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Uncountable Universal Locally Finite Groups

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INTRODUCTION

In this paper we solve some group-theoretic problems by the method of indiscernibles in infinitary logic. The problems under consideration come from the book "Locally Finite Groups," by Kegel and Wehrfritz [3], and concern a class of groups first studied by Philip Hall [2].

Hall introduced the notion of universal locally finite group, and proved the following results:

(a) Any locally finite group H of cardinal κ can be embedded in a universal locally finite group G of cardinal $\max(\kappa, \aleph_0)$;

(b) Any two countable universal locally finite groups are isomorphic.

In the light of this Kegel and Wehrfritz raised two questions. [3, pp. 182–183]. These are

I. Suppose $\kappa > \aleph_0$. Are any two universal locally finite groups of cardinal κ isomorphic?

II. Suppose H is a locally finite group of cardinal κ, where $\kappa > \aleph_0$. Is H embeddable in every universal locally finite group of cardinal κ?

We answer both questions in the negative for all uncountable κ. Concerning I, we show that there are 2^κ isomorphism types of universal locally finite groups of cardinal κ, if $\kappa > \aleph_0$. Moreover, if κ is regular and $\kappa > \aleph_0$, there are 2^κ pairwise nonembeddable universal locally finite groups of cardinal κ. Concerning II, we show, for example, that if $\kappa = 2^{\aleph_0}$ then a counterexample is given by taking $H = S_3^{\omega}$.

The definitive result on I was obtained by Shelah, after Macintyre had settled I, and II, as above stated.

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I. Solution of the Problems

1.1

We shall be dealing with locally finite groups, that is, groups all of whose finitely generated subgroups are finite.

We shall use the abbreviations:

f.g. = finitely generated;

l.f. = locally finite.

We record for future use two examples of large l.f. groups.

Lemma 1. (a) *Any torsion abelian group is* l.f.

(b) *If H is a finite group, and κ is a cardinal, H^κ is* l.f.

Proof. (a) is obvious from the structure theorem for f.g. abelian groups.

(b) The elements of H^κ are functions from κ to H. Any such function f determines a finite partition $\bigcup_{h\in H} X_{h,f}$, where $X_{h,f} = \{x \in \kappa : f(x) = h\}$. Now let $f_1, ..., f_n$ be arbitrary members of H^κ. The sets X_{h,f_j} $(j \leqslant n)$ generate a finite Boolean algebra B of subsets of κ. Suppose f is in the group generated by $f_1, ..., f_n$. Then it is obvious that $X_{h,f} \in B$. Since B and H are finite, there are only finitely many possibilities for f. This proves (b).

1.2

In [2] Hall introduced an important class of l.f. groups, namely the universal locally finite groups (to be abbreviated u.l.f. groups).

Definition. A group G is u.l.f. if

(a) G is l.f.;

(b) every finite group is embeddable in G;

(c) if H_1, H_2 are finite subgroups of G, and $\phi : H_1 \simeq H_2$ is an isomorphism, then there exists t in G such that $\phi(h) = t^{-1}ht$ for all h in H_1.

Kegel and Wehrfritz remark [3] that a u.l.f. group G is in some sense a universe in which to do finite group theory. If one thinks of large algebraically closed fields as universes for field theory, then the analogy is good, as will be pointed out below, for the u.l.f. groups G are precisely the existentially closed locally finite groups.

Hall proved the basic

THEOREM 1. (a) *If H is a* l.f. *group of cardinal κ, then H is embeddable in a* u.l.f. *G of cardinal* $\max(\kappa, \aleph_0)$.

(b) *If G_1, G_2 are* u.l.f. *groups of cardinal $\aleph_0$ then $G_1 \simeq G_2$.*

(c) *Any* u.l.f. *G is simple.*

We shall need the following direct consequence of (a) and Lemma 1:

LEMMA 2. *For any $\kappa \geq \aleph_0$ there are* u.l.f. *groups G of cardinal κ.*

Condition (b) comes from the following lemma, instantly suggestive to a model-theorist.

LEMMA 3. *Suppose G_1, G_2 are* u.l.f. *groups, and H_1, H_2 are finitely generated subgroups of G_1, G_2 respectively. Suppose $\phi : H_1 \simeq H_2$ is an isomorphism. Let Γ_1 be a finitely generated subgroup of G_1, with $H_1 \subseteq \Gamma_1$. Then ϕ extends to a monomorphism $\Gamma_1 \to G_2$.*

Proof. H_1 and H_2 are finite, since G_1 is l.f. Similarly Γ_1 is finite. By (b) of the definition, there exists a monomorphism $\psi: \Gamma_1 \to G_2$. So we have an isomorphism $\phi(H_1) \simeq \psi(H_1)$, sending $\phi(h)$ to $\psi(h)$ for each h in H_1. By (c) of the definition, there exists t in G_2 such that $t^{-1}\phi(h)t = \psi(h)$ for each h in H_1. Now consider the map

$$x \to t\psi(x)t^{-1}$$

from Γ_1 into G_2. If $x \in H_1$, $t\psi(x)t^{-1} = tt^{-1}\phi(x)tt^{-1}$, as required.

Theorem 1(b) follows from this by a standard back-and-forth argument.

But we obtain more model-theoretic information from the above. If L is a first-order logic then $L_{\infty,\omega}$ is the usual infinitary extension as in [4], and $\equiv_{\infty,\omega}$, $\prec_{\infty,\omega}$ are defined as usual. If we construe groups as structures for a logic L with the usual $\cdot$, $^{-1}$, e, then the above lemma immediately gives, by [4]:

LEMMA 4. (a) *If G_1, G_2 are* u.l.f. *then $G_1 \equiv_{\infty,\omega} G_2$.*

(b) *If further $G_1 \subseteq G_2$, then $G_1 \prec_{\infty,\omega} G_2$.*

1.3

Theorem 1(b) says that the class of u.l.f. groups is $\aleph_0$-categorical [4]. Problem 1 of Kegel and Wehrfritz asks if the class is κ-categorical for any $\kappa > \aleph_0$.

We must first axiomatize the class ULF of u.l.f. groups. It is obviously not axiomatizable in L, since the class is clearly not closed under ultraproducts (an ultraproduct will in general have an element of infinite order).

However, we have:

LEMMA 5. ULF *is axiomatizable by a single sentence of* $L_{\omega_1,\omega}$.

Proof. The proof is easier to see than to write down.

Enumerate as $\Delta_{m,n}$ $(n < \omega)$ the isomorphism-types of m-generator finite groups. Select for each m, n elements g_{jn} $(j \leqslant m)$ which generate a group $\langle g_{1n}, \ldots, g_{mn}\rangle$ of type $\Delta_{m,n}$. Then define an $L_{\omega_1,\omega}$-formula $\Phi_{m,n}(v_1, \ldots, v_m)$ "the multiplication table of type $\Delta_{m,n}$", by

$$\bigwedge \{\Psi(v_1, \ldots, v_m) \colon \langle g_{1n}, \ldots, g_{mn}\rangle \models \Psi(g_{1n}, \ldots, g_{mn}) \wedge \Psi \text{ is an atomic } L\text{-formula or the negation of an atomic } L\text{-formula}\}.$$

Then a group G is l.f. iff G satisfies the $L_{\omega_1,\omega}$-sentence

$$\bigwedge (\forall v_1, \ldots, v_m) \bigvee_n \Phi_{m,n}(v_1, \ldots, v_m).$$

The condition that every finite group is embeddable in G is equivalent to G satisfying the sentence

$$\bigwedge_{m,n} (\exists v_1, \ldots, v_m)\, \Phi_{m,n}(v_1, \ldots, v_m).$$

The final "conjugacy condition" for ULF is expressed by the sentence

$$\bigwedge_{m,n} (\forall x_1, \ldots, x_m, y_1, \ldots, y_m) \Big[\Phi_{m,n}(x_1, \ldots, x_m) \wedge \Phi_{m,n}(y_1, \ldots, y_m) \to (\exists t) \bigwedge_{j \leqslant m} t^{-1} x_j t = y_j\Big].$$

This proves the lemma.

1.4

Now we fix an $L_{\omega_1,\omega}$-sentence σ axiomatizing ULF, and, for definiteness we fix an admissible countable fragment L^* [4] with σ a sentence of L^*.

Before settling the main problem, let us justify the earlier claim that ULF is the class of existencially closed l.f. groups. We have:

(i) Every l.f. group is embeddable in a member of ULF;

(ii) If $G_1 \subseteq G_2$, each G_i in ULF, then $G_1 \prec_{\infty,\omega} G_2$;

(iii) If $G \in$ ULF and $H \prec_{\infty,\omega} G$ then $H \in$ ULF.

So by [9] ULF is the class of existentially closed l.f. groups.

1.5

From Lemma 2, σ has models of all infinite cardinals, so we are in a position to apply the technique of indiscernibles in L^*. For this one should consult [4].

Let G be a group, and A a subset of G. Enlarge L in the standard way by adding constants for elements of A. In this way we get $L(A)$, and G is naturally enriched to an $L(A)$-structure G_A.

We make the following

DEFINITION. Suppose $x, y \in G$. We say x is A-equivalent to y ($x \sim_A y$) if x and y satisfy exactly the same atomic $L(A)$-formulas in G_A.

Obviously $\sim_A$ is an equivalence relation. The equivalence classes will be called A-types.

Since σ has models of all infinite cardinals, we can apply [4, Theorem 21] and the usual Ehrenfeucht–Mostowski method [1] to get

THEOREM 2. *For each $\kappa \geqslant \aleph_0$ there is a model E_κ of σ such that*

(i) *E_κ has cardinal κ, and*

(ii) *For each countable subset A of E_κ, E_κ has only countably many A-types.*

Now fix E_κ as above.

1.6

The only group-theoretical observation needed is very trivial.

LEMMA 6. *There is an* l.f. *group H of cardinal $\aleph_1$ such that if $H \subseteq G$ then G has a countable subset A such that G has at least $\aleph_1$ A-types.*

Proof. Let $\Gamma = S_3{}^\omega$, i.e., the direct product of countably many copies of S_3, the symmetric group on three letters. Select in S_3 an element β of order 3 and an element α of order 2. Then $\alpha\beta \neq \beta\alpha$. Write as $\mathbb{Z}_2$ the subgroup of S_3 generated by α. Then Γ has a countable abelian subgroup A, where $A = \mathbb{Z}_2^{(\omega)}$, the direct *sum* of ω copies of $\mathbb{Z}_2$.

We claim that Γ has $2^{\aleph_0}$ A-types. First we define some special elements $a_n (n < \omega)$ in A. a_n is the unique function $f\colon \omega \to S_3$ such that $f(m) = e$ if $m \neq n$, and $f(n) = \alpha$. Now let X be any subset of ω, and let C_X be the unique function $g\colon \omega \to S_3$ such that $g(m) = e$ if $m \in X$, and $g(m) = \beta$ if $m \notin X$. Note immediately that $C_X a_n = a_n C_X$ if and only if $n \in X$.

It follows that if X and Y are distinct subsets of ω then C_X and C_Y have distinct A-types in Γ. Indeed, if Γ_1 is any group containing the subgroup of Γ generated by A, C_X and C_Y then C_X and C_Y have distinct A-types in Γ_1.

Now select $\aleph_1$ distinct subsets X_λ $(\lambda < \omega_1)$ of ω, and let H be the subgroup of Γ generated by A and the X_λ. Γ is l.f. by Lemma 1(b), so H is l.f. Obviously H has cardinal $\aleph_1$. The lemma is proved.

1.7

We can now solve Problems I and II.

First, II.

THEOREM 3. *There is a* l.f. *group H of cardinal $\aleph_1$ such that for each $\kappa \geqslant \aleph_1$ there is a* u.l.f. *G of cardinal κ such that H is not embeddable in G.*

Proof. Immediate from Theorem 2 and Lemma 6.

Note. $H \subseteq S_3{}^\omega$, and so is solvable of class 2.

Next, I.

THEOREM 4. *For each cardinal $\kappa \geqslant \aleph_1$ there are several nonisomorphic* u.l.f. *groups of cardinal κ.*

Proof. Take E_κ as in Theorem 2. Then H is not embeddable in E_κ. But by Theorem 1(a), H is embeddable in some u.l.f. of cardinal κ. This proves the theorem.

1.8

Let $\mu(\kappa)$ be the number of isomorphism-types of u.l.f. groups of cardinal κ. Obviously $\mu(\kappa) \leqslant 2^\kappa$. We now prove:

THEOREM 5. $\mu(\kappa) = 2^\kappa$ *if* $\kappa \geqslant \aleph_1$.

For this we must change to a more general context. In [7, 8] it is proved:

THEOREM 6. *Let $\psi \in L_{\lambda^+,\omega}$, $L' \subset L$, K the class of L'-reducts of models of ψ, and $\Delta \in L'_{\infty,\omega}$ a set of formulas closed under finitary connectives. Then* (1)$\leftrightarrow$(2)$\leftrightarrow$(3)$\to$(4) *where*:

(1) *For every μ there is a model $M \in K$ and $A \subset |M|$ such that $|A| \geqslant \mu$ and in M the number of Δ-types realized by (finite) sequences over A is $> |A|^M$.*

(2) *There is* $\phi(\bar{x}, \bar{y}) \in \Delta$ *such that: for every* μ *there are* $M \in K$ *and* $\bar{a}_i \in M (i < \mu)$ *such that* $M \models \phi[\bar{a}_i, \bar{a}_j]$ *iff* $i < j$.

(3) *As (2) for* $\phi(\bar{x}, \bar{y}) \in L'_{\lambda^+, \omega}$ *which is in the closure of* Δ *under subformulas.*

(4) *For each* $\mu > \lambda$ *there are* 2^{μ} *nonisomorphic models in K of cardinality* μ, *and when* μ *is regular, there are* 2^{μ} *such models, no one of which has an embedding into another preserving subformulas of* Δ.

A straightforward generalization of Lemma 6 gives

LEMMA 7. *For each* μ, *there is a* u.l.f. *group* G, *and* $A \subset G$, $|A| = \mu$, *and in G over A more than* μ *quantifier free types are realized.*

We can conclude:

For each uncountable λ there are 2^{λ} pairwise nonisomorphic u.l.f. groups of cardinality λ.

This is Theorem 5.

We also get:

THEOREM 8. *For each regular uncountable* κ *there are* 2^{κ} *pairwise non-embeddable* u.l.f. *groups of cardinal* κ.

1.9

We conclude by mentioning some problems.

Problem A. Which l.f. groups H are embeddable in all u.l.f. groups of cardinal $\geqslant$ card(H).

By Hall, all countable l.f. groups are embeddable in all u.l.f. groups. We do not know if all torsion abelian groups of cardinal $\aleph_1$ are embeddable in all uncountable u.l.f. groups. From our work above we know that not all 2-step solvable groups of exponent 6 and cardinal $\aleph_1$ are embeddable in all uncountable u.l.f. groups.

We note that the analogous problem for f.g. subgroups of existentially closed groups has a pleasant answer [5].

Our second problem relates to the notion of homogeneous-universal model.

Problem B. For which κ is there a u.l.f. group G of cardinal κ in which all l.f. groups of cardinal $\leqslant \kappa$ are embeddable?

In connection with homogeneity, let us note that the amalgamation property fails for l.f. groups [6].

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