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STABILITY AND OMITTING TYPES

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ABSTRACT

We give a complete solution to the following question: when does a superstable theory have a model of power κ omitting a partial type q? In particular, for fixed q, if there is such a model of power $\aleph_1$ then there is one of power $2^{\aleph_0}$; and if there is a model omitting q of power $\beth_\omega^+$, then there are arbitrarily large ones. For stable theories, a model of power $\aleph_{\omega+1}$ omitting q implies one of power $2^{\aleph_0}$, and this is sharp. Several improvements and some negative results are listed in the introduction.

1. Introduction

Questions of omitting types test our ability to find symmetries in models and to build models preserving given symmetries. As the existence of many symmetries and the ability to construct models are two of the main features of stability, it is not surprising that there is a connection. The simplest omitting types question is the Hanf number: given a class of countable theories, its Hanf number is the least κ such that if a theory in the class has arbitrarily large models below κ omitting a given countable set of incomplete types, then it has arbitrarily large such models. Here is a table demonstrating the increasing power of the structure theories for the various stability classes:

Class of theories	Hanf number	
stable	$\beth_{\omega_1}$	
superstable	$\beth_\omega^{++}$	
superstable unidimensional	$\beth_\omega$	[Sh1]
superstable, NDOP, NOTOP	$\beth_1^{++}$	
ω-stable	$\aleph_2$	[Mo2]

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The lower bound for stable theories is of course the general bound from [Mo1]; we give examples showing one cannot do better. For superstable theories, the Hanf number was annoyingly open. It turns out that this was an accident, as the techniques of [Sh1] suffice with only slight modification for the general superstable case. We will not repeat the proof, but rather give a different and more powerful one, enabling us to say precisely when a superstable theory T has a model of power κ omitting a given type q. It turns out that the only influence of the cardinal κ is the set of finite symmetries it forces the model to have. We first note the special cases corresponding to the existence of all symmetries, and no symmetries.

THEOREM 1.1. *The Hanf number for omitting types is* $\beth_\omega^{++}$ *for countable superstable theories.*

THEOREM 1.2. *Let T be countable and superstable, Q a countable collection of partial types of T. If there exists a model of T of power $\aleph_1$ omitting each type in Q, then such a model exists in power $\beth_1$.*

Actually we will prove the special case 1.2 separately, as it involves some issues not present above the continuum. We will also prove:

THEOREM 1.3. *Let T be countable and stable, Q a countable collection of partial types of T. If there exists a model of T of power $\aleph_{\omega+1}$ omitting each type in Q, then such a model exists in power $\beth_1$.*

We will give examples showing that the number $\aleph_{\omega+1}$ in 1.3 is sharp.

The following was originally defined by Shelah (in an inessentially different form) in order to characterize theories having $(\lambda,2^\omega)$-models. (See [Sc].) He showed there that for regular κ, condition 1.4(b) below was equivalent to: $(\kappa,2^\omega)\to(\lambda,2^\omega)$.

DEFINITION. A combinatorial identity is an equivalence relation I on the set of $\leq n$-tuples of a set F with n elements, such that:

(a) If $(\bar a,\bar b)\in I$ then $\bar a,\bar b$ have the same length.

(b) If $(\bar a,\bar b)\in I$, and $\bar a',\bar b'$ are subsequences of $\bar a,\bar b$ in corresponding positions, then $(\bar a',\bar b')\in I$.

Given a cardinal κ, I is a combinatorial identity of κ ($I\in\mathrm{Id}(\kappa)$) if for every structure M of power κ (in a countable language), there exists a 1–1 map $j:F\to M$ such that if $(\bar a,\bar b)\in I$ then $j(\bar a),j(\bar b)$ have the same type in M.

THEOREM 1.4. *Given infinite cardinals $\kappa<\lambda$, the following are equivalent:*

(a) *Every countable superstable theory with a model of power κ omitting a given countable set of types has such a model of power λ.*

(b) $\kappa>\omega$, *and for some regular* $\kappa_r\leq\kappa$, $\mathrm{Id}(\kappa_r)=\mathrm{Id}(\lambda)$.

In fact our results give, for any superstable T, an analytic criterion for the existence of a model of power κ omitting a given countable set Q of types. (Considered as a subset of the appropriate Baire space, the set of Q's omitted in some model of T of power κ is analytic in the sense of descriptive set theory, uniformly in T and $\mathrm{Id}(\kappa)$.) One can do better than this, and in addition observe some compactness phenomena, if one measures size by dimension rather than cardinality. If p_i $(i \in I)$ are stationary types, $\bigotimes_i p_i$ denotes the type of an independent sequence $(a_i : i \in I)$ such that $a_i \vDash p_i$. (It has infinitely many variables.) p^α stands for $\bigotimes_{i<\alpha} p$. We say that $\dim_p(M) \geq \alpha$ if M realizes p^α. The criterion for the existence of a model of p-dimension κ omitting a type q is of G_δ-type: for any $I \in \mathrm{Id}(\kappa)$, every I-symmetric generalized formula extends to another one that visibly contradicts q. The definitions will be given in §2. At this point we can state the following version.

THEOREM 1.5. *Let T be superstable, p a stationary type, Q a countable collection of partial types. Suppose that for each $I \in \mathrm{Id}(\lambda)$ and $q \in Q$ there exists a model M of T omitting q such that $I \in \mathrm{Id}(\dim_p(M))$. Then there exists a model N of T omitting each $q \in Q$ with $\dim_p(N) = \lambda$.*

COROLLARIES (*T superstable, countable, Q a countable set of partial types, p stationary*).

(a) *If for each $q \in Q$, T has a model M omitting q with $\dim_p(M) = \kappa$, then it has such a model omitting every $q \in Q$ at once with $\dim_p(M) = \kappa$.*

(b) *If $\mathrm{Id}(\kappa) = \mathrm{Id}(\lambda)$ and T has a model with $\dim_p(M) = \kappa$, omitting q, then it has such a model with $\dim_p(M) = \lambda$.*

Note that (a) is not trivial even when Q has two elements.

In order to combine the omitting types results with many-cardinal theorems, we need the notion of a combinatorial identity of several cardinals. Define: $I \in \mathrm{id}(\kappa_1, \ldots, \kappa_n)$ if I is an equivalence relation on $(F_1 \cup \cdots \cup F_n)^{\leq m}$, $m = \sum \mathrm{card}(F_i)$, F_i's disjoint finite sets, such that I satisfies (a) and (b) of the definition of a combinatorial identity, and for every structure M in a countable language, if $P_1, \ldots, P_n$ are subsets of M and $\mathrm{card}(P_i) = \kappa_i$, then there exists a map $j : (F_1 \cup \cdots \cup F_n) \to M$ such that $j(x) \in P_i$ if $x \in F_i$, and $\mathrm{tp}(j(\bar{x})) = \mathrm{tp}(j(\bar{y}))$ if $(\bar{x}, \bar{y}) \in I$.

The fuller version of the theorem describes when there exists a model omitting a countable set of types but realizing $\bigotimes_{i<\omega} p_i^{\lambda_i}$, where p_i are stationary types, λ_i uncountable cardinals. It allows us to state the following corollary:

COROLLARY 1.6. *Let α be a countable ordinal, $(\lambda_i : i < \alpha)$, $(\kappa_i : i < \alpha)$ two continuous, increasing sequences of cardinals, such that $\kappa_i = \omega$ iff $\lambda_i = \omega$, and $\kappa_i =$*

$\lim(\kappa_j : j < i)$ *iff* $\lambda_j = \lim(\lambda_j : 1 < i)$. *Let Q be a countable set of partial types of the countable superstable theory T, and let P_i ($i < \alpha$) be unary predicates of T. Suppose* $\mathrm{Id}(\lambda_i : i \in J) = \mathrm{Id}(\kappa_i : i \in J)$ *for every finite $J \subseteq \alpha$. If T has a model N omitting each $q \in Q$ with* $\mathrm{card}(P_i^N) = \kappa_i$, *then it has a model N' omitting each $q \in Q$ with* $\mathrm{card}(P_i^N) = \lambda_i$.

We also give examples showing that the results cannot be improved in various directions; for example, there exists a superstable theory T and a countable set Q of partial types such that T has a model omitting Q of each power $\aleph_n$, but not of power $\aleph_\omega$. As most of the examples throughout the paper are of a similar nature, we wrote one out in detail (5.1(a)), the rest more sketchily.

§5 contains results in the opposite direction, solving a problem from [Ku]. An elementary extension N of a model M is called a small extension if for every finite subset A of M, every type realized in N over A is realized in M over A. Kueker raised the question of the Hanf number for this notion: in this case it is the least cardinal λ such that whenever M is superstable, if M has a small extension of every power below λ, then it has a small extension of arbitrary size. (The corresponding number for ω-stable theories is $\aleph_1$.) The answer is $\beth_{\omega_1}$. To prove this we give a combinatorial characterization of $\beth_\alpha$ ($\alpha < \omega_1$) using partition theorems with finite homogeneous sets; this may be of interest somewhere else.

The main open problems appear to be the Hanf number of omitting a single complete type in a superstable theory, and the omitting-type behavior of stable theories between consecutive $\beth$'s. For the first question our results leave only three possibilities: $\beth_\omega$, $\beth_\omega^+$, and $\beth_\omega^{++}$. We do not know which one is correct.

Throughout this paper, T is stable and countable. We assume a general knowledge of [Sh2], including the conventions related to $\mathbf{C}$ and $\mathbf{C}^{\mathrm{eq}}$ (the latter will not be used deeply, except in quoting [BuSh].) For the most part only the basic properties of independence and of 1-isolation are assumed. $A \downarrow B \mid C$ means that A, B are independent over C.

2. Stable systems

Consider the problem of finding a large structure in a given class, say an elementary class with an omitting types condition. One must do two things: find sets with enough symmetries to be blown up to the required size; and show that a model in the class in question can be built around the expanded set. For theories with Skolem functions, the second problem is trivial; thus the work in Morley's theorem of [Mo1], for instance, concentrates on finding indiscernibles. For stable theories, it is well known that large indiscernibles *sets* exist in abundance. They can be blown up, but the second problem then becomes difficult.

In the ω-stable context, one also has prime models over arbitrary sets, an adequate substitute for Skolem functions. But in the stable (or superstable) framework, the closest approach is Lachlan's 1-atomic models. Since the first-order type of a set of elements does not determine the type of a 1-atomic model over it, one must find sets that are symmetric with respect to more than first order properties, specifically symmetric with respect to an attempt to build a model around them. Such sets will be found using a mix of stability-theoretic and combinatorial methods.

This section develops the technical results regarding stable theories that are needed in this paper. They center around the notion of a stable system. Let S be a nonempty collection of subsets of a finite set F such that if $a \subseteq b$ and $b \in S$ then $a \in S$. An S-system is an indexed collection $(M(s): s \in S)$ of submodels of $\mathbf{C}$ such that $M(s) \subseteq M(t)$ if $s \subseteq t \in S$. It is called stable if the models are as free as possible from each other with respect to the given constraints; i.e. for each $s \in S$, $M(s) \downarrow \{M(t): t \in S,\ t \text{ not} \geq s\} \mid \{M(t): t \in S,\ t < s\}$. Stable systems enjoy a certain transfer principle:

Proposition 2.1. *Suppose $(M(s): s \in S)$ is a stable system. Let $a_i \in M(\{i\})$, $b_s \in M(s)$, and suppose $\models \theta(\bar{a}, \bar{b})$, then there exists a formula $\varphi(\bar{x})$ true of $\bar{a}$ such that in any S-system $\bar{N}$, if $a_i' \in N(\{i\})$ and $\models \varphi(\bar{a}')$ then there exist $b_s' \in N(s)$ $(s \in S)$ with $\models \theta(\bar{a}', \bar{b}')$.*

We will prove a more delicate version of this, in which we are told how to choose the b_s's (2.2, 2.3). Together with the existence results for stable systems (2.6, 2.7), this is the basis of the proof of Theorems 1.2 and 1.3. To prove 1.4 we need a stronger version, that allows us to transfer a system of elements with partial symmetries, leaving the symmetries intact. This cannot be expected to work in an arbitrary stable system. We do it for 1-systems, defined below.

Definition. Let S be a set of subsets of the finite set F, closed under subsets. S is considered as a partially ordered index set; we write $s < t$ for $s \subset t$. An S-condition is a finite set Φ of formulas in variables x_i $(i \in F)$ and $y_{s,j}$ $(s \in S, j$ in some index set depending on $s)$, satisfying (a) and (b) below. The variables $y_{s,j}$ are jointly denoted as y_s. Let $\Phi(s)$ be the set of formulas in Φ with variables from among $\{x_i : i \in s\} \cup \{y_t : t \leq s\}$. Let $\Phi(<s) = \bigcup\{\Phi(t) : t < s\}$, except if $s = \{i\}$ let $\Phi(<s)$ be the set of formulas in Φ with sole variable x_i.

(a) $\Phi = \bigcup_{s \in S} \Phi(s)$.

(b) For each $s \in S$, $\Phi(<s) \vdash (\exists y_s) \bigwedge\!\!\!\bigwedge \Phi(s)$.

Lemma 2.2. *Suppose Φ is an S-condition, $\overline{M} = (M(s): s)$ an S-system, $a_i \in M(\{i\})$, and each formula in Φ in the x-variables alone is true of $\bar{a}$. Then there exist*

$b_s \in M(s)$ such that $\models \Phi(\bar{a}, \bar{b})$. Moreover, if $S' \subseteq S$ is downward closed and $b_s \in M(s)$ is given for $s \in S'$, and $\models \Phi(s)((a_i : i \in s), (b_t : t \leq s))$ for each $s \in S'$, then we can choose $b_s \in M(s)$ for $s \notin S'$ such that $\models \Phi(\bar{a}, \bar{b})$.

PROOF. Maximize S' and b_s $(s \in S')$ such that the condition holds. If $S' \neq S$, choose $s \in S - S'$ of minimal size. Then $\models \Phi(<s)((a_i : i \in s), (b_t : t < s))$. By (b), there exists $b_s \in M(s)$ such that $\models \Phi(s)((a_i : i \in s), (b_t : t \leq s))$. So S' can be enlarged to $S' \cup \{s\}$, a contradiction. Thus $S = S'$. By (a), $\models \Phi(\bar{a}, \bar{b})$.

DEFINITION. A type q is locally (or 1-) isolated if for every $\varphi(x, \bar{y}) \in L$ there exists $\theta \in q$ such that for all $\bar{b}$, if $\varphi(x, \bar{b}) \in q$ then $\theta \vdash \varphi(x, \bar{b})$. A model M is 1-atomic over a set $A \subseteq M$ if for every $\bar{a}$ from M, $\mathrm{tp}(\bar{a}/A)$ is 1-isolated. $\overline{M} = (M(s) : s \in S)$ is a 1-system if it is an S-system, and for each s, $M(s)$ is 1-atomic over $\bigcup\{M(t) : t < s\}$, and $M(\{i\})$ is 1-atomic over $M(\emptyset) \cup \{a_i\}$ for $i \in F$, and $\{a_i : i \in F\}$ is independent over M_0. (It follows that $\overline{M}$ is a stable system; this is evident from the proof of Proposition 2.3.)

PROPOSITION 2.3. (a) *Let $(M(s) : s \in S)$ be a 1-system, $S \subseteq P(F)$, and let $a_i \in M(\{i\})$ $(i \in F)$, $b_s \in M(s)$ $(s \in S)$. Suppose $\models \theta(\bar{a}, \bar{b})$. Then there exist $c_s \in M(s)$ $(s \in S)$ and an S-condition $\Phi(\bar{x}; \bar{y}, \bar{z})$ such that $\models \Phi(\bar{a}, \bar{b}, \bar{c})$ and $\Phi \vdash \theta$. (The $\bar{z}$'s are to be considered as y-type variables.)*

(b) *Let $(M(s) : s \in S)$ be an S-system, $S \subseteq P(F)$, and let $a_i \in M(\{i\})$ $(i \in F)$, $b_s \in M(s)$ $(s \in S)$. Suppose $\models \theta(\bar{a}, \bar{b})$. Then there exists an S-condition $\Phi(\bar{x}; \bar{y}, \bar{z})$ such that $\theta \in \Phi$, and every $\varphi \in \Phi$ in the x-variables alone is true of $\bar{a}$.*

REMARK. Φ may have parameters from $M(\emptyset)$. But in 2.3(b), if $M_0 \subseteq M(\emptyset)$, $\theta \in L(M_0)$, and each $a_i \perp M(\emptyset) \mid M_0$, then the parameters of Φ can be restricted to M_0.

PROOF OF 2.3(a). We use induction on $\mathrm{card}(S)$. Let s^* be a largest element of S. Let $S^- = S - \{s^*\}$. We assume s^* is not a singleton; there is a slight difference of detail in the other case, but it is easier. Let $\Delta = \{\theta\}$ with special variable y_{s^*}. By hypothesis, $\mathrm{tp}_\Delta(b_{s^*}/\bigcup\{M(t) : t < s^*\})$ is isolated by some formula $\varphi(y_{s^*})$. Adding some b_t's if necessary, we may assume the parameters of φ are $(b_t : t < s^*)$.

CLAIM. φ isolates a Δ-type over $\bigcup\{M(t) : t \in S^-\}$.

PROOF. Suppose not. So there exists a Δ-formula φ' over $\bigcup\{M(t) : t \in S^-\}$ such that $\models \theta'$, where $\theta' = (\exists y_{s^*})(\varphi \,\&\, \varphi') \,\&\, (\exists y^*)(\varphi \,\&\, \sim\varphi')$. Again we may assume the parameters of φ' are $(b_t : t \in S^-)$; so we can write $\varphi' = \varphi'(y_{s^*}, b_t : t \in S^-)$, $\theta' = \theta'(b_t : t \in S^-)$. Applying the induction hypothesis to S^- and θ', we get an

S^--condition Φ' with $\theta'(y_t : t \in S^-) \in \Phi'$. Let $\alpha(\bar{x})$ be the conjunction of all formulas in the x-variables in Φ'. Since $\{a_i : i \in F\}$ is independent over $M(\emptyset)$, there exists $a'_i \in M(\emptyset)$ $(i \in F - s^*)$ such that $\vDash \alpha(a'_i : i \in F)$, where a'_i is a_i if $i \in s^*$. Let $M'(s) = M(s \cap s^*)$ for $s \in S^-$. Then $(M'(s) : s \in S^-)$ is an S^--system, so by Lemma 2.2 there exist $b'_t \in M'(s)$ $(t \in S^-)$, with $b'_t = b_t$ if $t < s^*$, such that $\vDash \Phi'(b'_t : t \in S^-)$. In particular, $\vDash \theta'(b'_t : t \in S^-)$, so φ does not isolate a Δ-type over $\{b'_t : t \in S^-\}$. But $\{b'_t : t \in S^-\} \subseteq \bigcup\{M(t) : t < s^*\}$, a contradiction.

In particular, $\varphi(y_{s^*}, b_t : t < s^*) \vDash \theta(a_i(i \in F), y_{s^*}, b_t : t \in S^-)$. Let $\theta'(x_i : i \in F, y_s : s \in S^-)$ be a formula such that $\vDash \theta'(a_i : i \in F, b_s : s \in S^-)$, $\theta'(x_i : i \in F, y_s : s \in S^-) \vdash (\exists y_{s^*})\varphi(y_{s^*}, y_t : t < s^*)$, and

$$\theta'(x_i : i \in F, y_s : s \in S^-) \ \&\ \varphi(y_{s^*}, y_t : t < s^*) \vdash \theta(y_{s^*}, y_t : t \in S^-, x_i(i \in F)).$$

By the induction hypothesis, find an S^--condition Φ' with $\theta' \in \Phi'$, such that $\vDash \Phi'(a_i : i \in F, b_s : s \in S^-)$ (after swallowing the c_s's into the b_s's once again). Let $\Phi = \Phi' \cup \theta(y_{s^*}, y_t : t \in S^-, x_i(i \in F))$. One sees immediately that Φ is an S-condition.

Proof of 2.3(b). We proceed as in the proof of 2.3(a): use induction on card(S); let s^* be a largest element; again we will treat the non-singleton case, the other case being different only in that $M \cup \{a_i\}$ (where $s = \{i\}$) replaces $\bigcup\{M(t) : t < s\}$. Let $S^- = S - \{s^*\}$, $\Delta = \{\theta\}$ again. Now $\mathrm{tp}_\Delta(b_{s^*}/\bigcup\{M(t) : t < s^*\}$ need not be isolated. However, $M(s^*) \downarrow \bigcup\{M(t) : t \in S^-\} \,|\, \bigcup\{M(t) : t < s^*\}$, so by the open mapping theorem one can find $\varphi(y_{s^*}) \in L(\bigcup\{M(t) : t < s^*\})$ such that $\vDash \varphi(b_{s^*})$, and for any $b'_{s^*} \in M(s^*)$, if $\vDash \varphi(b_{s^*})$ then $\vDash \theta(b'_s : s \in S)$ (the $'$ applies to b_{s^*} only). Choose $\varphi'(y_{s^*})$ stronger than $\varphi(y_{s^*})$ with parameters in $\bigcup\{M(t) : t < s^*\}$ such that φ does isolate a Δ-type over $\bigcup\{M(t) : t < s^*\}$. As in the proof of (a), φ isolates a Δ-type over $\bigcup\{M(t) : t \in S^-\}$. Except for the fact that φ is not true of b^* (which we are no longer claiming), we find ourselves in precisely the same situation as in the last paragraph of the proof of (a), and so we finish.

Proof of the Remark. Since each $a_i \downarrow M(\emptyset) \,|\, M_0$ and the a_i's are independent over $M(\emptyset)$, they are independent from $M(\emptyset)$ over M_0 as a tuple, so we may use the open mapping theorem.

Lemmas 2.3(b) and 2.2 yield Proposition 2.1. However, we will have to use the lemmas rather than the proposition. We proceed to prove symmetric versions of Lemmas 2.2 and 2.3(a). We know of no plausible symmetric parallel to 2.3(b); this is one of the main reasons we cannot generalize Theorem 1.4 to the stable case. (Note that below $\beth_1$, where there are no symmetries, we do have a generalization; the higher price there reflects the difference between 2.6 and 2.7.)

If I is a combinatorial identity on F, write $s \equiv t(\text{mod } I)$ if for some $i, a_1, \ldots, a_i, b_1, \ldots, b_i$, $s = \{a_1, \ldots, a_i\}$, $t = \{b_1, \ldots, b_i\}$, and $(\bar{a}, \bar{b})$ are in I. Note that if I is a combinatorial identity of any cardinal, given $\bar{a}$ and t, $\bar{b}$ is determined uniquely.

DEFINITION. (1) Let $\overline{M} = (M(s) : s \in S)$ be a 1-system, Φ an S-condition, and e a function carrying the variables of Φ to elements of $\bigcup M(s)$. $(\overline{M}, \Phi, e)$ is an S-system if the variables of Φ_s go to $M(s)$ under e, and $\models \Phi(e(x_i), e(y_s) : i, s)$.

(2) An S-system $(\overline{M}, \Phi, e)$ is I-symmetric (via $\bar{h}$) if for each pair $s, t \in S$ with $s \equiv t(\text{mod } I)$, we are given maps $h(s,t), h_{\text{var}}(s,t)$ such that:

(a) $h(s,t) : M(s) \to M(t)$ is an isomorphism.

(a′) $h_{\text{var}}(s,t)$ takes the variables of Φ_s to those of Φ_t. It takes x-variables to x-variables and y-variables to y-variables; and Φ_s, Φ_t differ only by this change of variables.

(b) $h(s,s)$ is the identity; (b′) $h_{\text{var}}(s,s)$ is the identity.

(c) If $s = \{u_1, \ldots, u_k\}$, $t = \{v_1, \ldots, v_k\}$, $s' = \{u_{i_1}, \ldots, u_{i_n}\}$, $t' = \{v_{i_1}, \ldots, v_{i_n}\}$, $(\bar{u}, \bar{v}) \in I$ and $s, t \in S$, then $h(s', t')$ is the restriction of $h(s,t)$ to $M(s')$; and similarly (c′) for h_{var}.

(d) $h(s,t) \circ h(t,u) = h(s,u)$ whenever all are defined, and likewise (d′) h_{var}.

Φ is I-symmetric if h_{var} exists satisfying (a–d).

$\overline{M}$ is I-symmetric if h exists satisfying (a′–d′).

(e) $e \circ h_{\text{var}}(s,t) = h(s,t) \circ e$.

PROPOSITION 2.4. *Suppose $(M(s) : s \subseteq F)$ is an I-symmetric 1-system, Φ is an I-symmetric $P(F)$-condition, and one is given $e_0(x_i)$ $(i \in F)$ so that* (e) *holds for the x-variables and $\models \varphi((e_0(x_i) : i \in F)$ for each $\varphi \in \Phi$ in the x-variables alone. Then there exists e extending e_0 such that $(\overline{M}, \Phi, e)$ is an I-symmetric S-system.*

PROOF. Assume e has been defined on the x-variables and on the y_s-variables for $s \in [F]^{<i}$, agreeing with e_0 on the x-variables, and so that (e) holds for s, t of size $<i$, and $\models \Phi_s(e(y_t) : t \leq s, x_i : i \in s)$ for $s \in [F]^{<i}$. Let R be a subset of $[F]^i$ such that for every i-tuple $\bar{u}$ from F there is a unique $s \in R$ such that $\bar{u}$ is I-equivalent to some enumeration of s. Find $e(y_s)$ $(s \in R)$ so that $e(y_s) \in M(s)$ and $\models \Phi_s(e(x_i, e(y_t) : i \in s,\ t \leq s)$. Now define $e(y_s)$ on all $s \in [F]^i$ so that (e) holds for s, t of size i; there is a unique way to do it. By symmetry, $\models \Phi_s(e(x_i), e(y_t) : i \in s,\ t \leq s)$ for all $s \in [F]^i$. This suffices by the definition of an S-condition.

PROPOSITION 2.5. *Let $(M(s) : s \in S)$ be a* 1*-system, Φ an S-condition and e an assignment so that $(\overline{M}, \Phi, e)$ is an I-symmetric S-system; and suppose $\models \theta(e(\bar{x}), e(\bar{y}))$. Then there exists Φ' extending Φ and e' extending e so that $\Phi' \vdash \theta$, and $(\overline{M}, \Phi', e')$ is an I-symmetric S-system.*

PROOF. It is easier to prove something slightly stronger. Let $S^k = \{s \in S : \operatorname{card}(s) \leq k\}$. Call Ψ an S-k-precondition if it satisfies the definition of an S-condition, with (b) replaced by:

(b_k) For each $s \in S$ with $\operatorname{card}(s) > k+1$, $\Phi(<s) \vdash (\exists y_s) \bigwedge \Phi(s)$.

Call $(\overline{M}, \Psi, f)$ an I-symmetric S-k-pre-system if it satisfies the full definition of an I-symmetric system, except that Ψ need only be an S-k-precondition rather than fully an S-condition.

PROPOSITION 2.5 (technical version). *Let $(\overline{M}, \Phi, e)$ be an I-symmetric S-k-pre-system, and let θ be a formula involving only variables x_i and y_s $(s \in S^k)$ such that $\vDash \theta(e(x_i) : i \in F, e(y_s) : s \in S^k)$. Then there exists an I-symmetric S-system $(\overline{M}, \Phi', e')$ with $\Phi' \supseteq \Phi$, $e' \supseteq e$, and $\Phi \vdash \{\theta\}$.*

PROOF. Induction on k. As usual the details are slightly different when $k = 1$, and we assume $k > 1$. Let θ' be the conjunction of the following: θ; all formulas of the form $(\exists y_s) \bigwedge \Phi(s)$, where $s \in S^{k+1}$; and $\bigwedge \Phi(s)$ for $s \in S^k$. Let e^k be the restriction of e to the variables x_i and y_s $(s \in S^k)$. By 2.3(a) it is possible to find an S-condition $\Phi^* \vdash \theta'$ and e^* extending e^k such that $(\overline{M}, \Phi^*, e^*)$ is an S-system. Say $(\overline{M}, \Phi, e)$ is I-symmetric via $\bar{h}, \bar{h}_{\mathrm{var}}$. For $s, t \in S^k$ with $t \equiv s(\operatorname{mod} I)$ and z a y_t-variable of Φ^* not occurring in Φ, let $y_{s,(t:z)}$ be a new variable (of type y_s). If z does occur in Φ, let $y_{s,(t:z)}$ be $h_{\mathrm{var}}(t,s)(z)$. Let y'_s be the union of the variables $y_{s,(t:z)}$ over all appropriate t, z. Note that whenever $s, t \in S^k$, $t \equiv s(\operatorname{mod} I)$, $s' \subseteq s$, and $y_{s',(u:z)}$ is defined and occurs in Φ, we have $h_{\mathrm{var}}(s,t)(y_{s',(u:z)}) = y_{t',(u:z)}$, where t' is the subset of t corresponding to $s' \subseteq s$ (as in (c) of the definition of I-symmetry). Use this equation to define $h_{\mathrm{var}}(s,t)(y_{s',(u:z)})$ for new variables also. Let $e^*(y_{s,(t,z)}) = h(t,s)(e(z))$. So h_{var} extends the previous definition, e^* extends e^k, and one checks easily that (a–e) are satisfied. Let $\Psi = \{\varphi(y_{s,(u:z)}) : \varphi(z) \in \Phi^*_u\}$. Then $(\overline{M}, \Phi \cup \Psi, e \cup e^*)$ is an I-symmetric S-$(k-1)$-presystem. Induction does the rest.

The following are the existence lemmas for stable systems in the stable and superstable cases.

LEMMA 2.6. *Let N be a model of a stable theory T, and for $i \in \{1, \ldots, n\}$ let M_i be an elementary submodel of N in the language enriched with a predicate for $M_1, \ldots, M_{i-1}$. For $s \in S$ let $M(s) = \bigcap\{M_i : i \notin s\}$. Then $(M(s) : s \subseteq \{1, \ldots, n\})$ is a stable system.*

PROOF ([Sh2]). Induction on n. Checking that $M(s) \perp M(\not\geq s) \mid M(<s)$ involves two statements, according to whether or not $n \in s$.

(a) If $A \downarrow B \mid C$ in N, and M is an elementary submodel of (N,B), then $A \cap M \downarrow B \mid (C \cap M)$.

(b) If $A \downarrow B \mid C$ in N, and M is an elementary submodel of (N,A,B,C), then $A \downarrow (B \cup M) \mid (C \cup (A \cap M))$.

(a) is clear e.g. from the 2-rank characterization of forking. (b) follows formally from: $A \cup B \downarrow M \mid (A \cap M \cup B \cap M)$, a special case of (a).

Definition. Let $M \subseteq N \subseteq \mathbf{C}$. Then $M \subseteq_{na} N$ if whenever a formula $\varphi(x,\bar{a})$ with parameters in M has a non-algebraic solution in N, it has one in M.

We will show below that whenever $(M(s) : s \in S)$ is a stable system in a superstable theory, $M(s) \subseteq_{na} N$ $(s \in S)$, F a finite subset of N, $A = \bigcup_s M(s) \cup F$, then there exists $N' \subseteq N$, $A \subseteq N'$, N' 1-atomic over A. One concludes immediately:

Lemma 2.7. *Let $M \subseteq N$, $M \subseteq_{na} N$, and let $I \subseteq N$ be independent over M. Then there exists a stable system $(M(s) : s \in [I]^{<\omega})$ such that $M(\emptyset) = M$, $a \in M(\{a\})$ for $a \in I$ and $M(\{a\})$ is 1-atomic over $M \cup \{a\}$, and $M(s)$ is 1-atomic over $\bigcup\{M(t) : t < s\}$ if* $\operatorname{card}(s) \geq 2$; *and $M(s) \subseteq N$.*

For the record, we also recall:

Lemma 2.8. *Let T be countable, M a model of T of regular cardinality κ.*

(a) *If T is superstable and $\kappa > \omega$ then there exists a countable $B \subseteq M$ and an independent set I over B, $I \subseteq M$,* $\operatorname{card}(I) = \kappa$. *If $\kappa > 2^{\aleph_0}$, I can be taken to be a Morley sequence over B.*

(b) *If T is stable and $\kappa > \omega_1$, then there exists $B \subseteq M$,* $\operatorname{card}(B) < \kappa$, *and an independent set I over B, $I \subseteq M$,* $\operatorname{card}(I) = \kappa$.

Proof. We prove (b) for example. Let $M = \{a_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$ of cofinality ω_1, find $\beta(\alpha) < \alpha$ such that $\operatorname{tp}(a_\alpha / \{a_\gamma : \gamma < \alpha\})$ does not fork over $\{a_\gamma : \gamma < \beta\}$. By Fodor's lemma, there exists a stationary S such that $\beta(\alpha) = \beta$ for $\alpha \in S$. Let $B = \{a_\gamma : \gamma < \beta\}$, $I = \{a_\alpha : \alpha \in S\}$.

Lemma 2.9. *The following are equivalent*:

(a) $M \leq_{na} N$.

(b) *For every finite $F \subseteq M$, $\varphi \in L(M)$, $b \in \varphi^N$, and $\theta(x,b)$ such that θ forks over F, there exists $b' \in \varphi^M$ such that $\theta(x,b')$ forks over F.*

(c) *Let $F \subseteq M$ be finite, $b \in N$, $a \downarrow Mb \mid F$, and suppose $\varphi(x,a)$ & $\theta(x,a,b)$ is consistent and forks over $F \cup \{a\}$. Then there exists $b' \in M$ and θ' such that $\varphi(x,a)$ & $\theta'(x,a,b')$ is consistent and forks over $F \cup \{a\}$.*

Proof. (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are downhill. (For (b) $\Rightarrow$ (a) let $\theta(x,b) = (x = b)$.)

(a) $\Rightarrow$ (c): Let $q = \operatorname{stp}(a/M)$. Define an equivalence relation by: zEz' iff

$(d_q y)(\forall x)(\varphi(x,y) \Rightarrow. \theta(x,y,z) \equiv \theta(x,y,z'))$. As $\theta(x,a,b)$ forks over $M \cup \{a\}$, it is clear that $b/E \notin M$. Choose $b' \in M$ such that $\vDash(d_q y)(\exists x)(\varphi(x,y)\ \&\ \theta(x,y,b'))$, and $b'/E \notin \operatorname{acl}(F)$. Let $r = \operatorname{stp}(b'/F)$, and let $\alpha(x,y) = (d_r z)\theta(x,y,z)$. Then α is over $\operatorname{acl}(F)$, so it is not the case that $\vDash(d_q y)(\forall x)(\varphi(x,y) \Rightarrow. \theta(x,y,z) \equiv \alpha(x,y))$. Let $\theta'(x,y,z) = [\theta(x,y,z) \not\equiv \alpha(x,y)]$. Then $\theta'(x,a,b')$ satisfies the requirements.

We proceed to prove the statement preceding Lemma 2.7. It is not true in general, even for a superstable theory, that there exists a 1-isolated model over a given set A contained in a given model $N \supseteq A$. In [BuSh] it was shown that this is true if one also assumes NDOP, and if A is the union of a certain tree of models. We need to know this without the NDOP assumption, and dealing with arbitrary stable systems, rather than trees. We will recover on the way Theorem C of [BuSh], as well as the results of §6 there (see remark (b) at the end of the section). T is assumed superstable.

Definition. Let $q = \operatorname{tp}(a/B)$, $X = \{c \in \mathbf{C} : c \vDash q\}$. q is *homogeneous* if there exists $B' \supseteq B$, an ∞-definable group H and a definable transitive action of H on X (defined over B'), so that H acts on X as a group of automorphisms over B'.

Recall also that $q = \operatorname{tp}(a/B)$ is *c-isolated* if for some $\varphi \in q$, there is no type q' over B with $\varphi \in q'$ and $R^\infty(q') < R^\infty(q)$.

The following lemma appears in [BuSh] as 6.10.

Lemma 2.10. *Let q be homogeneous and c-isolated. Then q is 1-isolated.*

Proof. Let X, H, φ be as in the definition of homogeneity and c-isolation. By [H2, Thm2′] there exist definable X', H' and a definable action of H' on X', extending the action of H on X; moreover we may take $X' \subseteq \varphi^{\mathbf{C}}$. By the choice of φ, every type over B' of elements of X' has maximal R^∞, hence is generic. By [H2, 2.2d], for each finite Δ, there are only finitely many restrictions to Δ of generic types; hence in this case only finitely many Δ-types over B in X'. So in fact every type over B' inside X' is 1-isolated.

Fix a model $N \subseteq \mathbf{C}$ of T for the rest of the section. The notion of m-isolation defined below is relative to N, and all sets are assumed to be subsets of N.

Definition. $q = \operatorname{tp}(a/B)$ is m-isolated in N if *either*

(m_1) q homogeneous and c-isolated, *or*

(m_2) for some $\varphi \in q$, for every model $M \leq_{na} N$ with $M \subseteq B$, every strong type q' over B realized in N with $\varphi \in q'$ is orthogonal to M.

Note that in either case, if $M \leq_{na} N$ and $M \subseteq B$, then $aB \trianglelefteq B \,|\, M$ (in case (1) by 2.10). Hence if B is m-constructible over A and $M \leq_{na} N$, $M \subseteq A$, then $B \trianglelefteq A \,|\, M$.

LEMMA 2.11. *Let $A \subseteq N$ and suppose $A \not\leq_{na} N$. Then there exists $d \in N - A$ such that d/A is m-isolated. If A is a model, d can be chosen m_2-isolated.*

PROOF. If $A \neq \mathrm{acl}(A)$, choose $d \in \mathrm{acl}(A) - A$. Assume $A = \mathrm{acl}(A)$ from now on. If A is not a model, find $\varphi \in L(A)$ such that $\varphi^N \neq \emptyset$ but $\varphi^N \cap A = \emptyset$; if A is a model but $A \not\leq_{na} N$, find a finite $F \subseteq A$ and $\varphi \in L(A)$ such that $\varphi^N \not\subseteq \mathrm{acl}(F)$, but $\varphi^N \cap A \subseteq \mathrm{acl}(F)$. In either case choose φ of least possible R^∞. If φ satisfies (m_2) we are done (choose any $d \in \varphi^N - A$). Otherwise there exist $M \subseteq A, M \leq_{na} N$, and $c \in \varphi^N - A$ such that $\mathrm{tp}(c/A) \not\perp M$. Fix M and c.

Pick $c_1 \in \mathrm{dcl}(Ac)$ with $q = \mathrm{stp}(c_1/A)$ semi-regular. Let

$$\Psi = \{\psi(x,b) \colon \text{there exists a strong type } r \text{ with } \psi \in r \text{ and } r \not\perp q\}.$$

For a given $\psi(x,y)$, $\{b : \psi(x,b) \notin \Psi\}$ is an ∞-definable set ([H1]). As $q = \mathrm{stp}(c_1/A)$ is semi-regular and $\not\perp M$, if q' is conjugate to q over M then $r \perp q$ iff $r \perp q'$. Thus Ψ is invariant under $\mathrm{Aut}(\mathbf{C}/M)$. Combining the two observations, we see that: (*) if $\psi(x,b) \in \Psi$ then for some $\varphi \in \mathrm{tp}(b/M)$, for all b', if $\models \varphi(b')$ then $\psi(x,b') \in \Psi$.

Choose $\theta(x) \in L(M) \cap \Psi$ of least possible R^∞.

Case 1. $c_1 \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} \theta^{\mathbf{C}} \mid A$. In this case A cannot be a model. For $\theta \in \Psi$, so for some $e \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} c_1 \mid A$, and $a \in \theta^{\mathbf{C}}$, $c_1 \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} a \mid Ae$. Find $\alpha(x,y,z) \in \mathrm{tp}(c_1,a,e/A)$ such that $\alpha(c_1,y,z)$ forks over A. We have: $\models (d_q x)(\exists y)(\theta(y) \mathbin{\&} \alpha(x,y,e))$. If A were a model, then for some $e' \in A$, $\models (d_q x)(\exists y)(\theta(y) \mathbin{\&} \alpha(x,y,e'))$, so $\models (\exists y)(\theta(y) \mathbin{\&} \alpha(c_1,y,e'))$, whence $c_1 \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} \theta^{\mathbf{C}} \mid A$. Thus by the choice of φ when A is not a model, it c-isolates a type.

By [H1], there exists $d \in \mathrm{dcl}(Ac_1) - A$ such that $\mathrm{tp}(d/A)$ is θ-internal; and by [H2], letting $X = \{x : \mathrm{tp}(x/A) = \mathrm{tp}(d/A)\}$, $G = \mathrm{Aut}(X/A \cup \theta^{\mathbf{C}})$ is ∞-definable. By the case assumption (and the fact that $A = \mathrm{acl}(A)$), G acts transitively on X. Thus $\mathrm{stp}(d/A)$ is homogeneous, and is c-isolated since $\mathrm{tp}(c/A)$ is. Hence d is m_1-isolated.

Case 2. $c \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} \theta^{\mathbf{C}} \mid A$. Find a finite $E \subseteq \theta^N$ that $c \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} E \mid A$; choose E as small as possible. Say $cE \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} A \mid F'$, with $F \subseteq F'$, and choose a finite $F_0 \subseteq M$ such that $F' \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} M \mid F_0$, and θ is over F_0.

CLAIM. Let $e \in E$, $E' = A \cup E - \{e\}$. Then $e \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} E' \mid M$.

PROOF. By the minimality of E, $c \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} E' \mid A$, so $c \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} e \mid E'$. Suppose $e \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} E' \mid M$. Find $\theta'(x,e') \in \mathrm{tp}(e/E')$ such that $\theta'(x,e')$ forks over M, and $\theta' \Rightarrow \theta$. As $e \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.3ex\hbox{$|$}} c \mid E'$, $\theta' \in \Psi$. By (*) and Lemma 2.9(b), there exists $e'' \in M$ such that $\theta'(x,e'') \in \Psi$ and $\theta(x,e'')$ forks over F_0. This contradicts the minimality of $R^\infty(\theta)$.

By the claim, if $E = \{e_1, \dots, e_n\}$, then $e_k/Ae_1, \dots, e_{k-1} \,|\, M$ for each k. By transitivity, $A, e_1, \dots, e_n$ are independent over M. So $A \downarrow E \,|\, M$. Now go in the other direction: $c \not\downarrow E \,|\, A$. Find $\varphi' \in \operatorname{tp}(c/AE)$ such that φ' forks over A. By 2.9(c), there exists $\varphi'' \in L(M)$ such that φ'' & φ is consistent, and φ'' forks over F'. Let c' be any realization (in N) of φ'' & φ. Then $c' \not\downarrow M \,|\, F'$, so $c' \notin \operatorname{acl}(F)$. Also $R^\infty(c'/A) < R^\infty(c'/F') \leq R^\infty(\varphi) = R^\infty(c/A)$. This contradicts the minimality of $R^\infty(\varphi)$, showing that case 2 cannot in fact occur, and proving the lemma.

COROLLARY. *For every $B \subseteq N$ there exists an na-submodel of N containing B and m-constructible over B.*

LEMMA 2.12. *Let $M \subseteq N$ be m-constructible over $\bigcup_{s \in S} M(s) \cup F$, where $(M(s) : s \in S)$ is a stable system of models such that $M(s) \leq_{na} N$, and $F \subseteq N$ is finite. Then M is 1-atomic over $\bigcup_{s \in S} M(s) \cup F$.*

PROOF. We show by induction on the length of the construction that if A is m-constructible over $\bigcup_{s \in S} M(s) \cup F$, then it is 1-atomic over $\bigcup_{s \in S} M(s) \cup F$. Let a/A be m-isolated. If a/A is m_1-isolated, then by 2.10 it is 1-isolated. In the other case, $\operatorname{stp}(a/A) \perp M(s)$ for each $s \in S$. Let F' be a finite set containing F such that $a \downarrow A \,|\, F'$, and let $B = \bigcup_s M(s) \cup F'$. Then $\operatorname{stp}(a/B) \perp M(s)$ for each $s \in S$; and it suffices to show under this hypothesis that $\operatorname{tp}(a/B)$ is 1-isolated.

Suppose $\operatorname{tp}(a/B)$ is not 1-isolated; so for some $\Delta = \{\alpha(x, \bar{y})\}$, no $\varphi \in \operatorname{tp}(a/B)$ isolates $\operatorname{tp}_\Delta(a/B)$. Let $\beta(\bar{y}) = (d_r x)\alpha(x, \bar{y})$, where $r = \operatorname{stp}(a/B)$. Let L_1 be the language L enriched with a constant for each element of F', and a predicate P_s for $M(s)$ $(s \in S)$. Consider the following L_1-type:

$$\Gamma(x, \bar{y}) = \{\bar{y} \text{ is a tuple from } \textstyle\bigcup_s P_s \cup F'\} \cup \{\varphi(x) : \varphi \in r\} \cup \{\alpha(x, \bar{y}) \not\equiv \beta(\bar{y})\}.$$

Then Γ is finitely satisfiable in $(\mathbf{C}, F', M(s) : s \in S)$. Hence Γ is realized in an elementary extension $(\mathbf{C}^*, F', M^*(s) : s \in S)$ of this structure, say by $a', \bar{b}'$. For $T \subseteq S$, let $M\{T\} = \bigcup_{s \in T} M(s)$, $M^*\{T\} = \bigcup_{s \in T} M^*(s)$, $M^*\{<s\} = M^*\{\{s' : s' \leq s\}\}$, $M^*\{\downarrow s\} = M^*\{\{s' : s \text{ is not } \leq s'\}\}$. Then we have:

(i) $\mathbf{C} \downarrow M^*\{T\} \,|\, M\{T\}$ for each downward-closed $T \subseteq S$.

(ii) $M^*(s) \downarrow M^*(\downarrow s) \,|\, M^*(<s)$.

(iii) $a' \vDash r$ and $a' \not\downarrow (\bigcup_s M^*(s) \cup F') \,|\, F'$.

From (i) and (ii) we can conclude:

(i′) $M^*(s) \downarrow \mathbf{C} \,|\, M^*(\downarrow s) \cup M(s)$ (with $T = \{s' : s \leq s' \Rightarrow s = s'\}$).

(ii′) $M^*(s) \downarrow M^*(\downarrow s) \,|\, M^*(<s) \cup M(s)$.

Hence

(iii) $M^*(s) \downarrow M^*\{\downarrow s\} \cup \mathbf{C} \,|\, M^*(<s) \cup M(s)$.

We will now get a contradiction by proving that for every subsystem T of S, $a' \downarrow M^*\{T\} \cup M\{S\} \mid F'$. Use induction on card$(T)$. Let s be a maximal element of T, $T' = T - \{s\}$. By (i), as stp(a/F') is based on $\mathbf{C}$ and orthogonal to $M(s)$, stp$(a/F') \perp M^*(s)$. Hence stp$(a/F') \perp (M^*(<s) \cup M(s))$. But by (iii), $M^*\{T'\} \cup M\{S\} \cup F' \downarrow M^*(s) \mid (M^*(<s) \cup M(s))$; and by induction stp$(a'/F')$ is parallel to stp$(a'/M^*\{T'\} \cup M\{S\} \cup F')$. By the orthogonality,

$$a' \downarrow M^*(s) \mid (M^*\{T'\} \cup M\{S\} \cup F'),$$

and transitivity finishes the induction step.

Corollary. *Let T be superstable, N a model, $M(s) \subseteq_{na} N$, $(M(s): s \in S)$ a stable system, and $F \subseteq N$ finite. Then there exists $M \subseteq_{na} N$, $\bigcup_{s \in S} M(s) \cup F \subseteq M$, with M l-atomic over $\bigcup_{s \in S} M(s) \cup F$.*

Remarks. (a) In fact we proved j-isolation and not only 1-isolation everywhere.

(b) If it happens that T has NDOP and A contains a maximal tree of na-submodels of N, then no type in N over A can be m_2-isolated; so an m-construction is an m_1-construction. This recovers §6 of [BuSh].

(c) In general, homogeneous types need not be c-isolated; indeed there are often homogeneous types over models. But in an ω-stable theory, or in a context where for some reason the group H acting on q is definable, q is (even outright) isolated.

3. Models of size continuum

We prove here Theorems 1.2 and 1.3, as well as

Proposition 3.1. *Theorem 1.3 is false if $\aleph_{\omega+1}$ is replaced by $\aleph_n < 2^{\aleph_0}$ $(n \le \omega)$.*

The proof of 3.1 shows that the elementary submodel structure of stable models can be arbitrarily complicated; in particular there exists a stable Jonsson model in every cardinal in which there exists a Jonsson model at all (while there are no superstable Jonsson models of any uncountable power).

Theorems 1.2 and 1.3 have a "downward" part, yielding countable information from the existence of a large model omitting the types in Q, and an "upward" part, using this information to get a model of cardinality $\beth_1$. The upward part is the same, Proposition 3.3; but it has a considerably simpler form if T is superstable. We assume from now on that T is stable and countable.

Proposition 3.2. *Let T be a superstable, countable, complete, first order theory, Q a countable set of partial types. Then the following are equivalent:*

(a) *T has a model of cardinality $\beth_1$ omitting each $q \in Q$.*

(b) *T has a model of cardinality $\aleph_1$ omitting each $q \in Q$.*

(c) *There exists a countable M and an independent set I over M such that no $q \in Q$ is isolated over $M \cup I$, and for each $a \in I$ and each finite Δ, $\{b \in I: \mathrm{tp}_\Delta(b/M) = \mathrm{tp}_\Delta(a/M)\}$ is infinite.*

REMARKS. (1) Suppose M is a countable model of a superstable theory containing an infinite Morley sequence J, and omitting q. Then if $I \cup J$ is a Morley sequence and $I \perp M \mid J$, it is easy to see that q is not isolated over $M \cup I$. Thus (c) holds. The fact that (b) follows in this special case was proved by Steinhorn, and was improved to (a) by Newelski under certain additional conditions on T and on the set-theoretic universe [Ne].

(2) The density condition in (c) cannot be weakened to a condition using formulas rather than Δ-types. To see this, let $T_n = \{\eta \in {}^n\omega : n = 0 \text{ or } \eta(0) \geq n\}$, $T = \bigcup_n T_n$, and build an example with equivalence relations E_η ($\eta \in T$), and constants c_η ($\eta \in T - (0)$). For each $\eta \in T_n$ with $\eta(0) > n$, $\{c_{\eta^\frown n} : n < \omega\}$ lie in distinct E_η-classes, and $E_{\eta^\frown n}$ refines $c_{\eta^\frown n}/E_\eta - \{c_{\eta^\frown n}\}$ into infinitely many classes. Let $M = \{c_\eta : \eta\}$ be the prime model. Find I such that for each η, there is a unique $x \in I$ such that $(x,x) \in E_\eta$ but $(x, c_\tau) \notin E_\eta$ for any τ. Let $Q = \{q\}$, $q = q(x,y) = \{x \neq y\} \cup \{(x,c_\nu) \in E_\eta \equiv (y,c_\nu) \in E_\eta : \nu,\eta\}$.

PROPOSITION 3.3. *Let T be stable, countable. Let M be a model of T, I an infinite independent set over M, Q a countable set of partial types. Suppose $I(a,\Delta) = \{a' \in I : \mathrm{tp}_\Delta(a/M) = \mathrm{tp}_\Delta(a'/M)\}$ is infinite for $a \in I$, and*:

(*) *If $c_1, \ldots, c_n \in I$, $\Delta \subseteq L$ finite, and $K \subseteq M$ is finite, then there exist $a_i \in I(c_i;\Delta)$ and a stable system $(M(s) : s \in P(\{1,\ldots,n\}))$ with $a_i \in M(\{i\})$, $a_i \perp M(\varnothing) \mid (M(\varnothing) \cap M)$, $K \subseteq M(\varnothing)$, and $M(\varnothing) \perp M \mid (M(\varnothing) \cap M)$; and $M(\{1,\ldots,n\})$ omits each type in Q.*

Then T has a model of power $\beth_1$ omitting each type in Q.

LEMMA 3.4. (a) *If T is superstable, then* 3.2(b) *implies* 3.2(c), *and* 3.2(c) *implies the hypothesis of Proposition* 3.3.

(b) *If T is stable, and has a model of power $\aleph_{\omega+1}$ omitting each $q \in Q$, then the hypothesis of Proposition* 3.3 *holds.*

Proposition 3.2 *follows, as do Theorems* 1.2 *and* 1.3.

PROOF. (a) Let N be a model of power $\aleph_1$ omitting each $q \in Q$. By 2.8 there exists a countable $M_0 \subseteq N$ and an uncountable independent sequence J over M_0 with $J \subseteq N$. Let

$$I = \{a \in J : \text{for each } \Delta,\ \{b \in J : \mathrm{tp}_\Delta(b/M_0) = \mathrm{tp}_\Delta(a/M_0)\} \text{ is uncountable}\}.$$

Then $J - I$ is countable, and M_0, I satisfy 3.2(c). If $a_1, \ldots, a_n \in I$ are distinct, let N be a countable model containing $M_0 \cup \{a_1, \ldots, a_n\}$ and omitting each $q \in Q$. Then by 2.7 a stable system $M(s) : s \subseteq \{1, \ldots, n\}$ can be found, with $M(\emptyset) = M_0$ in fact, and $M(s) \subseteq N$. This gives (*).

(b) Let N be a model of cardinality $\lambda^+ = \aleph_{\omega+1}$ omitting each $q \in Q$. By 2.8, there exists $M \subseteq N$, $\mathrm{card}(M) = \lambda$, and $J \subseteq N$, I independent over M, $\mathrm{card}(J) = \lambda^+$. As in (a) we can get $I \subseteq J$ such that if $a \in I$ and Δ is finite, then $I(a, \Delta)$ has cardinality λ. Given $a_1', \ldots, a_n'$ from I and finite Δ and K, let $I_i = I(a_i', \Delta)$. Choose $N_1, \ldots, N_n$ and $a_1, \ldots, a_n$ as follows. Given $N_0, \ldots, N_{i-1}$, let N_i be an elementary submodel of N in the language enriched with predicates for $M, K, I_1, \ldots, I_n$, $N_1, \ldots, N_{i-1}$, such that:

(a) $\mathrm{card}(N_i) = \aleph_{n-i}$,

(b) $\mathrm{card}(N_i \cap N_1 \cap \cdots \cap N_{i-1} \cap I_j) = \aleph_{n-i}$ for each $j \geq i$,

(c) $a_1, \ldots, a_{i-1} \in N_i$,

and choose $a_i \in I_i \cap N_1 \cap \cdots \cap N_{i-1}$ such that $a_i \perp N_i \,|\, M$. (All but $\aleph_{n-i}$ elements of I_i have this property, while by (b) for $i - 1$, $\mathrm{card}(I_i \cap N_1 \cap \cdots \cap N_{i-1}) = \aleph_{n+1-i}$.)

For $s \subseteq \{1, \ldots, n\}$, let $M(s) = \bigcap \{N_i : i \notin s\}$. (So $M(\{1, \ldots, n\}) = N$.) By 2.6, this is a stable system. Clearly $a_i \in M(\{i\})$. Since N_i is an elementary submodel of N in the language with a predicate for M, a special case of 2.6 says that $N_i \perp M \,|\, N_i \cap M$. In particular, $M(\emptyset) \perp M \,|\, N_i \cap M$. So $Cb(M(\emptyset)/M) \subseteq N_i$ for each i. Thus $Cb(M(\emptyset)/M) \subseteq \bigcap_i N_i = M(\emptyset)$. So $M(\emptyset) \perp M \,|\, M(\emptyset) \cap M$. It remains to prove that $a_i \perp M(\emptyset) \,|\, (M(\emptyset) \cap M)$. By the choice of a_i, $a_i \perp N_i \,|\, M$. So $a_i \perp M(\emptyset) \,|\, M$. By the previous independence equation and transitivity, we have what we want.

$[A]^\lambda$ denotes the set of subsets of A of size λ.

Proof of Proposition 3.3. By a Löwenheim–Skolem argument, we may assume M, I are countable. Let x_η ($\eta \in 2^\omega$) and $y_{s,m}$ (s a finite subset of 2^ω, $m < \omega$) be variables. ($y_{t,m}$ counts as a y_s-variable if $t \leq s$.) We seek a consistent type Σ in these variables such that:

(a) For each $s \in [2^\omega]^{<\omega}$, each n-tuple $y_1, \ldots, y_n$ of y_s-variables, and each $\varphi(x_i (i \in s), \bar{y}, z) \in L$, either $\sim(\exists z)\varphi$ is in Σ, or else for some y_s-variable y_{n+1}, $\varphi(x_i (i \in s), \bar{y}, y_{n+1})$ is in Σ.

(b) For each variable y and each $q \in Q$, for some $\theta \in q$, $\sim\theta(y) \in \Sigma$.

(c) If $\eta \neq \nu$, then $x_\eta \neq x_\nu$ is in Σ.

We will build Σ in ω steps. At a finite stage r we will have a $P(2^r)$-condition Φ_r (over M) in the variables x_i ($i \in 2^r$) and $y_{s,m}$ ($s \subseteq 2^r$, with the same convention on y_s-variables).

NOTATION. (α) If $s \subseteq 2^\omega$, let $s|r = \{\eta|r : \eta \in s\}$; if $y = y_{s,m}$, let $y|r = y_{s|r,m}$. We will use this notation only when $\mathrm{card}(s|r) = \mathrm{card}(s)$.

(β) Let Φ_r^x be the set of formulas in Φ_r mentioning only x-variables.

(γ) If $r_1 < r$ and η^* extends η for η in 2^{r_1}, let $\Phi_{r_1}^*$ be the result of replacing x_η by x_{η^*} and $y_{\eta_1,\dots,\eta_n,m}$ by $y_{\eta_1^*,\dots,\eta_n^*,m}$ in Φ_{r_1}.

The intention is to let $\Sigma = \{\psi(x_{\eta_1},\dots,x_{\eta_n},y_1,\dots,y_m) : \{\eta_1,\dots,\eta_m\} = s$, each y_i is an s-variable, and for some $r < \omega$ such that $\eta_1|r,\dots,\eta_n|r$ are distinct, $\varphi(x_{\eta_1}|r,\dots,x_{\eta_n}|r, y_1|r,\dots,y_m|r)$ is in $\Phi_r\}$. The consistency of this Σ follows from the following requirements on the Φ_r's:

(i) There exist distinct $a_\eta \in I(\eta \in 2^r)$ such that $\models \Phi_r^x((a_\eta : \eta \in 2^r))$.

(ii) If $r_1 < r$ and η^* extends η for η in 2^{r_1}, then $\Phi_r \supseteq \Phi_{r_1}^*$.

In order to guarantee (a) and (b), we also need:

(iii) For each $\varphi(u_1,\dots,u_n,z) \in L$, for some $r < \omega$, for each n-tuple $(w_1,\dots,w_n)$ of variables of the form x_i $(i \in 2^r)$ or $y_{s,m}$ $(s \subseteq 2^r,\ m < r)$, either $\sim(\exists z)\varphi(w_1,\dots,w_n,z)$ is in Φ_r, or for some variable w of Φ_r, $\varphi(w_1,\dots,w_n,w)$ is in Φ_r.

(iv) For each m, and each $q \in Q$, for some $r < \omega$, for each k-tuple $\bar\eta$ of distinct elements of 2^r, there exists $\theta \in q$ such that $\sim\theta(y_{\bar\eta,m})$.

Thus modulo trivial bookkeeping, we have to prove the following claims.

(1) Let Φ_r be a $P(2^r)$-condition satisfying (i), and let $R > r$. Then there exists a $P(2^R)$-condition Φ_R satisfying (i) such that the pair satisfies (ii).

(2) Let Φ be a $P(F)$-condition, and suppose $a_i \in I$ and $\models \Phi^x(a_i : i \in F)$. Let $\varphi(y_1,\dots,y_n,z)$ be a formula, where $y_1,\dots,y_n$ are variables of Φ. Then there exists a $P(F)$-condition $\Phi' \supseteq \Phi$ and $a_i' \in I$ such that $\models \Phi^x(a_i' : i \in F)$, and either $\sim(\exists z)\varphi \in \Phi'$, or $\varphi(y_1,\dots,y_n,y) \in \Phi'$ for some variable y of Φ'.

(3) Let Φ be a $P(F)$-condition, and suppose $a_i \in I$ and $\models \Phi^x(a_i : i \in F)$. Let $q \in Q$ and let y be a variable of Φ. Then there exists a $P(F)$-condition $\Phi' \supseteq \Phi$ and $a_i' \in I$ such that $\models \Phi'^x(a_i' : i \in F)$, and $\sim\theta(y) \in \Phi'$ for some $\theta \in q$.

((2) and (3) need to be applied several times in succession to achieve (iii) and (iv).)

PROOFS. (1) Let Φ' be the union of Φ_r^* over all functions $* : 2^r \to 2^R$ such that $\eta^* \supseteq \eta$. One verifies immediately that Φ' is a $P(2^R)$-condition. We need only to check that there exist distinct elements a_η $(\eta \in 2^n)$ satisfying the x-formulas in Φ'. Let φ be the conjunction of the x-formulas in Φ, and let $a_\eta \in I$ be distinct elements such that $\models \varphi(a_\eta : \eta \in 2^{r_1})$. Find a sufficiently large finite Δ so that if $\mathrm{tp}_\Delta(a_\eta'/M) = \mathrm{tp}_\Delta(a_\eta/M)$ and then a_η''s are independent then $\models \varphi(a_\eta' : \eta \in 2^{r_1})$. By the density assumption (preceding (*)), we can choose distinct a_τ $(\tau \in 2^r)$ such that $\mathrm{tp}_\Delta(a_\tau/M) = \mathrm{tp}_\Delta(a_{\tau|R}/M)$ for each τ. This clearly meets the requirement.

(2) and (3) are very similar; let us prove (3). Let Δ be a large finite set of for-

mulas, so that if $\mathrm{tp}_\Delta(a_i'/M) = \mathrm{tp}_\Delta(a_i/M)$ and $a_i' \in I$ then $\models \Phi^x(a_i' : i \in F)$. By the hypothesis, replacing the a_i's by other elements if necessary, there exists a stable system $(M(s) : s \in P(\{1,\dots,n\}))$ with $a_i \in M(\{i\})$, $a_i \downarrow M(\emptyset) \mid (M(\emptyset \cap M)$, the parameters in Φ are in $M(\emptyset)$, $M(\emptyset) \downarrow M \mid (M(\emptyset) \cap M)$; and $M(\{1,\dots,n\})$ omits each type in Q. Use Lemma 2.2 to find $b_{s,m} \in M(s)$ such that $\models \Phi(a_i : i \in F,$ $b_{s,m} : s \subseteq F)$. Find θ such that $\models \sim\theta(b)$, where b is the $b_{s,m}$ or a_i corresponding to y. (Note that θ is over $\emptyset$.) By Lemma 2.3(b) there exists a $P(\{1,\dots,n\})$-condition $\Phi' \supset \Phi \cup \{\sim\theta\}$ such that $\models \Phi'^x(a_i : i \in F)$. By the remark following 2.3, Φ' may be chosen with parameters in $M(\emptyset) \cap M$. This finishes the proof.

Definition. a is next-to-definable from B in a model M ($a \in ndf(B;M)$) if there exists a formula $\varphi(\bar{x}) \in L(B)$ such that $M \models (\exists \bar{x})\varphi(\bar{x})$, and for all $\bar{c} \in M$, if $M \models \varphi(\bar{c})$ then a is definable from $\bar{c}$.

Proposition 3.1 is immediate from

Lemma 3.5. (a) *For each $n < \omega$ there exists a model M of a superstable theory with* $\mathrm{card}(M) = \aleph_n$, *such that*:

($*_n$) *for any $s \subseteq M$ with* $\mathrm{card}(s) = n + 1$, *there is $a \in s$ such that a is next-to-definable over $s - \{a\}$.*

Moreover, there exists a countable set of partial types Q omitted by M, such that every model omitting each type in Q has the same property.

(b) *No structure of cardinality $\aleph_{n+1}$ has* ($*_n$).

Proof. (b) Let $\mathrm{card}(M) = \aleph_n$. As in the proof of Lemma 3.4(b) one can find elementary submodels N_i of M ($i = 1,\dots,n$) (with $\mathrm{card}(N_i) = \aleph_{n-i}$) and elements a_i such that $a_i \in N_j$ iff $i \neq j$. But if a is next-to-definable from $\bar{b}$ in M, then every elementary submodel of M containing $\bar{b}$ must also contain a. Thus $s = \{a_1,\dots,a_n\}$ demonstrates the failure of ($*_{n-1}$).

To prove (a), note first that for each set X of cardinality at most $\aleph_n$ there exists $f: \omega \times [X]^n \to X$, such that for any $s \in [X]^{n+1}$, for some $x \in s$ and some $k \in \omega$, $x = f(k, s - \{x\})$. For $n = 0$ this is clear. Assume it is true below n. Define $f: \omega \times [\omega_n]^n \to \omega_n$ as follows. For each $\alpha < \omega_n$, $\mathrm{card}(\alpha) < \aleph_n$; so one can find $f_\alpha : \omega \times [\alpha]^{n-1} \to \alpha$ such that for each $s \in [\alpha]^n$, for some $x \in s$ and $k < \omega$, $x = f(s - \{x\})$. Given $s \in [\omega_n]^n$, let α be the greatest element of s, and let $f(k,s) = f_\alpha(k, s - \{\alpha\})$. It is easy to see by induction that this works. Of course, the model $(X, f(k,\ ) : k < \omega)$ is not stable. But starting with the trivial structure on ω_n (pure equality), and applying the following claim ω times, one gets a stable structure with the required property.

Claim. Let M be stable, $\mathrm{card}(M) = \aleph_n$. Then there exists a stable structure M^* such that:

(a) $M = P^{M^*}$ for some predicate P of the language of M^*.
(b) M^* induces no new first-order structure on M. (Every M^*-0-definable subset of M is M-0-definable.)
(c) For some formula $\varphi(x_1,\dots,x_n,y)$ of M, for any $s \in [M]^{n+1}$, for some enumeration $a,a_1,\dots,a_n$ of s, $M^* \models (\exists y)\varphi(a_1,\dots,a_n,y)$, and if $M^* \models \varphi(a_1,\dots,a_n,b)$ then a is definable from b in M^*.
(d) The cardinality of M^* is $\aleph_n$.

NOTE. There exists a partial type q (in $(n+1)!+n+1$ variables), omitted by M^*, such that if (N^*,N) is elementarily equivalent to (M^*,M) and omits q then (c) holds in (N^*,N). (The type has variables $x_0,\dots,x_n$ and y_e (e a permutation of $0,\dots,n$); it says that for each e, either $\sim(\exists y)\varphi(x_{e(1)},\dots,x_{e(n)},y)$, or $\varphi(x_{e(1)},\dots,x_{e(n)},y_e)$ and $x_{e(0)}$ is not definable over y_e.)

PROOF OF CLAIM. Let $f:\omega\times[M]^n\to M$ be the function constructed above. For $s\in[M]^n$, let G_s be a set of functions on ω into M, such that:
(i) G_s is dense: for each $k<\omega$ and $f_0:k\to M$, for some $g\in G_s$, $f_0=g|k$.
(ii) For each $g\in G_s$ and $k<\omega$, for some $m<\omega$, $g(m)=f(k,s)$.
(iii) $\mathrm{card}(G_s)=\aleph_n$.
(Let G_s' be the set of all g satisfying (ii); note that it is dense, and take a dense subset G_s of cardinality $\aleph_n$.)

Let $E=\{(s,g):s\in[M]^n,\ g\in G_s\}$, $\pi(s,g)=s$, $\pi_m(s,g)=g(m)$. Then the structure M^* consisting of $M\cup E$ with the structure of M, together with the relation $m\in\pi(s,g)$ and the functions π_m, meets the requirements. It is stable as every restriction to a finite language is interpretable in M.

4. Omitting types in superstable theories

The core of this section is Theorem 4.1 and Proposition 4.3. 4.1 gives a syntactical criterion for the existence of models of a superstable theory, of prescribed dimension, omitting a given type (or set of types). The cardinal enters the syntactical picture only through its set of combinatorial identities; Theorem 1.5 is an immediate corollary. 4.3 shows a converse, that it is indeed necessary to consider all the combinatorial identities; this is needed for Theorem 1.4 (b → a), as well as for Theorem 1.1. Theorems 4.2 and Proposition 4.4 are generalizations of 4.1 and 4.3 to the many-cardinal context; for example, 4.4 shows that there are no theorems of the form $(\kappa_\alpha:\alpha<\mu)\to\kappa$ (for superstable omitting types) other than those that follow formally from 1.4. We conclude by deducing 1.1, 1.4, and 1.6.

DEFINITION. Let $\bar F=(F_1,\dots,F_n)$ be a sequence of disjoint finite sets, $F=\bigcup_i F_i$, and let p_i $(1\le i\le n)$ be stationary types, $S\subseteq P(F)$. For $\nu\in F$, let $i(\nu)$

be the i such that $\nu \in F_i$. An S-condition Φ with x-variables x_ν $(\nu \in F)$ is called $\bar{p}$-compatible if each $\varphi(x_\nu : \nu \in F)$ is in the nonforking extension of $\bigotimes_{\nu \in F} p_{i(\nu)}(x_\nu)$ to the set of parameters; i.e. if $\bar{b}$ is the set of parameters of Φ, $a_\nu \vDash p_{i(\nu)}$ and $\{a_\nu : \nu \in F\} \cup \{\bar{b}\}$ is independent, then $\vDash \varphi(a_\nu : \nu \in F)$.

$\mathrm{Id}(\lambda_1, \ldots, \lambda_n; \bar{F})$ is the set of combinatorial identities of $(\lambda_1, \ldots, \lambda_n)$ on the set $F_1 \cup \cdots \cup F_n$, so divided. Note that if $\mathrm{card}(F_i) > \lambda_i$ for some i, then $\mathrm{Id}(\lambda_1, \ldots, \lambda_n; \bar{F}) = \varnothing$. $P(\bar{F})$ is the set of all subsets of $\bigcup_i F$.

THEOREM 4.1. *Let T be countable and superstable, Q a countable set of incomplete types, p a type over a countable model M. Then the following are equivalent:*

(a) *T has a model $N \supseteq M$, $\dim_p(N) \geq \lambda$, such that N omits each $q \in Q$.*

(b) *For each $I \in \mathrm{Id}(\lambda, F)$, each $q \in Q$, every I-symmetric, p-compatible $P(F)$-condition Φ, and each variable y of Φ, there exists an I-symmetric, p-compatible $P(F)$-condition $\Phi' \supseteq \Phi$ and $\theta \in q$ with $\Phi' \vdash \sim\theta(y)$.*

THEOREM 4.2. *Let T, Q be as in 4.1, E a countable set of stationary types, and $\lambda(p)$ an uncountable cardinal for $p \in E$. Then (a),(b) are equivalent:*

(a) *T has a model omitting each $q \in Q$ and realizing $\bigotimes_{p \in E} p^{\lambda(p)}$.*

(b) *For each m, each m-tuple $p_1, \ldots, p_m$ from E, each $I \in \mathrm{Id}(\lambda; \bar{F})$, each $q \in Q$, each I-symmetric, $(p_1, \ldots, p_m)$-compatible $P(\bar{F})$-condition Φ, and each variable y of Φ, there exists a stronger such condition Φ' with $\Phi' \vdash \sim\theta(y)$ for some $\theta \in q$.*

PROOF OF THEOREMS 4.1 and 4.2. (a) $\Rightarrow$ (b). Let N be a model omitting each $q \in Q$ and containing a realization I of $\bigotimes_{p \in E} p^{\lambda(p)}$. Let M be a countable *na*-submodel of N. $\{a \in I : M \not\perp a\}$ is countable; we can discard it (using $\lambda(p) \geq \aleph_1$), so that $M \perp I$. Skolemize N partially as follows. Let L^* be a richer language containing many terms. Let S be the set of finite subsets of I, and use 2.8 to get a 1-system $(M(s) : s \in S)$ contained in N. Interpret the terms of L^* in N in such a way that every L^*-submodel of N is an L-elementary submodel, and the L^*-hull of $s \in S$ is $M(s)$. Let $p_1, \ldots, p_m$ be stationary types not in E, $I \in \mathrm{Id}(\lambda(p_1), \ldots, \lambda(p_m); \bar{F})$, $q \in Q$, and let Φ be an I-symmetric, $(p_1, \ldots, p_m)$-compatible $P(\bar{F})$-condition, $\bar{F} = (F_1, \ldots, F_m)$, $F_1, \ldots, F_m$ disjoint, $F = F_1 \cup \cdots \cup F_m$. Let $a_{i,\alpha}$ $(i = 1, \ldots, m,\ \alpha < \lambda(p_i))$ be distinct elements of I, with $a_{i,\alpha} \vDash p_i$. By the definition of a combinatorial identity, one can find among them a_j $(j \in F)$ such that $a_j \vDash p_k$ if $j \in F_k$, and if $((j_1, \ldots, j_i), (j'_1, \ldots, j'_i)) \in I$ then $\mathrm{tp}_{L^*}(a_{j_1}, \ldots, a_{j_i}/M) = \mathrm{tp}_{L^*}(a_{j'_1}, \ldots, a_{j'_i}/M)$. To prove the existence of Φ', we may identify F with the subset of I indexed by it, i.e. identify j with a_j. So $(M(s) : s \subseteq F)$ is an I-symmetric 1-system, and the hypothesis of 2.4 holds. (The isomorphisms $h(s,t) : M(s) \to M(t)$ are defined as follows. Let $s = \{a_1, \ldots, a_i\}$. Then there is a unique enumeration $(b_1,\dots,b_i)$ of t such that $((a_1,\dots,a_i),(b_1,\dots,b_i)) \in I$; and a unique L^*-atomic map $M(s) \to M(t)$ with $a_i \mapsto b_i$, namely $\tau(\bar a) \mapsto \tau(\bar b)$ (τ a term of L^*). The choice of the enumeration $a_1,\dots,a_i$ of s is irrelevant.) By 2.4, there exists e such that $e(x_a) = a$ $(a \in F)$, and $((M(s): s \subseteq F),\Phi,e)$ is an I-symmetric S-system. Now $M(F)$ omits q, so for some $\theta \in q$, $\vDash \sim\theta(e(y))$. By Proposition 2.5 there exists an I-symmetric $P(F)$-condition Φ' stronger than Φ, such that $\Phi' \vdash \sim\theta$, and $(\overline{M},\Phi',e')$ is a $P(F)$-system for some $e' \supseteq e$. Since $e'(x_a) \vDash p_{i(a)}$ (where $a \in F_{i(a)}$), Φ' is $\bigotimes_{a\in F} p_{i(a)}$-compatible.

(b) $\Rightarrow$ (a). We deal only with 4.1; this simplifies the notation considerably without really sacrificing any of the ideas.

Proof of 4.1. For each combinatorial identity $I \notin \mathrm{Id}(\lambda)$, find a countable structure on λ demonstrating the fact that it is not an identity of λ. Combining all these structures, and the ordering on λ, we get a countable structure on λ such that if $s \in [\lambda]^n$, then the equivalence relation $I(s) = \{(\bar\alpha,\bar\beta) \in s^{\le n} : \mathrm{tp}(\bar\alpha) = \mathrm{tp}(\bar\beta)\}$ is a combinatorial identity of λ. Let $\epsilon_1,\epsilon_2,\dots$ be all the formulas of the language of this structure, $\Delta(j) = \{\epsilon_1,\dots,\epsilon_j\}$, and for $\beta_1 < \cdots < \beta_i < \lambda$, let $F(i,j)(\{\beta_1,\dots,\beta_i\}) = \mathrm{tp}_{\Delta(j)}(\bar\beta)$. So for each $s \in S = [\lambda]^{<\omega}$, for some j, the equivalence relation induced on the set of increasing sequences from s by $F(i,j)$ $(i \le j)$ agrees with $I(s)$. In this situation call s j-resolved, and let $S^j = \{s \in S : s$ is j-resolved.$\}$ So $S = \bigcup_{j<\omega} S^j$.

We need to find the elementary diagram of a model realizing p^λ and omitting q. The variables will be x_i $(i < \lambda)$ and $y_{s,n}$ $(s \in S,\ n < \omega)$. At stage $j < \omega$, for each $s \in S^j$, we will have a $P(s)$-condition $\Phi^{s,j}$ in the variables x_i $(i \in s)$ and $y_{t,n}$ $(t \subseteq s,\ n < \omega)$. The complete diagram will be (the set of consequences of) the union over $j < \omega$ and $s \in S^j$ of $\Phi^{s,j}$. We will ensure:

(a) $\Phi^{s,j}$ is p-compatible.

(b) If $s,t \in S^j$, $\mathrm{card}(s) = \mathrm{card}(t) = i$, and $F(i,j)(s) = F(i,j)(t)$, then $\Phi^{s,j}$ and $\Phi^{t,j}$ differ only by an order-preserving change of variables. If $s \subseteq t$ then $\Phi^{s,j} = \Phi^{t,j}(s)$.

(c) If $s \in S^j$ and $j' > j$ then $\Phi^{s,j'} \vdash \Phi^{s,j}$.

(d) For each $q \in Q$ and $n < \omega$, for arbitrarily large $j < \omega$, for all $s \in S^j$, for some $\theta \in q$, $\sim\theta(y_{s,n}) \in \Phi^{s,j}$.

(e) For each $\varphi(u_1,\dots,u_n,z) \in L$, for arbitrarily large $j < \omega$, for all $s \in S^j$ and each n-tuple $(w_1,\dots,w_n)$ of variables of the form x_i $(i \in s)$ or $y_{t,k}$ $(t \subseteq s,\ k < j)$, either $\Phi^{s,j} \vdash \sim(\exists z)\varphi(w_1,\dots,w_n,z)$ or for some k', $\Phi^{s,j} \vdash \varphi(w_1,\dots,w_n,y_{s,k'})$.

Clearly it suffices to achieve (a)–(e). Again we show only how to preserve (a) and (b) while meeting (c) and also (d) for q,n or (e) for φ, where q,n or φ are handed to us by a bookkeeper. Suppose we have $\Phi^{s,j}$ for $s \in S^j$, and (a),(b) hold. For

$s \in S^{j+1}$, let $\Psi_1^s = \bigcup\{\Phi^{j,t} : t \in S^j, t \subseteq s\}$. Then it is easy to check that Ψ_0^s is a $P(s)$-condition, that (a),(b) hold for Ψ_0 with $j+1$, and $\Psi_0^s \vdash \Phi^{j,t}$ for $t \subseteq s$. To take care of (d) for q, n, let $C_i = \{F(i, j+1)(s) : s \in S^{j+1}\}$, $C = \bigcup_i C_i$, and enumerate C as $\{c_1, \ldots, c(r)\}$. Find $\Psi_1, \Psi_2, \ldots, \Psi_{c(r)}$ inductively, satisfying the same conditions as Ψ_0, with $\Psi_{k+1}^s \vdash \Psi_k^s$ ($s \in S^{j+1}$) and such that for all s, if $F(i, j+1)(s) = c_k$, then for some $\theta \in q$, $\sim\theta(y_{s,n}) \in \Psi_k^s$. To do this, assume we have Ψ_k, and let $c = c_{k+1}$, $c \in C_i$. Let $x^0, \ldots, x^{i-1}$ and $y^{s,m}$ ($s \subseteq \{0, \ldots, i-1\}$, $m < \omega$) be new variables (for temporary use). We have a $P(i)$-condition ${}^c\Psi$ on these variables: ${}^c\Psi$ is isomorphic to Ψ_k^s for any $s \in S^{j+1}$ with $F(i, j+1)(s) = c$, by the order-preserving change of variables. Similarly define ${}^cF(u) = d$ (for $u \subseteq \{0, \ldots, i-1\}$) iff for (any) $s \in S^{j+1}$ with $F(i, j+1)(s) = c$, the order preserving bijection $\{0, \ldots, i-1\} \to s$ carries u to a set t with $F(i, j+1)(t) = d$. ${}^c\Psi$ exists by (b), is p-compatible by (a), and is I-symmetric by (b) again. By hypothesis, there exists an I-symmetric, p-compatible $P(i)$-condition ${}^c\Psi' \vdash {}^c\Psi$, and $\theta \in q$, with $\sim\theta(y) \in {}^c\Psi'$, where $y = y^{\{0, \ldots, i-1\}}$. Let ${}^cS^{j+1} = \{s \in S^{j+1} : \mathrm{card}(s) = i' \leq i$, and for some $u \subseteq \{0, \ldots, i-1\}$, $F(i', j+1)(s) = {}^cF(u)\}$. For $s \in {}^cS^{j+1}$, $F(i', j+1)(s) = {}^cF(u)$, let Ψ'^s be the $P(i)$-condition obtained from ${}^c\Psi'(u)$ by the order-preserving change of variables $u \to s$. This definition is consistent (the choice of u does not matter); moreover, ${}^cS^{j+1}$ is closed downwards, and (a),(b) hold on ${}^c\Psi'^s$ where it is defined. This allows us to define, for $s \in S^{j+1}$, $\Psi'^s = \bigcup\{\Psi'(t) : t \subseteq u, t \in {}^cS^{j+1}\}$; (a) and (b) continue to hold. Let $\Psi_{k+1}^s = \Psi_k^s \cup \Psi'^s$. It is now easy to check that Ψ_{k+1}^s is a $P(s)$-condition, and (a),(b) hold. Letting $\Phi^{s,j+1} = \Psi_{c(r)}^s$, we achieve (d). (e) is similar; instead of the hypothesis one needs a lemma to the effect that for any p-compatible, I-symmetric $P(F)$-condition Φ, any variables $w_1, \ldots, w_n$ of Φ and any $\varphi(u_1, \ldots, u_n, z) \in L$, there exists a stronger p-compatible, I-symmetric $P(F)$-condition Φ' and a variable w' of Φ' such that $\Phi' \vdash \sim(\exists z)\varphi(\bar{w}, z)$ or $\Phi' \vdash \varphi(\bar{w}, w')$. This is easy to prove directly, or it can be deduced from 2.4, 2.5, and the obvious existence of fully symmetric systems meeting the hypothesis of 2.4 and with $e_0(x_i) \vDash p$ (not necessarily omitting any type). This finishes the construction, and the proof of 4.1.

PROPOSITION 4.3. *There exists a complete, countable, superstable, unidimensional theory T with the following property. For every combinatorial identity I, there exists a partial type $q(I)$ such that T has a model of power κ omitting $q(I)$ if and only if I is not a combinatorial identity of κ.*

If I is the canonical identity separating $\beth_n$ from $\beth_n^+$ (given by the Erdös–Rado theorem) then $q(I)$ is a complete type.

T is a many-sorted theory with sorts $S_0, S_1, \ldots$. However, for any particular I,

$q(I)$ mentions only finitely many sorts, so there exists a single-sorted theory (depending on I) with the same property.

Proof. Let $(V,+,V_1,V_2,\dots)$ be an Abelian group of exponent 2 with a strictly descending chain of subgroups of finite index. For $k<\omega$, let $V^{(k)}$ be the set of linearly independent subsets of V of power k. For $s\in V^{(k)}$, let $V_s=V\times\{s\}$, $S_k=\bigcup\{V_s:s\in V^{(k)}\}$, $\pi_k=\{(v,(w,s))\in V\times S_k:v\in s\}$,

$$R_k=\{(v,(v_1,s),(v_2,s)):v,v_1,v_2\in V,\ v+v_1=v_2,\ s\in V^{(k)}\}$$

(this gives the action of V on V_s without specifying the zero element of V_s), $E_{k,m}=\{((v_1,s),(v_2,t))\in S_k^2:v_1-v_2\in V_m\}$. Let

$$M=(V,+,V_m,S_k,\pi_k,E_{k,m},R_k:k,m<\omega).$$

M is interpretable in $(V,+,V_m:m<\omega)$ (it is so given). So M is superstable; by virtue of the π_k's, it is unidimensional. Let $T=\mathrm{Th}(M)$.

It will be convenient to consider also the theory T' of a certain expansion M' of M: for each m, each $k<\omega$ and each coset C of V_m, adjoin the predicate $P_{k,m,C}=\{(v,s)\in S_k:v\in C\}$. Note that any model of T expands to a model of T'. Hence for any partial type q of T, T has a model of power λ omitting q if and only if T' does.

Now let I be a combinatorial identity on n. The partial type $q=q(I)$ will have variables y_i for $i\in n$ and x_s for $s\subset n$. q will say that $\{y_0,\dots,y_{n-1}\}\in V^{(n)}$, $x_s\in V_{\{y_i:i\in s\}}$, and $x_sE_{k,m}x_t$ whenever $m<\omega$ and $(s,t)\in I$. Let $q'(I)=q(I)\cup\{y_i\in V_m:i<n,\ m<\omega\}$. It is easy to see that if I is the Erdös–Rado identity ($I=\{(s,t):\mathrm{card}(s)=\mathrm{card}(t)\}$), then $q'(I)$ is complete.

If $I\subseteq P(n)^2$ is a combinatorial identity of λ and M is a model of T of power λ, we must show M realizes $q(I)$, and even $q'(I)$ provided $\lambda>2^{\aleph_0}$ (as is the case if a nontrivial Erdös–Rado identity applies to λ). Let J be any basis for V^M as a vector space over the 2-element field; if $\lambda>2^{\aleph_0}$, choose $J\subseteq\bigcap_m V_m^M$. Then $\mathrm{card}(J)=\lambda$. Expand M to a model of T', choose any $a_s\in V_s$ for $s\in[J]^{<n}$, and define c on $[J]^{<n}$ by:

$$c(s)=\{(m,C):a_s\in P_{k,m,c}(k=\mathrm{card}(s))\}.$$

As $\lambda\to I$, there exist $b_1,\dots,b_n\in J$ such that $c(\{b_i:i\in s\})=c(\{b_i:i\in t\})$ if $(s,t)\in I$. Letting $c_s=a_{\{b_i:i\in s\}}$, this translates to: $(b_1,\dots,b_n,c_s:s\subset n)$ realizes q. If $\lambda>2^{\aleph_0}$, it follows that M realizes q'.

Now suppose I is not a combinatorial identity of λ. Let $\mathbf{C}$ be a large saturated model of T', $V=V^{\mathbf{C}}/(V^\circ)^{\mathbf{C}}$ where $V^\circ=\bigcap_m V_m$; and if M is an elementary submodel of $\mathbf{C}$, let $\bar{V}(M) = V^{\mathbf{C}}/(V^{\circ \mathbf{C}} + V^M)$. Define c_M on the set of finite independent subsets of V^M as follows. Given $s \in [V^M]^{(k)}$, choose $a \in V_s^M$. Find $a' \in V^{\mathbf{C}}$ such that for each m and each coset C of V_m, $a' \in C$ iff $a \in P_{k,m,C}$. This determines $a + V^{\circ \mathbf{C}}$ uniquely. The choice of a' is determined up to the action of V^M, so $a + V^{\circ \mathbf{C}} + V^M$ is determined from s alone. Let $c_M(s) = a + V^{\circ \mathbf{C}} + V^M$. Given any M_0, and any function c on the finite independent subsets of V^{M_0} into $\bar{V}(M_0)$, there exists an elementary submodel M of $\mathbf{C}$ with $V^M = V^{M_0}$ and $c_M = c$. (Simply choose an appropriate $a_s \in M_s$ for each s; let $M = V^{M_0} \cup \{b \in S_k$: for some $s \subseteq V^{M_0}$ and $a \in V^{M_0}$, $b \in V_s$ and $(a, a_s, b_s) \in R_k\}$.)

Start with any countable elementary submodel of $\mathbf{C}, M_0'$. Let M_0 be a model with $V^{M_0} = V^{M_0'} + S$, where $\operatorname{card}(S) = \lambda$, $S \subseteq V^{\circ \mathbf{C}}$. Then V^{M_0}/V° is countable, so $\bar{V}(M_0)$ has cardinality continuum. Thus there exists $c: [V^{M_0}]^{<n} \to \bar{V}(M_0)$ demonstrating that I is not a combinatorial identity of λ. Ignore c on non-independent subsets of V^{M_0}; and find M with $V^M = V^{M_0}$ and $c_M = c$ (on independent subsets). Then q is omitted in M.

Question. This shows that the Hanf number for omitting a single complete type for countable, superstable theories is at least $\beth_\omega$. By Theorem 1.1, it is at most $\beth_\omega^{++}$. Which of the three remaining possibilities holds? (If the answer is $\beth_\omega$, a proof would be interesting.)

Proposition 4.4. *Let $(\kappa_\alpha : \alpha < \mu)$ be an increasing sequence of cardinals with limit κ.*

(a) *Suppose* $\operatorname{Id}(\kappa_\alpha) \neq \operatorname{Id}(\kappa)$ *for any α. Then there exists a superstable T and a countable set Q of partial types such that T has a model omitting each $q \in Q$ in each power κ_α, but not in κ.*

(b) *Suppose* $\operatorname{Id}(\kappa_\alpha) \neq \operatorname{Id}(\kappa^+)$ *for any α. Then there exists a superstable T and a countable set Q of partial types such that T has a model omitting each $q \in Q$ in power κ, but not in power κ^+.*

Proof. Let T_0 be the unidimensional theory of the proposition.

(a) If μ is uncountable then for some $\alpha_0 < \mu$, for all $\alpha > \alpha_0$, $\operatorname{Id}(\kappa_\alpha) = \operatorname{Id}(\kappa_{\alpha_0})$; so choosing $I \in \operatorname{Id}(\kappa) - \operatorname{Id}(\kappa_{\alpha_0})$, we can let $T = T_0$, $Q = \{q(I)\}$. If $\mu = \aleph_0$, let $(V, +, V_1, V_2, \ldots, c_0, c_1, \ldots)$ be a countable group of exponent 2 with a decreasing chain of subgroup of finite index, such that $\bigcap V_n = (0)$, with constants (c_n) naming each element of V. For each $v \in V - \{0\}$, let M_v be a model of T_0; the M_V's are to be taken pairwise disjoint and disjoint from V, and we will impose no further relations on them. Let $I_n \in \operatorname{Id}(\kappa_{n+1}) - \operatorname{Id}(\kappa_n)$, $q_n = q(I_n)$. If $q_n = q_n(\bar{y})$, let $q_n^*(\bar{y}, x)$ be the partial type asserting that $x \in V - (0)$, $\bar{y}$ is from M_x, and $\bar{y}$

realizes q_n in M_x considered as a model of T_0. Let Q consist of the following partial types:

(a) $\{x \in V\} \cup \{x \neq c_0, x \neq c_1, \dots\}$;

(b) $\{x_1, x_2 \in V - (0), x_1 \neq x_2\} \cup q_1^*(\bar{y}_1, x_1) \cup q_2^*(\bar{y}_2, x_2)$;

(c) $\{x \in V - V_m\} \cup q_m^*(x, \bar{y})$.

If a model N omits (a), then it has no nonstandard elements of V; if it omits the type (b), then $\mathrm{card}(M_v^N) < \kappa_2$ for all but one $v \in V^N$; and if it also omits the types in (c), then $\mathrm{card}(M_v)^N < \kappa_{m+1}$, where $x \notin V_m$. So any model omitting each $q \in Q$ has cardinality $<\kappa_{m+1}$ for some m. Conversely, it is easy to build a model of cardinality κ_m, for any m.

(b) This time it is the countable case that is trivial, so assume $\mu > \aleph_0$, and choose $\alpha_0 < \mu$, such that $\mathrm{Id}(\kappa_\alpha) = \mathrm{Id}(\kappa_{\alpha_0})$ for all $\alpha \geq \alpha_0$. We may assume $\alpha_0 = 0$, and $\kappa_0 \geq \mu$. We will use the following theory. Let M be a model of T_0. For $a \in V(M)$, let M_a be another model of T_0, with the M_a's disjoint and disjoint from M. Let $I \in \mathrm{Id}(\kappa^+) - \mathrm{Id}(\kappa_0)$. Then there exists a partial type q, such that $(M, M_a : a \in M)$ omits q if and only if M and each M_a omit I. Clearly this has a model of power κ but not κ^+.

Proof of 1.4, (b) $\Rightarrow$ (a). Let M be a model of a superstable theory of power κ, a regular cardinal. If $\kappa > 2^{\aleph_0}$ then by 2.8 M contains a Morley sequence of power κ over some countable submodel, so 1.5 applies. If $\omega < \kappa < \lambda \leq 2^{\aleph_0}$ then 1.2 applies. If $\kappa = 2^{\aleph_0} < \lambda$ then κ has no nontrivial combinatorial identities while λ does have one, so the theorem holds vacuously.

(a) $\Rightarrow$ (b). The fact that $\kappa > \omega$ is obvious. From 4.3 it follows that $\mathrm{Id}(\kappa) = \mathrm{Id}(\lambda)$. In the regular case there is nothing more to prove. If κ is singular, then by 4.4(b) there exists $\kappa' < \kappa$ with $\mathrm{Id}(\kappa') = \mathrm{Id}(\kappa^+)$. But $\kappa < \kappa^+ \leq \lambda$, so $\mathrm{Id}(\kappa) = \mathrm{Id}(\kappa^+) = \mathrm{Id}(\lambda)$. Thus $\mathrm{Id}(\kappa'^+) = \mathrm{Id}(\lambda)$.

Corollary to 1.4. *If κ is singular of cofinality μ, $\omega < \mu \leq 2^{\aleph_0}$, and T has models of arbitrarily large p-dimension below κ omitting q, then it has a model of p-dimension κ omitting q.*

Question. Can we delete the condition on the cofinality? Equivalently, if κ is singular of uncountable cofinality, is there always $\kappa' < \kappa$ such that $\mathrm{Id}(\kappa') = \mathrm{Id}(\kappa)$?

Proof of 1.1. This is a special case of 1.4, since $\mathrm{Id}(\lambda) = \mathrm{Id}(\lambda')$ if λ, λ' are above $\beth_\omega$, but not if one is above and one below.

Proof of 1.6. Let P be a 0-definable set. A stationary type p is called foreign to P if for some (or all) models N, if a realizes the non-forking extension of p to

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N, then the partial type $\Pi =$ "$P(x)$ and $x \notin N$" is not a consequence of any consistent formula over $N \cup \{a\}$. It follows that Π is not realized in any 1-atomic model over $N \cup \{a\}$. It is also easy to check that if M is a countable model, I is independent over M, and M' is 1-atomic over $M \cup I$, then

$$\aleph_0 + \mathrm{card}(P^M) = \aleph_0 + \mathrm{card}\{b \in I : \mathrm{tp}(b/M) \text{ is not foreign to } p\}.$$

Also note that if K is a set of types over M, all foreign to P, and p is a limit point of K in the Δ-type topology (i.e. for every Δ, for some $p \in K$, p and p' have the same restriction to a Δ-type) then p is foreign to K.

We may assume each $\kappa_i > \omega$. Let α^* be the least $\beta < \alpha$ such that $\kappa_\beta \geq 2^{\aleph_0}$. We want to find nonalgebraic stationary types $p_{\beta,n,m}$ ($\beta < \alpha$, $n < \omega$) with the following properties. For $\beta < \alpha$, let $a_n(\beta) = \beta$ if β is a successor; if β is a limit ordinal, let $a_n(\beta)$ be an increasing sequence of ordinals approaching β. Note that $\lambda_\beta < 2^{\aleph_0}$ if $\beta < \alpha^*$. We need:

(a) If $\delta < a_n(\beta)$ then $p_{\beta,n,m}$ is foreign to P_δ; and $P_\beta \in p_{\beta,n,m}$.

(b) Each $p_{\beta,n,m}$ is based on a countable model M with $M \subseteq_{na} N$; and $\bigotimes_{\beta,n,m} p_{\beta,n,m}^{a_n(\beta)}$ is realized in N over M.

(c) (Density). For each finite Δ and each β, n, m, there exist infinitely many m' such that $p_{\beta,n,m}$ and $p_{\beta,n,m'}$ have the same restriction to a Δ-type over M.

Note that if $p'_{\beta,n,m}$ ($m < \omega$) satisfy the density condition over a model M', and $M' \subseteq M$, and $p_{\beta,n,m}$ is the nonforking extension of $p'_{\beta,n,m}$ to M, then $p_{\beta,n,m}$ ($m < \omega$) satisfy the density condition over M. This remark makes it easy to find the required types, by induction on β, using 2.8(a). (To achieve density, choose first $p_{\beta,n,i}$ ($i < \omega_1$), and then refine.)

Assuming (a),(b),(c) are achieved, we proceed in two steps. Note that by (a) $\Rightarrow$ (b) of 4.2,

(*) For each finite sequence $(\beta_1, n_1, m_1), \ldots, (\beta_k, n_k, m_k)$, letting $p^i = p_{\beta_i, n_i, m_i}$, $\lambda^i = a_{n_i}(\beta_i)$, for each $I \in \mathrm{Id}(\lambda^1, \ldots, \lambda^n; \bar{F})$, for each $q \in Q$, each I-symmetric, $(p^1, \ldots, p^n)$-compatible $P(\bar{F})$-condition Φ, and each variable y of Φ, there exists a stronger such condition Φ' with $\theta \in q$, $\sim\theta(y) \in \Phi'$ (some θ).

Mimicking the proof of Theorem 1.2, and noting that there are only countably many $p_{\beta,n,m}$'s altogether, we find a new collection of types $p_{\beta,n,m}$ over M satisfying (*) but with m ranging over 2^ω rather than ω; and $p_{\beta,n,m}$ is in the closure (Δ-type topology) of $\{p_{\beta,n,m} : m < \omega\}$. Thus by the initial remark, our new collection also satisfies (a). By 4.2 (b) $\Rightarrow$ (a), there exists a model N' realizing $\bigotimes\{p_{\beta,n,i} : i < \max(\lambda_{a_n(\beta)}, 2^{\aleph_0})\}$ while omitting each $q \in Q$. In addition, we can demand that $M \subseteq_{na} N'$; this just amounts to adding some partial types (omitted in N) to Q. Thus we may assume N is 1-atomic over $M \cup I$, where I is the independent sequence realizing $\bigotimes\{p_{\beta,n,i} : i < \max(\lambda_{a_n(\beta)}, 2^{\aleph_0})\}$. By the remark in the fourth sentence of the proof, this gives the cardinalities we want.

5. Larger Hanf numbers

Proposition 5.1. (a) *The Hanf number for small extensions of a countable superstable theory T is $\beth_{\omega_1}$.*

(b) *The Hanf number for omitting an arbitrary set of types is $\beth_{\beth_1^+}$, even for superstable theories.*

(c) *The Hanf number for omitting a countable set of incomplete types in a stable theory is $\beth_{\omega_1}$.*

Lemma 5.2. *Let S be a set of sentences of $L_{\omega_1,\omega}$.*

(a) *There exists a countable universal theory T' in a relational language, and a set Q' of partial atomic types of T' such that for every cardinal κ, T' has a model of power κ omitting each type in Q' iff S has a model of power κ.*

(b) *There exists a countable superstable T and a set Q of partial types of T such that for every cardinal κ, T has a model of power κ omitting each type in Q iff S has a model of power κ.*

Proof. (a) First reduce to the question of omitting a set of types Q'' for some first order theory T''; Skolemize it; and make every definable relation atomic. Let T' be the universal part of the resulting theory, and find Q' such that if A omits Q, then the Skolem-hull of A omits Q''.

(b) (Sketch) Let $V = (2^{\omega\times\omega}, +, \pi_{i,j}, 2)$ where $+$ is pointwise addition modulo 2, and $\pi_{i,j}(v) = v(i,j)$. Let I be an infinite set, $I \cap V = \emptyset$. For $s \in I^{<\omega}$, let A_s be a copy of V, presented with the action of V on A_s by translation as well as copies of the functions $\pi_{i,j}$, but without the 0 element. Let $S_n = \bigcup\{A_s : s \in I^n\}$, $T = \mathrm{Th}(V, I, S_1, S_2, \ldots)$ with the evident structure.

Call a model N of this theory proper if: (a) $v \in V^N$ and $\pi_{i,j}(v) = 0$ for all i,j then $v = 0$; (b) $v \in V^N$, then for large enough i, for all j, $\pi_{i,j}(v) = 0$; and for each i, for large enough j, $\pi_{i,j}(v) = 0$; (c) $s \in I^n \cap N$, $a \in A_s$, then for each i, *either* for large enough j, $\pi_{i,j}(a) = 0$, *or* for large enough j, $\pi_{i,j}(a) = 1$.

Let T', Q' be from (a), and let $R_1, R_2, \ldots$ list the atomic formulas of T', with R_i $k(i)$-ary. Given a model N of T, we attempt to interpret T' in N as follows: the underlying set is to be I^N; and $R_n(s)$ holds (for $s \in I^{k(n)}$) iff for every $a \in A_s$, for large enough j, $\pi_{n,j}(a) = 1$. We require (d): under this interpretation, one has a model of T', omitting each atomic type in Q'. It is clear that each of

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(a),(b),(c),(d) is an omitting-types requirement; and that every infinite model of T' can be obtained from a model of T in this way.

5.1(b) follows immediately from Lemma 5.2.

PROOF OF 5.1(c). Show by induction on λ (a $\beth$): there exists a stable M_λ of cardinality λ and a countable set of types Q_λ omitted by M such that no model of cardinality $>\lambda$ omits each type in Q_λ. For $\lambda = \aleph_0$ or λ a limit cardinal of cofinality ω, this is easy. If $\lambda = 2^\rho$, let $M = M_\rho$. Let C_i ($i < \rho$) be a dense set of functions $\omega \to M$, such that if $i \neq j$ then $C_i \cap C_j = \emptyset$. (Find the C_i's inductively; as $\bigcup_{j<i} C_i$ has cardinality $<2^\rho$ there is still $f: \omega \to M$ with $f \mid n$ prescribed and $f \notin \bigcup_{j<i} C_i$). Let $h: [I]^2 \to \rho$ be a coloring with no homogeneous set of size 3. For $i \neq j \in I$, let $S(\{i,j\})$ be a copy of $C_{h(i,j)}$. Let $M_\lambda = M'_\rho \cup I' \cup (\bigcup\{S\{i,j\} : i \neq j \in I)$, with the various "evaluation" functions, and the function $s \to \{i,j\}$ if $s \in S\{i,j\}$. The fact that the elements of S are "functions" (i.e. no two have the same "value at n" for each n) and that there is no homogeneous set of size 3 can now be expressed by partial types. By the Erdös–Rado theorem, these types cannot be omitted in a model of power λ^+.

REMARK. The Hanf number for omitting types is $\beth_\omega^{++}$ for stable theories with the property:

(*) Whenever I is an indiscernible subset of a model M, there exists a model $N \subseteq M$, $I \subseteq N$ with N/I $\aleph_1$-isolated.

Hence the above gives an example of a stable theory without (*).

It remains to prove 5.1(a). The coding of Lemma 5.2 cannot be used because it yields models realizing many types. An inductive construction of the type of 5.1(c) cannot work either, because if M is superstable of regular cardinality κ, then it has an elementary submodel M' of power κ such that every definable set of M' has power κ or $\aleph_0$ (unlike in the stable case, where any model omitting the required types contained traces of its construction, by having definable subsets of power μ for cofinally many $\mu < \kappa$). So we seem to need a direct construction. This seems to require a combinatorial characterization of $\beth_\alpha$ (distinguishing it from $\beth_\alpha^+$) via a partition theorem that involves only finite homogeneous sets (so that their non-existence can be expressed by omitting types).

Homogeneity for a tree of colorings

We will use structured collections c_η of colorings, where c_η colors $r(\eta)$-sets in countably many colors. Note that if $r(\eta)$ is bounded then already $\beth_\omega$ has infinite homogeneous sets for all the c_η's at once, and if they are not then it takes a cardinal much larger than $\beth_{\omega_1}$ to have simultaneous homogeneous sets for them. We use an intermediate notion, where the homogeneous set is homogeneous for only finitely many of the colorings, but their identity is not known in advance.

The set indexing the colorings will be a countable well-founded tree T, endowed with a positive integer-valued function r on T. A (T,r)-coloring of X is a collection c_η $(\eta \in T)$, c_η being an $r(\eta)$-coloring on X with countably many colors. If η is not a maximal node in T, it is required that the colors of c_η are immediate successors of η in T. A subset H of X is called homogeneous for $\bar{c}$ if there exists a branch b of T such that for each $l \le \text{length}(b)$, $(l \ge 1)$, and each $r(b(l))$-element subset s of H, $c_{b(l-1)}(s) = b(l)$. H is called a proper homogeneous set if $\text{card}(H) > r(b(l))$ for each l $(0 \le l < \text{length}(b))$. If H is a proper homogeneous set, then b is clearly determined by H and $b(0)$; in the interesting case, T has a unique root, so b is determined by H.

We define an ordinal rank on tagged trees (T,r) as follows. Given $\eta \in T$, let T_η $(T_{>\eta})$ be the subtree consisting of all elements of T (strictly) above η. If T has more than one root, let

$$\text{rk}(T,r) = \sup\{\text{rk}(T_\eta, r|T_\eta) : \eta \text{ a root of } T\}.$$

If T has a unique root η, let $\text{rk}(T,r) = \text{rk}(T_{>\eta}, r|T_{>\eta}) + r(\eta) - 1$. If $T = \emptyset$, $\text{rk}(T,r) = \emptyset$. As T is well-founded, this defines a countable ordinal.

Write $(\lambda) \Rightarrow (T)$ if every T-coloring on λ has a proper homogeneous set.

Proposition 5.3. *Let* $\alpha < \omega_1$.

(a) $\beth_\alpha^+ \Rightarrow (T)$ *for every tree* T *of rank* $\le \alpha$. *In fact every* T*-coloring has an infinite homogeneous set.*

(b) *There exists a tree* T *of rank* α *such that* $\beth_\alpha \not\Rightarrow (T)$.

Proof. (a) Let $\bar{c}$ be a T-coloring on an $X = \beth_\alpha^+$. Choose a root η of T. Let $b = r(\eta) - 1$, $T' = T_{>\eta}$, $\gamma = \text{rk}(T', r|T')$. Then $\text{rk}(T) \ge \gamma + b$. By the Erdös-Rado theorem (or if $b = 0$ by the pidgeon-hole principle), as $\text{card}(X) \ge \beth_{\gamma+b}^+$, there exists a subset Y of X of cardinality $\beth_\gamma^+$ such that Y is homogeneous for c_η. Let $\eta^+ = \eta\hat{\ }n$, where n is the homogeneous color for Y, and let $T'' = T_{\ge\eta^+}$. By induction on the foundation rank of T, there exists an infinite subset Z of Y, homogeneous for $\bar{c}|T''$. It follows that Z is homogeneous for $\bar{c}$.

(b) Recursively in α, we will build a tree T of rank α, a tagging r on T, and a (T,r)-coloring $\bar{c}$ of a set of size $\beth_\alpha$ with no proper homogeneous sets. If $\alpha = \beta + n$, β a limit ordinal, T will have a unique root η, and $r(\eta) = n + 1$.

If $\alpha = 0$ we take any singleton tree $T = \{\eta\}$, $r(\eta) = 1$, and we take the 1-coloring on ω to be the identity. The case of α *a limit ordinal* is similar.

If $\alpha = \beta + 1$, *with* β *limit.* Let β_n be a sequence of ordinals approaching β. Each one has its tagged tree (T_n, r_n) and a corresponding coloring $(c^n_\tau : \tau \in T_n)$ of X_n with no proper homogeneous sets. Without loss of generality the T_n's are disjoint. Let T be the tree with a unique root η, such that $T_{>\eta}$ is the union of the T_n's. Define r on T so as to agree with each r_n on T_n, and $r(\eta) = 2$. Note that $X =_{\rm def} \prod_n X_n$ has cardinality $\prod_n \beth_{\beta_n} = \beth^\omega_\beta = \beth_\alpha$. Define a (T,r)-coloring $\bar{c}$ on X as follows. Let $c_\eta(\{x,y\})$ be the largest integer n such that $x|n \neq y|n$. If $\tau \in T_n$, let $c_\tau(\{x_1,\ldots,x_{r(\tau)}\}) = c^n_\tau(\{x_1(n),\ldots,x_{r(\tau)}(n)\})$. Suppose H is a proper homogeneous subset for this coloring. Then $\mathrm{card}(X) \geq 3$. Let n be the homogeneous color for c_η. Then if $x \neq y \in H$, then $x(n) \neq y(n)$; and it is easy to check that $\{x(n) : x \in H\}$ is a proper homogeneous subset for $\bar{c}^n$. This contradicts the choice of $\bar{c}^n$.

The remaining case is $\alpha = \beta + n$, $n > 2$. By induction, there exists a tagged tree (T', r') of rank $\beta + n - 1$ with a unique root η and $r'(\eta') = n$, and a (T',r')-coloring $\bar{d}$ on $\lambda = \beth_{\beta+n-1}$ with no proper homogeneous subset. Let $X = {}^\lambda 2$, and order it lexicographically. Given $x, y \in X$, let (x,y) be the smallest $\alpha < \lambda$ such that $x(\alpha) \neq y(\alpha)$. We define a preliminary 3-coloring c_0 on T: Given $x < y < z$ in X, let the color of $\{x,y,z\}$ be the order type of $((x,y),(x,z),(y,z))$ in λ. As $n \geq 2$ we can consider c_0 as an $n+1$-coloring, by ignoring all but the first three elements (in the ordering of X). Given any m-coloring d on λ, let d^* be the $m+1$-coloring on X given as follows: if $x_0 < \cdots < x_n \in X$,

$$d^*\{x_0,\ldots,x_n\} = d(\{(x_0,x_1),(x_1,x_2),\ldots,(x_{n-1},x_n)\}).$$

Let T be the following tagged tree. It has unique root η, and $r(\eta) = n+1$. The second level of T is the set of pairs (a,b), where a is a color of d_η, and b is a possible order type of $n+1$ elements in a well-ordering. $T_{\geq(a,b)}$ is a copy of $T'_{\geq a}$ as a tree; the restriction of r to $T_{\geq(a,b)}$ is obtained from the restriction of r' to $T'_{\geq a}$ by adding 1 everywhere. Define a (T,r)-coloring $\bar{c}$ of X as follows: Let c_η be the $n+1$-coloring combining c_0 and $d^*_{\eta'}$, i.e. $c_\eta(s) = (c_0(s), d^*_{\eta'}(s))$. If $\tau \in T_{\geq(a,b)}$ corresponds to $\tau' \in T'_{\geq a}$, let $c_\tau = d^*_{\tau'}$.

Suppose H is a proper homogeneous set for $\bar{c}$. So H has at least four elements, and the order type in λ of $((x,y),(x,z),(y,z))$ is fixed for $x < y < z$ in H. Consider (x,y) as a function of x, y for $x < y$; I claim that it depends on at most one of its variables. In other words, if $x < y < z$ then $(x,y) = (x,z)$ *or* $(x,z) = (y,z)$. Let $x < y < z < w$ be four elements of H. If $(x,y) < (x,z)$ then $(x,y) = (y,z)$. By homogeneity, $(x,y) = (y,w)$, so $(y,z) = (y,w)$ and $(x,y) = (x,z)$, a contradiction. So $(x,y) \geq (x,z)$. If $(x,y) = (x,z)$ we are done. If $(x,y) > (x,z)$ then $(x,z) =$

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(y,z), so again the claim holds. Say for definiteness the first possibility occurs; so $(x,y) = \varphi(x)$ for $x < y \in H$. Let

$$H' = \{\varphi(x) : x \in H, \text{ and } x \text{ is not the greatest element of } H\}.$$

Then $\mathrm{card}(H') = \mathrm{card}(H) - 1$, and one can verify that H' is a proper homogeneous set for d. (Given an n-element subset s' of H' for example, let $s' = \{\varphi(x_0),\dots,\varphi(x_{n-1})\}$ with $x_0 < \cdots < x_{n-1} \in H$, and choose $x_n \in H$ with $x_{n-1} < x_n$. Then

$$c_{\eta'}(s') = c_{\eta'}\{\varphi(x_0),\dots,\varphi(x_{n-1})\} = c_{\eta'}\{(x_0,x_1),\dots,(x_n,x_{n+1})\} = d(\{x_0,\dots,x_n\}),$$

so it does not depend of the choice of s'.) This contradiction proves the proposition.

Proof of 5.1(a). We give the construction in detail. Our task is to find, for each $\alpha < \omega_1$, a countable superstable M with a small extension N of power $\beth_\alpha$, but with no small extension of power $\beth_\alpha^+$. Call a model small if, for each n, it realizes only finitely many n-types. Then N is small iff it is a small extension of some countable model. So it suffices to find, for each $\alpha < \omega_1$, a small superstable N_α of power $\beth_\alpha$ such that every model of $T_\alpha = \mathrm{Th}(N_\alpha)$ realizes some type (over $\emptyset$) omitted in N_α.

Let (I,r) be a tagged tree, X a set of cardinality $\beth_\alpha$, and $\bar{c}$ an (I,r)-coloring of X with no proper homogeneous subsets. The language of T_α will have a unary predicate D, a unary predicate B_i for each $i \in I$, a relation symbol f_i for $i \in I$, and a unary predicate P_m for $m < \omega$. The axioms will say that f_i is the graph of a function from B_i onto $[D]^{r(i)}$, the set of $r(i)$-element subsets of D. We will treat f_i as a function. For $i \in I$ and $u \in [D]^{r(i)}$, let $B_i(u) = \{x \in B_i : f_i(x) = u\}$. D and the B_i's will be pairwise disjoint, and $P_m \cap D = \emptyset$ for each m. Aside from these restrictions, everything is independent: given $i \in I$ and two disjoint finite sets F, F' of integers, an axiom will say that for all u from $[D]^k$, there exists $y \in B_i(u)$ such that $P_k(y)$ holds for $k \in F$ and fails for $k \in F'$. It is not difficult to check that once one specifies the number of predicates $B_{k,i}$ for each k, one has a description of a complete superstable theory, of finite rank. (D is strongly minimal, and B_i has infinity-rank $r(i)+1$ for each i.)

Given a model N of T_α and $x \in N$, define $f_x : \omega \to 2$ by $f_x(m) = 1$ if $N_\alpha \models P_{k,m}(x)$, $f_x(m) = 0$ otherwise. f_x will be called the P-type of x (in N). In the construction of N_α we will ensure that only countably many P-types are realized as the P-types of elements of N_α. It is easy to deduce from this property that for every finite subset C of N_α, only countably many types over C are realized in N_α; so N_α is small.

Each N_α will also omit each type of the form:

$$(y_1 \neq y_2) \;\&\; B_i(y_1) \;\&\; B_i(y_2) \;\&\; (f_i(y_1) = f_i(y_2)) \;\&\; \bigwedge_m (P_m(y_1) \equiv P_m(y_2)),$$

as well as the type: $\sim D(x) \;\&\; \bigwedge_i \sim B_i(x)$. In any model omitting these types, each element y of a $B_i(u)$ is determined by its P-type, so $B_i(u)$ has cardinality at most continuum. If the model omits every type that the N_α's omit, then it realizes only countably many P-types, so in fact each $B_i(u)$ is countable. Thus the cardinality of each model N we will be concerned with will be equal to the cardinality of D^N. So we have only to find N_α such that if N' is elementarily equivalent to N_α and omits every type omitted by N_α, then $D^{N'}$ has cardinality at most $\beth_\alpha$.

Let J be the set of colors mentioned in any of the colorings c_η. Given $j \in J$, choose $\gamma_j^0 : \omega \to 2$ in such a way that if $j \neq j'$ then $\{m : s_j(m) \neq s_{j'}(m)\}$ is infinite. Let

$$\Gamma_j = \{f : \omega \to 2\text{: for all but finitely many } m < \omega, f(m) = s_j(m)\}.$$

So the Γ_j's are pairwise disjoint. Every P-type of an element of N_α outside D will be in some Γ_j; so N will satisfy the requirement of realizing only countably many P-types. $N = N_\alpha$ will be the disjoint union of D^N and B_i^N $(i \in I)$. Let $D^N = X$. For $i \in I$ with $r(i) = k$, B_i^N is the disjoint union over $u \in [X]^k$ of the $B_i(u)$'s. $B_i(u)$ is a countable set, with distinguished subsets $P_m \cap B_i(u)$; choose these subsets in such a way that the set of P-types of elements in $B_i(u)^N$ is precisely $\Gamma(c_i(u))$. The choice of the Γ_j's ensures that N is indeed a model of T_α.

Given a branch b of T (b is necessarily finite, and there are only countably many possibilities for b), and given a choice of $\gamma_j \in \Gamma_j$ for $j = b(i)$, some i, we define a partial type $q = q_{b,\bar\gamma}$. Let $h = 1 + \max(r(b(1) : l \leq \text{length}(b)\}$. q_b has variables x_i $(1 \leq i \leq h)$ and $y_{1,s}$, where $1 \leq l \leq \text{length}(b)$, and s is a subset of $\{1,\ldots,h\}$ of cardinality $r(l)$. Let x_s denote $\{x_m : m \in s\}$. The partial type asserts that each $x_i \in D$, and the x_i's are distinct; that $y_{l,s} \in B_{b(l)}(x_s)$; and that the P-color of $y_{l,s}$ is $\gamma_{b(l+1)}$. $q_{b,\bar\gamma}$ is omitted in N_α; if it were realized by $(a_1,\ldots,a_h,\ldots)$, then $\{a_1,\ldots,a_h\}$ would be a proper homogeneous set for $\bar c$.

Conversely, let N' be elementarily equivalent to N, omitting every type omitted in N, and suppose $D^{N'}$ has cardinality $>\beth_\alpha$. Given i, and given $u \in [D']^i$, choose any $y_{i,u} \in B_i(u)$, and let $d_i(u)$ be the element j of J such that the P-color of y is in Γ_j. (Every P-color realized in N' is realized in N, hence is in some Γ_j.) This coloring has some proper homogeneous set $\{a_1,\ldots,a_h\} \subseteq D^{N'}$; $\bar a$ together with the corresponding $y_{i,u}$'s give a realization of some $q_{b,\bar\gamma}$, a contradiction. This finishes the proof.

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