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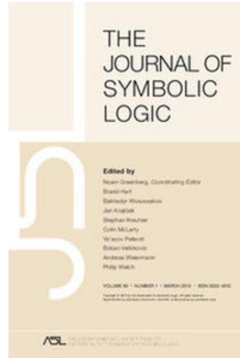

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Δ^1_3-SETS OF REALS

HAIM JUDAH AND SAHARON SHELAH

Abstract. We build models where all Δ^1_3-sets of reals are measurable and (or) have the property of Baire and (or) are Ramsey. We will show that there is no implication between any of these properties for Δ^1_3-sets of reals.

§0. Introduction. In this paper we continue the study of the connections between measurability, categoricity, and Ramsey property over the projective sets. We will concentrate on the Δ^1_3-sets of reals. This direction of research was stated by R. M. Solovay when he proved the following:

0.1. THEOREM (Solovay [SO]). *If there is a model with an inaccessible cardinal, then there is a model where every projective set of reals is measurable, has the property of Baire, etc.*

0.2. THEOREM (Solovay [JU1]). (i) *Every $\underset{\sim}{\Sigma}^1_2$-set of reals is Lebesgue measurable iff* $(\forall r)(\{s: s$ *is Random over* $L[r]\}$ *has measure* $1)$.

(ii) *Every $\underset{\sim}{\Sigma}^1_2$-set of reals has the property of Baire iff* $(\forall r)(\{s: s$ *is Cohen over* $L[r]\}$ *is comeager*$)$.

(iii) *Martin's axiom implies every $\underset{\sim}{\Sigma}^1_2$-set of reals is Lebesgue measurable and has the property of Baire.* (*For our purposes* MA *is* $MA + \neg$CH.)

The following has been known since the time of Gödel's work on the constructible universe.

0.3. THEOREM [JS2]. *$V = L$ implies that there is a Δ^1_2-set of reals which is not Lebesgue measurable and does not have the property of Baire.*

For the Δ^1_2-set of reals we were able to give a characterization of measurability and of the Baire property, in Solovay's fashion, by showing in [JS2] the following:

0.4. THEOREM. (i) *Every $\underset{\sim}{\Delta}^1_2$-set of reals is Lebesgue measurable iff* $(\forall r\exists s)$ $(s$ *is Random over* $L[r])$.

(ii) *Every $\underset{\sim}{\Delta}^1_2$-set of reals has the property of Baire iff* $(\forall r\exists s)(\{s$ *is Cohen over* $L[r])$.

The first theorem concerning the measurability of the Δ^1_3-set of reals was established by S. Shelah [SH2] where he showed

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0.5. THEOREM. *If* ZF *is consistent, then* ZFC+ *"Every $\underset{\sim}{\Delta}^1_3$-set of reals is measurable" is consistent.*

It was natural to ask if MA implies the measurability of the $\underset{\sim}{\Delta}^1_3$-sets of reals. This was answered by the following

0.6. THEOREM. (i) *(Harrington-Shelah [HS]). If every $\underset{\sim}{\Delta}^1_3$-set of reals has the Baire property and* MA + $\neg$CH *holds, then ω_1 is a weakly compact cardinal in L.*

(ii) *(Judah-Shelah [JS]). If every $\underset{\sim}{\Delta}^1_3$-set of reals is measurable and* MA + $\neg$CH *holds, then ω_1 is a weakly compact cardinal in L.*

A model where every $\underset{\sim}{\Delta}^1_3$-set of reals has the property of Baire was given by Shelah when he proved

0.7. THEOREM [SH1]. *If* ZF *is consistent then* ZFC+ *"Every projective set has the Baire property" is also consistent.*

The first result connecting two of these regularity properties was the following

0.8. THEOREM (Bartoszynski-Raisonnier-Stern [RS]). *If every $\underset{\sim}{\Sigma}^1_2$-set of reals is Lebesgue measurable, then every Σ^1_2-set of reals has the property of Baire.*

By using 0.4 it is not hard to see that 0.8 does not generalize to Δ^1_2-sets of reals. And it was very natural to ask if the measurability of the Δ^1_3-sets of reals implies the Baire property for the $\underset{\sim}{\Delta}^1_3$-sets of reals. In this paper we will show that this does not happen if we assume the existence of a measurable cardinal.

The converse direction of 0.8 is false: $\Sigma^1_2(\mathrm{B}) \not\Rightarrow \Sigma^1_2(\mathrm{L})$, and for $\underset{\sim}{\Delta}^1_3$-sets it was proved by Judah [JU2], that

0.9. THEOREM. *If* ZF *is consistent, then there exists a model where every $\underset{\sim}{\Delta}^1_3$-set of reals has the Baire property and there is a $\underset{\sim}{\Delta}^1_2$-sets of reals, which is not measurable.*

At the end of the 1960s, the set theorists found a new regularity property for sets of reals, namely, the Ramsey Property. Instead of viewing the reals as sequences of 0's and 1's, we can view them as infinite subsets of the set of natural numbers. We call this space $[\omega]^\omega = \{x \subseteq \omega: |x| = \infty\}$. We say that a set $A \subseteq [\omega]^\omega$ is Ramsey if there exists $b \in [\omega]^\omega$ such that

$$[b]^\omega = \{x \subseteq b: |x| = \infty\} \subseteq A \quad \text{or} \quad [b]^\omega \subseteq \sim A.$$

At this time the following was known:

0.10. THEOREM (Galvin-Prikry). *Every Borel set is Ramsey.*

0.11. THEOREM (Silver). (i) *Every analytic set is Ramsey.*

(ii) MA *implies every $\underset{\sim}{\Sigma}^1_2$-set of reals is Ramsey.*

0.12. THEOREM (Mathias-Solovay [ST2]). *If there is an inaccessible cardinal, then there is a model where every projective set is Ramsey.*

The first author, in his Ph.D. thesis, made a study of the possible implications between measurability, the Baire property, and the Ramsey property for Σ^1_2-sets of reals. In [JU1] it was proved that the only implication provable in ZFC is Theorem 0.8.

In another paper, [JS2], we found a characterization of the statements "Every $\underset{\sim}{\Sigma}^1_2$-set of reals is Ramsey" and "Every $\underset{\sim}{\Delta}^1_2$-set of reals is Ramsey." To establish this characterization we need the following

0.13. DEFINITION. Let M be a model for set theory. Let r be a subset of ω. We say that r is Ramsey over M if for every $\pi: [\omega]^2 \to 2$, if $\pi \in M$ then there is $n \in \omega$ such that $\pi \restriction [r - n]^2$ is constant.

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0.14. THEOREM. (i) *Every $\underset{\sim}{\Sigma}^1_2$-set of reals is Ramsey iff for every $s \in [\omega]^\omega$ there is $r \in [\omega]^\omega$, r is Ramsey over $L[s]$.*

(ii) *Every $\underset{\sim}{\Sigma}^1_2$-set of reals is Ramsey iff every Δ^1_2-set of reals is Ramsey.*

Models where every $\underset{\sim}{\Delta}^1_3$-set of reals is Ramsey were studied in [JU1]. There the following was proved:

0.15. THEOREM. *If* ZF *is consistent, then there is a model for* ZFC *where every Δ^1_3-set of reals is Lebesgue measurable, has the property of Baire and is Ramsey.*

In [JS] we also proved

0.16. THEOREM. *If every Δ^1_3-set of reals is Ramsey and* MA + $\neg$CH *holds, then ω_1 is a weakly compact cardinal in L.*

It is an open problem whether from ZF it is possible to build a model where every projective set is Ramsey.

In this paper we will give models where one of the above mentioned properties holds and the other two fail. Thus no implication holds between Lebesgue measurability, the Baire property and the Ramsey property for Δ^1_3-sets of reals.

The parallel questions for Σ^1_3-sets of reals is open. We think that a new idea on building models for "projective measurability" is necessary to solve this interesting question.

This paper is organized as follows. In §1 we will show that a specific countable support iteration does not add Cohen reals. This is used to build models where Δ^1_3 (Baire) fails (Δ^1_3 (Baire) means "Every $\underset{\sim}{\Delta}^1_3$-set of reals has the property of Baire, etc."). The reader who is familiar with [JU2], where models for $\underset{\sim}{\Delta}^1_3$ (Ramsey) were built, should note that in this previous work we used explicitly the fact that the iteration was with finite support. If we want to avoid Cohen reals, this support is forbidden. In §2 we will see that countable support iteration of "simple forcing notions" (called Souslin) satisfies a kind of absoluteness lemma for $\underset{\sim}{\Delta}^1_3$-formulas. This will be the main technical device in order to build models for Δ^1_3 (Ramsey). In §3 we build the models. The authors are grateful to T. Bartoszynski for useful comments.

§1. Adding no Cohen reals.

1.0. DEFINITION. (a) Solovay real forcing is the following poset:

$$B = \{p : p \subseteq [0,1] \text{ and } \mu(p) > 0\}.$$

Let $p, q \in B$; we say $p \leq q$ iff $q \subseteq p$.

(b) Mathias real forcing is the following poset:

$$P = \{p : (\exists s \in [\omega]^{<\omega})(\exists a \in [\omega]^\omega)(p = (s, a) \,\&\, \sup(s) < \inf(a))\}.$$

Let $(s, a), (t, b) \in P$; we say $(s, a) \leq (t, b)$ iff $s \subseteq t \subseteq a \cup s$, $b \subseteq a$.

1.1. THEOREM. *Let $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha\rangle$ be a countable support iterated forcing let $S \subseteq \alpha$, and assume that*

(a) *If $i \in S$ then $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is Solovay real forcing",*

(b) *If $i \notin S$ then $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is Mathias real forcing".*

Let $P_\alpha = \text{Lim}\,\bar{Q}$, then no real in V^{P_α} is Cohen over V.

PROOF. We make the following conventions:

(a) For $p, q \in B$ and $n \in \omega$ we define two partial orderings

$p \le_n^* q$ if $p \supseteq q$ and $\mu(p - q) \le \mu(p)/2^n$;

$p \le_n q$ if $p = q$.

(b) For $(s, a), (t, b) \in P$ and $n \in \omega$ we define $(s, a) \le_n^* (t, b)$ iff (def) $(s, a) \le_n (t, b)$ iff (def) $t = s$ and the first n members of a are in b.

(c) For $p, q \in P_\alpha$, $F \in [\alpha]^{<\omega}$, and $n \in \omega$, we define $p \le_{F,n}^* q$ ($p \le_{F,n} q$) iff

(i) $p \le q$,

(ii) $(\forall \xi \in F)(p | \xi \Vdash$ "$p(\xi) \le_n^* q(\xi)$ ($p(\xi) \le_n q(\xi)$)".

The proof of the theorem is based on the following

1.2. LEMMA. *Suppose that $p \in P_\alpha$, $F \in [\alpha]^{<\omega}$ and $n \in \omega$ are given. Let τ be a P_α-name for a natural number. Then there is $p \le_{F,n}^* q$ and $k \in \omega$ such that $q \Vdash$ "$\tau \neq \hat{k}$".*

We will prove the theorem using the lemma; later we will prove the lemma.

Let σ be a P_α-name for an element of ω^ω and let p_0 be any element of P_α. By using 1.2 define by induction sequences $\langle p_n : n \in \omega \rangle$, $\langle F_n : n < \omega \rangle$ and a function $f \in \omega^\omega$ such that

(i) $p_n \le_{F_n, n}^* p_{n+1}$ for $n \in \omega$,

(ii) $\forall \xi \in \text{supp}(p_n)(\exists j \in \omega)(\xi \in F_j)$,

(iii) $F_n \subset F_{n+1}$ for $n \in \omega$,

(iv) $p_{n+1} \Vdash$ "$\sigma(\hat{n}) \neq f(\hat{n})$".

Let q be the limit of $\langle p_n : n < \omega \rangle$. Clearly $q \Vdash (\forall n)(\sigma(n) \neq f(n))$, therefore $q \Vdash$ "σ is not Cohen over V". (Recall that a Cohen real is infinitely often equal to any old real.)

PROOF OF 1.2. Suppose that $p \in P_\alpha$, $F \in [\alpha]^{<\omega}$ and $n \in \omega$ are given. Let τ be a P_α-name for a natural number. We will give an inductive construction of the desired condition. Let $0 = \beta_0 < \beta_1 < \cdots < \beta_m = \alpha$ be an enumeration of the set $F \cup \{\xi + 1 : \xi \in F\} \cup \{0, \alpha\}$. Fix a big enough natural number k, and define by downward induction:

(a) a sequence of P_{β_l}-names $\langle \underset{\sim}{a}_j^l : j < k \rangle$ of real numbers,

(b) a condition $q_l \in P_{\beta_{l+1}} \backslash P_{\beta_l}$, a condition $p_l \in P_{\beta_l}$ such that $p | \beta_l \le_{F,n} p_l$ and $p_l \vee q_l$ is a condition in $P_{\beta_{l+1}}$ and if $\beta_l \in F$ then

$$p_l \Vdash p_{l+1}(\beta_l) \le_n^* q_l$$

(therefore $p_l \Vdash p(\beta_l) \le_n^* q_l$),

(c) ε_l a positive real number, depending only on F, and satisfying

$$p_l \Vdash \text{“} \sum_{j<k} \underset{\sim}{a}_j^{l+1} < m - l + 2 \text{”}.$$

We start the induction by defining

$$\underset{\sim}{a}_j^m = \begin{cases} 1 & \text{if } \tau = j, \\ 0 & \text{otherwise}, \end{cases}$$

$$\varepsilon_m = 1/2.$$

We assume we have already defined

$$\langle \underset{\sim}{a}_j^{l+1} : j < k \rangle, \quad \varepsilon_{l+1}, \quad q_{l+1}, \quad p_{l+1}.$$

Then we have three different cases:

Case 1. $\beta_l \notin F$. Then let $q_l \in P_{\beta+1} \setminus P_{\beta_l}$ and let $\langle \underset{\sim}{a}_j^l : j < k \rangle$ be such that

$$q_l \Vdash_{P_{\beta_{l+1}} \setminus P_{\beta_l}} \text{“} \sum_{j<k} |\underset{\sim}{a}_j^{l+1} - \underset{\sim}{a}_j^l| < \hat{\varepsilon}_l \text{”},$$

where $\varepsilon_l = \frac{1}{2}\varepsilon_{l+1}$ and $\langle \underset{\sim}{a}_j^l : j < k \rangle$ is a P_{β_l}-name. The only problem is that q_l is only a name for a condition and $p_{l+1} \mid \beta_l$ does not know its support. However, over $p_{l+1} \mid \beta_l$ we have that the support of q_l is a countable set. Therefore, by [BA] there exists $p_l \in P_{\beta_l}$ and $A \in [\alpha]^\omega$ such that $p_{l+1} \mid \beta_l \leq_{F,n} p_l \Vdash$ “$\text{supp}(q_l) \subseteq \hat{A}$”.

Case 2. $\beta_l \in F - S$. In this case $P_{\beta_{l+1}} \setminus P_{\beta_l}$ is Mathias forcing, and it is well known (see [BA]) that this forcing notion satisfies the Laver property; namely, if K is a finite set of ordinals and τ is a term and p a condition, then there exists q and H such that

$$p \leq_n q, \quad |H| = 2^n, \qquad q \Vdash \text{“}\tau \in \hat{H}\text{”}.$$

Now let $\varepsilon_l = 2^{-n-1} \cdot \varepsilon_{l+1}$. Then, using the Laver property and a rational approximation for $\langle \underset{\sim}{a}_j^{l+1} : j < k \rangle$, we can find a P_{β_l}-name $\langle \underset{\sim}{a}_{j,s}^l : s < 2^n, j < k \rangle$ and $q_l \in P_{\beta_{l+1}} \setminus P_{\beta_l}$ satisfying

$$p_{l+1} \mid \beta_l \Vdash \text{“} p_{l+1}(\beta_l) \leq_n^* q_l \text{”},$$

$$q_l \Vdash_{P_{\beta_{l+1}} \setminus P_{\beta_l}} \text{“}(\exists s < 2^n)\left(\sum_{j<k} |\underset{\sim}{a}_j^{l+1} - \underset{\sim}{a}_{j,s}^l| < \varepsilon_l \right)\text{”}.$$

Let $\underset{\sim}{a}_j^l = \text{Max}\{\underset{\sim}{a}_{j,s}^l : s < 2^n\}$.

Case 3. $\beta_l \in F \cap S$. Then $P_{\beta_{l+1}} \setminus P_{\beta_l}$ is a measure algebra. Working in $V^{P_{\beta_l}}$ we can view $\underset{\sim}{a}_j^{l+1}$ as a measurable function. Let $\varepsilon_l = 2^{-n-1}\varepsilon_{l+1}$ and

$$\underset{\sim}{a}_j^l = \left(\int_{p_{l+1}(\beta_l)} \underset{\sim}{a}_j^{l+1} dx \right) \Big/ \mu(p_{l+1}(\beta_l)),$$

and let $q_l = p_{l+1}(\beta_l)$, $p_l = p_{l+1} \mid \beta_l$.

We will check that condition (c) holds. W.l.o.g. $P_{\beta_l} = \varnothing$. $p(0) = 1$. Therefore we have

$$(*) \qquad \Vdash \text{“} \sum_{j<k} \underset{\sim}{a}_j^{l+1} < m - l + 1 \text{”}.$$

In this case $\underset{\sim}{a}_j^l = \int \underset{\sim}{a}_j^{l+1} dx$.

We should show $\sum_{j<k} a_j^l < m - l + 2$.

By $(*)$ $\mu(\{x : \sum_{j<k} a_j^{l+1}(x) < m - l + 1\}) = 1$. And this says that

$$\int \left(\sum_{j<k} \underset{\sim}{a}_j^{l+1} \right) dx < m - l + 2.$$

And thus $\sum_{j<k} a_j^l = \int (\sum_{j<k} \underset{\sim}{a}_j^{l+1}) \, dx < m - l + 2$.

This completes the inductive definition. At the end $\langle a_j^0 : j < k \rangle$ is a sequence of reals in the ground model satisfying $\sum_{j<k} \underset{\sim}{a}_j^0 < m + 2$. Clearly we may start with k big enough relative to ε_0. Therefore there exists $k' < k$ such that $\underset{\sim}{a}_{k'}^0 < \varepsilon_0$.

Now we go back up defining by induction $\langle \bar{p}_l : l \leq m \rangle$ satisfying

$$\bar{p}_l \in P_{\beta_l}, \qquad \bar{p}_{l+1} \restriction \beta_l = \bar{p}_l.$$

If $\beta_l \in F$ then $p_{l+1} \restriction \beta_l \Vdash p(\beta_l) \leq^*_n \bar{p}_{l+1}(\beta_l)$, $\bar{p}_l \Vdash \underset{\sim}{a}^l_{k'} < \varepsilon_l$.

The induction

$l = 0$. trivial.

$l + 1$. $\bar{p}_l \Vdash \underset{\sim}{a}^l_{k'} < \varepsilon_l$.

Case 1. $\beta_l \notin F$. Then $\bar{p}_{l+1} = \bar{p}_l \vee q_l$ where q_l comes from the previous Case 1.

Case 2. $\beta_l \in F \backslash S$. Then $\bar{p}_{l+1} = \bar{p}_l \vee q_l$ where q_l comes from the previous Case 2.

Case 3. $\beta_l \in F \cap S$. Then we have that

$$\underset{\sim}{a}^l_j = \left(\int_{p_{l+1}(\beta_l)} \underset{\sim}{a}^{l+1}_{k'} dx \right) \Big/ \mu(p_{l+1}(\beta_l))$$

and

$$\bar{p}_l \Vdash \underset{\sim}{a}^l_{k'} < 2^{-n-1} \cdot \varepsilon_l.$$

By standard computation, working in $V^{P_{\beta_l}}$ over $\bar{p}_l$ we can find q in Solovay real forcing such that $p_{l+1}(\beta_l) \leq^*_n q \Vdash \underset{\sim}{a}^{l+1}_k < \varepsilon_{l+1}$. (Recall $p_{l+1}(\beta_l) = p_l(\beta_l)$.) Let $\bar{p}_{l+1} = \bar{p}_l \vee q$. ($q = \{x : \underset{\sim}{a}^{l+1}_{k'}(x) < \varepsilon_l\} \cap p_l$ (β_l works).)

This concludes the induction. We have

$$p \leq^*_{F,n} \bar{p}_m, \qquad \bar{p}_m \Vdash \underset{\sim}{a}^m_{k'} < 1/2$$

but this implies $\bar{p}_m \Vdash \tau \neq k'$. □

§2. On Proper Souslin Forcing. In this section we will prove a theorem about Proper Souslin Forcing. The definition of Proper Souslin Forcing can be found in Judah-Shelah [JS1]. We recall the following facts about Proper Souslin Forcing. We say that $P = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \kappa \rangle$ is Proper Souslin if for each $\alpha < \kappa$ the following holds:

(i) P_α is a countable support iteration.

(ii) $\Vdash_{P_\alpha}$ "Q_α is Souslin" (a forcing notion is Souslin if the set of conditions is Souslin (i.e., $\underset{\sim}{\Sigma}^1_1$), the order is Souslin and the set of incompatible pairs is Souslin).

(iii) In V^{P_α} the following holds: if N is a countable model for some sufficiently rich part of ZFC and the parameters of the definition of $\underset{\sim}{Q}_\alpha$ are in N, then for every $p \in N \cap \underset{\sim}{Q}_\alpha$ there is $q \in \underset{\sim}{Q}_\alpha$ such that $p \leq_{\underset{\sim}{Q}_\alpha} q$ and q is N-generic for $\underset{\sim}{Q}_\alpha$.

In [JS1] we show that if P is Proper Souslin Forcing then the same as above holds for P; namely, if N is a countable model for some sufficient rich part of ZFC and $P \in N$, then for every $p \in N \cap P$ there is $q \in P$ such that $p \leq_P q$ and q is N-generic for p.

2.1. THEOREM. *Let P be a Proper Souslin Forcing and let $\theta(r)$ be a Π^1_2-formula with parameter a. Then the following are equivalent for a P-name $\underset{\sim}{r}$ for a real*

(i) $p \Vdash \theta(\underset{\sim}{r})$,

(ii) *If N is a model for a sufficiently rich part of* ZFC *and* $\{P, a, p\} \subseteq N$ *then*

$$N \models p \Vdash \text{“}\theta(\underset{\sim}{r})\text{”}.$$

PROOF. Assume $\neg$(i). Then there is q such that $q \Vdash \neg\theta(r)$. Let N be elementary submodel of $(H(2^{2^{\aleph_0}}), \varepsilon)$ containing q, p, a, P, etc. Then

$$N \models q \Vdash \neg\theta(r).$$

Therefore $N \models p \nVdash \theta(r)$.

Now assume $\neg$(ii). Let N be a countable model for a rich part of ZFC, $\{P, a, p\} \subseteq N$. Assume also that $N \models p \nVdash \theta(r)$. Therefore we can find p' in $N \cap P$ such that

$$N \models p' \Vdash \neg\theta(r).$$

By the proper Souslin condition, there exists q extending p', which is N-generic. Let G be generic containing q. Then $N \cap G$ is generic over N. Therefore,

$$N[G] \models \neg\theta(r[N \cap G]).$$

But $r[N \cap G] = r[G]$, and Σ^1_2-formula are up-absolute. So $V[G] \models \neg\theta(r[G])$, a contradiction. □

§3. Models for Δ^1_3-sets. Let us make the following abbreviations:

$\underset{\sim}{\Delta}^1_3(M)$ iff every $\underset{\sim}{\Delta}^1_3$-set of reals is measurable.

$\underset{\sim}{\Delta}^1_3(B)$ iff every $\underset{\sim}{\Delta}^1_3$-set of reals has the property of Baire.

$\underset{\sim}{\Delta}^1_3(R)$ iff every $\underset{\sim}{\Delta}^1_3$-set of reals is Ramsey.

We start by showing that ω_1-iteration with countable support of Mathias reals produces extensions where $\underset{\sim}{\Delta}^1_3(R)$ holds.

3.1. THEOREM. *Let $\bar{Q} = \langle P_i; \underset{\sim}{Q}_i : i < \omega_1\rangle$ be a countable support iteration of Mathias reals, and let $P = \lim \bar{Q}$. Then $V^P \models \Delta^1_3(R)$.*

PROOF. Let (φ, ψ) be a pair of Σ^1_3-formulas with parameter a such that

$$V^P \models \text{“}\varphi(x) \leftrightarrow \neg\psi(x)\text{”}.$$

By properness we may assume that $a \in V$. Let x_0 be the first Mathias real, and without loss of generality we may assume $V^P \models$ “$\varphi(x_0)$”. Therefore, $V^P \models \exists y\theta(x_0, y)$, when θ is a Π^1_2-formula. By properness and Π^1_2-absoluteness we have that there exists $\alpha < \omega_1$ such that $V^{P_\alpha} \models \exists y\theta(x_0, y)$. Let τ be a P_α-name, and $p \in P_\alpha$ such that

$$p \Vdash_{P_\alpha} \text{“}\theta(x_0, \tau)\text{”}.$$

Now let $x \subseteq x_0$ be such that if G_x is the generic object, for Mathias forcing, generated by x we have that $p(0) \in G_x$ (this means that the finite part of $p(0)$ is the initial part of x). Clearly by [JU2] it will be enough to show that $V^P \models$ “$\varphi(x)$”. For this we come back to P_α.

From the previous statements we have that the following is true in $V[x_0]$, (w.l.o.g. $\alpha = \bigcup\alpha \neq 0$):

(*1) if we force α-many times Mathias reals with countable support and $p | [1, \alpha)$ belongs to the generic object G, then $\theta(x_0, \tau[G])$.

Therefore we have that the same is true in $V[x]$, namely,

(*2) if we force α-many times Mathias reals with countable support and $p | [1, \alpha)$ belongs to the generic object G, then $\theta(x_0, \tau[G])$.

(Remember that here $p|[1,\alpha)$ and τ depend on x.) We want to show that (*2) is absolute, i.e., we want to show that for every β big enough (i.e., $x \in V^{P_\beta}$)

(**) in V^{P_β} if we force α-many times Mathias reals with countable support and $p|[1,\alpha)$ belongs to the generic object G, then $\theta(x_0, \tau[G])$.

Claim. (*2) $\Rightarrow$ (**).

Proof. If (**) is false there is N as in 2.1 such that

(i) $N \in V^{P_\beta}$,

(ii) $N \models p|[1,\alpha) \nVdash$ "$\theta(x,\tau)$",

(iii) $\alpha \in N$, $x \in N$, etc.

But $x \in V^{P_\beta}$, therefore $V^{P_\beta} = V^{P_x * \underset{\sim}{Q}}$, where P_x is the algebra generated by a name for x. Then we can see that in $V[x]$, N has a $Q[x]$-name and is countable; therefore, there exists, in $V[x]$, $H \subseteq \underset{\sim}{Q}[x]$, such that

$$N[H] \models p|[1,\alpha) \nVdash \text{“}\theta(x,\tau)\text{”}.$$

By Theorem 2.1 this is a contradiction with (*2). □

Now the theorem is clear, because if β is such that $x \in V^{P_\beta}$, then in $V^{P_{\beta+\alpha}}$ we can find some y witnessing $\theta(x, y)$. The rest is easy. □

3.2. THEOREM. *Let $\bar{Q} = \langle P_i; \underset{\sim}{Q}_i : i < \omega_1 \rangle$ be a countable support iteration satisfying*

(i) *If i is odd then $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is Mathias real".*

(ii) *If i is even then $\Vdash_{P_i}$ "$\underset{\sim}{Q}_i$ is Random real".*

Let $P = \lim_{\to} \bar{Q}$. Then $V^P \models \Delta^1_3(R)$.

PROOF. Like 3.1. □

3.3. THEOREM. Con(ZF) *implies* Cons(ZFC + $\Delta^1_3(R) + \neg\Delta^1_2(M) + \neg\Delta^1_2(B)$).

PROOF. Let $V = L$ and let P be as in Theorem 3.1. Then $V^P \models$ "$\Delta^1_3(R)$". Because P satisfies the Laver condition (see [SH]) we have that

$$V^P \models \text{“No real is Random over } V\text{”}$$

and

$$V^P \models \text{“No real is Cohen over } V\text{”}.$$

Therefore, by [JS], we have that $V^P \models \neg\Delta^1_2(M) + \neg\Delta^1_2(B)$. □

3.4. THEOREM. Cons(ZFC + *there exists a measurable cardinal*) *implies*

$$\text{Cons(ZFC} + \Delta^1_3(R) + \Delta^1_3(M) + \neg\Delta^1_3(B)).$$

PROOF. Let $V = L[\mu]$ and P be as in Theorem 3.2. Then $V^P \Vdash \Delta^1_3(R)$. By §1.1 we have that no real in V^P is Cohen over V. Then using the Σ^1_3-well order of $2^\omega \cap V$ we can show, as in [JS2], that $V^P \models \neg\Delta^1_3(B)$.

Now we will show that $V^P \models \Delta^1_3(M)$. Suppose φ and ψ are Σ^1_3-formulas with parameter a such that

$$V^P \models \text{“}\varphi(x) \leftrightarrow \neg\psi(x)\text{”}.$$

By properness we may assume that $a \in V$. Let r be a name for the first Random real. Without loss of generality, we may assume that $V^P \models \varphi(r)$; but φ is a Σ^1_3-formula and P is a forcing notion of size less than the measurable cardinal, so by [MS] we have $V[r] \models \varphi(r)$. Now let $\varphi(r) = \exists y \theta(r, y)$. Therefore there is τ a Random forcing name such that $V[r] \models \theta(r, \tau[r])$.

Therefore there exists B, a positive set, such that, in V, $B \Vdash \theta(\underset{\sim}{r}, \tau)$. Let b be a real number satisfying $B, \tau \in L[b]$. Then it is not hard to show that $L[b] \models$ "$B \Vdash \theta(\underset{\sim}{r}, \tau)$".

But it is well known that $\omega_1^{L[b]} < \omega_1$; therefore, almost every real in V^P is Random over $L[b]$. Let r_0 be a Random real over $L[b]$ such that $r_0 \in B$. Then $L[b][r_0] \models \theta(r_0, \underset{\sim}{\tau}[r_0])$. Then by Shoenfield's lemma $V^P \models \theta(r_0, \underset{\sim}{\tau}[r_0])$. Therefore $V^P \models \varphi(r_0)$. □

QUESTION. Could Theorem 3.4 be proved using ZFC?

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