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The primal framework II: Smoothness†

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Abstract

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Let $(\mathbf{K}, \leqslant, \text{cpr})$ be a class of models with a notion of 'strong' submodel and of canonically prime model (cpr) over an increasing chain. We show under appropriate set-theoretic hypotheses that if $\mathbf{K}$ is not smooth (there are incompatible models over some chains), then $\mathbf{K}$ has many models in certain cardinalities. On the other hand, if $\mathbf{K}$ is smooth, we show that in reasonable cardinalities $\mathbf{K}$ has a unique homogeneous-universal model. In this situation we introduce the notion of type and prove the equivalence of saturated with homogeneous-universal.

This is the second in a series of articles developing abstract classification theory for classes that have a notion of prime models over independent pairs and over chains. It deals with the problem of smoothness and establishing the existence and uniqueness of a 'monster model'. We work here with a predicate for a canonically prime model. In a forthcoming paper, entitled "Abstract classes with few models have 'homogeneous-universal' models", we show how to drop this predicate from the set of basic notions and still obtain results analogous to those here.

Experience with both first-order logic and more general cases has shown the advantages of working within a 'monster' model that is both 'homogeneous-

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universal' and 'saturated'. Fraïssé [6] for the countable case and Jónsson [9] for arbitrary cardinalities gave algebraic conditions on a class **K** of models that guaranteed the existence of a model that is homogeneous and universal for **K**. Morley and Vaught [10] showed that if **K** is the class of models of a first-order theory, then the algebraic conditions of homogeneity and universality are equivalent to model-theoretic conditions of saturation. First-order stability theory works within the fiction of a monster model $\mathcal{M}$. Such a fiction can be justified as 'a saturated model in an inaccessible cardinal', by speaking of 'class' models, or by asserting the existence of a function f from cardinals to cardinals such that any set of data (collection of models sets, types etc.) of cardinality μ can be taken to exist in a sufficiently saturated model of cardinality $f(\mu)$. This paper culminates by establishing the existence and uniqueness of such a monster mode for classes **K** of the sort discussed in [3] that do not have the maximal number of models. We avoid some of the cardinality complications of [10] by specifying closure properties on the class of models.

One obstacle to the construction that motivates an important closure condition is the failure of 'smoothness'. Is there a unique compatibility class of models embedding a given increasing chain? It is easy for a model to be compatibility prime (i.e., prime among all models in a joint embeddability class over a chain) without being absolutely prime over the chain. This 'failure of smoothness' is a major obstacle to the uniqueness of a monster model. Our principal result here shows that this situation implies the existence of many models in certain cardinalities. We can improve the 'certain' by the addition of appropriate set-theoretic hypotheses.

The smoothness problem also arose in [17]. Even when the union of a chain is in **K**, it does not follow that it can be **K**-embedded in every member of **K** which contains the chain. The argument showing this situation implies many models is generalized here. However, we have further difficulties. In the context of [17] once full smoothness (unions of chains are in the class and are absolutely prime) is established in each cardinality, one can prove a representation theorem as in [16] to recover a syntactic (omitting types in an infinitary language) definition for the abstractly given class. From this one obtains full information about the Löwenheim-Skolem number of **K** and in particular that there are models in all sufficiently large powers. The examples exhibited in Section 2 show this is too much to hope for in our current situation. Even the simplest case we have in mind, $\aleph_1$-saturated models of strictly stable theories, gives trouble in λ if $\lambda^{\omega} > \lambda$. This illustrates one of the added complexities of the more general situation. Many properties that in the first order case hold on a final segment of the cardinals hold only intermittently in the general case. This greatly complicates arguments by induction and presents the problem of analyzing the spectrum where a given property holds.

This paper depends heavily on the notations established in [3, Sections 1 and 2]; we do not use the results of [3, Section 3]. Reference to [17] is helpful since we are generalizing the context of that paper but we do not expressly rely on any of the results there. Some arguments are referred to analogous proofs in [17, 20].

Section 1 of this paper recapitulates the properties of canonical prime models over chains and contains some examples illustrating the efficacy of the notion. Sections 2 and 4 fix some basic notations and assumptions. Section 2 deals with downward Löwenheim-Skolem phenomena; the upwards Löwenheim-Skolem theorem is considered in Section 3. In Section 5 we describe the combinatorial principles used in the paper. Section 7 introduces a useful game and some axioms on double chains that are used to show Player I has a winning strategy if **K** is not smooth.

In Section 6 we consider two important ideas. First we discuss the notion '**K** codes stationary sets' — a particularly strong form of a nonstructure result for **K**. Then we consider two different ways that a model might code a stationary set. Dealing with canonically prime models over chains rather than just unions of chains introduces subtleties into the decomposition of models that make the process of 'taking points of continuity' more complicated than in earlier studies.

Section 8 contains the main technical results of this paper, showing that with appropriate set theory, if **K** is not smooth, then it codes stationary sets.

In Section 9 we assume that **K** is smooth. We are then able to (i) construct and prove the categoricity of a monster model, (ii) introduce the notion of a type, and (iii) recover the Morley–Vaught equivalence of saturation with homogeneous-universality. We conclude in Section 10 with a discussion of further problems.

1. Prime models over chains

We begin by reviewing the discussion in [3] of prime models over chains.

Let $\mathfrak{M} = \langle M_\alpha, f_{\alpha,\beta} \colon \alpha < \beta < \delta \rangle$ be an increasing chain of members of **K**. An embedding f of $\mathfrak{M}$ into a structure M is a family of maps $f_i : M_i \rightarrow M$ that commute with the $f_{i,j}$. As for any diagram, there is an equivalence relation of 'compatibility over $\mathfrak{M}$'. Two triples $(\mathfrak{M}, f, M)$ and $(\mathfrak{M}, g, N)$, where f (g) is a **K**-embedding of $\mathfrak{M}$ into M (N), are compatible if there exists an M' and f_1 (g_1) mapping M (N) into M' such that $f_1 \circ f$ and $g_1 \circ g$ agree on $\mathfrak{M}$ (i.e., on each M_i). This relation is transitive since **K** has the amalgamation property.

Now M is *compatibility prime* over $(\mathfrak{M}, f)$ if it can be embedded over f into every model compatible with it. In [3, Section 2.3] we introduced the relation cpr for canonically prime, characterized it by an axiom Ch1, and then asserted the existence of canonically prime models by axiom Ch2. Before stating the basic characterization and existence axioms used in this paper we need some further notation.

1.1. Definition. (i) The chain $\langle M_i, f_{i,j} : i < j < \beta \rangle$ is *essentially* **K**-*continuous at* $\delta < \beta$ if there is a model M'_δ that is canonically prime over $\mathfrak{M}_\delta$ and compatible with M_δ over $\langle M_i, f_{i,j} : i < j < \delta \rangle$.

(ii) The chain $\mathfrak{M}$ is *essentially* **K**-*continuous* if for each limit ordinal $\delta < \beta$, $\mathfrak{M}$ is essentially **K**-continuous at δ.

We will use the following slight variants on the existence and characterization axiom in [3, Section 2.3]. Note that by Axiom Ch1′, the M'_δ in Definition 1.1 can be $\leq_{\mathbf{K}}$-embedded in M_δ over $\mathfrak{M} \mid \delta$.

1.2. Axioms for canonically prime models

Axiom Ch1′. $\text{cpr}(\mathfrak{M}, M, f)$ implies (i) $\mathfrak{M}$ is an essentially **K**-continuous chain;
(ii) M is compatibility prime over $\mathfrak{M}$ via f.

Axiom Ch2′. If $\mathfrak{M}$ is essentially **K**-continuous, there is a canonically prime model over $\mathfrak{M}$.

Clearly, if **K** satisfies Ch1′ and Ch2′, each essentially **K**-continuous chain can be refined to a **K**-continuous chain. Examples 1.3 and 1.5 show the necessity of introducing the predicate cpr rather than just working with models that satisfy the definition of compatibility prime.

1.3. Example. Fix a language with ω_1 unary predicates L_i (for level) and a binary relation $<$. Let **K** be the collection of structures isomorphic to structures of the form $\langle A, L_i, < \rangle$ where A is a subset of ${}^{<\omega_1}\lambda$ closed under initial segment and containing no uncountable branch, $<$ is interpreted as initial segment, and $L_i(f)$ holds if $f \in A$ has length i. Now for $M, N \in \mathbf{K}$, let $M \leq N$ if $M \subseteq N$ and every ω-chain in M that is unbounded in M remains unbounded in N.

Let j_α denote the sequence $\langle k : k < \alpha \rangle$ and let M_i be the member of **K** whose universe is $\{j_\alpha : \alpha < i\}$. Then $\mathfrak{M} = \langle M_i : i < \omega_1$ and i is not a limit ordinal$\rangle$ is a **K**-increasing chain of members of **K**. (We do not include the M_i for limit i, since if i is a limit ordinal, M_i is not a **K**-submodel of M_{i+1}.) Now in the natural sense for 'compatibility prime models' this chain is continuous. For each limit i, M_{i+1} is compatibility prime in its compatibility class over $\mathfrak{M} \mid (i+1)$ the compatibility class of models with a top on the chain. But the union of this chain is not a member of **K** and has no extension in **K**.

So for this example if we tried to introduce 'prime'-models over chains by definition, Axiom Ch2′ would fail. If in this context we define cpr to mean union, then the chain is not even essentially **K**-continuous and so Axiom Ch2′ does not require the existence of a cpr-model over $\mathfrak{M}$.

Here is an example where the definition of cpr is somewhat more complicated.

1.4. Example. Let **K** be the class of triples $\langle T, <, Q \rangle$ where T is a tree partially ordered by $<$ that has ω_1 levels and such that each increasing ω-sequence has a unique least upper bound. Q is a unary relation on T such that if $\{t(i) : i < \omega_1\}$ is an enumeration in the tree order of a branch, then $\{i : t(i) \in Q\}$ is not stationary.

For $M, N \in \mathbf{K}$, write $M \leqslant N$ if M is a substructure of N in the usual sense and each element of M has the same level (height) in N.

If $\mathfrak{M}$ is an increasing chain of $\mathbf{K}$-models of length μ, the canonically prime model over $\mathfrak{M}$ will be the union of the chain if $\mathrm{cf}(\mu)$ is uncountable. If $\mathrm{cf}(\mu) = \aleph_0$, the canonically prime model will be the union plus the addition of limit points for increasing ω-sequences but with no new elements added to Q.

These examples may seem sterile. Note however, one of the achievements of first-order stability theory is to reduce the structure of quite complicated models to trees $\lambda^{<\omega}$. It is natural to expect that trees of greater height will arise in investigating infinitary logics. Moreover, it is essential to understand these 'barebones' examples before one can expect to deal with more complicated matters. In particular, in this framework we expect to discuss the class of $\aleph_1$-saturated models of a strictly stable theory. We cannot expect to reduce the structure of models of such theories to anything simpler than a tree with ω_1-levels. The following example provides another reason for introducing the predicate cpr.

1.5. Example. Let T be the theory REI_ω of countably many refining equivalence relations with infinite splitting [2, p. 81]. Let $\mathbf{K}$ be the class of $\aleph_1$-saturated models of T and define $M \leqslant N$ if no E_ω-class of M is extended in N. (E_ω denotes the intersection of the E_i for finite i.) Now there are many choices for the interpretation of the predicate cpr; one choice is induced from the κ-saturated prime model for each uncountable κ. Suppose $\mathfrak{M}$ is $\mathbf{K}$-increasing chain and M is prime among the κ-saturated models over $\bigcup\mathfrak{M}$. Let M' be the restriction of M to the set of $x \in \mathfrak{M}$ such that for each n there is a $y \in \bigcup\mathfrak{M}$ with $xE_n y$. (Note that if $\mathfrak{M} = \langle M_i : i < \omega \rangle$, the models prime among the κ- and μ-saturated models respectively containing $\bigcup\mathfrak{M}$ are incompatible over $\mathfrak{M}$ if $\mu \neq \kappa$.)

Thus, the canonically prime model becomes canonical only with the addition of the predicate cpr. There are a number of reasonable candidates in the basic language and we have to add a predicate to distinguish one of them. The last example shows that we should demand that cpr models are compatible. This property was not needed in [3] but we need it here to prove smoothness. Its significance is explained in Definition 4.4.

1.6. Axiom Ch4. Let $\mathfrak{M}$ be a $\mathbf{K}$-continuous chain and suppose both $\mathrm{cpr}(\mathfrak{M}, M)$ and $\mathrm{cpr}(\mathfrak{M}, N)$ hold. Then M and N are compatible over $\mathfrak{M}$.

2. Adequate classes

This paper can be considered as a reflection on the construction of a homogeneous universal model as in [6, 9, 10]. These constructions begin with a class $\mathbf{K}$ that satisfies the amalgamation and joint embedding properties. They have assumptions of two further sorts: Löwenheim–Skolem properties and closure under unions of chains.

We deal with these assumptions in two ways. Some are properties of the kinds of classes we intend to study; we just posit them. For others we are able to establish within our context a dichotomy between the property holding and a nonstructure result for the class. Most of this paper is dedicated to the second half of the dichotomy; in this section we sum up the basic properties we are willing to assume.

We begin by fixing the language.

2.1. Vocabulary. Recall that each class $\mathbf{K}$ is a collection of structures of fixed vocabulary (i.e., similarity type) $\tau_{\mathbf{K}}$. We define a number of invariants below. We will require that the cardinality of $\tau_{\mathbf{K}}$ is less than or equal to any of our invariants. If we did not make this simplifying assumption, we would have to modify each invariant to the maximum of the current definition and $|\tau_{\mathbf{K}}|$. This would complicate the notation but not affect the arguments in any essential way.

As usual we denote by $\mathbf{K}_\lambda$ ($\mathbf{K}_{<\lambda}$) the collection of members of $\mathbf{K}$ with cardinality λ ($<\lambda$). In the next axiom we introduce a cardinal $\chi_1(\mathbf{K})$.

2.2. $\chi_1(\mathbf{K})$ introduced

Axiom S0. $\chi_1(\mathbf{K})$ is a regular cardinal greater than or equal to $|\tau_{\mathbf{K}}|$.

Now let us consider Löwenheim–Skolem phenomena. In the first-order case, the upwards Löwenheim–Skolem property is deduced from the compactness theorm; the downwards Löwenheim–Skolem property holds by the ability to form elementary submodels by adding *finitary* Skolem functions. In Section 3 we show that an upwards Löwenheim–Skolem property can be derived from the basic assumptions of [3].

The finitary nature of the Skolem functions in the first-order case guarantees that the hull of a set of power $\lambda > \chi_1(\mathbf{K})$ has power λ. Since we now deal with essentially infinitary functions, we cannot make this demand for all λ. If there are κ-ary functions it is likely to fail in cardinals of cofinality κ. We assume a downwards Löwenheim–Skolem property in many but not all cardinals. We justify this assumption in two ways. First the condition holds for the classes (most importantly, $\aleph_1$-saturated models of strictly stable theories) that we intend to consider. Secondly, the assumption holds for any class where the models can be generated by κ-ary Skolem functions for some κ that depends only on $\mathbf{K}$ and the similarity type.

2.3. Definition. $\mathbf{K}$ has the *λ-Löwenheim–Skolem property* if for each $M \in \mathbf{K}$ and $A \subseteq M$ with $|A| \leqslant \lambda$ there exists an N with $A \subseteq N \leqslant M$ and $|N| \leqslant \lambda$.

Replacing the two occurrences of $\leq\lambda$ in the definition of the λ-Löwenheim–Skolem property by $<\lambda$, we obtain the *$(<\lambda)$-Löwenheim–Skolem property*. If $\mu=\lambda^+$, then the λ-Löwenheim–Skolem property and the $(<\mu)$-Löwenheim–Skolem property are equivalent.

Note that $\mathbf{K}$ may have the λ-Löwenheim–Skolem property and fail to have the λ'-Löwenheim–Skolem property for some $\lambda'>\lambda$.

$\mathrm{LS}(\mathbf{K})$ denotes the least λ such that $\mathbf{K}$ has the λ-Löwenheim–Skolem property.

2.4. Downward Löwenheim–Skolem property

Axiom S1. There exists a χ such that for every λ, if $\lambda^\chi=\lambda$, then $\mathbf{K}$ has the λ-Löwenheim–Skolem property.

Notation. (i) $\chi_1(\mathbf{K})$ denotes the least such χ that is $\geq|\tau_{\mathbf{K}}|$.

(ii) $\chi_{\mathbf{K}}=\sup(\chi_1(\mathbf{K}),2^{\mathrm{LS}(\mathbf{K})})^+$.

2.5. Example. Examination of Example 1.5 shows that as stated it does not satisfy the λ-Löwenheim–Skolem property for any λ. An appropriate modification is to consider the class $\mathbf{K}^\mu$ of models of T that are $\aleph_1$-saturated but each E_ω class has less than μ elements. Then $\mathbf{K}^\mu$ satisfies the λ-Löwenheim-Skolem property for any $\lambda\geq\mu$ and we are able to apply our main results.

We easily deduce from the $<\lambda$-Löwenheim–Skolem property the following decomposition of members of $\mathbf{K}_\lambda$. Note that no continuity requirement is imposed on the chain.

2.6. Proposition. *If $\mathbf{K}$ satisfies the $<\lambda$-Löwenheim–Skolem property, λ is regular, and $M\in\mathbf{K}$ has cardinality λ, then M can be written as $\bigcup_{i<\lambda}M_i$ where each M_i has power less than λ and $M_i\leq M_j\leq M$ for $i<j<\lambda$.*

We describe chains by a pair of cardinals (size, cofinality) bounding the size of the models in the chain and the cofinality of the chain.

2.7. Notation. A (λ,κ)-chain is a $\mathbf{K}$-increasing chain ($i<j$ implies $M_i\leq M_j$) of cofinality κ of $\mathbf{K}$-structures which each have cardinality λ.

We define in the obvious way variants on notations of this sort such as a $(<\lambda,\kappa)$ chain. Unfortunately, different decisions about $<$ versus $\leq$ are required at different points and the complications of notation are needed.

2.8. Definition. (i) A chain $\mathfrak{M}$ is *bounded* if for some $M\in\mathbf{K}$ there is a $\mathbf{K}$-embedding of $\mathfrak{M}$ into M.

(ii) $\mathbf{K}$ is *$(\leq\lambda,\kappa)$-bounded* if every $(\leq\lambda,\kappa)$-chain is bounded.

To assert $\mathbf{K}$ is $(\leq\lambda,\kappa)$-bounded imposes a nontrivial condition even in the presence of Axiom Ch2$'$ because there is no continuity assumption on the chain.

Moreover, a demand of boundedness is not comparable to a demand for the Löwenheim–Skolem property; it is a demand that a certain abstract diagram have a concrete realization. It is easy to construct examples of abstract classes where boundedness fails if there is a bound on the size of the models in the class. We describe several more interesting examples in paragraph 4.3.

2.9. Alternatives. Here is a natural further notion. $\mathbf{K}$ is $\mathbf{K}$-*weakly bounded* (with appropriate parameters) if every $\mathbf{K}$-continuous chain is bounded. We will not actually have to consider this notion because the existence of canonically prime models implies that $\mathbf{K}$ is $\mathbf{K}$-weakly bounded.

2.10. Definition. λ is $\mathbf{K}$-*inaccessible* if for any $\mathbf{K}$-diagram $\mathfrak{M}$ with the sum of the cardinalities of the $M \in \mathfrak{M}$ less than λ, if there is a $\mathbf{K}$-embedding f of $\mathcal{M}$ into $N \in \mathbf{K}$, then there is an $N' \in \mathbf{K}$ with $|N'| < \lambda$ and an embedding f' of $\mathcal{M}$ into N' such that $\operatorname{rng} f'$ satisfies all the independence relations that $\operatorname{rng} f$ does and such that N and N' are compatible over $\mathcal{M}$ via f, f'.

This definition is slightly stronger than the one in [17] and we report this fact in the following proposition. It is strictly weaker than assuming the λ-Löwenheim–Skolem property.

2.11. Proposition. *If λ is $\mathbf{K}$-inaccessible then*

(i) *any free amalgam $\langle M_0, M_1, M_2\rangle$ with $|M_1|, |M_2| < \lambda$ can be extended to $\langle M_0, M_1, M_2, M_3\rangle$ with $|M_3| < \lambda$;*

(ii) *any $(<\lambda, <\lambda)$ chain which is bounded is bounded by a model with cardinality less than λ.*

Since λ^+ is $\mathbf{K}$-inaccessible if $\mathbf{K}$ satisfies the λ-Löwenheim–Skolem property, we deduce from Axiom S1 the following proposition. The first clause shows there are an abundance of $\mathbf{K}$-inaccessible cardinals. For many of the results of this paper it suffices for λ to be $\mathbf{K}$-inaccessible rather than requiring the $<\lambda$-Löwenheim–Skolem property.

2.12. Lemma. *Suppose λ is greater than $\chi_{\mathbf{K}}$ and $\mathbf{K}$ satisfies Axiom* S1.

(i) *If $\lambda^{\chi_1(\mathbf{K})} = \lambda$, then λ^+ is $\mathbf{K}$-inaccessible.*

(ii) *If λ is a strongly inaccessible cardinal, then λ is $\mathbf{K}$-inaccessible.*

2.13. Lemma. *If $\mathfrak{M}$ is a $(<\lambda, <\lambda)$ chain, λ is $\mathbf{K}$-inaccessible and M is canonically prime over $\mathfrak{M}$, then $|M| < \lambda$.*

The following examples show that some of the classes we want to investigate have models only in intermittent cardinalities.

2.14. Examples. Let **K** be the class of $\aleph_1$-saturated models of a countable strictly stable theory T.

(i) If $\lambda^\omega > \lambda$, then there are sets with power λ which are contained in no member of **K** with power λ.

(ii) If T is the model completion of the theory of countably many unary functions, there is no member of **K** with power λ if $\lambda^\omega > \lambda$.

We modify our notion of adequate class from [3, Section 3] to incorporate these ideas.

2.15. Adequate Class. We assume in this paper Axiom Groups A and C, Axioms D1 and D2 from group D (all from [3]), Axioms Ch1', Ch2' and Ch4 from Section 1, and Axioms S0 and S1 from this section. A class **K** satisfying these conditions is called *adequate.*

One of our major uses of the Löwenheim–Skolem property is to guarantee the existence of **K**-inaccessible cardinals as in Lemma 2.12. We now note that this conclusion can be deduced from very weak model theory and a not terribly strong 'large cardinal' hypothesis. We begin by describing the set-theoretic hypothesis.

2.16. Definition. We say ∞ is *Mahlo* if for any class C of cardinals that is closed and unbounded in the class of all cardinals, there is a regular cardinal μ such that $C \cap \mu$ is an unbounded subset of μ.

In fact, the μ of the definition could be taken as strongly inaccessible since the strong limit cardinals form a closed unbounded class. The assertion that ∞ is Mahlo is not provable in ZFC. Using Proposition 2.11 we have the next theorem.

2.17. Theorem. *Suppose ∞ is Mahlo and that* **K** *is a class of τ-structures that is closed under isomorphism, satisfies axiom* C1 *(existence of free amalgamations of pairs) and is $(<\infty,<\infty)$-bounded. Then the class of* **K**-*inaccessible cardinals is unbounded. In fact, it has nonempty intersection with any closed unbounded class of cardinals.*

Proof. For any cardinal λ, let $J(\lambda)$ be the least cardinal such that for any diagram $\mathfrak{M}$ such that the sum of the cardinalities of structures in $\mathfrak{M}$ is less than λ and for any N that bounds $\mathfrak{M}$ there is an N' with $|N'| < J(\lambda)$ and an embedding of $\mathfrak{M}$ into N' preserving all the independence relations among structures from $\mathfrak{M}$ that hold in N. Now the set $C = \{\lambda\colon \mu < \lambda \text{ implies } J(\mu) < \lambda\}$ is closed and unbounded. Since ∞ is Mahlo, there is an inaccessible cardinal χ with $C \cap \chi$ unbounded in χ. But then χ is **K**-inaccessible. □

It is easy to vary this argument to show there are actually a proper class of **K**-inaccessibles and indeed that that class is 'stationary'.

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3. Upwards Löwenheim–Skolem phenomena

As the examples in paragraph 2.14 show, it is impossible to get the full Löwenheim–Skolem–Tarski phenomenon — models in all sufficiently large cardinals — in the most general situation we are studying. Nevertheless we can establish an upwards Löwenheim–Skolem theorem. We show that $\chi_{\mathbf{K}}$ is a Hanf number for models of **K**.

These results generalize (with little change in the proof) and imply [18, Fact V.1.2].

3.1. Remark. The real significance of the following theorem is that it does not rely on axiom C7 (disjointness). With the axiom the second part of the following theorem is trivial. We asserted in [3] that the use of C7 was primarily to ease notation; this argument keeps us true to that assertion.

Recall that we have assumed for simplicity that $|\tau_{\mathbf{K}}| \leq \chi_{\mathbf{K}}$.

3.2. Theorem. *Suppose* **K** *has the χ-Löwenheim–Skolem property and there is a member M of* **K** *with cardinality greater than 2^{χ}. Then*

(i) *there exist $\langle M_0, M_1, M_2, M_3\rangle$ such that* $\mathrm{NF}(M_0, M_1, M_2, M_3)$ *and there is a nontrivial (i.e., not the identity on M_1) isomorphism of M_1 onto M_2 over M_0;*

(ii) *there exist arbitrarily large members of* **K**.

Proof. The proof of conclusion (i) is exactly as in [18]. That is, since $|M| > 2^{\chi}$, by the χ-Löwenheim–Skolem property, we can fix $M_0 \leq M$ with $|M_0| \leq \chi$ and choose for each $c \in M - M_0$ an N_c with $M_0 \leq N_c \leq M$, $c \in N_c$, and $|N_c| \leq \chi$. Expand the language L of **K** to L' by adding names for $\{d\colon d \in M_0\}$ and let L'' contain one more constant symbol. There are at most 2^{χ} isomorphism types of L''-structures $\langle N_c, c\rangle$ satisfying the diagram of M_0 so there are $c_1 \neq c_2 \in M$ with $\langle N_{c_1}, c_1\rangle \approx \langle N_{c_2}, c_2\rangle$. Thus there is an isomorphism f from N_{c_1} onto N_{c_2} *over* M_0 with $f(c_1) = c_2$. Applying axiom C2 (existence of free amalgams), we can choose M_3 and $g\colon N_{c_1} \approx M_1$ over M_0 with $\mathrm{NF}(M_0, M_1, M, M_3)$. Now by monotonicity we have both $\mathrm{NF}(M_0, M_1, N_{c_1}, M_3)$ and $\mathrm{NF}(M_0, M_1, N_{c_1}, M_3)$. Let c denote $g(c_1)$. Now not both $g^{-1}(c) = c_1$ and $f \circ g^{-1}(c) = c_2$ equal c. So one of N_{c_1} and N_{c_2} can serve as the required M_2.

Our proof of the existence of arbitrarily large models actually only relies on conclusion (i). Let $c \in M_1$ be such that the isomorphism f of M_1 and M_2 moves c. For any λ, we define by induction on $\alpha \leq \lambda$ a **K**-continuous sequence of models M^{α} such that $|M^{\lambda}| \geq \lambda$ as required. As an auxiliary in the construction we define f^{α} and N^{α} such that f^{α} is a nontrivial isomorphism between M_3 and N^{α}. We demand $\mathrm{NF}(M_0, N^{\alpha}, M^{\alpha}, M^{\alpha+1})$.

For $\alpha = 0$, let $M^0 = M_3$. At stage $\alpha + 1$ we define f^{α}, N^{α} and $M^{\alpha+1}$ by invoking the existence axiom to obtain $\mathrm{NF}(M_0, N^{\alpha}, M^{\alpha}, M^{\alpha+1})$ and $f^{\alpha}\colon M_3 \to N^{\alpha}$. For limit α, choose M^{α} canonically prime over its predecessors.

To obtain the cardinality requirement is suffices to show that if $\alpha < \lambda$, then $f^\alpha(c) \notin M^\alpha$. Fix α and let A_1 denote $f^\alpha(M_1)$ and A_2 denote $f^\alpha(M_2)$. We have $\mathrm{NF}(M_0, A_1, M^\alpha, M^{\alpha+1})$ and $\mathrm{NF}(M_0, A_2, M^\alpha, M^{\alpha+1})$ by the construction. Again from the construction $g^\alpha = f^\alpha \mid M_2 \circ f \circ (f^\alpha \mid M_1)^{-1}$ is an isomorphism between A_1 and A_2 over M_0. By the weak uniqueness axiom C5 (see [3, Proposition 1.1.7]), g^α extends to an isomorphism g_α between A_1 and A_2 which fixes M^α pointwise. Now, if $f^\alpha(c) \in M^\alpha$, $g^\alpha(f^\alpha(c)) = g_\alpha(f^\alpha(c)) = (f^\alpha(c))$. But $g^\alpha(f^\alpha(c)) = f^\alpha(f(c))$ (by the definition of g^α) so f fixes c. This contradiction yields conclusion (ii). □

Noticing that the existence of a nontrivial map implies the existence of a nontrivial amalgamation and that only conclusion (i) was used in the proof of conclusion (ii), we can reformulate the theorem as follows.

3.3. Corollary. *Suppose* **K** *does not have arbitrarily large models. Then all members M of* **K** *have cardinality less than $\chi_{\mathbf{K}}$. Moreover, if $N \leqslant M \in \mathbf{K}$, there is no nontrivial automorphism of M fixing N.*

Proof. Note the definition of $\chi_{\mathbf{K}}$ (paragraph 2.4) and apply Theorem 3.2 with χ as $\mathrm{LS}(\mathbf{K})$. Thus the models of a class with a bound on the size of its models are all 'almost rigid'. These arguments give some more local information. □

3.4. Definition. The structure M is a *maximal* model in **K** if there is no proper **K**-extension of M.

3.5. Corollary. (i) *If $|M| > 2^\chi$ and* **K** *has the χ-Löwenheim–Skolem property, then M is not a maximal model in* **K**.

(ii) *Thus if $|M| \geqslant \chi_{\mathbf{K}}$, M is not a maximal model.*

(iii) *If particular if μ is* **K***-inaccessible and $\chi_{\mathbf{K}} \leqslant |M| < \mu$, then M has a proper extension of cardinality less than μ.*

Proof. The first two propositions just restate the previous results in this language. For the third observe that for each $\alpha < \lambda \leq \mu$ in the construction for the second part of Theorem 3.2, $|M_\alpha| < \mu$ by the definition of **K**-inaccessibility and Lemma 2.13. □

4. Tops for chains

We discuss in this section several requirements on a model that bounds a chain. Shelah has emphasized (e.g., [16, 17]) that the Tarski union theorem has two aspects. One is the assertion that the union of an elementary chain is an elementary extension of each member of the chain and thus a member of any elementary class containing the chain; the second is the assertion that the union is an elementary submodel of any elementary extension of each member of the chain. First we consider the second aspect.

4.1. Definition. (i) The class **K** is $(<\lambda, \kappa)$-*smooth* if there is a unique compatibility class over every $(<\lambda, \kappa)$-chain.

(ii) **K** is *smooth* if it is $(<\infty, <\infty)$-smooth.

Note that if the class **K** is $(<\lambda, \kappa)$-*smooth*, then the canonically prime model over any essentially **K**-continuous $(<\lambda, \kappa)$ chain is absolutely prime. Moreover, if **K** is smooth, every **K**-increasing chain is essentially **K**-continuous.

The next two notions represent the first aspect of Traski's theorem; the third unites both aspects.

4.2. Closure under unions of chains. (i) **K** is $(<\lambda, \kappa)$-*closed* if for any $(<\lambda, \kappa)$-chain $\mathfrak{M}$ inside a structure N (not necessarily in **K**), the union of $\mathfrak{M}$ is in **K** and for each i, $M_i \leqslant \bigcup \mathfrak{M}$.

(ii) **K** is $(<\lambda, \kappa)$-*weakly closed* if for any $(<\lambda, \kappa)$ **K**-continuous chain $\mathfrak{M}$ inside a structure N (not necessarily in **K**), the union of $\mathfrak{M}$ is in **K** and for each i, $M_i \leqslant \bigcup \mathfrak{M}$.

(iii) **K** is *fully* $(<\lambda, \kappa)$-*smooth* if the union of every **K**-continuous $(<\lambda, \kappa)$-chain inside a structure N (not necessarily in **K**) is in **K** and is absolutely prime over the chain.

The "inside N" in these definitions is perhaps misleading. We have not asserted $N \in \mathbf{K}$, so this is not an a priori assumption of boundedness. In fact N must exist as the union of $\mathfrak{M}$. If **K** is (λ, κ)-closed it is (λ, κ)-bounded as the union serves as the bound.

If **K** is $(<\lambda, \kappa)$-*smooth* and $(<\lambda, \kappa)$-*weakly closed*, then for any $(<\lambda, \kappa)$ **K**-continuous chain $\mathfrak{M}$ inside N the union of $\mathfrak{M}$ is the canonically prime model over $\mathfrak{M}$ and **K** is *fully* $(<\lambda, \kappa)$-*smooth*.

If **K** is the class of $\aleph_1$-saturated models of a strictly stable countable theory, **K** is not closed under unions of countable cofinality but is closed under unions of larger cofinality. This property of a class being closed under unions of chains with sufficiently long cofinality is rather common. For example, any class definable by Skolem functions with infinite but bounded arity will have this property. We can rephrase several properties of some of the examples in Section 1 in these terms.

4.3. Examples. (i) The class **K** of Example 1.3 is $(\infty, \geqslant\aleph_2)$-bounded, $(<\infty, \geqslant\aleph_2)$-closed, and even fully $(<\infty, \geqslant\aleph_2)$-smooth. But **K** is not $(<\aleph_1, \aleph_1)$-bounded and not $(<\aleph_1, \aleph_0)$- or $(<\aleph_1, \aleph_1)$-smooth.

(ii) Example 1.5 shows that for **K** the class of $\aleph_1$-saturated models of REI_ω and a particular choice of $\leqslant$, the class **K** is not smooth. For, e.g., the prime $\aleph_1$-saturated model over a chain and the prime $\aleph_2$-saturated model over the same chain may be incompatible.

Note that for any strictly stable countable theory and any uncountable κ, if **K** is the class of κ-saturated models of a countable strictly stable theory, $\leqslant$ denotes elementary submodel, and cpr means prime among the κ-saturated models, then **K** is smooth. In this case χ_1 is $\aleph_0$.

(iii) Consider again the class $\mathbf{K}^\mu$ discussed in Example 2.5. The union of a countable **K**-chain may determine an E_ω class that is not realized. So $\mathbf{K}^\mu$ is not $(<\mu, \omega)$-closed. But it is $(<\mu, [\aleph_1, <\mu])$-closed and $(<\infty, \infty)$-bounded.

Our basic argument will establish a dichotomy between the following weakening of smoothness and a nonstructure theorem.

4.4. Definition. The class **K** is $(<\lambda, \kappa)$-*semismooth* if for each $(<\lambda, \kappa)$ **K**-continuous chain $\mathfrak{M}$, each compatibility class over $\mathfrak{M}$ contains a canonically prime model over $\mathfrak{M}$.

The distinction between smooth and semi-smooth quickly disappears in the presence of Axiom Ch4.

4.5. Lemma. *If* **K** *is semismooth and satisfies Axiom* Ch4, *then* **K** *is smooth.*

Proof. Axiom Ch4 asserts that all M satisfying cpr$(\mathfrak{M}, M)$ are compatible. Since each compatibility class contains such an M, there is only one compatibility class. $\square$

4.6. Remark. By an argument similar to the main results of this paper (but much simpler) we can show for a proper class of λ that a class **K** that has prime models over independent pairs and is closed under unions of chains (of any length) is fully $(<\lambda, <\chi_1(\mathbf{K}))$-smooth unless **K** codes stationary subsets of λ (see Section 6). To establish this result we need the axioms about independence of pairs enumerated in [3] and that there is a proper class of **K**-inaccessible cardinals. The last condition can be guaranteed by assuming a Löwenheim–Skolem property like Axiom S1 or by assuming ∞ is Mahlo as in Theorem 2.17.

This situation is 'half way' between the situation in [17] and that considered here. We replace 'closed under substructure' by the existence of 'prime models over independent pairs' but retain taking limits by unions.

5. Some variants on $\square$

We discuss in this section some variants on Jensen's combinatorial principle $\square$ which will be useful in model-theoretic applications. We begin by establishing some notation.

5.1. Notation. (i) For any set of ordinals C, acc$[C]$ denotes the set of accumulation points of C — the $\delta \in C$ with $\delta = \sup C \cap \delta$. The nonaccumulation points of C, $C -$ acc$[C]$, are denoted nacc$[C]$.

(ii) For any set of ordinals S, $C^\kappa(S)$ denotes the elements of S with cofinality κ.

(iii) For any cardinal λ, $\mathrm{sing}(\lambda)$, the set of *singular ordinals* less than λ is the collection of limit ordinals less than λ that are not regular cardinals.

(iv) In the following δ always denote a limit ordinal.

The following definition is a version to allow singular cardinals of the $\square$ principle for κ^+ in [5]. We refer to it as 'full $\square$'. This principle has been deduced only from strong extensions of ZFC such as V = L [5]. We will also use here weaker versions, obtained by relativizing to a stationary set, that are provable in ZFC.

When $\lambda = \mu^+$, Jensen called the condition here a $\square$ on μ. The version here also applies to limit ordinals and, since we will deal with inaccessibles, seems preferable.

5.2. Definition. The sequence $\langle C_\delta : \delta \in \mathrm{sing}(\lambda)\rangle$ *witnesses* that λ satisfies $\square$ if it satisfies the following conditions.
- Each C_δ is a closed unbounded subset of δ.
- $\mathrm{otp}(C_\delta) < \delta$.
- If $\alpha \in \mathrm{acc}[C_\delta]$, then $C_\alpha = C_\delta \cap \alpha$.

Now we proceed to the relativized versions of $\square$. The relativization is with respect to two subsets, S and S^+. It is in allowing the C_α to be indexed by S^+ rather than all of λ that this principal weakens those of Jensen and can be established in ZFC.

We will consider two relativizations. In Section 8 we will see that the two set-theoretic principles will allow us two different model-theoretic hypotheses for the main result. They, in fact correspond to two different ways of assigning invariants to models.

5.3. Definition. We say that $S^+ \subseteq \lambda$ and $\langle C_\alpha : \alpha \in S^+\rangle$ *witness* that the subset S of $C^\kappa(\lambda)$ satisfies $\square^a_{\lambda,\kappa}(S)$ if $S \subseteq S^+$ and the following conditions hold.

(i) S is stationary in λ.

(ii) For each $\alpha \in S^+$, $C_\alpha \subseteq S^+ - S$.

(iii) If $\alpha \in S^+$ is not a limit ordinal, C_α is a closed subset of α.

(iv) If $\delta \in S^+$ is a limit ordinal, then

(a) C_δ is a club in δ;

(b) $\mathrm{otp}(C_\delta) \leqslant \kappa$;

(c) $\mathrm{otp}(C_\delta) = \kappa$ if and only if $\delta \in S$;

(d) all nonaccumulation points of C_δ are successor ordinals.

(v) For all $\beta \in S^+$, if $\alpha \in C_\beta$, then $C_\alpha = C_\beta \cap \alpha$.

5.4. Definition. $\square^a_{\lambda,\kappa}$ *holds* if for some subset $S \subseteq \lambda$, $\square^a_{\lambda,\kappa}(S)$ holds.

5.5. Fact. Suppose $\lambda > \kappa$ are regular cardinals. If λ is a successor of a regular cardinal greater than κ or $\lambda = \mu^+$ and $\mu^\kappa = \mu$, then for any stationary $S \subseteq C^\kappa(\lambda)$ there is a stationary $S' \subseteq S$ such that $\square^a_{\lambda,\kappa}(S')$ holds.

The proof with $\mu^\kappa = \mu$ is on [15, p. 276] (see also the appendix to [20]); for regular μ see [11, Theorem 4.1].

Fact 5.5 is proved in ZFC; if we want to make stronger demands on the stationary set S, we must extend the set theory.

5.6. Definition. The subset S of a cardinal λ is said to *reflect in* $\delta \in S$ if $S \cap \delta$ is stationary in δ. We say S *reflects* if S reflects in some $\delta \in S$.

Thus a stationary set that does not reflect is extremely sparse in that its various initial segments are not stationary.

5.7. Fact. If $\lambda > \kappa$ are regular, λ is not weakly compact, and $V = L$, then for any stationary $S_0 \subseteq C^\kappa(\lambda)$ there is a stationary $S \subseteq S_0$ that does not reflect such that $\square^a_{\lambda,\kappa}(S)$ holds.

This is a technical variant on the result of [4]. Although this result follows from $V = L$, it is also consistent with various large cardinal hypotheses.

We now consider the other relativization of $\square$. For it we need a new filter on the subsets of λ.

5.8. Definition. Let the stationary subset S of the regular cardinal λ index the family of sets $C^* = \{C_\delta : \delta \in S\}$ where each $C_\delta \subseteq \delta = \sup C_\delta$. Then $\mathrm{ID}(C^*)$ denotes the collection of subsets B of λ such that there is a cub C of λ satisfying: for every $\delta \in B \cap S$, C_δ is not contained in C. We denote the dual filter to $\mathrm{ID}(C^*)$ by $\mathrm{FIL}(C^*)$.

It is easy to verify that $\mathrm{ID}(C^*)$ is an ideal. Note that $B \notin \mathrm{ID}(C^*)$ if and only if for every club C, there is a $\delta \in B$ with $C_\delta \subseteq C$.

5.9. Definition $[\square^b_{\lambda,\kappa,\theta,R}(S, S_1, S_2)]$. Suppose $\theta \neq \kappa$, λ and R are four regular cardinals with $\theta < \lambda$, $\kappa^+ < \lambda$, $R \leqslant \lambda$ and S is a subset of λ containing all limit ordinals of cofinality $<R$. We say $\square^b_{\lambda,\kappa,\theta,R}(S, S_1, S_2)$ *holds* if the following conditions are satisfied for some $C^* = \langle C_\delta : \delta \in S \rangle$.

(i) $C^* = \langle C_\delta : \delta \in S \rangle$ is a sequence of subsets of λ satisfying
  - (a) C_δ is a closed subset of δ;
  - (b) $C_\delta \subseteq S$;
  - (c) if δ is a limit ordinal, then C_δ is unbounded in δ;
  - (d) if δ' is an accumulation point of C_δ, then $C_{\delta'} = C_\delta \cap \delta'$;
  - (e) if $\alpha < \delta_1, \delta_2$ and $\alpha \in C_{\delta_1} \cap C_{\delta_2}$, then $C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$.

(ii) S_1 and S_2 are disjoint subsets of $C^\kappa(S)$ with union $S_0 \subseteq S$.
(a) If $\delta \in S_0$, $\mathrm{otp}(C_\delta) = \kappa$.
(b) If $\beta \in \mathrm{nacc}[C_\delta]$ and $\delta \in S_0$, then $\mathrm{cf}(\beta) = \theta$.
(iii) The ideal $\mathrm{ID}(C^*)$ is nontrivial; S_1 and S_2 are not in $\mathrm{ID}(C^*)$.

5.10. Remark. Note that if $\delta \in S_0$ and $\beta \in C_\delta$, then $\mathrm{cf}(\beta) < \kappa$ or $\mathrm{cf}(\beta) = \theta$ so $C_\delta \cap S_0 = \emptyset$.

5.11. Definition [$\square^b_{\lambda,\kappa,\theta,R}$]. We say $\square^b_{\lambda,\kappa,\theta,R}$ *holds* if there exist subsets $S \subset \lambda$ and $C_\beta \subseteq S$ for $\beta \in S$ that satisfy the following conditions.

(i) S contains each of a family $\langle T_i : i < \lambda \rangle$ of sets; each $T_i \subseteq C^\kappa(\lambda)$ and the T_i are pairwise disjoint and not in $\mathrm{ID}(C^*)$.
(ii) For each $A \subseteq \lambda$ there exist $S_1, S_2 \subseteq S$ such that
(a) $S_1 = \bigcup_{i \in A} T_i$ and $S_2 = \bigcup_{i \notin A} T_i$ and
(b) $\square^b_{\lambda,\kappa,\theta,R}(S, S_1, S_2)$ holds with $C^* = \langle C_\beta : \beta \in S \rangle$.

Now the set-theoretic strength required for these combinatorial principles can be summarized as follows.

5.12. Theorem. (i) *If λ is a successor of a regular cardinal, θ and κ are regular cardinals with $\theta^+ < \lambda$ and $\kappa^+ < \lambda$, then $\square^b_{\lambda,\kappa,\theta,\aleph_0}$ is provable in* ZFC.

(ii) *If λ is a successor cardinal, $\theta < \lambda$, and $\kappa < \lambda$, then $\square^b_{\lambda,\kappa,\theta,R}$ is provable in* ZFC + V = L *for any $R \leqslant \lambda$.*

Proof. Case (i) is proved in [18, III.6.4, III.7.8 F(3)] and in [11]. For Case (ii) consult [18, III.7.8G]. □

5.13. Alternative set-theoretic hypotheses. There are a number of refinements on conditions sufficient to establish $\square^b_{\lambda,\kappa,\theta,R}$.

(i) If λ is a successor of a regular cardinal or even just 'not Mahlo', Theorem 5.12(ii) can be strengthened by replacing "V = L", by "there is a square on λ". See [18, III.7.8H].

(ii) In fact the conclusion of Theorem 5.12(ii) holds for any λ that is not weakly compact (similar to [4]).

(iii) The existence of a function F such that $\square^b_{\lambda,\kappa,\theta,R}$ holds for any regular $\lambda > F(\lambda + \kappa + R)$ and $\theta < \lambda$ is consistent with ZFC + there is a class of supercompact cardinals.

6. Invariants

As a first approximation we say a class **K** has a nonstructure theorem if for many λ, **K** has 2^λ models of power λ. But this notion can be refined. For some classes **K** it is possible to code stationary subsets of regular λ by models of **K** in a uniform and absolute way while other classes have many models for less uniform reasons. The distinction between these cases is discussed in [14, 19]. In the stronger situation we say, informally, that **K** *codes stationary sets*. We do not give a formal general definition of this notion, but the two coding functions we describe below sm, sm_1 epitomize the idea.

The basic intention is to assign to each model a stationary set so that at least modulo some filter on subsets of λ nonisomorphic models yield distinct sets. Historically (e.g., [17]) to assign such an invariant one writes the model M as an ascending chain of submodels and asks for which limit ordinals is the chain continuous. The replacement of continuity by **K**-continuity in this paper makes this procedure more difficult. We can succeed in two ways. Either we add an additional axiom about canonically prime models and proceed roughly as before or we work modulo a different filter. Both of these solutions are expounded here.

Naturally we are mainly interested in classes **K** that are *reasonably absolute.* That is, the property that a structure M is in **K** should be preserved between V and L and between V and reasonable forcing extensions of V. Of course a first-order class or a class in a $L_{\infty,\lambda}$ meets this condition (a reasonable forcing in this context would preserve the family of sequences of length $<\lambda$ or ordinals $\leqslant\lambda$). But somewhat less syntactic criteria are also included. For example, if **K** is the class of $\aleph_1$-saturated models of a strictly stable theory, membership in **K** is preserved if we do not add countable sets or ordinals.

Clearly, **K** codes stationary sets implies **K** has 2^λ models of power λ. But it is a stronger evidence of nonstructure in several respects. First, the existence of many models is preserved under any forcing extension that does not add bounded subsets of λ and does not destroy the stationarity of subsets of λ. Secondly, the existence of many models on a proper class of cardinals is not such a strong requirement; for example, a multidimensional (unbounded in the nomenclature of [2]) theory has $2^{\aleph_\alpha}$ models of power $\aleph_\alpha$ whenever $\aleph_\alpha = \alpha$. However, the class of models of a first-order theory codes stationary sets only if T is not superstable or has the dimensional order property or has the omitting types order property.

6.1. Definition. (i) A representation of a model M with power λ (with λ regular) is an increasing chain $\mathfrak{M} = \langle M_i : i < \lambda \rangle$ of **K**-substructures of M such that each M_i has cardinality less than λ and $\bigcup \mathfrak{M} = M$.

(ii) the representation is *proper* if $\bigcup \mathfrak{M} \mid \delta \leqslant M$ implies $M_\delta = \bigcup \mathfrak{M} \mid \delta$.

We showed in Proposition 2.6 that if λ is a regular cardinal greater than $\chi_1(\mathbf{K})$ and **K** satisfies the λ-Löwenheim–Skolem property, then each model of power λ has a representation. We will not however have to invoke the Löwenheim–Skolem property in our main argument because we analyze models that are constructed with a representation. Using axiom A3, it is easy to perturb any given representation into a proper representation.

We now show how to define invariant functions in our context. For the first version we need an additional axiom.

6.2. Axiom Ch5. For every **K**-continuous chain $\mathfrak{M} = \langle M_i, f_{i,j} : i, j < \delta \rangle$ and every unbounded $X \subseteq \delta$, a **K**-structure M is canonically prime over $(\mathfrak{M}, f)$ if and only if M is canonically prime over $(\mathfrak{M} \restriction X, f \restriction X)$.

We refer to this axiom by saying that *canonically prime models behave on subsequences.*

6.3. Definition. Let $\mathfrak{M}$ be a representation of M. Let $\mathrm{sm}(\mathfrak{M}, M)$ denote the set of limit ordinals $\delta < \lambda$ such that for some X unbounded in δ a canonically prime model N over $\mathfrak{M} \restriction X$ is a **K**-submodel of M.

6.4. Lemma. *If $\mathfrak{M}$ and $\mathfrak{N}$ are representations of M and Axiom* Ch5 *holds, then* $\mathrm{sm}(\mathfrak{M}, M) = \mathrm{sm}(\mathfrak{N}, M)$ *modulo the closed unbounded filter on λ.*

Proof. Since $|M|$ is regular, there is a cub C on λ such that every $\delta \in C$ is a limit ordinal and $\bigcup \mathfrak{M} \restriction \delta = \bigcup \mathfrak{N} \restriction \delta$ for $\delta \in C$. Now we claim that for $\delta \in C$, $\delta \in \mathrm{sm}(\mathfrak{M}, M)$ if and only if $\delta \in \mathrm{sm}(\mathfrak{N}, M)$. To see this choose an increasing sequence L_i alternately from $\mathfrak{M}$ and $\mathfrak{N}$. Since cpr behaves on subsequences, the canonically prime model over the common subsequence of $\mathfrak{L}$ and $\mathfrak{M}$ is a **K**-submodel of M if and only if the canonically prime model over $\mathfrak{L}$ is, and similarly for the common subsequence of $\mathfrak{L}$ and $\mathfrak{N}$. Thus, $\delta \in \mathrm{sm}(\mathfrak{N}, M)$ if and only if $\delta \in \mathrm{sm}(\mathfrak{N}, M)$. $\square$

This lemma justifies the following definition.

6.5. Definition. Denote the equivalence class modulo $\mathrm{cub}(\lambda)$ of $\mathrm{sm}(\mathfrak{M}, M)$ for some (any) representation $\mathfrak{M}$ of M by $\mathrm{sm}(M)$. We call $\mathrm{sm}(M)$ the *smoothness set of M.*

We now will describe a second way to assign invariants to models. This approach avoids the reliance on Axiom Ch5 at the cost of complicating (but not increasing the strength of) the set theory. Recall from Section 5 the ideal $\mathrm{ID}(C^*)$ assigned to a family of sets $C^* = \{C_\beta : \beta \in S\}$. Fix for the following definition and arguments subsets S, S_1, S_2 satisfying $\square^b_{\lambda,\kappa,\theta,R}(S, S_1, S_2)$. We define a second invariant function with C^* as a parameter. It distinguishes models modulo $\mathrm{ID}(C^*)$.

6.6. Definition. Fix a subset S of λ and $C^* = \{C_\beta : \beta \in S\}$. Let $\mathfrak{N}$ be a representation of M. Let $\mathrm{sm}_1(\mathfrak{N}, C^*, M)$ denote the set of $\delta \in S$ such that

(i) for every $\gamma \in \mathrm{nacc}[C_\delta]$, $N_\gamma = \bigcup_{\alpha<\gamma} N_\alpha$;

(ii) $\mathfrak{N} \mid C_\delta$ is **K**-continuous;

(iii) there is an N'_δ canonically prime over $\mathfrak{N} \mid \mathrm{nacc}[C_\delta]$ that can be **K**-embedded into M over $\mathfrak{N} \mid \mathrm{nacc}[C_\delta]$.

We are entitled to choose an N'_δ canonically prime over $\mathfrak{N} \mid \mathrm{nacc}[C_\delta]$ in condition (iii) because condition (i) guarantees that $\mathfrak{N} \mid \mathrm{nacc}[C_\delta]$ is **K**-continuous.

6.7. Lemma. *If $\mathfrak{M}$ and $\mathfrak{N}$ are proper representations of M, then*

$$\mathrm{sm}_1(\mathfrak{N}, C^*, M) = \mathrm{sm}_1(\mathfrak{M}, C^*, M) \textit{ modulo } \mathrm{FIL}(C^*).$$

Proof. Let X_1 denote $\mathrm{sm}_1(\mathfrak{N}, C^*, M)$ and X_2 denote $\mathrm{sm}_1(\mathfrak{M}, C^*, M)$. Without loss of generality we can assume the universe of M is λ. There is a cub C containing only limit ordinals such that for $\delta \in C$, $\delta = \bigcup_{\alpha<\delta} M_\alpha = \bigcup_{\alpha<\delta} N_\alpha$.

To show $X_1 = X_2 \bmod \mathrm{FIL}(C^*)$, it suffices to show there is a $Y \in \mathrm{FIL}(C^*)$ such that $X_1 \cap Y = X_2 \cap Y$. Let $Y = \{\delta: C_\delta \subseteq C\}$.

Suppose $\delta \in Y \cap X_1$. If $\alpha \in \mathrm{nacc}[C_\delta]$, then $\delta \in Y$ implies $\alpha \in C$ which in turn implies $\alpha = \bigcup_{i<\alpha} N_i = \bigcup_{i<\alpha} M_i$. Now $\delta \in X_1$ implies $N_\alpha = \bigcup_{i<\alpha} N_i$ and $N_\alpha \leq M$ so $\bigcup_{i<\alpha} N_i \leq M$ and thus $\bigcup_{i<\alpha} M_i \leq M$. But then by properness $M_\alpha = \bigcup_{i<\alpha} M_i$. Thus, $M_\alpha = N_\alpha$. That is, $\mathfrak{N} \mid \mathrm{nacc}[C_\delta] = \mathfrak{M} \mid \mathrm{nacc}[C_\delta]$. So $\delta \in X_1$ if and only if $\delta \in X_2$. $\square$

In view of the previous lemma we make the following definition.

6.8. Definition. For any M in the adequate class **K** and some (any) proper representation $\mathfrak{M}$ of M, $\mathrm{sm}_1(C^*, M) = (\mathrm{sm}_1(\mathfrak{M}, C^*, M)/\mathrm{FIL}(C^*))$.

7. Games, strategies and double chains

We will formulate one of the main model-theoretic hypotheses for the major theorem deriving nonstructure from nonsmoothness in terms of the existence of winning strategies for a certain game. In this section we describe this game and show how to derive a winning strategy for it from the assumption that **K** is not smooth.

7.1. Definition. A play of Game 1 (λ, κ) lasts κ moves. Player I chooses models L_i and Player II chooses models P_i subject to the following conditions. At move β, (i) Player I chooses a model L_β in **K** of power less than λ that is a proper **K**-extension of all the structures P_γ for $\lambda < \beta$. If β is a limit ordinal less than κ, L_β must be chosen canonically prime over $\langle P_\gamma: \gamma < \beta \rangle$;

(ii) Player II chooses a model P_β in **K** of power less than λ that is a **K**-extension of L_β.

Any player who is unable to make a legal move loses. Player I wins the game if there is a model $P_\kappa \in \mathbf{K}$ that extends each P_β for $\beta < \kappa$ but the sequence $\langle P_i : i \leq \kappa \rangle$ is not essentially **K**-continuous.

In order to establish that nonsmoothness implies a winning strategy for Player I we need to consider certain properties of double chains. We introduce here some notation and axioms concerning this kind of diagram.

7.2. Definition. (i) $\mathfrak{M} = \langle \mathfrak{M}^0, \mathfrak{M}^1 \rangle = \{\langle M_i^0, M_i^1 \rangle : i < \delta\}$ is a *double chain* if each $M_i^0 \leq M_i^1$ and $\mathfrak{M}^0, \mathfrak{M}^1$ are **K**-increasing chains. We say $\mathfrak{M}$ is (separately) (**K**)-continuous if each of $\mathfrak{M}^0$ and $\mathfrak{M}^1$ is (**K**)-continuous.

(ii) $\mathfrak{M}$ is a *free double chain* if for each $i < j < \delta$, $M_j^0 \downarrow_{M_i^0} M_i^1$ inside M_{j+1}^1.

(iii) $\mathfrak{M} = \langle \mathfrak{M}^0, \mathfrak{M}^1 \rangle = \{\langle M_i^0, M_j^1 \rangle : i \leq \delta + 1, j < \delta\}$ is a **K**-*continuous augmented double chain* inside N if $i < \delta$ implies $M_i^0 \leq M_i^1$, and $\mathfrak{M}^0, \mathfrak{M}^1$ are increasing **K**-continuous chains inside N.

(iv) An augmented double chain is *free* inside N if for each $i < \delta$,

$$M_i^1 \underset{M_i^0}{\downarrow} M_{\delta+1}^0 \quad \text{inside } N.$$

We extend the existence axiom Ch2′ for a prime model over a chain to assert the compatibility of the prime models guaranteed for each sequence in a double chain.

7.3. Axioms concerning double chains

DC1. If $\mathfrak{M}$ is an essentially **K**-continuous free double chain and M_1 is canonically prime over $\mathfrak{M}^1$, then there is an M_0 that is canonically prime over $\mathfrak{M}^0$ such that M_0 and M_1 are compatible over $\mathfrak{M}^0$.

DC2. If $\mathfrak{M}$ is an essentially **K**-continuous free augmented double chain of length δ in M, then there is an N with $m \leq N$ and an $M_\delta^1 \leq N$ such that

$$M_\delta^1 \underset{M_\delta^0}{\downarrow} M_{\delta+1}^0 \quad \text{inside } N$$

and the chain $\mathfrak{M}^1 \cup \{M_\delta^1\}$ is essentially **K**-continuous.

We will refer to versions of these axioms for chains of restricted length; we may denote the variant of the axiom for chains of length less than κ as DCi($<\kappa$).

Note that it would be strictly stronger in DC2 to assert that M_δ^1 is canonically prime over $\mathfrak{M}^1$ since under DC2 as stated the canonically prime model over $\mathfrak{M}^1$ inside M_δ^1 need not contain M_δ^0.

Since we are going to use these axioms to establish smoothness we indicate some relationships between the properties. **K** is $(<\infty, \kappa)$-smooth means that every **K**-continuous chain of cofinality κ has a single compatibility class over it — necessarily there will be a canonically prime model in that class. DC1 would hold if there were many compatibility classes over a chain but each had a canonically prime model (i.e., **K** is semismooth). In particular it holds at κ if **K** is $(<\infty, \kappa)$-smooth (sometimes read smooth at κ). Thus, the following lemma is easy.

7.4. Lemma. *If* **K** *is an adequate class that is* $(<\infty, \kappa)$*-smooth, then* **K** *satisfies* DC1 *for chains of cofinality* κ.

Now we come to the main result of this section.

7.5. Lemma. *Let* **K** *be an adequate class that is* $(<\lambda, \kappa)$*-bounded and suppose* **K** *is* $(<\lambda, <\kappa)$*-smooth but not* $(<\lambda, \kappa)$*-smooth. Suppose further that* **K** *satisfies* DC1 *and* DC2 *and* $\lambda \geq \chi_{\mathbf{K}}$ *is* **K***-inaccessible. Then* Player I *has a winning strategy for Game* 1 (λ, κ).

Proof. Since **K** is $(<\lambda, <\kappa)$-smooth, we can choose a counterexample $\mathfrak{N} = \langle N_i : i < \kappa \rangle$ to $(<\lambda, \kappa)$-smoothness that is essentially **K**-continuous. Then $\mathfrak{N}$ is bounded by two models N_κ and N'_κ with cardinality $<\lambda$ that are incompatible over $\mathfrak{N}$. If there exist M_κ and M'_κ canonically prime over $\mathfrak{N}$ and embeddable in N_κ and N'_κ, respectively, axiom Ch4 requires that M_κ and M'_κ are compatible. But then so are N_κ and N'_κ. From this contradiction we conclude without loss of generality that each $N_i \leq N_\kappa$ but that no canonically prime model over $\mathfrak{N}$ can be **K**-embedded into N_κ. That is, **K** is not semismooth (Definition 4.4). Now Players I and II will choose models $\langle L_i : i < \kappa \rangle$ and $\langle P_i : i < \kappa \rangle$ for a play of Game 1.

We describe a winning strategy for Player I. The construction requires auxiliary models P'_i, N^*_i, and L'_i and isomorphisms $\alpha_i : L'_i \to L_i$. They will satisfy the following conditions.

(i) $\mathfrak{P}'$ and $\mathfrak{L}'$ are essentially **K**-continuous sequences and the α_i are an increasing sequence of maps.

(ii) $P'_i \downarrow_{N_i} N_\kappa$ inside N^*_{i+1}.

(iii) L'_{i+1} is prime over $P'_i \cup N_{i+1}$ inside N^*_{i+1}.

(vi) α_i is an isomorphism between L'_i and L_i mapping P'_j onto P_j for $j < i$.

(v) The N^*_i form an essentially **K**-continuous sequence with $N_i \leq N^*_i$.

Let $L_0 = N_0$. Each successor stage is easy. Player II has chosen $P_i \in \mathbf{K}_{<\lambda}$ to extend L_i. For Player I's move, apply axiom D1 (existence of free amalgamations) to first choose N^*_{i+1} to extend N^*_i and P'_i with $P'_i \approx P_i$ by an isomorphism $\hat{\alpha}_i$ extending α_i and with $P'_i \downarrow_{N_i} N_\kappa$ inside N^*_{i+1} to satisfy condition (ii). Then choose L'_{i+1} to satisfy (iii) by the existence of free amalgamations (axiom D1). Finally choose L_{i+1} and α_{i+1} extending $\hat{\alpha}_i$ to satisfy condition (iv). As λ is **K**-inaccessible, N^*_{i+1} and L_{i+1} can be chosen in $\mathbf{K}_{<\lambda}$. At a limit ordinal $\delta < \kappa$, let $\tilde{N}_\delta$ be canonically prime over $\langle N^*_i : i < \delta \rangle$. Then $\langle \langle N_i : i \leq \delta \rangle \cup \{N_\kappa\}, \langle L'_i : i < \delta \rangle \rangle$ is a free augmented double chain inside $\tilde{N}_\delta$. (Strictly speaking, this is proved by induction on $\beta < \delta$. Use the base extension axiom to pass from $P'_i \downarrow_{N_i} N_\kappa$ to

$L'_{i+1} \downarrow_{N_{i+1}} N_\kappa$.) By DC2 there is an N^*_δ **K**-extending $\tilde{N}_\delta$ and an $L'_\delta \leqslant N^*_\delta$ with $L'_\delta \downarrow_{N_\delta} N_\kappa$ such that $\mathfrak{L} \restriction \delta \cup \{L'_\delta\}$ is essentially **K**-continuous. Extend $\langle \alpha_i : i < \delta \rangle$ to map L'_δ to L_δ.

Now we show that this strategy wins for Player I. Since $\mathfrak{P}$ and $\mathfrak{P}'$ are isomorphic, it suffices to show that there is no P'_κ with $\mathfrak{P}' \cup \{P'_\kappa\}$ essentially **K**-continuous. Suppose for contradiction that such a P'_κ exists. Since $\langle \mathfrak{N} \cup \{N_\kappa\}, \mathfrak{P}' \rangle$ is a free double chain inside N^*_κ, by DC1 the canonically prime model N' over $\mathfrak{N}$ can be embedded in P'_κ inside some extension of N^*_κ. But then N_κ and N' are compatible over $\mathfrak{N}$ contrary to assumption. $\square$

7.6. Remarks. (i) Instead of assuming Axiom Ch4 (part of the definition of adequate) we could have assumed that **K** was not $(<\lambda, \kappa)$-semismooth.

(ii) It is tempting to think that by choosing the minimal length of a sequence witnessing nonsmoothness, we could apply Lemma 7.4 and avoid assuming DC1. However, DC1 is applied for chains of length κ so this ploy is effective.

(iii) Why is L_{i+1} a proper extension of P_i? Since L_{i+1} was chosen as an amalgam of P'_i and N_κ, this is immediate if we assume the disjointness axiom (C7). To avoid this hypothesis we can demand that each model in the construction have cardinality $>\chi_{\mathbf{K}}$ and so not be maximal (by Corollary 3.5). That is why we assumed $\lambda \geqslant \chi_{\mathbf{K}}$.

(iv) Note that DC1 is used to derive the contradiction at the end of the proof; DC2 is used to pass through limit stages of the construction. Thus in the important case when $\kappa = \omega$ we have the following lemma.

7.7. Lemma. *Let* **K** *be an adequate class that is not* $(<\lambda, \omega)$*-smooth. Suppose that* **K** *satisfies* DC1. *Then Player* I *has a winning strategy for Game* 1 (λ, ω).

The choice of L_i according to the winning strategy of Player I depends only on the sequence $\langle L_j, P_j \rangle$, for $j < i$ (not, for example, on some guess about the future of the game).

In the remainder of this section we consider a third axiom DC3 on double chains. The following axiom bears the same relation to DC2 that C5 bears to C2.

7.8. Weak uniqueness for prime models over double chains

DC3. Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are essentially **K**-continuous augmented double chains that are free in M and N, respectively, and f is an isomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$. Suppose also that $M^1_\delta \downarrow_{M^0_\delta} M^0_{\delta+1}$ inside M and $N^1_\delta \downarrow_{N^0_\delta} N^0_{\delta+1}$ inside N. Then there is an $\hat{M} \in \mathbf{K}$ and **K**-embeddings h_0 of M and h_1 of N into $\hat{M}$ with $h_1 \circ f = h_0 \restriction \mathfrak{M}$.

Just as [3, Lemma 1.1.8] rephrased the weak uniqueness axiom for amalgamation over vees we can reformulate DC3 as follows.

7.9. Lemma. *Assume* DC2 *and* DC3. *Suppose that* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are essentially* **K**-*continuous augmented double chains that are free in* M *and* N, *respectively, and* f *is an isomorphism from* $\mathfrak{M}$ *onto* $\mathfrak{N}$. *Suppose also that for some* $M_\delta^1 \leq N$ $M_\delta^1 \downarrow_{M_\delta^0} M_{\delta+1}^0$ *inside* M.

Then there exist a model $\hat{N}$ *and an isomorphism* $h: M \mapsto \hat{N}$ *such that* $h \supseteq f$ *and* $h(M_\delta^1) \downarrow_{N_\delta^0} N_{\delta+1}^0$ *inside* $\hat{N}$.

Question 1. If DC2 and DC3 hold and **K** is not smooth, does Player I have a winning strategy for Game 1?

8. Nonsmoothness implies many models

We show in this section that if the class **K** is not smooth, then **K** codes stationary sets. These results involve several tradeoffs between set theory and model theory. The main result is proved in ZFC. Here there are two versions; one uses $\square^a$ and requires the hypothesis that cpr behaves on subsequences. The second uses $\square^b$ and replaces "cpr behaves on subsequences" with stronger hypotheses concerning the closure of **K** under unions of chains. By working in L we can reduce our assumptions on which chains are bounded in both cases.

8.1. Invariants modulo the cub filter

In this subsection we show if **K** is not smooth, then for many λ we can code stationary subsets of λ by assigning invariants in the cub filter by the function sm. Our general strategy for constructing many models is this. We build a model M^W for each of a family of 2^λ stationary subsets W of S that are pairwise distinct modulo the cub filter. The key point of the construction is that, modulo cub(λ), we can recover W from M^W as $\lambda - \mathrm{sm}(M^W)$.

We need one more piece of notation.

8.1. Notation. Fix a square sequence $\langle C_\alpha : \alpha \in S \rangle$. Suppose Player I has a winning strategy for Game 1 (λ, κ). In the proof of Theorem 8.2 and some similar later results we define a **K**-increasing sequence $\mathfrak{M}$. We describe here what is meant by saying a certain M_α is chosen by playing Player I's strategy on $\mathfrak{M} \mid C_\alpha$.

Let $\langle c_\beta : \beta < \beta_0 \rangle$ enumerate C_α. We regard $\mathfrak{M} \mid C_\alpha$ as two sequences $\langle \mathfrak{L}, \mathfrak{P} \rangle$ by setting for any ordinal $\delta + n$ with δ a limit ordinal and $n < \omega$:

$$L_{\delta+n} \text{ is } M_{c_{\delta+2n}}, \qquad P_{\delta+n} \text{ is } M_{c_{\delta+2n+1}}.$$

We say M_β for $\beta = \alpha$ or $\beta \in C_\alpha$ is chosen by Player I's winning strategy on $M \mid C_\alpha$ if the sequence $\langle \mathfrak{L}, \mathfrak{P} \}$ associated with $C_\alpha \cap \beta \cup \{\beta\}$ is

(i) an initial segment of a play of Game 1 (λ, κ) and

(ii) Player I's moves in this game follow his winning strategy.

Here is the technical version of the main result with the parameters and reliance on the axioms enunciated in Sections 2 and 4 stated explicitly.

Although the assumption that λ is **K**-inaccessible is weaker than the assumption that **K** satisfies the λ-Löwenheim–Skolem property, it plays the role of the Löwenheim–Skolem property in the following construction. We assume $\lambda \geqslant \chi_{\mathbf{K}}$ and apply Corollary 3.5 to avoid the appearance of maximal models in the construction. We do not have to assume that **K** is $(<\lambda, <\kappa)$-smooth for this argument. We are able to substitute the existence of Player I's winning strategy. This is not much of a saving as we assumed that **K** is $(<\lambda, <\kappa)$-smooth in proving the existence of the winning strategy but it does clarify the roles of the various hypotheses.

8.2. Theorem. *Fix regular cardinals $\kappa < \lambda$. Suppose the following conditions hold.*

(i) **K** *is an adequate class.*

(ii) *Player* I *has a winning strategy for Game* 1 (λ, κ).

(iii) λ *is a* **K**-*inaccessible cardinal, for some stationary $S \subseteq \lambda$, $\square^a_{\lambda,\kappa}(S)$ holds and* $\lambda \geqslant \chi_{\mathbf{K}}$.

(iv) **K** *is* $(<\lambda, <\lambda)$-*bounded.*

(v) cpr *behaves on subsequences* (*Axiom* Ch5).

(vi) **K** *is* $(<\lambda, \lambda)$-*closed.*

Then for any stationary $W \subseteq S$ there is a model M^W and a representation $\mathfrak{M}^W$ with $W \subseteq \lambda - \mathrm{sm}(M^W, \mathfrak{M}^W)$ and $S^+ - W \subseteq \mathrm{sm}(M^W, \mathfrak{M}^W)$.

Proof. Fix S^+ and $C^* = \langle C_i : i \in S^+ \rangle$ to witness $\square^a_{\lambda,\kappa}(S)$. Without loss of generality, $0 \in C_0$. Fix also a stationary subset W of S. For each $\alpha < \lambda$ we define a model M^W_α. The model $M^W = \bigcup_{\alpha<\lambda} M^W_\alpha$ constructed in this way is in **K** by condition (vi) and will satisfy the conclusion.

Each of these conditions depends indirectly on W, but since we are constructing each M^W separately, we suppress the dependence on W to avoid notational confusion in the construction.

For each $\alpha < \lambda$ we define M_α to satisfy the following requirements.

(i) $|M_\alpha| \geqslant \chi_{\mathbf{K}}$ (to avoid maximal models).

(ii) $\mathfrak{M}\,(=\mathfrak{M}^W)$ is an increasing sequence of members of $K_{<\lambda}$ which is essentially **K**-continuous at δ if $\delta \in (S^+ - W)$.

(iii) If $\alpha \in W$, then $\mathfrak{M}$ is not essentially **K**-continuous at α.

(iv) $M\,(=M^W) = \bigcup_{\alpha<\lambda} M_\alpha$.

The construction proceeds by induction. There are a number of cases depending on whether $\alpha \in W$, S, etc.

Case I. $\alpha \in (W \cup \bigcup_{\delta \in W} C_\delta)$. If $\alpha \in W$, let $\beta = \alpha$; otherwise fix $\beta > \alpha$ with $\alpha \in C_\beta$. (The choice of β does not matter because of the coherence condition in the definition of $\square^a_{\lambda,\kappa}$.) Let $\langle c_\gamma : \gamma < \beta_0 \rangle$ enumerate C_β. $M_\alpha \in \mathbf{K}_{<\lambda}$ is chosen by

Player I's winning strategy for $\mathfrak{M} \mid C_\beta$ if $\alpha \in \operatorname{acc}(C_\beta)$ or $\alpha = c_{\delta+2n}$. If $\alpha \in C_\beta$ and $\alpha = c_{\delta+2n+1}$, then M_α can be chosen as any extension of $M_{C_{\delta+2n}}$ with cardinality $<\lambda$.

Case II. $\alpha \in S^+ - (W \cup \bigcup_{\delta \in W} C_\delta)$. Then $C_\alpha \subseteq S^+ - S$ so $\mathfrak{M} \mid C_\alpha$ is **K**-continuous. Choose M_α to be canonically prime over $\mathfrak{M} \mid C_\alpha$ (which is the same as canonically prime over $\mathfrak{M} \mid \alpha$ since cpr behaves on subsequences). M_α can be chosen in $\mathbf{K}_{<\lambda}$ by invoking Lemma 2.13 since λ is **K**-inaccessible. (The **K**-continuity can be verified by induction on α. Let γ be a putative least counterexample. Then γ satisfies Case I or Case II. In Case I, M_γ is chosen canonically prime over $\mathfrak{M} \mid \gamma$ (by the definition of a winning strategy) and so is canonically prime over $\mathfrak{M} \mid C_\gamma$ since canonically prime models behave on subsequences. In Case II, M_γ is chosen canonically prime over $\mathfrak{M} \mid \gamma$ as required.)

Case III. $\alpha \notin S^+$. Choose M_α to bound $\mathfrak{M} \mid \alpha$ by $(<\lambda, <\lambda)$-boundedness and with $|M_\alpha| < \lambda$ since λ is **K**-inaccessible (apply Lemma 2.13).

Case IV. Any successor ordinal not already done. Say, $\beta = \gamma + 1$. Choose $M_\beta \in \mathbf{K}_{<\lambda}$ as a proper **K**-extension of M_α by Corollary 3.5.

The cases in the construction are easily seen to be disjoint (using (ii) of the definition of $\square^a_{\lambda,\kappa}$) and inclusive. If $\delta \in W$ is a limit ordinal, the canonically prime model over $\mathfrak{M} \mid \operatorname{nacc}[C_\delta]$ is not compatible with M_δ since Player I played a winning strategy on $M \mid C_\alpha$. So, since cpr behaves on subsequences, neither is the canonically prime model on $\mathfrak{M} \mid A$ for any A unbounded in δ. Thus $\delta \notin \operatorname{sm}(\mathfrak{M}, M^W)$. All other limits $\delta \in S^+$ are in $\operatorname{sm}(\mathfrak{M}, M^W)$ and we finish. $\square$

The next theorem rephrases Theorem 8.2 to avoid technicalities. It shows that reasonable **K** that are not smooth have many models in all sufficiently large successor cardinals. In fact we have the stronger result that **K** codes stationary subsets of such cardinals.

8.3. Theorem (ZFC). *Let* **K** *be an adequate class and suppose that* **K** *satisfies* DC1, DC2 *and* cpr *behaves on subsequences* (*Axiom* Ch5). *Suppose there exist* κ, λ_1 *with* $\kappa < \lambda_1$ *such that* **K** *is not* (λ_1, κ)*-smooth. Then for every* **K***-inaccessible* $\lambda > \sup(\chi_{\mathbf{K}}, \lambda_1)$ *such that*

(i) λ *is a successor of a regular cardinal,*

(ii) **K** *is* $(<\lambda, <\lambda)$*-bounded,*

(iii) **K** *is* $(<\lambda, \lambda)$*-closed,*

K *has* 2^λ *models in power* λ.

Proof. **K** is not $(<\lambda_1, \kappa)$-smooth trivially implies **K** is not $(<\lambda, \kappa)$-smooth. Fix the minimal such κ. We assumed DC1 and DC2, so by Lemma 7.5 Player I has a

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winning strategy in Game 1 (λ, κ). By Fact 5.5, there is a stationary $S \subseteq C^{\kappa}(\lambda)$ such that $\Box^{a}_{\lambda,\kappa}$ holds. The result now follows from the previous theorem, choosing 2^{λ} stationary sets $W \subseteq S$ that are distinct modulo the cub ideal. (In more detail, let V and W be two of these stationary sets. Then $\mathrm{sm}(M^{W}, \mathfrak{M}^{W}) \triangle \mathrm{sm}(M^{V}, \mathfrak{M}^{V}) \supseteq W \cup V$. Thus by Lemma 6.4, $(M^{W}) \not\approx (M^{V})$.) $\Box$

8.4. Remark. If we add the requirement $\lambda^{\chi_1(\mathbf{K})} = \lambda$, we can deduce that λ is $\mathbf{K}$-inaccessible from Lemma 2.12. Applying Lemma 7.7 we could omit DC2 from the hypothesis list if $\kappa = \omega$.

The assumption in Theorem 8.2 that $\mathbf{K}$ is $(<\lambda, <\lambda)$-bounded is used only for the construction of the M_{α} for $\alpha \notin S^{+}$. We can weaken this model-theoretic hypothesis at the cost of strengthening the set-theoretic hypothesis. We noted in Fact 5.7 that the set-theoretic hypotheses of the next theorem follow from V = L.

8.5. Theorem. *Suppose $\Box^{a}_{\lambda,\kappa}(S)$ holds for an S that does not reflect. Then the hypothesis that $\mathbf{K}$ is $(<\lambda, <\lambda)$-bounded can be deleted from Theorem* 8.2.

Proof. The only use of this hypothesis is the construction of M_{α} for $a \notin S^{+}$. In this case we make our construction more uniform by demanding for $\alpha \notin S^{+}$ that M_{α} is canonically prime over $\langle M_{\beta}: \beta < \alpha\rangle$. If S does not reflect in α, then there is a club $C \subseteq \alpha$ with $C \cap S = \emptyset$. By induction, for each $\delta \in C$, we have M_{δ} canonically prime over $\langle M_{\beta}: \beta < \delta\rangle$. Thus the chain $\langle M_{\delta}: \delta \in C\rangle$ is $\mathbf{K}$-continuous and we can choose M_{α} canonically prime over it. By Axiom Ch5, M_{α} is canonically prime over $\langle M_{\beta}: \beta < \alpha\rangle$ as required. $\Box$

Recall that $\mathbf{K}$ is $(<\lambda, [\mu,\lambda])$-bounded if every chain with cofinality between μ and λ inclusive of models that each have cardinality $<\lambda$ is bounded. In a number of the examples we have adduced (paragraph 4.3), $\mathbf{K}$ is $(<\infty,(\mu, <\infty])$-bounded for appropriate μ. Thus the model-theoretic hypothesis of the following theorem is reasonable. The existence of stationary sets that do not reflect in δ of small cofinality is provable if V = L and is consistent with large cardinal hypotheses.

8.6. Theorem. *Fix κ, $\mu < \lambda$. Suppose $\Box^{a}_{\lambda,\kappa}(S)$ holds for some stationary subset S of λ that satisfies*

if S reflects in δ, then $\mathrm{cf}(\delta) \geq \mu$.

Then the hypothesis that $\mathbf{K}$ is $(<\lambda, <\lambda)$-bounded can be replaced in Theorem 8.2 *by assuming that $\mathbf{K}$ is $(<\lambda, [\mu,\lambda])$-bounded.*

Proof. Again we must construct M_{α} for $\alpha \notin S^{+}$. If S reflects in α, $\mathrm{cf}(\alpha) \geq \mu$ so $\mathfrak{M} \mid \alpha$ is bounded. Since λ is $\mathbf{K}$-inaccessible (Definition 2.10), we can choose M_{α} to bound $\mathfrak{M} \mid \alpha$ and with $|M_{\alpha}| < \lambda$. If S does not reflect in α, write α as a limit of ordinals β_i of cofinality $<\mu$. By induction, M_{β_i} is canonically prime over $\mathfrak{M} \mid \beta_i$ and taking M_{α} canonically prime over the M_{β_i} (using Axiom Ch5 and Lemma 2.13) we finish. $\Box$

8.2. Invariants modulo ID(C^*)

In this subsection we shown if **K** is not smooth, then for many λ we can code stationary subsets of λ by assigning invariants modulo the ideal ID(C^*) by the function sm_1. We now replace the hypothesis that cpr behaves on subsequences by assuming **K** is weakly $(<\lambda, \theta)$-closed for certain θ; we use $\square^b$ rather than $\square^a$ but these have the same set-theoretic strength.

Again we first give the technical version of the main result with the parameters and reliance on the axioms enunciated in Sections 2 and 4 stated explicitly. We need to vary the meaning of the phrase, "a winning strategy against $\mathfrak{M} \mid C_\alpha$" by changing the game played on C_α.

8.7. Notation. Fix a square sequence $\langle C_\alpha : \alpha \in S\rangle$. Suppose Player I has a winning strategy for Game 1 (λ, κ). We modify our earlier notion of what is meant by saying a certain M_α is chosen by playing Player I's strategy on $\mathfrak{M} \mid C_\alpha$ to a notion that is appropriate for the proof of the next theorem.

Let $\langle c_\beta : \beta < \beta_0\rangle$ enumerate C_α. Denote $C_\alpha \cup \{\gamma+1 : \gamma \in \mathrm{nacc}[C_\alpha]\}$ by $\hat{C}_\alpha$. We attach to $\mathfrak{M} \mid C_\alpha$ two sequences $\langle L_i : i < \mathrm{otp}(C_\alpha)\rangle$ and $\langle P_i : i < \mathrm{opt}(C_\alpha)\rangle$ by setting

$$P_\gamma = M_{c_\gamma},$$

$$L_\gamma = M_{c_\gamma+1} \quad \text{if } \gamma \in \mathrm{nacc}[C_\alpha],$$

$$L_\gamma = M_{c_\gamma} \quad \text{if } \gamma \in \mathrm{acc}[C_\alpha],$$

We say M_β for $\beta = \alpha$ or $\beta \in C_\alpha$ is chosen by Player I's winning strategy on $M \mid C_\alpha$ if the sequence $\langle \mathfrak{L}, \mathfrak{P}\rangle$ associated with $C_\alpha \cap \beta \cup \{\beta\}$ is

(i) an initial segment of a play of Game 1 (λ, κ) and

(ii) Player I's moves in this game follow his winning strategy.

In defining this play of the game we have restrained Player II's moves somewhat (as $P_\gamma = L_\gamma$ if $\gamma \in \mathrm{acc}[C_\alpha]$). But this just makes it even easier for Player I to play his winning strategy. Lemma 2.13 and the definition of winning strategy for Player I are used as in the proof of Theorem 8.2 to establish the continuity and to work below λ in the following construction.

8.8. Theorem. *Fix regular cardinals κ, θ, R, λ with $\kappa \neq \theta$, $\theta < \lambda$, $R \leq \lambda$, and $\chi_{\mathbf{K}}$, $\kappa^+ < \lambda$. Suppose the following.*

(i) **K** *is an adequate class.*

(ii) *Player* I *has a winning strategy for Game* 1 (λ, κ).

(iii) λ *is a* **K**-*inaccessible cardinal and for some* $S, S_1, S_2 \subseteq \lambda$ *and* $C^{**} = \langle C_\alpha : \alpha \in S\rangle$, $\square^b_{\lambda,\kappa,\theta,R}(S, S_1, S_2)$ *is witnessed by* C^{**}. *Let* $S_0 = S_1 \cup S_2$ *and* $C^* = C^{**} \mid S_0$.

(iv) **K** *is* $(<\lambda, [R,\lambda])$-*bounded.*

(v) **K** *is* $(<\lambda, \theta)$-*closed.*

(vi) **K** *is* $(<\lambda, \lambda)$-*closed.*

Then there is a model M with $\mathrm{sm}_1(M, C^*) = (S_2/\mathrm{FIL}(C^*))$.

Proof. For each $\alpha<\lambda$ we define M_α as follows. We will guarantee

$$\mathrm{sm}_1(\mathfrak{M}, M, C^*) = S_2 \text{ modulo } \mathrm{FIL}(C^*).$$

The construction proceeds by induction. There are a number of cases depending on whether $\alpha \in S_1$, S_2, etc. Choose M_α by the first of the following conditions that applies to α.

Case I. $\alpha \in (S_1 \cup \bigcup_{\delta \mathbf{S}_1} \mathrm{acc}[C_\delta] \cup \bigcup_{\delta \in \mathbf{S}_1} \{\gamma+1\colon \gamma \in \mathrm{nacc}[C_\delta]\}$. Choose any $\beta \in S$ with $\alpha \in C_\beta$ (if α is in S, let $\beta=\alpha$). Apply Player I's winning strategy for Game 1 to $\mathfrak{M} \mid C_\beta$ to choose M_α. Again the coherence conditions in the definition of the square sequence guarantee that the particular choice of β is immaterial. Note that for α not in S_1, the definition of Player I winning the game guarantees that M_α is canonically prime over $C_\beta \cap \alpha$.

Case II. For any successor ordinals not yet covered, say $\beta=\gamma+1$, choose M_β as a **K**-extension of M_α. Thus in the rest of the cases we may assume α is a limit ordinal.

Case III. $\alpha \in \bigcup_{\delta \in \mathbf{S}_0} \mathrm{nacc}[C_\delta]$. Then $\mathrm{cf}(\alpha)=\theta$, so by the fifth hypothesis we may choose $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ provided that $\mathfrak{M} \mid \alpha$ is bounded. If $\alpha \in S$, $\mathfrak{M} \mid \alpha$ is bounded by the canonically prime model over $\mathfrak{M} \mid C_\alpha$ (which exists by the argument for Case V). If $\alpha \notin S$, then Definition 5.9 guarantees $\mathrm{cf}(\alpha) \geqslant R$. Choose M_α as a **K**-extension of M_β for each $\beta<\alpha$ by $(<\lambda, \geqslant R)$-boundedness.

Case IV. $\alpha \in S_2$. By Case III, $\mathfrak{M} \mid \mathrm{nacc}[C_\alpha]$ is **K**-continuous; choose M_α canonically prime over $\mathfrak{M} \mid \mathrm{nacc}[C_\alpha]$.

Case V. All remaining ordinals $\alpha \in S$. Our construction guarantees that $\mathfrak{M} \mid C_\alpha$ is continuous as $C_\alpha \subseteq S - S_1$. Choose M_α canonically prime over $\mathfrak{M} \mid C_\alpha$.

Case VI. α is a limit ordinal and $\alpha \notin S$. Then Definition 5.9 guarantees $\mathrm{cf}(\alpha) \geqslant R$. Choose M_α as a **K**-extension of M_β for each $\beta<\alpha$ by $(<\lambda, \geqslant R)$-boundedness.

Now we show that $\mathrm{sm}_1(\mathfrak{M}, C^*, M)$ intersects the stationary set S_0 in S_2. If $\alpha \in S_1$, the play of Game 1 guarantees that there is no M'_α such that $\mathfrak{M} \mid \mathrm{nacc}[C_\alpha] \cup \{M'_\alpha\}$ is essentially **K**-continuous. Thus by condition (iii) of Definition 6.6, $\alpha \notin \mathrm{sm}_1(\mathfrak{M}, C^*, M)$ But if $\alpha \in S_2$, α is in neither S_1 nor $\bigcup_{\beta \in S_0} \mathrm{nacc}[C_\beta]$ (since all elements of the second set have cofinality θ and $\mathrm{cf}(\alpha)=\kappa$). Thus, by Case IV of the construction M_α is canonically prime over $\mathfrak{M} \mid \mathrm{nacc}[C_\alpha]$ and since **K** is $(<\lambda, \lambda)$-closed, $M_\alpha \leqslant M$. Condition (ii) in the definition of sm_1 is guaranteed since $\mathfrak{M} \mid C_\alpha$ is **K**-continuous (as $C_\alpha \cap S_1 = \emptyset$ and all points of non-**K**-continuity are in S_1). Condition (i), $M_\gamma = \bigcup_{\beta<\gamma} M_\beta$ for

$\gamma \in \mathrm{nacc}[C_\alpha]$, is guaranteed by Case III of the construction. $\square$

8.9. Remark. At the cost of assuming that **K** is $(<k, <\kappa)$-smooth and $\theta < \kappa$, hypothesis (v) could be weakened to "**K** is weakly $(<\lambda, \theta)$-closed".

Again we rephrase the result to emphasize the salient hypotheses.

8.10. Theorem (ZFC). *Let* **K** *be an adequate class satisfying* DC1 *and* DC2. *Fix* $\lambda > \chi_{\mathbf{K}}$ *and* θ *such that*

(i) λ *is* **K**-*inaccessible*;

(ii) λ *is a successor of a regular cardinal and* θ^+ *is less than* λ;

(iii) *and* **K** *is*

(a) $(<\lambda, <\lambda)$-*bounded*;

(b) *weakly* $(<\lambda, \theta)$-*closed*;

(c) $(<\lambda, \lambda)$-*closed*.

If **K** *has fewer than* 2^λ *models with cardinality* λ *and* $\kappa^+ > \lambda$, *then* **K** *is* $(<\lambda, <\kappa)$-*smooth*.

Proof. Fix κ with $\kappa^+ < \lambda$ such that **K** is not $(<\lambda, \kappa)$-smooth. By Lemma 7.5 Player I has a winning strategy in Game 1 (λ, κ). By Theorem 5.12(i), $\square_{\lambda,\kappa,\theta,\aleph_0}$ holds. Now a very slight variant of the proof of Theorem 8.8 shows there exist 2^λ models M_i with the $\mathrm{sm}_1(M_i)$ distinct modulo $\mathrm{FIL}(C^*)$. (Namely, since **K** is $(<\lambda, <\lambda)$-bounded, we do not need to worry about the cofinality of α in Case VI.) $\square$

This shows that if **K** is not smooth at some κ, then there will be many models in power λ for many $\lambda > \kappa$ satisfying certain model-theoretic hypotheses.

If V = L, we can waive the boundedness hypothesis.

8.11. Theorem (V = L). *Let* $\lambda \geqslant \chi_{\mathbf{K}}$ *and not weakly compact be* **K**-*inaccessible. Suppose the adequate class* **K** *satisfies* DC1, DC2 *and is*

(i) *weakly* $(<\lambda, \mu)$-*closed, for some* $\mu < \lambda$;

(ii) $(>\lambda, \lambda)$-*closed*.

If **K** *has fewer than* 2^λ *models in power* λ *and* $\kappa^+ < \lambda$, *then* **K** *is* $(<\lambda, \kappa)$-*smooth*.

Proof. Since V = L, Theorem 5.12 implies $\square^b_{\lambda,\kappa,\mu,\lambda}$ holds. The result now follows from Theorem 8.8 taking μ as θ and λ as R. We observed after Definition 4.2 that $(<\lambda,\lambda)$-closed implies $(<\lambda,\lambda)$-bounded. $\square$

We have shown in this section that each of the variants of $\square$ discussed in Section 5 suffice to show that a nonsmooth **K** codes stationary subsets of λ for many λ.

9. The monster model

We begin this section by recapitulating the assumptions that we will make in developing the structure theory. Then we show that under these assumptions we can prove the existence of a monster model and prove the equivalence between 'homogeneous-universal' and 'saturated'.

9.1. Notation. $\mathscr{D}$ denotes a compatibility class of $\mathbf{K}$.

9.2. Definition. We say $\mathbf{K}$ satisfies the *joint embedding property* if for any $A, B \in \mathbf{K}$, there is a $C \in \mathbf{K}$ and $\mathbf{K}$-embeddings of A and B onto C.

At this point we extend the axioms from Section 2 by adding the smoothness hypothesis we have justified in the last few sections. We have shown that under reasonable set-theoretic hypotheses the failure of these smoothness conditions allows us to code stationary subsets of λ for a proper class of λ.

9.3. Assumptions. We assume in this section Axiom Groups A and C, Axioms D1, D2 and L1 from group D, (all from [3]), axioms Ch1′, Ch2′, Ch4 from Section 1 and Axioms S0 and S1 from Section 2 and the following smoothness conditions. $\mathbf{K}$ is $(<\infty, \leqslant\chi_1(\mathbf{K}))$-smooth and $(<\infty, >\chi_1(\mathbf{K}))$-fully smooth. Thus, we assume $\mathbf{K}$ is $(<\infty, <\infty)$-smooth. Finally we assume that $\mathbf{K}$ satisfies the joint embedding property. We call such a class *fully adequate.*

The assumption of the joint embedding property is purely a notational convenience. We have just restricted from $\mathbf{K}$ to a single compatibility class in $\mathbf{K}$. Thus, the notions that in [17] are written, e.g., $(\mathscr{D}, \mu)$-homogeneous here become $(\mathbf{K}, \mu)$-homogeneous with no loss in generality. We could in fact drop the $\mathbf{K}$ altogether.

This class is called fully adequate because (modulo V = L, see Conclusion 9.7(ii)) any fully adequate class either has a unique homogeneous-universal model or many models. We did not include Axiom Ch5, 'cpr behaves on subsequences', since we can rely on Theorem 8.8 to obtain smoothness without that hypothesis. We did assume that $\mathbf{K}$ is $(<\lambda, \mu)$-closed for large μ, so the other hypotheses of that theorem are fulfilled.

The following observation allows us to perform the required constructions.

9.4. Lemma. *If* $\mathbf{K}$ *satisfies the* $<\lambda$*-Löwenheim–Skolem property,* $\lambda \geqslant \chi_1(\mathbf{K})$*,* $\langle M_i : i<\kappa\rangle$ *is a chain of models inside* M *with each* $|M_i| < \lambda$ *and* $\mathrm{cf}(\kappa) > \lambda$*, then there is a canonically prime model* M' *over* $\langle M_i : i<\kappa\rangle$ *with* $|M'| < \lambda$*.*

Proof. If $\mathrm{cf}(\kappa) \geqslant \chi_{\mathbf{K}}$, $M' = \bigcup_{i<\kappa} M_i$ is the required model. If not, note that by the $<\lambda$-Löwenheim–Skolem property there is an $N \leqslant M$ containing the union with $|N| < \lambda$. By $(<\infty, \leqslant\chi_{\mathbf{K}})$-smoothness any canonically prime model over $\langle M_i : i < \kappa \rangle$ can be embedded in N. $\square$

9.5. Definition. The model M is $(\mathbf{K},\mu)$-*homogeneous* if

(i) for every $N_0 \leqslant M$ and every N_1 with $N_0 \leqslant N_1 \in \mathbf{K}$ and $|N_1| \leqslant \mu$ there is a $\mathbf{K}$-embedding of N_1 into M over N_0;

(ii) every $N \in \mathbf{K}$ with cardinality less than μ can be $\mathbf{K}$-embedded in M.

There is a certain etymological sense in labeling this notion a kind of saturation. The argument for homogeneity is that naming $\mathbf{K}$ describes the level of universality and we need only indicate the homogeneity again. In any case Shelah established this convention some twenty years ago in [12]. We identify this algebraic notion with a type realization notion in Theorem 9.11.

9.6. Lemma. *Suppose* $\mathbf{K}$ *is* $(<\mu, <\mu)$-*bounded and* $(<\mu, \mu)$-*closed.*

(i) *If* $\mu \geqslant \chi_{\mathbf{K}}$ *is regular,* $\mathbf{K}$-*inaccessible and satisfies* $\mu^{<\mu} = \mu$, *there is a* $(\mathbf{K}, \mu)$-*homogeneous model of power* μ.

(ii) *If, in addition,* $\mathbf{K}$ *is* $(<\mu, \leqslant\mu)$-*smooth, this* $(\mathbf{K}, \mu)$-*homogeneous model is unique up to isomorphism.*

Proof. We define an increasing chain $\langle M_i : i < \mu \rangle$ by induction; the union of the M_i is the required model. Let M_0 be any element of $\mathbf{K}_{<\mu}$. Fix an enumeration $\langle N_\beta : \beta < \mu \rangle$ of all isomorphism types in $\mathbf{K}_{<\mu}$. There are only μ such since $\mu^{<\mu} = \mu$. Given M_i, with $|M_i| < \mu$ we define M_{i+1} as a bound for a sequence $M_{i,j}$ with $j < |i| + |M_i|^{|M_i|} = \alpha < \mu$. First let $G_j = \langle A_j, B_j, f_j \rangle$ for $j < \alpha$ be a list of all triples such that f_j is an isomorphism of A_j onto a $\mathbf{K}$-submodel of M_i and $A_j \leqslant B_j$ and $B_j \approx N_\beta$ for some $\beta < i$. (Note that the B_j are specified only up to isomorphism; a given isomorphism type of A_j will occur many times in the list depending on various embeddings f_j into M_i.) Now, $M_{i,0} = M_i$; $M_{i,j+1}$ is the amalgam of $M_{i,j}$ and B_j over A_j (via f_j and the identity map). If δ is a limit ordinal less than α, $M_{i,\delta}$ is any bound of $\langle M_{i,j} : j < \delta \rangle$ with $|M_{i,\delta}| > \mu$. M_{i+1} is a bound for the $M_{i,j}$. By regularity of μ for limit $\delta < \mu$, each $M_{i,\delta}$ and M_δ have cardinality less than μ.

It is easy to see that M is homogeneous since if $f : N_0 \mapsto M$ is a $\mathbf{K}$-embedding and $N_0 \leqslant N_1$ with $|N_1| < \mu$, f was extended to a map into some $M_{i,j}$ at some stage in the construction and $M_{i,j} \leqslant M$.

The uniqueness of the $(\mathbf{K}, \mu)$-homogeneous model now follows by the usual back and forth argument to show any two $(\mathbf{K}, \mu)$-homogeneous models M and N of power μ are isomorphic. But smoothness is crucial. At a limit stage δ, one takes the canonically prime model M_δ over an initial segment of the sequence of submodels of M and embeds it as a submodel N_δ of N. In order to continue the induction we must know M_δ is a strong submodel of M and this is guaranteed by smoothness. $\square$

9.7. Conclusion. (i) For any fully adequate $\mathbf{K}$ that is $(<\infty,<\infty)$-bounded there is (in some cardinal μ) a unique $(\mathbf{K},\mu)$-homogeneous model.

(ii) If $V=L$, we can omit the boundedness hypothesis (by Theorem 8.11).

(iii) We will call the unique $(\mathbf{K},\mu)$-homogeneous model, $\mathcal{M}$, the monster model. From now on all sets and models are contained in $\mathcal{M}$.

9.8. Remark. This formalism encompasses the constructions by Hrushovski [7] of $\aleph_0$-categorical stable psuedoplanes. An underlying (but unexpressed) theme of his constructions is to generalize the Fraïssé–Jónsson construction by a weakening of homogeneity. He does not demand that any isomorphism of finite substructures extend to an automorphism but only an isomorphism of submodels that are 'strong substructures' (where strong varies slightly with the construction). This is exactly encapsuled in the formalism here. This viewpoint is pursued in [1]. Of course in Hrushovski's case the real point is the delicate proof of amalgamation and ω is trivially $\mathbf{K}$-inaccessible. We assume amalgamation and worry about inaccessibility and smoothness in larger cardinals.

9.9. Definition. (i) The *type* of $\bar{a}$ over A (for $\bar{a}, A \subseteq \mathcal{M}$) is the orbit of $\bar{a}$ under the automorphisms of $\mathcal{M}$ that fix A pointwise. We write $p=\mathrm{tp}(\bar{a};A)$ for this orbit.

(ii) p is a *k-type* if $\lg(\bar{a})=k$.

(iii) The *type* of B over A (for $B, A \subseteq \mathcal{M}$) is the type of some (fixed) enumeration of B.

(iv) $S^k(A)$ denotes the collection of all k-types over A.

We will often write p, q, etc. for types. This notion is really of interest only when $\lg(\bar{a}) \leqslant \mu$; despite the suggestive notation, k-type, we may deal with types of infinite length. We will write $S(A)$ to mean $S^k(A)$ for some $k<\mu$ whose exact identity is not important at the moment.

9.10. Definition. (i) The type $p \in S(A)$ is *realized* by $\bar{c} \in N$ with $A \subseteq N \leqslant \mathcal{M}$ if $\bar{c}$ is member of the orbit p.

(ii) $N \leqslant \mathcal{M}$ is *$(\mathbf{K},\lambda)$-saturated* if for every $M \leqslant N$ with $|M|<\lambda$, every 1-type over M is realized in N.

9.11. Theorem. *Let $\lambda \geqslant \chi_{\mathbf{K}}$ be $\mathbf{K}$-inaccessible. Then M is $(\mathbf{K},\lambda)$-saturated if and only if M is $(\mathbf{K},\lambda)$-homogeneous.*

The proof follows that of [20, Proposition 2.4] line for line with one exception. If we consider those stages δ in the construction where $\mathrm{cf}(\delta)<\chi_1(\mathbf{K})$, we cannot form M_δ just by taking unions. However, any canonically prime model over the initial segment of the construction will work by smoothness and Lemma 9.4.

10. Problems

Question 2. Can one give more precise information on the class of cardinals in which an adequate class **K** has a model?

In Example 1.5 we gave a definition of $\leqslant$ on the class of $\aleph_1$-saturated models of the theory $T = \mathrm{REI}_\omega$ under which this class is not $(<\aleph_1, \omega)$-smooth. Thus, by our main result, T has the maximum number of $\aleph_1$-saturated models in power λ (if, e.g., $\lambda = \mu^+$).

Question 3. Define $\leqslant$ on the class of $\aleph_1$-saturated models of a strictly stable with didop [13] (or perhaps if **K** is not finitely controlled in the sense of [8]) so the class is not smooth.

There are strictly stable theories with fewer than the maximal number of $\aleph_1$-saturated models in most λ. See [2, Example 8, p. 8].

Question 4. Formalize the notion of coding a stationary set to encompass the examples we have described and clarify the distinctions described at the beginning of Section 6.

We have developed this paper entirely in the context of cpr models. In a forthcoming work we replace this fundamental concept by axioms for winning games similar to Game 1 (λ, κ) and establish smoothness in that context. The cost is stronger set theory (but $V = L$ suffices).

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