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Many simple cardinal invariants

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Summary. For $g < f$ in ω^ω we define $\boldsymbol{c}(f, g)$ be the least number of uniform trees with g-splitting needed to cover a uniform tree with f-splitting. We show that we can simultaneously force $\aleph_1$ many different values for different functions (f, g). In the language of [Bl]: There may be $\aleph_1$ many distinct uniform Π^0_1 characteristics.

0 Introduction

Blass [Bl] defined a classification of certain cardinal invariants of the continuum, based on the Borel hierarchy. For example, to every Π^0_1 formula $\varphi(x, y) = \forall n R(x \restriction n, y \restriction n)$ (R recursive) the cardinal

$$\kappa_\varphi := \min\{\mathscr{B} \subseteq {}^\omega\omega : \forall x \in {}^\omega\omega \exists y \in \mathscr{B} : \varphi(x, y)\}$$

is the "uniform Π^0_1 characteristic" associated to φ.

Blass proved structure theorems on simple cardinal invariants, e.g., that there is a smallest Π^0_1 characteristic (namely, $Cov(\mathscr{M})$, the smallest number of first category sets needed to cover the reals), and also that the Π^0_2-characteristics can behave quite chaotically. He asked whether the known uniform Π^0_1 characteristics ($\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{r}, Cov(\mathscr{M})$) are the only ones or (since that is very unlikely) whether there could be a reasonable classification of the uniform Π^0_1 characteristics – say, a small list that contains all these invariants.

In this paper we give a strong negative answer to this question: For two Π^0_1 formulas φ_1, φ_2 we say that φ_1 and φ_2 define "potentially nonequal characterictics" if $\kappa_{\varphi_1} \neq \kappa_{\varphi_2}$ is consistent. We say that φ_1 and φ_2 define "actually different characteristics", if $\kappa_{\varphi_1} \neq \kappa_{\varphi_2}$.

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We will find a family of Π^0_1-formulas indexed by a real parameter (f, g), and we will show not only that there is a perfect set of parameters which defines pairwise potentially nonequal Π^0_1-characteristics, but we produce a single universe in which (at least) $\aleph_1$ many cardinals appear as Π^0_1-characteristics. (In fact it is also possible to produce a universe where there is a perfect set of parameters defining pairwise actually different Π^0_1-characteristics. See [Sh470]).

If we want more than countably many cardinals, we obviously have to use the boldface pointclass. But the proof also produces many lightface uniform Π^0_1 characteristics.

For more information on cardinal invariants, see [Bl, vDo, Va].

From another point of view, this paper is part of the program of finding consistency techniques for a large continuum, i.e., we want $2^{\aleph_0} > \aleph_2$ and have many values for cardinal invariants. We use a countable support product of forcing notions with an axiom A structure.

We will use invariants that were implicitly introduced in [Sh326, Sect. 2], where it was proved that $\boldsymbol{c}(f, g)$ and $\boldsymbol{c}(f', g')$ (see below) may be distinct.

0.1 Definition. If $f \in {}^\omega\omega$, we say that $\bar{B} = \langle B_k : k \in \omega\rangle$ is an f-slalom if for all k, $|B_k| = f(k)$. We write $h \in \bar{B}$ for $h \in \prod_n B_n$, i.e., $\forall n\, h(n) \in B_n$ (see Fig. 1). This is a Π^0_1-formula in the variables h and $\bar{B}$.

Some authors call the set $\{h : h \in \bar{B}\}$ a "belt", or "uniform tree".
For example, $\prod_n f(n)$ is an f-slalom, because we identify the number $f(n)$ with the set of predecessors, $\{0, \ldots, f(n) - 1\}$.

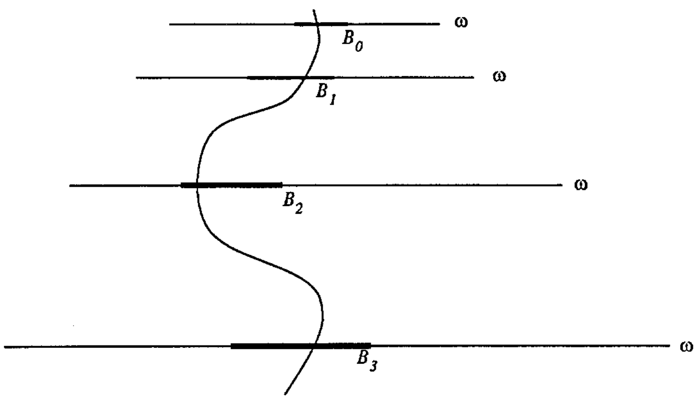

Fig. 1. A slalom

0.2 Definition. Assume $f, g \in {}^\omega\omega$. Assume that $\mathscr{B}$ is a family of g-slaloms, and $\bar{A} = \langle A_k : k \in \omega\rangle$ is an f-slalom.

We say that $\mathscr{B}$ covers $\bar{A}$ iff:

$$(*) \qquad \text{for all } s \in \bar{A} \text{ there is } \bar{B} \in \mathscr{B} \text{ such that } \quad s \in \bar{B}$$

0.3 Definition. Assume $f, g \in {}^\omega\omega$. Then we define the cardinal invariant $\boldsymbol{c}(f, g)$ to be the minimal number of g-slaloms needed to cover an f-slalom.

(Clearly this makes sense only if $\forall k f(k), g(k) > 0$, so we will assume that from now on.)

This is a uniform Π^0_1-characteristic. [Strictly speaking, we are not working in ${}^\omega\omega$, but rather in ${}^\omega([\omega]^{<\omega})$, but a trivial coding translates $\boldsymbol{c}(f,g)$ into a "uniform Π^0_1 characteristic" as defined above.]

Some relations between these cardinal invariants are provable in ZFC: For example, if $g < g' < f' < f$, then $\boldsymbol{c}(f',g') \leqq \boldsymbol{c}(f,g)$. Also, $\boldsymbol{c}(f^2,g^2) \leqq \boldsymbol{c}(f,g)$.

We will show that if (f,g) is sufficiently different from (f',g'), then the values of $\boldsymbol{c}(f,g)$ and $\boldsymbol{c}(f',g')$ are quite independent, and moreover: if $\langle (f_i,g_i) : i < \omega_1 \rangle$ are pairwise sufficiently different, then almost any assignment of the form $\boldsymbol{c}(f_i,g_i) = \kappa_i$ will be consistent.

Similar results are possible for the "dual" version of $\boldsymbol{c}(f,g)$: $\boldsymbol{c}^d(f,g) :=$ the smallest family of g-slaloms $\bar{B}$ such that for every h bounded by f there are infinitely many k with $h(k) \in B_k$, and for the "tree" version (a g-tree is a tree where every node in level k has $g(k)$ many successors) (see [Sh470].

We thank Tomek Bartoszynski for pointing out the following known results about the cardinal characteristics $\boldsymbol{c}(f,g)$:

For example, Lemma 1.11 follows from Theorem 3.17 in [CN]: Taking $\kappa = \alpha = \omega$, $\beta = n$, and letting $\mathscr{S} \subseteq n^\omega$ be a family of ω-large oscillation, then no family of $n-1$-slaloms of size $< 2^{\aleph_0}$ can cover $\mathscr{S}$. Indeed, whenever F is a function on S such that for each $s \in \mathscr{S}$, $F(s)$ is a $n-1$-slalom covering s, then F has to be finite-to-one and in fact at most $n-1$-to-one.

Also, since $\boldsymbol{c}(f, f-1)$ is the size of the smallest family of functions below f which does not admit an "infinitely equal" function, i.e.,

$$\boldsymbol{c}(f,f-1) = \min\left\{ |G| : G \subseteq \prod_n f(n) \,\&\, \forall h \in \prod_n f(n) \exists g \in G \forall^\infty n\, f(n) \neq g(n) \right\}$$

by [Mi] we have that the minimal value of $\boldsymbol{c}(f,f-1)$ is the smallest size of a set of reals which does not have strong measure zero.

Also, note that if r is a random real over V in $\prod_n f(n)$, and if $\sum_{n=1}^{\infty} 1/f(n) = \infty$, then $\prod_n (1 - 1/f(n)) = 0$, so r cannot be covered by any $f-1$-slalom from V.

Conversely, if $\sum_{n=1}^{\infty} 1/f(n) < \infty$, then for any function $h \in \prod_n f(n) \cap V$ there is a condition forcing that h is covered by the $f-1$-slalom $(\{0,\ldots,f(k)-1\} - \{r(k)\} : k \in \omega)$.

Thus, if we add κ many random reals with the measure algebra, a easy density argument shows that in the resulting model we have

$$\boldsymbol{c}(f,f-1) = \begin{cases} \kappa = 2^{\aleph_0} & \text{if } \sum_{n=1}^{\infty} 1/f(n) = \infty \\ \aleph_1 & \text{otherwise (use any } \aleph_1 \text{ many of the random deals).} \end{cases}$$

That already shows that we can have at least two distinct values of $\boldsymbol{c}(f,g)$ and $\boldsymbol{c}(f',g')$.

Contents of the paper. In Sect. 1 we prove results in ZFC of the form

"If (f,g) is in relation ... to (f',g'), then $\boldsymbol{c}(f,g) \leqq \boldsymbol{c}(f',g')$".

In Sect. 2 we define a forcing notion $Q_{f,g}$ that increases $\boldsymbol{c}(f,g)$. (I.e., in $V^{Q_{f,g}}$, the g-slaloms from V to not cover $\prod_n f(n)$.) Informally speaking, elements of $Q_{f,g}$ are perfect trees in which the size of the splitting is bounded by f, sometimes $= 1$, but often (i.e., on every branch), much bigger than g.

In Sect. 3 we show that, assuming $\{(f_\xi, g_\xi): \xi < \omega_1\}$ are sufficiently "independent", a countable support product $\prod_{\xi<\omega_1} Q_\xi^{\kappa_\xi}$ of such forcing notions will force $\forall \xi \boldsymbol{c}(f_\xi, g_\xi) = \kappa_\xi$.

We use the symbol ☺ to denote the end of a proof, and we write ☹ when we leave a proof to the reader.

1 Results in ZFC

1.1 Notation. Operations and relations on functions are understood to be pointwise, e.g., f/g, g^ε, $g < f$, etc. $\lfloor x \rfloor$ is the greatest integer $\leqq x$. $\lim f$ is $\lim_{k\to\infty} f(k)$.

We write $f \leqq^* g$ for $\exists n \forall k \geqq n f(k) \leqq g(k)$.

First we state some obvious facts:

1.2 Fact.

(1) $f \leqq g$ iff $\boldsymbol{c}(f, g) = 1$.

(2) $f \leqq^* g$ iff $\boldsymbol{c}(f, g)$ finite.

(3) If $A := \{k : g(k) < f(k)\}$ is infinite then $\boldsymbol{c}(f \restriction A, g \restriction A) = \boldsymbol{c}(f, g)$.

(4) If π is a permutation of ω, then $\boldsymbol{c}(f \circ \pi, g \circ \pi) = \boldsymbol{c}(f, g)$. ☹$_{1.2}$

(Strictly speaking, we define $\boldsymbol{c}(f, g)$ only for functions f, g defined on all of ω, so (3) should be formally rephrased as $\boldsymbol{c}(f \circ h, g \circ h) = \boldsymbol{c}(f, g)$, where h is a $1-1$ enumeration of A).

1.3 Convention. We will concentrate on the case where $\boldsymbol{c}(f, g)$ is infinite, so we will wlog assume that $g < f$. By (4), we may also wlog assume that g is nondecreasing.

In these cases we will have that $\boldsymbol{c}(f, g)$ is infinite, and moreover an easy diagonal argument shows the following fact:

1.4 Fact.

$\boldsymbol{c}(f, g)$ is uncountable. ☹$_{1.4}$

Furthermore, we have the following properties:

1.5 Fact.

(1) (Monotonicity) If $f \leqq^* f'$, $g \geqq^* g'$, then $\boldsymbol{c}(f, g) \leqq \boldsymbol{c}(f', g')$.

(2) (Multiplicativity) $\boldsymbol{c}(f \cdot f', g \cdot g') \leqq \boldsymbol{c}(f, g) \cdot \boldsymbol{c}(f', g')$.

(3) (Transitivity) $\boldsymbol{c}(f, h) \leqq \boldsymbol{c}(f, g) \cdot \boldsymbol{c}(g, h)$.

(4) (Invariance) $\boldsymbol{c}(f, g) = \boldsymbol{c}(f^-, g^-)$ (where f^- is the function defined by $f^-(n) = f(n+1)$.

(5) (Monotonicity II) If $A \subseteq \omega$ is infinite, then $\boldsymbol{c}(f \restriction A, g \restriction A) \leqq \boldsymbol{c}(f, g)$. ☹$_{1.5}$

1.6 Remark. (2) implies in particular $\boldsymbol{c}(f^n, g^n) \leqq \boldsymbol{c}(f, g)$. See 3.4 for an example of $\boldsymbol{c}(f^2, g^2) < \boldsymbol{c}(f, g)$.

The following inequalities need a little more work.

1.7 Lemma. (1) $\boldsymbol{c}(f \cdot \lfloor f/g \rfloor, f) = \boldsymbol{c}(f, g)$.

(2) $\boldsymbol{c}(f \cdot \lfloor f/g \rfloor, g) = \boldsymbol{c}(f, g)$.

(3) $\boldsymbol{c}(f \cdot \lfloor f/g \rfloor^m, g) = \boldsymbol{c}(f, g)$ *for all* $m \in \omega$.

Proof. (2) follows from (1) using transitivity, and (3) follows from (2) by induction, so we only have to prove (1).

Proof of (1): By monotonicity we only have to show $\leqq$. So let $(N, \in)$ be a reasonably closed model of a large fragment of ZFC (say, $(N, \in) < (H(\chi^+), \in)$, where $\chi = 2^{\mathfrak{c}}$) of size $\boldsymbol{c}(f, g)$ such $\prod_n f(n)$ is covered by the set of all g-slaloms from N.

Define h by $h(k) := f(k) \cdot \lfloor f(k)/g(k) \rfloor$. We can find a family $\langle B_k^i : i < f(k), k \in \omega \rangle$ in N such that for all k, $\{0, \ldots, h(k) - 1\} = \bigcup_{i<f(k)} B_k^i$, where $|B_k^i| \leqq f(k)/g(k)$. We have to show that the set of f-slaloms from N covers $\prod_k h(k)$.

So let x be a function satisfying $\forall k\, x(k) \in \bigcup_{i<f(k)} B_k^i$. We can define a function $y \in \prod_n f(n)$ such that for all $k, x(k) \in B_k^{y(k)}$. So there is some g-slalom $\bar{C} \in N$ such that for all $k, y(k) \in C_k$.

Define $\bar{A} = \langle A_k : k \in \omega \rangle$ by $A_k := \bigcup_{i \in C_k} B_k^i$. Then $|A_k| \leqq |C_k| \cdot |B_k^i| \leqq g(k) \cdot f(k)/g(k) = f(k)$, so $\bar{A}$ is an f-slalom in N, and for all $k, x(k) \in A_k$. $\circledcirc_{1.7}$

1.8 Lemma. *Assume that $f > g > 0$. Assume that $\langle w_i : i \in \omega \rangle$ is a partition of ω into finite sets, and for each i there are $\bar{H}^i = \langle H_l^i : l \in w_i \rangle$ satisfying* (a)–(c). *Then $\boldsymbol{c}(f', g') \leqq \boldsymbol{c}(f, g)$.*

(a) $\operatorname{dom} H_l^i = f'(i) = \{0, \ldots, f'(i) - 1\}$

(b) $\operatorname{rng} H_l^i \subseteq f(l) = \{0, \ldots, f(l) - 1\}$

(c) *Whenever $\langle u_l : l \in w_i \rangle$ satisfies*

$$u_l \subseteq f(l), \qquad |u_l| \leqq g(l)$$

then $\{n < f'(i) : \forall l \in w_i\, H_l^i(n) \in u_l\}$ has cardinality $\leqq g'(i)$.

Proof. To any g-slalom $\bar{B} = \langle B_l : l \in \omega \rangle$ we can associate a g'-slalom $\bar{B}^* = \langle B_i^* : i \in \omega \rangle$ by letting

$$B_i^* := \{n < f'(i) : \forall l \in w_i\, H_l^i(n) \in w_l\}$$

Conversely, to any function $x \in \prod_i f'(i)$ we can define a function x^* in $\prod_n f(n)$ by

$$\text{if } l \in w_i, \quad \text{then } x^*(l) = H_l^i(x(i)).$$

It is easy to check that if x^* is in $\bar{B}$ then x is in $\bar{B}^*$. The result follows. $\circledcirc_{1.8}$

1.9 Corollary. *Assume $0 = n_0 < n_1 < \ldots$, and let*

$$f'(i) := f(n_i) \cdot f(n_i + 1) \ldots f(n_{i+1} - 1)$$
$$g'(i) := g(n_i) \cdot g(n_i + 1) \ldots g(n_{i+1} - 1)$$

Then $\boldsymbol{c}(f', g') \leqq \boldsymbol{c}(f, g)$.

Proof. Identify the set of numbers less than $f(n_i) \cdot f(n_i + 1) \ldots f(f_{i+1} - 1)$ with the cartesian product $\prod_{n_i \leqq k < n_{i+1}} f(k)$, and let

$$H_l^i : \prod_{n_i \leqq k < n_{i+1}} f(k) \to f(l)$$

be the projection onto the l-coordinate. We leave the verification of 1.8(c) to the reader. $\circleddash_{1.9}$

1.10 Lemma. *If g is constant, $f(k) \geqq 2^k$, then $\boldsymbol{c}(f, g) = \boldsymbol{c}$.*

Proof. Let $\forall k g(k) = n$, $f(k) = 2^k$. Assume that $\prod\limits_l {}^l2$ can be covered by $< \boldsymbol{c}$ many g-slaloms.

For any $\eta \in {}^\omega 2$, the sequence $\bar{\eta} := \langle \eta \restriction l : l \in \omega \rangle$ is in $\prod\limits_l {}^l 2$. But any g-slalom can contain only n many such $\bar{\eta}$, i.e. for any g-slalom $\bar{B} = \langle B_l : l \in \omega \rangle$ we have

$$|\{\eta \in {}^\omega 2 : \forall l \eta \restriction l \in B_l\}| \leqq m$$

Since there are continuum many η we need continuum many g-slaloms to cover $\prod\limits_l f(l)$ $\left(\text{or equivalently, } \prod\limits_l {}^l 2\right)$. $\circleddash_{1.10}$

1.11 Lemma. *If f and g are constant with $f > g$, then $\boldsymbol{c}(f, g) = \boldsymbol{c}$.*

Proof. Using monotonicity wlog we assume that $f(k) = n + 1$, $g(k) = n$ for all k. We will use 1.8. Let $\omega = \bigcup\limits_{i \in \omega} w_i$ be a partition of ω where $|w_i| = n^{2^i}$.

Let $f'(i) = 2^i$, $g'(i) = n$, and let $\langle H_l^i : l \in w_i \rangle$ enumerate all functions from 2^i to n.

We plan to show $\boldsymbol{c}(f, g) \geqq \boldsymbol{c}(f', g')$ (so $\boldsymbol{c}(f, g) = \boldsymbol{c}$ by 1.10). We want to apply 1.8, so fix a sequence $\langle u_l : l \in w_i \rangle$, where $u_l \subseteq f(l)$ and $|u_l| \leqq g(l)$.

To show that the hypotheses of 1.8 are satisfied, fix i_0 and let

$$A := \{x < f'(i_0) : \forall l \in w_{i_0}\, H_l^{i_0}(x) \in u_l\}$$

and assume A has cardinality $> g'(i_0) = n$. So let $x_0, \ldots, x_n$ be distinct elements of A. Let $H : f'(i_0) \to n + 1$ be a function satisfying

$$\forall j \leqq n\, H(x_j) = j$$

H is one of the functions $\{H_l^{i_0} : l \in w_{i_0}\}$, say $H = H_{l_0}^{i_0}$. Let $j_0 \notin u_{l_0}$, then also

$$x_{j_0} \notin \{x < f'(i_0) : H_{l_0}^{i_0}(x) \in u_{l_0}\} \supseteq A,$$

contradicting $x_{j_0} \in A$. $\circleddash_{1.11}$

1.12 Corollary. *If $f > g$, and $\liminf\limits_{k \to \infty} g(k) < \infty$, then $\boldsymbol{c}(f, g) = \boldsymbol{c}$.*

Proof. This follows from 1.11, using monotonicity and monotonicity II. $\circleddash_{1.12}$

We can now extend 1.7 as follows:

1.13 Theorem. *If for some $\varepsilon > 0$, $g^{1+\varepsilon} \leqq f$, then for all n, $\boldsymbol{c}(f^n, g) = \boldsymbol{c}(f, g)$.*

Proof. First we consider a special case: Assume that $g^2 \leqq f$. Then we get

$$\boldsymbol{c}(f, g) \leqq \boldsymbol{c}(f^2, g) \leqq \boldsymbol{c}(f^2, f) \cdot \boldsymbol{c}(f, g) \leqq \boldsymbol{c}(f^2, g^2) \cdot \boldsymbol{c}(f, g) = \boldsymbol{c}(f, g)$$

Now we use this result on (f, g), then on (f^2, g), etc., to get

$$\boldsymbol{c}(f, g) = \boldsymbol{c}(f^2, g) = \boldsymbol{c}(f^4, g) = \boldsymbol{c}(f^8, g) = \ldots$$

and use monotonicity to get the general result under the assumption $g^2 \leqq f$.

Now we consider the general case $g^{1+\varepsilon} \leqq f$:

If g does not diverge to infinity, we have already (by 1.12) $\boldsymbol{c}(f, g) = \boldsymbol{c}$. Otherwise we can find some $\delta > 0$ such that for almost all k,

$$\frac{f(k)}{g(k)} \geqq g(k)^\delta + 1\,,$$

so

$$\left\lfloor \frac{f(k)}{g(k)} \right\rfloor \geqq g(k)^\delta$$

Now choose m such that $m \cdot \delta > 1$. Then $\lfloor f(k)/g(k) \rfloor^m \geqq g$. By 1.7, $\boldsymbol{c}(f \cdot \lfloor f/g \rfloor^m, g) = \boldsymbol{c}(f, g)$ and so by monotonicity also $\boldsymbol{c}(f \cdot g, g) = \boldsymbol{c}(f, g)$. Since $g^2 \leqq f \cdot g$, we can apply the result from the special case above to get $\boldsymbol{c}(f, g) = \boldsymbol{c}(f^n \cdot g^n, g)$ so in particular, $\boldsymbol{c}(f^n, g) = \boldsymbol{c}(f, g)$. $\circledcirc_{1.13}$

If f is not much bigger than g, the assumption in 1.7 and 1.13 may be false. For these cases, we can prove the following:

1.14 Lemma. (1) $\boldsymbol{c}(2f - g, f) = \boldsymbol{c}(f, g)$.
(2) $\boldsymbol{c}(2f - g, g) = \boldsymbol{c}(f, g)$.
(3) $\boldsymbol{c}(f + m(f - g), g) = \boldsymbol{c}(f, g)$ *for all* $m \in \omega$.

Proof. The proof is similar to the proof of 1.7. Again we only have to show (1). Let $(N, \in)$ be a reasonably closed model of a large fragment of ZFC (say, $(N, \in) \prec (H(\chi^+), \in)$, where $\chi = 2^{\mathfrak{c}}$) of size $\boldsymbol{c}(f, g)$ such $\prod_n f(n)$ is covered by the set of all g-slaloms from N.

Define h by $h(k) := f(k) + f(k) - g(k)$. We can find a family $\langle B^i_k : i < f(k), k \in \omega \rangle$ in N such that for all k, $\{0, \ldots, h(k) - 1\} = \bigcup_{i < f(k)} B^i_k$, where $|B^i_k| = 2$ for $l < f(k) - g(k)$, and $|B^i_k| = 1$ otherwise. We have to show that the set of f-slaloms from N covers $\prod_k h(k)$.

So let x be a function satisfying $\forall k x(k) \in \bigcup_{i < f(k)} B^i_k$. We can define a function $y \in \prod_n f(n)$ such that for all $k, x(k) \in B^{y(k)}_k$. So there is some g-slalom $\bar{C} \in N$ such that for all $k, y(k) \in C_k$.

Define $\bar{A} = \langle A_k : k \in \omega \rangle$ by $A_k := \bigcup_{i \in C_k} B^i_k$. Thus A_k is the union of $g(k)$ many sets, of which at most $f(k) - g(k)$ are pairs, and the others singletons. Thus $|A_k| \leqq g(k) + (f(k) - g(k)) = f(k)$, so $\bar{A}$ is an f-slalom in N, and for all $k, x(k) \in A_k$. $\circledcirc_{1.14}$

Similar to the proof of 1.13 we now get:

1.15 Lemma. (1) *If* $2g \leqq f$, *then for all* $n, \boldsymbol{c}(nf, g) = \boldsymbol{c}(f, g)$.
(2) *If for some* $\varepsilon > 0$, $(1 + \varepsilon)g \leqq f$, *then for all* $n, \boldsymbol{c}(nf, g) = \boldsymbol{c}(f, g)$. $\circledcirc_{1.15}$

2 The forcing notion $Q_{f,g}$

2.1 Definition. We fix sequences $\langle n_k^- : k \in \omega\rangle$ and $\langle n_k^+ : k \in \omega\rangle$ that increase very quickly and satisfy $n_0^- \ll n_0^+ \ll n_1^- \ll n_1^+ \ll \ldots$. In particular, we demand

(1) For all k $\prod_{j<k} n_j^- \leqq n_k^-$

(2) $\lim_{k\to\infty} \dfrac{\log n_k^+}{\log n_k^-} = 0$.

(3) $n_k^- \cdot n_k^+ < n_{k+1}^-$.

We will only consider functions f, g satisfying $n_k^- \leqq g(k) < f(k) \leqq n_k^+$. This is partly justified by 1.9, and it also helps to keep the formulation of the main theorem relatively simple.

2.2 Definition. Let $X \neq \emptyset$ be finite, $c, d \in \omega$. A (c, d)-complete norm on $\boldsymbol{P}(X)$ is a map

$$\| \, \| : \boldsymbol{P}(X) - \{\emptyset\} \to \omega$$

mapping any nonempty $a \subseteq X$ to a number $\|a\|$ such that

$$\begin{aligned}&\text{whenever } a = a_1 \cup \ldots \cup a_c \subseteq X, \text{ then for some}\\ &i_1, \ldots, i_d \in \{1, \ldots, c\}, \qquad \|a_{i_1} \cup \ldots \cup a_{i_d}\| \geqq \|a\| - 1\,.\\ &(|a| \text{ is the cardinality of the set } a)\end{aligned}$$

A natural (c, d)-complete norm is given by $\|a\| := \log_{c/d} |a|$. c-complete means $(c, 1)$-complete.

2.3 Definition. We call (f, g, h) *progressive*, if f, g, h are functions in ${}^\omega\omega$, satisfying

(1) For all $k, n_k^- \leqq g(k) < f(k) \leqq n_k^+$

(2) For all $k, n_k^- \leqq h(k)$

(3) $\lim_k \log \dfrac{f(k)}{g(k)} \Big/ \log h(k) = \infty$.

We call (f, g) progressive, if there is a function h such that (f, g, h) is progressive (or equivalently, if (f, g, n^-) is progressive, where n^- is the function defined by $n^-(k) = n_k^-$).

2.4 Remark. For example, if f and g satisfy (1), then (f, g, g) is progressive iff $\log f / \log g \to \infty$. $\ominus_{2.4}$

In 2.6 we will define a forcing notion $Q_{f,g,h}$ for any progressive (f, g, h). First we recall the following notation:

2.5 Notation. ${}^{<\omega}\omega = \bigcup_n {}^n\omega$ is the set of finite sequences of natural numbers. For $s \in {}^{<\omega}\omega$, $|s|$ is the length of s.

A tree p is a nonempty subset of ${}^{<\omega}\omega$ with the properties
$\forall \eta \in p \forall k < |\eta| : \eta \restriction k \in p$
$\forall \eta \in p : \mathrm{succ}_p(\eta) \neq \emptyset$, where

$$\mathrm{succ}_p(\eta) := \{\nu \in p : \eta \subset \nu, |\eta| + 1 = |\nu|\}\,.$$

A branch b of p is a maximal linearly $\subseteq$-ordered subset of p. Every branch b defines a function $\bar{b} : \omega \to \omega$ by $\bar{b} = \bigcup b$. We usually identify b and $\bar{b}$, so we write $b \restriction k$

(instead of $(\bigcup b) \restriction k$) for the k-th element of b. The set of all branches of p is written as $[p]$.

For $\eta \in p$, we let
$$p^{[\eta]} := \{\nu \in p : \nu \subseteq \eta \text{ or } \eta \subseteq \nu\}$$

We let
$$\begin{aligned} \mathrm{split}(p) &:= \{\eta \in p : |\mathrm{succ}_p(\eta)| > 1\} && \text{(the splitting nodes of } p\text{)} \\ \mathrm{split}_n(p) &:= \{\eta \in \mathrm{split}(p) : |\{\nu \subset \eta : \nu \in \mathrm{split}(p)\}| = n\} && \text{(the } n\text{-th splitting level)} \end{aligned}$$

and we define the stem of p to be the unique element of $\mathrm{split}_0(p)$.

2.6 Definition. Assume f, g, h are as in 2.3. Then we define for all k, and for all sets x
$$\|x\|_k := \left\lfloor \frac{\log(|x|/g(k))}{\log h(k)} \right\rfloor$$

and we define the forcing notion $Q_{f,g}$ (or more accurately, $Q_{f,g,h}$) to be the set of all p satisfying

(1) p is a perfect tree.
(2) $\forall \eta \in p \forall i \in \mathrm{dom}(\eta) \, \eta(i) < f(i)$.
(3) $\forall \eta \in \mathrm{split}_n(p) \, \|\mathrm{succ}_p(\eta)\|_{|\eta|} \geqq n$.

We let $p \leqq q$ ("q extends p") iff $q \subseteq p$.

2.7 Remark. If we define
$$p \sqsubseteq_k q \quad \text{iff} \quad p \leqq q \quad \text{and} \quad \mathrm{split}_k(p) \subseteq q$$

then $Q_{f,g,h}$ satisfies axiom A, and is in fact strongly ${}^\omega\omega$-bounding, i.e., for any name of an ordinal, $\underset{\sim}{\alpha}$, for any p and for any n there is a finite set A and a condition $q \sqsupseteq_n p$, $q \Vdash \underset{\sim}{\alpha} \in A$. However, it will be more convenient to use the relation $\leqq_n$ that is based on *levels* rather than *splitting levels.*

2.8 Definition. For $p, q \in Q, n \in \omega$ we define
$$p \leqq_n q \quad \text{iff} \quad p \leqq q \quad \text{and} \quad p \cap {}^{\leqq n}\omega \subseteq q$$

2.9 Notation. We will usually write $\|\eta\|_p$ instead of $\|\mathrm{succ}_p(\eta)\|_{|\eta|}$.

2.10 Remark. This forcing is similar to the forcing in [Shelah 326], but note the following important difference: Whereas in [Shelah 326] all nodes above the stem have to be splitting points, we allow many nodes to have only one successor, as long as there "many" nodes with high norm.

2.11 Remark.

(1) The norm $\|\cdot\|_k$ is $h(k)$-complete (hence also n_k^--complete).
(2) If $c/d \leqq h(k)$, then the norm is (c, d)-complete.
(3) If $\|a\|_k > 0$, then $|a| > g(k)$.
(4) $\|f(k)\|_k \to \infty$ (so $Q_{f,g,h}$ is nonempty). $\ominus_{2.11}$

We will see in the next section that this forcing (and any countable support product of such forcings) is proper and ${}^\omega\omega$-bounding. For the moment, we only show why this forcing is useful in connection with $\boldsymbol{c}(f, g)$:

2.12 Fact. Any generic filter $G \subseteq Q_{f,g}$ defines a "generic branch"

$$r := \bigcup_{p \in G} \mathrm{stem}(p)$$

that avoids all g-slaloms from V.

Proof. Let $\bar{B} = \langle B_k : k \in \omega \rangle$ be a g-slalom in V, and let $p \in Q_{f,g}$ be a condition. Let $\eta \in p$ be a node satisfying $\|\eta\|_p > 0$. Let $k := |\eta|$. Then $|\mathrm{succ}_p(\eta)| > g(k)$ by 2.11(3), so there is $i \notin B_k$, $\eta^\frown i \in p$. So $p^{[\eta^\frown i]} \Vdash r(k) = i \notin B_k$. $\circledcirc_{2.12}$

3 The construction

In this section we will prove the following theorem:

3.1 Theorem (CH). *Assume that* $(f_\xi, g_\xi : \xi < \omega_1)$ *is a sequence of progressive functions, witnessed by functions* h_ξ (see 2.3).

Let $(\kappa_\xi : \xi < \omega_1)$ be a sequence of cardinals satisfying $\kappa_\xi^\omega = \kappa_\xi$ such that whenever $\kappa_\xi < \kappa_\zeta$, then

$$\lim_{k \to \infty} \min\left(\frac{f_\zeta(k)}{g_\xi(k)}, \frac{f_\xi(k)}{g_\xi(k)} \Big/ h_\zeta(k) \right) = 0$$

(or informally: either $f_\zeta \ll g_\xi$, or $f_\xi / g_\xi \ll h_\zeta$, or a combination of these two condition holds).

Then there is a proper forcing notion P not collapsing cardinals nor changing cofinalities such that

$$\Vdash_P \forall \xi : \boldsymbol{c}(f_\xi, g_\xi) = \kappa_\xi$$

For the proof we use a countable support product of the forcing notions Q_{f_ξ, g_ξ, h_ξ} described in the previous section.

3.2 Remark. The theorem is of course also true (with the same proof) if we have countably or finitely many functions to deal with.

If we are only interested in 2 cardinal invariants $\boldsymbol{c}(f', g')$, $\boldsymbol{c}(f, g)$, then we can phrase the theorem without the auxiliary functions h as follows: If (f, g) and (f', g') are progressive, and satisfy

$$\min\left(\frac{f'}{g}, \frac{\log(f/g)}{\log(f', g')} \right) \to 0$$

then $\boldsymbol{c}(f, g) < \boldsymbol{c}(f', g')$ is consistent.

In particular, this shows that our result is quite sharp: For example, if for some function d we have $\lim d = \infty$, $f' = f^d$, $g' = g^d$ (and (f, g), (f', g') are progressive with the same n_k^-, n_k^+), then $\boldsymbol{c}(f, g) < \boldsymbol{c}(f', g')$ is consistent. On the other hand, $\boldsymbol{c}(f^n, g^n) \leqq \boldsymbol{c}(f, g)$ for every fixed n.

Proof. Choose h' such that $\log h' \approx 2\log(f/g)$ whenever $\dfrac{f'}{g} \geqq \dfrac{\log(f/g)}{\log(f'/g')}$. (f', g', h') is progressive, and the assumptions of the theorem are satisfied. (Recall that (f, g) is progressive, hence $\log f/g \gg \log n^-$, so h' will satisfy $h'(k) \geqq n_k^-$). $\circledcirc_{3.2}$

A similar simplified formulation of 3.1 is possible when we deal with only countably many functions.

3.3 Example. There is a family $\langle f_\xi, g_\xi, g_\xi : \xi < \boldsymbol{c}\rangle$ of continuum many progressive functions such that for any $\zeta \neq \xi$, $\min\left(\frac{f_\xi}{g_\zeta}, \frac{f_\zeta}{g_\xi}\right) \to 0$. [In particular, under CH we may choose any family $(\kappa_\xi : \xi < \omega_1)$ of cardinals satisfying $\kappa_\xi^\omega = \kappa_\xi$ and get an extension where $\boldsymbol{c}(f_\xi, g_\xi) = \kappa_\xi$.]

Proof. Let $\ell_k := \left\lfloor \frac{1}{2}\sqrt{\log \frac{\log n_k^+}{\log n_k^-}} \right\rfloor$. (Here, "log" can be the logarithm to any (fixed) base, say 2.) Then $\lim_{k\to\infty} \ell_k = \infty$, and by invariance (1.5(4)) we may assume $\ell_k \geqq 1$ for all k.

Let $T \subseteq 2^{<\omega}$ be a perfect tree such that for all k we have $|T \cap 2^k| = \ell_k$, say, $T \cap 2^k = \{s_1(k), \ldots, s_{\ell_k}(k)\}$. For any $x \in [T]$ (i.e., $x \in 2^\omega$, $\forall k\, x \restriction k \in T$) we now define functions f_x, g_x, h_x by:

If $x \restriction k = s_i(k)$, then

$$f_x(k) = (n_k^-)^{\ell_k^{2i}}$$
$$h_x(k) = g_x(k) = (n_k^-)^{\ell_k^{2i-1}}$$

We leave the verification that (f_x, g_x, h_x) is indeed progressive to the reader. [Recall 2.4, and also note that $\log\log f_x(k) \leqq 2\ell_k \log \ell_k + \log\log n_k^- < \log\log n_k^+$. Finally, note that if $x \neq y$, then for almost all k we have

$$\min\left(\frac{f_x(k)}{g_y(k)}, \frac{f_y(k)}{h_x(k)}\right) \ll \frac{1}{n_k^-}.\Big] \quad ☺_{3.3}$$

3.4 Example. It is consistent to have $\boldsymbol{c}(f^2, g^2) < \boldsymbol{c}(f, g)$ (for certain f, g).

Proof. Let $\ell_k := \left\lfloor \frac{1}{6} \log \frac{n_k^+}{n_k^-} \right\rfloor$. Assume $\ell_k > 0$ for all k. Then, letting

$$f(k) := (n_k^-)^{3\ell_k}$$
$$g(k) := (n_k^-)^{2\ell_k}$$
$$h(k) := n_k^-,$$

we have that (f, g, h) and (f^2, g^2, h) are progressive, and $\lim \frac{f}{g^2} = 0$, so we can apply the theorem. $☺_{3.4}$

3.5 Definition. Let κ be a disjoint union $\kappa = \bigcup_{\xi<\omega_1} A_\xi$, where $|A_\xi| = \kappa_\xi$.

For $\alpha < \kappa$, let Q_α be the forcing Q_{f_ξ, g_ξ, h_ξ}, if $\alpha \in A_\xi$, and let $P = \prod_{\alpha<\kappa} Q_\alpha$ be the *countable support product* of the forcing notions Q_α, i.e., elements of P are countable functions p with $\mathrm{dom}(p) \subseteq \kappa$, and $\forall \alpha \in \mathrm{dom}(p)\ p(\alpha) \in Q_\alpha$.

For $A \subseteq \kappa$, we write $P \restriction A := \{p \restriction A : p \in P\}$. Clearly $P \restriction A \lessdot P$ for any A. In particular, $Q_\alpha \lessdot P$.

We write $\underset{\sim}{r}_\alpha$ for the Q_α-name (or P-name) for the generic branch introduced by a generic filter on Q_α.

We say that q *strictly extends* p, if $q \geqq p$ and $\mathrm{dom}(q) = \mathrm{dom}(p)$.

3.6 Facts. Assume CH. Then

(1) each Q_α is proper and ${}^\omega\omega$-bounding.
(2) P is proper and ${}^\omega\omega$-bounding.
(3) P satisfies the $\aleph_2$-cc.
(4) Neither cardinals nor cofinalities are changed by forcing with P.

Proof of (1), (2): See below (3.23, 3.24).
Proof of (3): A straightforward Δ-system argument, using CH.
(4) follows from (2) and (3). $\circledcirc_{3.6}$

We plan to show that $\Vdash_P \boldsymbol{c}(f_\xi, g_\xi) = \kappa_\xi$ for all $\xi < \omega_1$.

3.7 Definition. If $p \in P$, $k \in \omega$, we let the level k of p be

$$\mathrm{Level}_k(p) := \{\bar\eta \colon \mathrm{dom}(\bar\eta) = \mathrm{dom}(p), \forall\alpha \in \mathrm{dom}(\bar\eta) \colon |\bar\eta(\alpha)| = k, \bar\eta(\alpha) \in p(\alpha)\}$$

We define the set of active ordinals at level k as

$$\mathrm{active}_k(p) := \{\alpha \in \mathrm{dom}(p) \colon |\mathrm{stem}(p(\alpha))| \leqq k\}$$

3.8 Remark. Sometimes we identify the set $\mathrm{Level}_k(p)$ with the set

$$\begin{aligned}&\{\bar\eta \colon \mathrm{dom}(\bar\eta) = \mathrm{active}_k(p), \forall\alpha \in \mathrm{dom}(\bar\eta) \colon |\bar\eta(\alpha)| = k\}\\ &\quad = \{\bar\eta \restriction \mathrm{active}_k(p) \colon \bar\eta \in \mathrm{Level}_k(p)\}\end{aligned}$$

3.9 Definition. We say that the k-th level is a splitting level of p (or "k is a splitting level of p") iff

$$\exists\alpha \in \mathrm{dom}(p) \exists\eta \in \mathrm{split}(p(\alpha)) \colon |\eta| = k$$

3.10 Definition. If $\bar\eta \in \mathrm{Level}_k(p)$, $\bar\eta' \in \mathrm{Level}_{k'}(p)$, $k < k'$, then we say that $\bar\eta'$ extends $\bar\eta$ iff for all $\alpha \in \mathrm{dom}(\bar\eta)$, $\bar\eta'(\alpha)$ extends (i.e., $\supseteq$) $\bar\eta(\alpha)$.

3.11 Definition. For $p, q \in P$, $k \in \omega$, we let

$$p \leqq_k p \quad \text{iff} \quad p \leqq q \text{ and } \forall\alpha \in \mathrm{dom}(p) \colon p(\alpha) \leqq_k q(\alpha) \text{ and } \mathrm{active}_k(p) = \mathrm{active}_k(q)$$

That is, we allow $\mathrm{dom}(q)$ to be bigger than $\mathrm{dom}(p)$, but for all new $\alpha \in \mathrm{dom}(q) - \mathrm{dom}(p)$ we require that $|\mathrm{stem}(q(\alpha))| > k$.

3.12 Definition. Let $A \subseteq P$. A set $D \subseteq P$ is

dense in A, if $\forall p \in A \exists q \in D \colon p \leqq q$
strictly dense in A, if $\forall p \in A \exists q \in D \colon p \leqq q$ and $\mathrm{dom}(p) = \mathrm{dom}(q)$

open in A, if $\forall p \in D \forall q \in A : (p \leqq q$ implies $q \in D)$
almost open in A, if $\forall p \in D \forall q \in A : (p \leqq q$ and $\mathrm{dom}(p) = \mathrm{dom}(q)$ implies $q \in D)$.

These definitions can also be relativized to conditions above a given condition p_0. If we omit A we mean $A = P$.

3.13 Definition. If $\bar{\eta} \in \mathrm{Level}_k(p)$, we let $q = p^{[\bar{\eta}]}$ be the condition defined by $\mathrm{dom}(q) = \mathrm{dom}(p)$, and

$$\forall \alpha \in \mathrm{dom}(q)\, q(\alpha) = p(\alpha)^{[\bar{\eta}(\alpha)]}$$

3.14 Definition. If $p \Vdash \underset{\sim}{x} \in V$, and $\bar{\eta} \in \mathrm{Level}_k(p)$, we say that $\bar{\eta}$ decides $\underset{\sim}{x}$ (or more accurately, $p^{[\bar{\eta}]}$ decides $\underset{\sim}{x}$) if for some $y \in V$, $p^{[\bar{\eta}]} \Vdash \underset{\sim}{x} = \check{y}$.

First we simplify the form of our conditions such that all levels are finite.

3.15 Fact. The set of all conditions p satisfying

$\boxed{\mathrm{I}}$ $\forall k |\mathrm{active}_k|(p)| < \omega$, and moreover:

$\boxed{\mathrm{II}}$ For any splitting level k there is exactly one pair (η, α) such that $|\mathrm{succ}_{p(\alpha)}(\eta)| > 1$.

is dense in P. $\circleddash_{3.15}$

3.16 Fact. If p is in the dense set given by (I) and (II), then the size of level k is $\leqq n_{k-1}^- \cdot n_{k-1}^+ < n_k^-$.

Proof. By induction. $\circleddash_{3.16}$

From now on we will only work in the dense set of conditions satisfying (I) and (II).

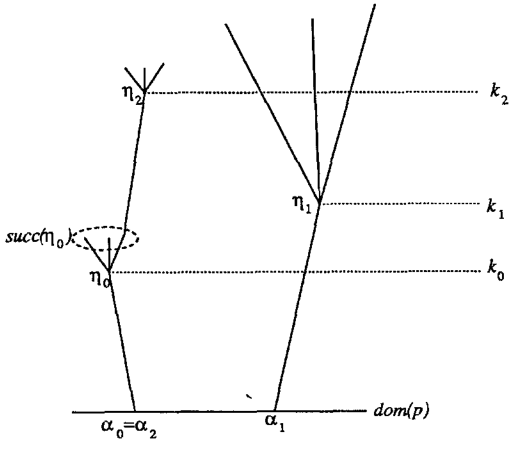

Fig. 2. A condition satisfying (I) and (II)

3.17 Notation. For p satisfying (I)–(II), we let $k_l = k_l(p)$ be the l-th splitting level. Let $\eta_l = \eta_l(p)$ and $\alpha_l = \alpha_l(p)$ be such that $|\eta_l(p)| = k_l(p)$, $\eta_l(p) \in \mathrm{split}(p(\alpha_l))$. We let $\zeta_l = \zeta_l(p)$ be such that $\alpha_l \in A_{\zeta_l}$.

We write $|p|_{k_l}$ for $|\eta_l\|_{p(\alpha_l)}$, i.e., for $\|\mathrm{succ}_{p(\alpha_l)}(\eta_l)\|_{\zeta_l,k_l}$. (See Fig. 2).

3.18 Definition. If p is a condition, $l \in \omega$, $\alpha^* := \alpha_l(p)$, $\eta^* := \eta_l(p)$, $\nu^* \in \mathrm{succ}_{p(\alpha^*)}(\eta^*)$, we can define a stronger condition q by letting $q(\alpha) = p(\alpha)$ for all $\alpha \neq \alpha^*$, and

$$q(\alpha^*) := \{\eta \in p(\alpha^*) : \text{If } \eta^* \subset \eta, \text{ then } \nu^* \subseteq \eta\}$$

In this case, we say that q was obtained from p by "pruning the splitting node η^*."

To simplify the notation in the fusion arguments below, we will use the following game:

3.19 Definition. For any condition $p \in P$, $G(P,p)$ is the following two person game with perfect information:

There are two players, the *spendthrift* and the *accountant*. A play in $G(P,p)$ last ω many moves (starting with move number 1). The *accountant* moves first. We let $p_0 := p$, $i_0 := 0$.

In the n-th move, the *accountant* plays a pair (η^n, α^n) with $\eta^n \in p_{n-1}(\alpha^n)$, $|\eta^n| = i_{n-1}$, and a number b_n.

Player *spendthrift* responds by playing a condition p_n and a finite sequence ν^n (letting $i_n := |\nu^n| + 1$) satisfying the following (see Fig. 3):

(1) $p_n \geqq_{i_{n-1}} p_{n-1}$.
(2) $\nu^n \in p_n(\alpha^n)$.
(3) $\|\nu^n\|_{p_n(\alpha^n)} > b_n$.
(4) $\nu^n \supset \eta^n$.
(5) For all $\alpha \in \mathrm{dom}(p_n) - \mathrm{dom}(p_{n-1})$, $|\mathrm{stem}(p_n(\alpha))| > |\nu^n|$.
(6) $|\mathrm{Level}_{|\nu^n|}(p_n)| = |\mathrm{Level}_{|\eta^n|}(p_n)| = |\mathrm{Level}_{|\eta^n|}(p_{n-1})|$

(Remember that all conditions p_n have to be in the dense set given by (I) and (II))

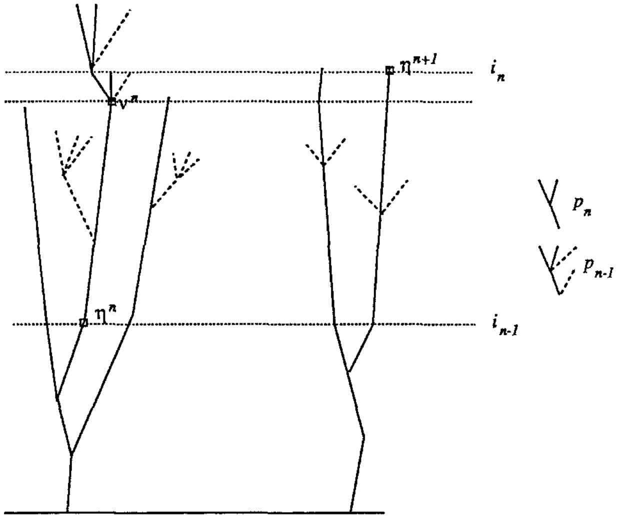

Fig. 3. Stage n

Player *accountant* wins iff after ω many moves there is a condition q such that for all n, $p_n \leqq q$, or equivalently, if the function q with domain $\bigcup_n \mathrm{dom}(p_n)$, defined by

$$q(\alpha) = \bigcup_{\alpha \in \mathrm{dom}(p_n)} p_n(\alpha)$$

is a condition. Note that we have $\eta_l(q) = \nu^l$, $\alpha_l(q) = \alpha^l$, since the only splitting points are the ones chosen by *spendthrift*.

3.20 Fact. Player *accountant* has a winning strategy in $G(P,p)$.

Proof. We leave the proof to the reader, after pointing out that a finitary bookkeeping will ensure that the limit of the conditions p_n is in fact a condition.

In particular, this shows that *spendthrift* has no winning strategy. Below we will define various strategies for the *spendthrift*, and use only the fact that there is a play in which the *accountant* wins. ☹$_{3.20}$

The game gives us the following lemma:

3.21 Lemma. *Assume that p is a condition satisfying (I)–(II). For each l let $\emptyset \neq F_{\eta_l} \subseteq \mathrm{succ}_{p(\alpha_l)}(\eta_l)$ be a set of norm $\|F_{\eta_l}\|_{k_l} \geqq \|\mathrm{succ}_{p(\alpha_l)}(\eta_l)\|/2$.*

Then there is a condition $q \geqq p$, $\mathrm{dom}(q) = \mathrm{dom}(p)$ such that for all l:

(∗) *If $\eta_l(p) \in q(\alpha_l(p))$, then $\mathrm{succ}_{q(\alpha_l(p))}(\eta_l(p)) \subseteq F_{\eta_l}$*

Proof. The condition q can be constructed by playing the game. In the n-th move, *spendthrift* first finds a $\eta^n \supset \nu^n$ satisfying $\eta^n(i) \in F_{\eta_i}$ whenever this is applicable, and $\|\mathrm{succ}_{p_{n-1}}(\eta^n)\| > 2b_n$. Then *spendthrift* obtains p_n by pruning (see 3.18) all splitting nodes of p_{n-1} whose height is between $|\eta^n|$ and $|\nu^n|$ and further thinning out the successors of η^n to satisfy $\mathrm{succ}_{p_n}(\eta^n) = F_{\eta^n}$. (Note that $F_{\eta^n} \subseteq \mathrm{succ}_{p_{n-1}}(\eta^n) = \mathrm{succ}_{p_0}(\eta^n)$.) In the resulting condition q the only splitting nodes will be the nodes η^n, so (∗) will be satisfied. ☹$_{3.21}$

(Note that in general $\eta_l(q) \neq \eta_l(p)$, and indeed $k_l(q) \neq k_l(p)$, since many splitting levels of p are not splitting levels in q anymore.)

3.22 Lemma. *Assume $\underset{\sim}{\tau}$ is a P-name of a function from ω to ω, or even from ω into ordinals. Then the set of conditions satisfying (I)–(II) is dense and almost open.*

$\boxed{\text{III}}$ *Whenever k is a splitting level, then every $\bar{\eta}$ in level $k+1$ decides $\underset{\sim}{\tau} \restriction k$.*

Proof of (III): We will use the game from 3.19. We will define a strategy for the *spendthrift* ensuring that the condition q the *accountant* produces at the end will satisfy (III).

In the n-th move, *spendthrift* finds a condition $r_n \geqq_{i_{n-1}} p_{n-1}$ such that for every $\bar{\eta} \in \mathrm{Level}_{i_{n-1}}(r_n)$ the condition $(p_n)^{[\bar{\eta}]}$ decides $\underset{\sim}{\tau} \restriction i_{n-1} + 10$. Then *spendthrift* finds $\eta^n \in r_n(\alpha^n)$ satisfying the rules and obtains p_n with $\eta^n \in p_n(\alpha^n)$ from r_n by pruning all splitting levels between i_{n-1} and $|\eta_n|$. ☹$_{3.22}$

Since all levels of q are finite, it is thus possible to find a finite sequence $\bar{B} = \langle B_k : k \in \omega\rangle$ in the ground model that will cover $\underset{\sim}{\tau}$. (I.e. $q \Vdash \underset{\sim}{\tau}(k) \in B_k$). The rest of this section will be devoted to finding "small" such sets B_k.

3.23 Corollary. *P is ${}^{\omega}\omega$-bounding and does not collapse ω_1.* ☹$_{3.23}$

3.24 Remark. Although it does not literally follow from 3.22, the reader will have no difficulty in showing that P is actually α-proper for any $\alpha < \omega_1$. ☺ Indeed, using the partial orders $\sqsubseteq_n$ from 2.7, it is possible to carry out straightforward fusion arguments, without using the game 3.19 at all. However, the orderings $\leqq_n$ are more easy to handle, since in induction steps we only have to take care of a single η^n, instead of a front.

3.25 Fact. $\Vdash_P \forall \tau \in {}^\omega\omega \exists B \subseteq \kappa$, B countable, $B \in V$, and $\tau \in V[G \upharpoonright B]$.

Proof. Let p be any condition and let $\underset{\sim}{\tau}$ be a name for a real. There is a stronger condition q satisfying (I), (II), and (III). Let $B := \mathrm{dom}(q)$. Clearly $q \Vdash \underset{\sim}{\tau} \in V[G \upharpoonright B]$. ☺$_{3.25}$

3.26 Corollary. *If* $\lambda = |A|^\omega$, *then* $\Vdash_{P\upharpoonright A} 2^{\aleph_0} \leqq \lambda$.

Proof. For each countable subset $B \subseteq A$, $\Vdash_{P\upharpoonright B} CH$. Since every real in $V[G]$ is in some such $V[G \upharpoonright B]$, the result follows. ☺$_{3.26}$

3.27 Fact and Notations. If P satisfies (II), then

(1) If $\bar\eta(\alpha_l) = \eta_l$ and $\nu \in \mathrm{succ}_{p(\alpha_l)}(\eta_l)$, then the requirement

$$\bar\eta^{+\nu}(\alpha_l) = \nu$$

uniquely defines an extension $\bar\eta^{+\nu}$ of $\bar\eta$ in $\mathrm{Level}_{k_l+1}(p)$.

(2) If $\bar\eta(\alpha_l) \neq \eta_l$, $\bar\eta$ has a unique extension $\bar\eta^+ \in \mathrm{Level}_{k_l+1}(p)$. To simplify the notation in 3.33 below, we also define for this case, for any $\nu \in \mathrm{succ}_{p(\alpha_l)}(\eta_l)$, $\bar\eta^{+\nu} := \bar\eta^+$.

3.28 Fact. The set of conditions satisfying (IV) is strictly dense (but not almost open) in the set of conditions satisfying (I)–(II).

$\boxed{\text{IV}}$ For all l:

$$|\mathrm{Level}_{k_l}(p)| < \min\left(\frac{\|p\|_{k_l}}{2}, n^-_{k_l}\right)$$

For the proof, note that $|\mathrm{Level}_{k_l}(p)| = |\mathrm{Level}_{k_{l-1}+1}(p)|$ ☺$_{3.28}$

3.29 Lemma. *Assume $\underset{\sim}{\tau}$ is a P-name of a function $\in {}^\omega\omega$, and $\Vdash_P \forall k \underset{\sim}{\tau}(k) < n^+_k$. Then the set of conditions satisfying (V) is strictly dense and almost open in the set give by (I),(II), (III), where*

$\boxed{\text{V}}$ *Whenever k is a splitting level, then every $\bar\eta$ in level k decides $\underset{\sim}{\tau} \upharpoonright k$.*

Proof. Fix p satisfying (I), (II), (III), (IV).

Let $k_l := k_l(p)$, etc. Let $m_l := |\mathrm{Level}_{k_l}|$.

Proof. We will use 3.21. For each $l \in \omega$, $F_{\eta_l} \subseteq \mathrm{succ}_{p(\alpha_l)}(\eta_l)$ will be defined as follows: Let $m_l := |\mathrm{Level}_{k_l}(p)|$, and let $\bar\eta^0, \ldots, \bar\eta^{m-1}$ enumerate $\mathrm{Level}_k(p)$. Find a sequence

$$\mathrm{succ}_{p(\alpha_l)}(\eta_l) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^m \quad \forall i \|F^{i+1}\|_k \geqq \|F^i\|_k - 1$$

such that for all i there exists x^i such that for all $\nu \in F^{i+1}$ we have $p^{[(\bar\eta^i)^{+\nu}]} \Vdash \underset{\sim}{\tau} \upharpoonright k = x$. It is possible to find such F^{i+1} since $\|\cdot\|_k$ is n^-_k-complete, and there are only $n^+_0 \cdot n^+_1 \ldots n^+_{k-1} < n^-_k$ many possible values of $\underset{\sim}{\tau} \upharpoonright k$.

Finally, let $F_{\eta_l} := F^m$. Applying 3.21 will yield the desired result. ☺$_{3.29}$

3.30 Remark. Note that (V) in particular implies

$\boxed{\text{Va}}$ Whenever k is not a splitting level, then every $\bar\eta$ in level k decides $\underset{\sim}{\tau}(k)$.

3.31 Proof that $\Vdash_P \boldsymbol{c}(f_\xi, g_\xi) \geqq \kappa_\xi$: (This proof is essentially the same as 2.12).

Recall that $\underset{\sim}{r}_\alpha$ is the generic real added by the forcing Q_α. Working in $V[G]$, let $\mathscr{B}$ be a family of less than κ_ξ many g_ξ-slaloms. We will show that they cannot cover $\prod f_\xi$, by finding an α such that $\underset{\sim}{r}_\alpha$ is forced not to be covered.

There exists a set $A \in V$ of size $< \kappa_\xi$ such that $\mathscr{B} \subseteq V[G \restriction A]$. Since $|A| < \kappa_\xi$ there is $\alpha \in A_\xi - A$.

Assume that $\bar{B}$ is a g_ξ-slalom in $V[G \restriction A]$ covering r_α. So in V there are a $P \restriction A$-name $\underset{\sim}{\bar{B}}$ and a condition p such that

$$\Vdash_{P\restriction A} \underset{\sim}{\bar{B}} \text{ is a } g\text{-slalom}$$

and

$$p \Vdash_P \underset{\sim}{\bar{B}} \text{ covers } r_\alpha$$

We can find a node η in $p(\alpha)$ with $\mathrm{succ}_{p(\alpha)}(\eta)$ having more than $g(|\eta|)$ elements. Increase $p \restriction A$ to decide $\underset{\sim}{B}_{|\eta|}$, then increase $p(\alpha)$ to make r_α avoid this set. ☺$_{3.31}$

3.32 Fact. Fix ξ^*. Then the set of conditions p satisfying

$\boxed{\text{VI}}$ For all l: If $\kappa_{\xi^*} < \kappa_{\zeta_l(p)}$, then

$$\min\left(\frac{f_{\zeta_l(p)}(k_l)}{g_{\xi^*}(k_l)}, \frac{f_{\xi^*}(k_l)}{g_{\xi^*}(k_l)} \Big/ h_{\zeta_l(p)}(k_l)\right) < \frac{1}{|\mathrm{Level}_{k_l}(p)|}$$

is dense almost open.

Proof. Write F_ζ for the function $\min\left(\frac{f_\zeta}{g_{\xi^*}}, \frac{f_{\xi^*}}{g_{\xi^*}} \Big/ h_\zeta\right)$. Recall that if $\kappa_\zeta < \kappa_{\xi^*}$, then F_ζ tends to 0. Fix a condition p. We will use the game $G(P,p)$. *spendthrift* will use the following strategy. Whenever $\alpha_n \in A_\zeta$ and $\kappa_\zeta < \kappa_{\xi^*}$, then *spendthrift* first find m_0 such that for all $m \geqq m_0$ we have $F_\zeta(m) < 1/|\mathrm{Level}_{h_{n-1}}(p_{n-1})|$. Now find $\nu^n \supseteq \eta^n$ of length $> m_0$ with a large enough norm, and play any condition p_n obeying the rules of the game. In particular, we must have $|\mathrm{Level}_{|\nu^n|}(p^n)| = |\mathrm{Level}_{|\eta^n|}(p^n)|$.

Clearly the condition resulting from the game satisfies the requirements. ☺$_{3.32}$

3.33 Proof that $\Vdash_P \boldsymbol{c}(f_\xi, g_\xi) \leqq \kappa_\xi$: Fix ξ. We will write f for f_ξ, etc. Let

$$A := \bigcup \{A_\zeta : \kappa_\zeta \leqq \kappa_\xi\}.$$

We will show that the g-slaloms from $V^{P\restriction A}$ already cover $\prod f$. This is sufficient, because $\Vdash_P (2^{\aleph_0})^{V^{P\restriction A}} \leqq |A| = \kappa_\xi$.

Let p_0 be an arbitrary condition. Let $\underset{\sim}{\tau}$ be a name of a function $< f$. Find a condition $p \geqq p_0$ satisfying (I)–(VI).

For each l we now define sets $F_{\eta_l} \subseteq \mathrm{succ}_{p(\alpha_l)}(\eta_l)$ as follows:

(1) If $\alpha_l \in A$, then $F_{\eta_l} = \mathrm{succ}_{p(\alpha_l)}(\eta_l)$.

(2) If $f_{\zeta_l}(k_l) \leqq g_\xi(k_l)/|\mathrm{Level}_{k_l}(p)|$, then again $F_{\eta_l} = \mathrm{succ}_{p(\alpha_l)}(\eta_l)$.

(3) Otherwise, we thin out the set $\mathrm{succ}_{p(\alpha_l)}(\eta_l)$ such that each $\bar{\eta}$ in $\mathrm{Level}_{k_l}(p)$ decides $\underset{\sim}{\tau}(k_l)$ up to at most $g(k_l)/|\mathrm{Level}_{k_l}(p)|$ many values.

Here is a more detailed description of case (3): Let $k = k_l$, $\zeta = \zeta_l$.

Note that if neither (1) nor (2) holds, the letting $c := f_\xi(k)$, $d := g_\xi(k)/|\mathrm{Level}_k(p)|$, we have $c/d \leqq h_\zeta(k)$.

Using (c, d)-completeness of the norm $\|\cdot\|_{\zeta,k}$ we define a sequence

$$\mathrm{succ}_{p(\alpha_l)}(\eta_l) = L(0) \supseteq L(1) \supseteq \ldots \supseteq L(|\mathrm{Level}_k(p)|)$$

as follows. Let $\bar\eta_0, \ldots, \bar\eta_{|\mathrm{Level}_k(p)|-1}$ be an enumeration of $\mathrm{Level}_k(p)$.

Given $L(i)$, we know that for each $\nu \in L(i)$ the sequence $\bar\eta_i^{+\nu}$, (i.e., the condition $p^{[\bar\eta_i^{+\nu}]}$) decides $\underset{\sim}{\tau}(k)$. (See 3.27.) Since there only $\leqq c$ many possible values for $\underset{\sim}{\tau}(k)$, we can use (c, d)-completeness to find a set $L(i+1) \subseteq L(i)$ and a set $C(i)$ such that

(a) $\|L(i+1)\| \geqq \|L(i)\| - 1$

(b) $|C(i)| \leqq d$.

(c) For every $\nu \in L(i+1)$, $p^{[\bar\eta_i^{+\nu}]} \Vdash \underset{\sim}{\tau}(k) \in C(i)$.

Now let F_{η_l} be $L(|\mathrm{Level}_k(p)|)$, and let

$$(\oplus) \qquad B_k := \bigcup_i C(i)\,.$$

So $|B_k| \leqq |\mathrm{Level}_k(p)| \cdot d \leqq g(k)$.

Clearly $\|F_{\eta_l}\|_{\zeta_l,k_l} \geqq \|p\|_{k_l} - |\mathrm{Level}_{k_l}(p)| > \frac{1}{2}\|p\|_{k_l}$.

This completes the definition of the sets F_{η_l}.

Let $q \leqq p$ be the condition defined from p using the F_{η_l} (see 3.21). We will find a $P \restriction A$-name for a g-slalom $\underset{\sim}{\bar B} = \langle \underset{\sim}{B}_k : k \in \omega\rangle$ such that

$$q \Vdash \underset{\sim}{\bar B} \text{ covers } \underset{\sim}{\tau}\,.$$

If k is not a splitting level, then every $\bar\eta$ in level k decides $\underset{\sim}{\tau}(k)$ by (Va). So in this case we can let

$$B_k := \{i : \exists\bar\eta \in \mathrm{Level}_k(p), p^{[\bar\eta]} \Vdash \underset{\sim}{\tau}(k) = i\}$$

This set is of size $\leqq |\mathrm{Level}_k(p)| < g(k)$, and clearly $q \Vdash \underset{\sim}{\tau}(k) \in B_k$.

If k is a splitting level, $k = k_l$, then there are three cases.

Case 1: $\alpha_l \in A$: We define $\underset{\sim}{B}_k$ to be a $P \restriction A$-name satisfying the following:

$$\Vdash_{P\restriction A} \underset{\sim}{B}_k = \{i : \exists\bar\eta \in \mathrm{Level}_{k+1}(p), V \models p^{[\bar\eta]} \Vdash \underset{\sim}{\tau}(k) = i, \bar\eta(\alpha_l) \subseteq r_{\alpha_l}\}$$

Thus, we only admit those $\bar\eta$ which agree with the generic real added by the forcing Q_{α_l}. Clearly $\Vdash_{P\restriction A} |B_k| \leqq \mathrm{Level}_k(p) < g(k)$, and $p \Vdash_P \underset{\sim}{\tau}(k) \in B_k$.

Case 2: $f_{\zeta_l}(k) \leqq g_\xi(k)/|\mathrm{Level}_k(p)|$.

So we have $|\mathrm{Level}_{k+1}(p)| \leqq f_{\zeta_l}(k) \cdot |\mathrm{Level}_k(p)| \leqq g(k)$, so we can let

$$B_k := \{i : \exists\bar\eta \in \mathrm{Level}_{k+1}(p), p^{[\bar\eta]} \Vdash \underset{\sim}{\tau}(k) = i\}$$

This set is of size $\leqq |\mathrm{Level}_{k+1}(p)| \leqq g(k)$, and again $p \Vdash \underset{\sim}{\tau}(k) \in B_k$.

Case 3: Otherwise. We have already defined B_{k_l} in ($\oplus$). By condition (c) above, $q \Vdash \underset{\sim}{\tau}(k) \in B_k$.

So indeed $q \Vdash$ "$\underset{\sim}{\bar{B}} = \langle \underset{\sim}{B}_k : k \in \omega \rangle$ is a g-slalom covering $\underset{\sim}{\tau}$" ☺$_{3.33}$ ☺$_{3.1}$ ☺$_{[\mathrm{GSh}\,448]}$

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