequence of ordinal functions has regular length which is larger than κ^+, then by the Trichotomy Theorem the sequence satisfies one of three structural conditions. It was of some interest to find out if the Trichotomy Theorem could hold also for sequences of length κ^+. It is shown that this is not the case.

1. Introduction

The Trichotomy Theorem [5], II,1.2, specifies three alternatives for the structure of an increasing sequence of ordinal functions modulo an ideal on an infinite cardinal κ – provided the sequence has regular length λ and λ is at least κ^{++}.

The natural context of the Trichotomy Theorem is, of course, pcf theory, where a sequence of ordinal functions on κ usually has length which is larger than $\kappa^{+\kappa}$. However, the trichotomy theorem has already been applied in several proofs to sequences of length κ^{+n}, $(n \geq 2)$ (see [4], [1] and [3]).

Therefore, a natural question to ask is, whether the Trichotomy Theorem is valid also for sequences of length κ^+, namely, whether the condition on the minimum length of the sequence can be lowered by one cardinal.

The first author was partially supported by NSF grant No. DMS-9622579
The second author was partially supported by the Israeli Foundation for Basic Science. Number 673 in list of publications

Below we show that the assumption $\lambda \geq \kappa^{++}$ in the Trichotomy Theorem is tight. For every infinite κ, we construct an ideal I over κ and $<_I$-increasing sequence $\overline{f} \subseteq \mathrm{On}^\kappa$ so that all three alternatives in the Trichotomy theorem are violated by $\overline{f}$.

2. The counter-example

Let κ be an infinite cardinal. Denote by On^κ the class of all functions from κ to the ordinal numbers.

Let I be an ideal over κ. We write $f <_I g$, for $f, g \in \mathrm{On}^k$, if $\{i < \kappa : f(i) \geq g(i)\} \in I$ and we write $f \leq_I g$ if $\{i < \kappa : f(i) > g(i)\} \in I$. A sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \mathrm{On}^\kappa$ is $<_I$*-increasing* if $\alpha < \beta < \lambda$ implies that $f_\alpha <_I f_\beta$ and is $<_I$*-decreasing* if $\alpha < \beta < \lambda$ implies that $f_\beta <_I f_\alpha$.

A function $f \in \mathrm{On}^k$ is a least upper bound mod I of a sequence $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \mathrm{On}^\kappa$ if $f_\alpha <_I f$ for all $\alpha < \lambda$ and whenever $f_\alpha <_I g$ for all α then $f \leq_I g$. A function $f \in \mathrm{On}^\kappa$ is an *exact upper bound* of $\overline{f}$ if $f_\alpha \leq f$ for all $\alpha < \lambda$, and whenever $g <_I f$, there exists $\alpha < \lambda$ such that $g <_I f_\alpha$. For subsets t, s of κ, write $t \subseteq_I s$ if $s - t \in I$.

The dual filter I^* of an ideal I over κ is the set of all complements of members of I. The relations $\leq_I$, $<_I$ and $\subseteq_I$ will also be written as $\leq_{I^*}$, $<_{I^*}$ and $\subseteq_{I^*}$.

Let us quote the theorem under discussion:

Theorem 1. *(The Trichotomy Theorem)*

Suppose $\lambda \geq \kappa^{++}$ is regular, I is an ideal over k and $\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is a $<_I$-increasing sequence of ordinal functions on k. Then $\overline{f}$ satisfies one of the following conditions:

1. *$\overline{f}$ has an exact upper bound f with $\mathrm{cf}\, f(i) > \kappa$ for all $i < \kappa$;*
2. *there are sets $S(i)$ for $i < \kappa$ satisfying $|S(i)| \leq \kappa$ and an ultra-filter U over k extending the dual of I so that for all $\alpha < \lambda$ there exists $h_\alpha \in \prod_{i<\kappa} S(i)$ and $\beta < \lambda$ such that $f_\alpha <_U h_\alpha <_U f_\beta$.*
3. *there is a function $g : \kappa \to On$ such that the sequence $\overline{t} = \langle t_\alpha : \alpha < \lambda \rangle$ does not stabilize modulo I, where $t_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$.*

Proofs of the Trichotomy Theorem are found in [5], II,1.2, in [3] or in the future version of [2].

Theorem 2. *For every infinite κ there exists an ultrafilter U over κ and a $<_U$-increasing sequence $\overline{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle \subseteq On^\kappa$ such that conditions 1, 2 and 3 in the Trichotomy Theorem fail for $\overline{f}$.*

Proof. Let $\lambda = \kappa^+$.

Let us establish some notation.

We recall that every ordinal has an expansion in base λ, namely can be written as a unique finite sum $\sum_{k\leq l}\lambda^{\beta_l}\alpha_l$ so that $\beta_{k+1}<\beta_k$ and $\alpha_k<\lambda$. We limit ourselves from now on to ordinalz $\zeta<\lambda^\omega$. For such ordinals, the expansion in base λ contains only finite powers of λ (that is, every β_k is a natural number).

We agree to write an ordinal $\zeta=\lambda^l\alpha_l+\lambda^{l-1}\alpha_{l-1}+\cdots+\alpha_0$ simply as a finite sequence $\alpha_l\alpha_{l-1}\ldots\alpha_0$. We identify an expansion with λ digits with one with $n>l$ digits by adding zeroes on the left. If $\zeta=\alpha_l\alpha_{l-1}\ldots\alpha_0$, we call α_k, for $k\leq l$, the *k-th digit* in the expansion of ζ.

For $\alpha<\lambda$ and an integer l, define:

$$A^l_\alpha=\{\alpha_l\alpha_{l-1}\ldots\alpha_0:\alpha_k<\alpha\text{ for all }k\leq l\}\tag{1}$$

A^l_α is the set of all ordinals below λ^ω whose expansion in base λ contains $l+1$ or fewer digits from α.

Fact 3. *For all $\alpha<\lambda$ and $l<\omega$,*

1. $\bigcup_{\alpha<\lambda}A^l_\alpha=\lambda^{l+1}$
2. *The ordinal* $\sum_{k=0}^l\lambda^k=\overbrace{\alpha\alpha\ldots\alpha}^{l+1}$ *is the maximal element in* $A^l_{\alpha+1}$
3. *if* $\zeta=\alpha_l\alpha_{l-1}\ldots\alpha_0\in A^l_{\alpha+1}$ *is not maximal in* $A^l_{\alpha+1}$*, then the immediate successor of ζ in $A^l_{\alpha+1}$ is obtained from ζ as follows: let k be the first $k\leq l$ for which $\alpha_k<\alpha$. Replace α_k by α_k+1 and replace α_m by 0 for all $m<k$*

Fix a partition $\{X_n:n<\omega\}$ of κ with $|X_n|=\kappa$ for all n. Let $n(i)$, for $i<\kappa$, be the unique n for which $i\in X_n$.

By induction on $\alpha<\kappa^+$, define $f_\alpha:\kappa\to\text{On}$ so that:

1. $f_\alpha(i)\in A^{n(i)}_{\alpha+1}-A^{n(i)}_\alpha$
2. For all $n,l<\omega$ and finite, strictly increasing, sequences $\langle\alpha_k:k\leq l\rangle\subseteq\lambda$ it holds that for every sequence $\langle\zeta_k:k\leq l\rangle$ which satisfies $\zeta_k\in A^n_{\alpha_k+1}-A_{\alpha_k}$, there are κ many $i\in X_n$ for which $\bigwedge_{k\leq l}f_{\alpha_k}(i)=\zeta_k$.

The first item above says that $f_\alpha(i)$ is an ordinal below λ^ω whose expansion in base λ has $\leq n(i)$ digits, at least one of which is α. The second item says that every possible finite sequence of values $\langle\zeta_k:k<l\rangle$ is realized κ many times as $\langle f_{\alpha_\kappa}(i):k<\lambda\rangle$ for an arbitrary increasing sequence $\langle\alpha_\kappa:k<l\rangle$.

The induction required to define the sequence is straightforward.

Define now, for every $\alpha<\kappa^+$, a function $g_\alpha:\kappa\to\text{On}$ as follows:

$$g_\alpha(i)=\min[(A^{n(i)}_{\alpha+1}\cup\{\lambda^{n(i)+1}\})-f_\alpha(i)]\tag{2}$$

Since $f_\alpha(i)<\lambda^{n(i)+1}$ for $i\in X_n$, the definition is good. If $f_\alpha(i)$ is not maximal in $A^{n(i)}_{\alpha+1}$, then $g_\alpha(i)$ is the *immediate successor of $f_\alpha(i)$ in $A^{n(i)}_{\alpha+1}$*. Let us make a note of that:

Fact 4. *There are no members of* $A^{n(i)}_{\alpha+1}$ *between* $f_\alpha(i)$ *and* $g_\alpha(i)$

We have defined two sequences:

$$\overline{f} = \langle f_\alpha : \alpha < \lambda \rangle$$
$$\overline{g} = \langle g_\alpha : \alpha < \lambda \rangle$$

Next we wish to find an ideal modulo which $\overline{f}$ is $<_I$-increasing and $\overline{g}$ is a $<_I$-decreasing sequence of upper bounds of $\overline{f}$.

Claim 5. *For every finite increasing sequence* $\alpha_0 < \alpha_1 < \cdots < \alpha_l < \lambda$ *there exists* $i < \kappa$ *such that for all* $k < l$

$$f_{\alpha_k}(i) < f_{\alpha_{k+1}}(i) < g_{\alpha_{k+1}}(i) < g_{\alpha_k}(i) \tag{3}$$

Proof. Suppose $\alpha_0 < \alpha_1 < \cdots < \alpha_l < \lambda$ is given and choose $n > l$. Let $\zeta_0 = \overbrace{\alpha_0\alpha_0 \ldots \alpha_0}^{l+1} \in A^n_{\alpha_0}$. Let ζ_{k+1} be obtained from ζ_k by replacing the first $l + 1 - k$ digits of ζ_κ by α_{k+1}:

$$\alpha_0\alpha_1 \ldots \alpha_k\alpha_{k+1} \ldots \alpha_l = \zeta_l$$
$$\vdots$$
$$\alpha_0\alpha_1 \ldots \alpha_{k-1}\alpha_k \ldots \alpha_k = \zeta_k$$
$$\vdots$$
$$\alpha_0\alpha_1 \;\ldots\ldots\; \alpha_1 \ldots \alpha_1 = \zeta_1$$
$$\alpha_0\alpha_0 \;\ldots\ldots\; \alpha_0 \ldots \alpha_0 = \zeta_0$$

Thus $\zeta_0 < \zeta_1 < \ldots < \zeta_l$ and $\zeta_k \in A^l_{\alpha_k} \subseteq A^n_{\alpha_k}$ is not maximal in $A^n_{\alpha_k}$ (because $n > l$). Let ξ_k be the immediate successor of ζ_k in $A^n_{\beta_k}$.

By Fact 3 above, we have

$$1\overbrace{0 \;\ldots \quad \ldots\; 0 \;\ldots \quad \ldots\; 0}^{l+1} = \xi_0$$
$$(\alpha_0 + 1)\, 0 \;\ldots\ldots\ldots\ldots\; 0 = \xi_1$$
$$\vdots$$
$$\alpha_0\alpha_1 \ldots (\alpha_{k-1} + 1)0 \ldots 0 = \xi_k$$
$$\vdots$$
$$\alpha_0\alpha_1 \;\ldots\ldots\; (\alpha_{l-1} + 1)0 = \xi_l$$

Therefore $\zeta_0 < \zeta_1 < \ldots \zeta_l < \xi_l < \xi_{l-1} < \ldots < \xi_0$. To complete the proof it remains to find some $i \in X_n$ for which $f_{\alpha_k}(i) = \zeta_\kappa$ for $k \le l$, and, consequently, by the definition (2) above, $g_{\alpha_k}(i) = \xi_k$. The existence of such $i \in X_n$ is guaranteed by the second condition in the definition of $\overline{f}$.

$\square$

For $\alpha < \beta < \lambda$, let

$$C_{\alpha,\beta} = \{i < \kappa : f_\alpha(i) < f_\beta(i) < g_\beta(i) < g_\alpha(i)\} \tag{4}$$

Claim 6. *$\{C_{\alpha,\beta} : \alpha < \beta < \kappa^+\}$ has the finite intersection property*

Proof. Suppose that $\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_l, \beta_l$ are given and $\alpha_k < \beta_k < \lambda$ for $k \leq l$. Let $\langle \gamma_m : m < m(*) \rangle$ be the increasing enumeration of $\bigcup_{k \leq l} \{\alpha_k, \beta_k\}$. To show that $\bigcap_{k \leq l} C_{\alpha_k, \beta_k}$ is not empty, it suffices to find some $i < \kappa$ for which the sequence $g_{\gamma_m}(i)$ is decreasing in m and $f_{\gamma_m}(i)$ is increasing in m. The existence of such an $i < \kappa$ follows from Claim 5. $\square$

Let U be any ultrafilter extending $\{C_{\alpha,\beta} : \alpha < \beta < \lambda\}$. Since for every $\alpha < \beta$ it holds that $f_\alpha <_U f_\beta <_U g_\beta <_U g_\alpha$, we conclude that $\overline{f}$ is $<_U$-increasing, that $\overline{g}$ is $<_U$-decreasing and that every g_α is an upper bound of $\overline{f}$ mod U.

Claim 7. *There is no exact upper bound of $\overline{f}$ mod U.*

Proof. It suffices to check that there is no $h \in \mathrm{On}^\kappa$ that satisfies $f_\alpha <_U h <_U g_\alpha$ for all $\alpha < \kappa^+$. Suppose, then, that $h \in \mathrm{On}^\kappa$ satisfies this. Since $h <_U g_0$, we may assume that $h(i) < \lambda^{n(i)+1}$ for all $i < \kappa$ (by changing h on a set outside of U).

Let $i < \kappa$ be arbitrary. Since $\bigcup_{\alpha<\lambda} A^{n(i)}_\alpha = \lambda^{n(i)+1}$, there is some $\alpha(i)$ so that $h(i) \in A^{n(i)}_{\alpha(i)}$. By regularity of λ it follows that there is some $\alpha(*) < \lambda$ such that $h(i) \in A^{n(i)}_{\alpha(*)}$ for all $i < \kappa$. By our assumption about h, $f_{\alpha(*)} <_U h <_U g_{\alpha(*)}$. Thus, there is some $i < \kappa$ for which $f_{\alpha(*)}(i) < h(i) < g_{\alpha(*)}(i)$. However, all three values belong to $A^{n(i)}_{\alpha(*)+1}$, while by Fact 4 there are no members of $A^{n(i)}_{\alpha(*)+1}$ between $f_{\alpha(*)(i)}$ and $g_{\alpha(*)}(i)$ – a contradiction. $\square$

Claim 8. *there are no sets $S(i) \subseteq On$ for $i < \kappa$ which satisfy condition 2 in the trichotomy for $\overline{f}$ and U.*

Proof. Suppose that $S(i)$, for $i < \kappa$, and $h_\alpha \in \prod_{i<\kappa} S(i)$ satisfy 2. in the Trichotomy Theorem. Find $\alpha < \lambda$ such that $S(i) \subseteq A^{n(i)}_\alpha$ for all i. Thus $f_\alpha <_U h_\alpha <_U g_\alpha$ – contradiction to 4. $\square$

Claim 9. *there is no $g : \kappa \to On$ such that $g, \overline{f}$ and the dual of U satisfy condition 3. in the Trichotomy Theorem.*

Proof. Let $g : \kappa \to On$ be arbitrary, and let $t_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$. As $\overline{f}$ is $<_U$-increasing, for every $\alpha < \beta < \lambda$ necessarily $t_\alpha \subseteq_U t_\beta$. Since U is an ultrafilter, every $\subseteq_U$-increasing sequence of sets stabilizes. $\square$

$\square$

References

1. James Cummings. Collapsing successors of singulars. Proc. Amer. Math. Soc., 125(9):2703–2709, 1997.
2. Menachem Kojman. The abc of pcf. Logic Eprints.
3. Menachem Kojman. Exact upper bounds and their uses in set theory. Ann. Pure. App. Logic, 92:267–282, 1998.
4. Menachem Magidor and Saharon Shelah. When does almost free imply free? (For groups, transversal etc.). Jour. Amer. Math. Soc., 7:769–830, 1994.
5. Saharon Shelah. Cardinal Arithmetics, volume 29 of Oxford Logic Guides. Oxford Science Publications, 1994.