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MORE ON REAL-VALUED MEASURABLE CARDINALS AND FORCING WITH IDEALS

BY

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ABSTRACT

(1) It is shown that if c is real-valued measurable then the Maharam type of $(c, \mathcal{P}(c), \sigma)$ is 2^c. This answers a question of D. Fremlin [Fr, (P2f)].

(2) A different construction of a model with a real-valued measurable cardinal is given from that of R. Solovay [So]. This answers a question of D. Fremlin [Fr, (P1)].

(3) The forcing with a κ-complete ideal over a set X, $|X| \geq \kappa$ cannot be isomorphic to Random×Cohen or Cohen×Random. The result for $X = \kappa$ was proved in [Gi-Sh1] but, as was pointed out to us by M. Burke, the application of it in [Gi-Sh2] requires dealing with any X. The application is: if A_n is a set of reals for $n < \omega$ then for some pairwise disjoint B_n (for $n < \omega$) we have $B_n \subseteq A_n$ but they have the same outer Lebesgue measure.

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Introduction

In Section 1 we deal with the Maharam types of real-valued measurable cardinals. The result (1) stated in the abstract and its stronger version are proved. The proofs are based on Shelah's strong covering lemmas and his revised power set operation.

In Section 2 a model is constructed with a real-valued measurable which is not obtained like the Solovay one by forcing random reals over a model with a measurable.

In Section 3, the result (3) stated in the abstract is proved.

Theorem 1.1 and the construction of Section 2 are due to the first author. Theorem 1.2 is joint and the result of Section 3 is due to the second author.

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1. On the number of Cohen or random reals

D. Fremlin asked the following in [Fr, (P2f)]:

If c is a real-valued measurable with witnessing probability ν, does it follow that the Maharam type of $(c, \mathcal{P}(c), \nu)$ is 2^c?

Or in equivalent formulation:

If c is a real-valued measurable, does the forcing with witnessing ideal isomorphic to the forcing for adding 2^c Random reals?

The next theorem provides the affirmative answer; see also 1.2.

THEOREM 1.1: *Suppose that I is a $2^{\aleph_0}$-complete ideal over $2^{\aleph_0}$ and the forcing with it (i.e., $\mathcal{P}(2^{\aleph_0})/I$) is isomorphic to the adding of λ-Cohen or λ-Random reals. Then $\lambda = 2^{2^{\aleph_0}}$.*

Proof: Suppose otherwise. Denote $2^{\aleph_0}$ by κ. Let $j\colon V \to N$ be a generic elementary embedding.

CLAIM 1: $j(\kappa) > (\lambda^+)^V$.

Proof: By a theorem of Prikry [Pr] (see also [Gi-Sh2] for a generalization), for every $\tau < \kappa$, $2^\tau = 2^{\aleph_0} = \kappa$. Then, in N, $2^\kappa = j(\kappa)$. But $(\mathcal{P}(\kappa))^V \subseteq N$, so $j(\kappa) \geq (2^\kappa)^V$. By [Gi-Sh2, Lemma 2.2], then $(2^\kappa)^V = \mathrm{cov}(\lambda, \kappa, \aleph_1, 2)$. So

$\mathrm{cov}(\lambda, \kappa, \aleph_1, 2) \geq \lambda^+$. Clearly,

$$\mathrm{cov}(\lambda, \kappa, \aleph_1, 2) \leq \mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2) \leq (\mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2))^N.$$

The last inequality holds since N is obtained by a c.c.c. forcing and so every countable set of ordinals in N can be covered by a countable set of V. By Shelah [Sh430, 3.2(2)], in N, $\mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2) < j(\kappa)$. Hence $\lambda^+ \leq \mathrm{cov}(\lambda, \kappa, \aleph_1, 2) \leq (\mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2))^N < j(\kappa)$. ∎of the claim.

By Shelah [Sh 430, 3.2(1), (2)], for every $i < 3$ there is $S_i = \langle s_{i\beta} \mid \beta < \kappa \rangle \subseteq [\kappa]^{\leq \aleph_i}$ unbounded of cardinality κ. We can assume by shrinking S_i that for every regular $\delta < \kappa$, if $\langle s_{i\beta} \mid \beta < \delta \rangle$ is unbounded in $[\delta]^{\leq \aleph_i}$, then for all γ, $\delta \leq \gamma < \kappa$, $s_{i\gamma} \nsubseteq \delta$. For $\alpha < \kappa$ and $i < 3$ let $S_i \restriction \alpha = \{P \in S_i | P \subseteq \alpha\}$. Fix a function $f \in {}^\kappa\kappa$ representing κ in a generic ultrapower and restrict everything to a condition forcing this. Without loss of generality $f(\alpha) \leq \alpha$ for every $\alpha < \kappa$.

Claim 2: *Let $i < 3$; then $\{\alpha < \kappa | S_i \restriction f(\alpha)$ is unbounded in $[f(\alpha)]^{\leq \aleph_0}$ and $|S_i \restriction f(\alpha)| = f(\alpha)\} \in I^*$ where I^* is the filter dual to I.*

Proof: We drop the index i for a while. $|S| = \kappa$ in V, so S is in a generic ultrapower. Suppose that in a generic ultrapower S is bounded. There will be some $t \subseteq \kappa$ countable such that for every $s \in S$, $s \nsupseteq t$. Using c.c.c. of the forcing we find a countable subset of κ in V, $t^* \supseteq t$. Since S is unbounded in V, some $s \in S$ contains t^*. Contradiction. Now, $j(S) \restriction \kappa = S$, since κ is regular, $S = \langle s_\beta \mid \beta < \kappa \rangle$ is unbounded in $[\kappa]^{\leq \aleph_i}$ and hence no $s_\gamma \subseteq \kappa$ for every γ, $\kappa \leq \gamma < j(\kappa)$. ∎of the claim.

Let N be a generic ultrapower. By [Gi-Sh1] there are in N at least κ Cohen (or random) reals over V.

Claim 3: *There exists a sequence $\langle r_\alpha \mid \alpha < \kappa \rangle$ of reals in V so that*

(1) *every real of V appears in $\langle r_\alpha \mid \alpha < \kappa \rangle$,*
(2) *for almost all $\alpha (\mathrm{mod}\, I)$, $\langle r_{\alpha+i} \mid i < f(\alpha) \rangle$ are $f(\alpha)$-Cohen (random) generic over $L[\langle S_\xi \restriction f(\alpha) \mid \xi < 3 \rangle, \langle r_\beta \mid \beta < f(\alpha) \rangle]$.*

Proof: Construct $\langle r_\alpha \mid \alpha < \kappa \rangle$ by induction. On nonlimit stages add reals in order to satisfy (1). For limit α's with $\xi_0 \restriction f(\alpha)$ unbounded in $[f(\alpha)]^{\leq \aleph_0}$, for every $\xi < 3$, add $f(\alpha)$-Cohen (or random) reals. It is possible since there are at least κ candidates in a generic ultrapower by [Gi-Sh1]. ∎of the claim.

Now work in N. $\mathrm{rng} f \restriction A$ is unbounded in κ, for every $A \notin I$. Let $j(\langle r_\alpha \mid \alpha < \kappa \rangle) = \langle r_\alpha \mid \alpha < j(\kappa) \rangle$ where $\langle r_\alpha \mid \alpha < \kappa \rangle$ is a sequence given by Claim 3.

Then, using Claim 3 in N we can find some $\alpha^* < j(\kappa)$ satisfying (2) of Claim 3 such that $j(S_\xi) \restriction f(\alpha^*)$ is unbounded in $[f(\alpha^*)]^{\leq\aleph_0}$, for every $\xi < 3$ and $j(f)(\alpha^*) \geq (\lambda^+)^V$. It is possible since by Claim 1, $(\lambda^+)^V < j(\kappa)$ and, in V, the range of f restricted to a set not in I is unbounded in κ.

The following will provide the contradiction and complete the proof of the theorem.

CLAIM 4: *$\langle r_{\alpha^*+i} \mid i < j(f)(\alpha^*)\rangle$ is a sequence of Cohen (random) reals over V.*

Proof: $\langle r_{\alpha^*+i} \mid i < j(f)(\alpha^*)\rangle$ is Cohen (random)-generic over

$$L[\langle j(S_\xi) \restriction j(f)(\alpha^*) \mid \xi < 3\rangle, \langle r_\beta \mid \beta < j(f)(\alpha^*)\rangle] =_{df} M(\alpha^*).$$

But $\langle r_\beta \mid \beta < \kappa\rangle$ is the list of all the reals of V. So all the reals of V are in $M(\alpha^*)$. Hence it is enough for every $Q \in ([j(f)(\alpha^*)]^{\leq\aleph_0})^V$ to find $P \in ([j(f)(\alpha^*)]^{\leq\aleph_0})^V \cap ([j(f)(\alpha^*)]^{\leq\aleph_0})^{M(\alpha^*)}$ so that $P \supseteq Q$. Then we will have that also $Q \in ([j(f)(\alpha^*)]^{\leq\aleph_0})^{M(\alpha^*)}$, which implies that Cohen or Random genericity over $M(\alpha^*)$ guarantees such genericity over V.

The following is more than enough.

SUBCLAIM 5: *$([j(f)(\alpha^*)]^{\leq\aleph_0})^V \cap ([j(f)(\alpha^*)]^{\leq\aleph_0})^{M(\alpha^*)}$ is unbounded in $[j(f)(\alpha^*)]^{\leq\aleph_0}$ of $V[G]$, where G is the generic subset producing j.*

Proof: Denote $j(f)(\alpha^*)$ by τ. We like to apply Shelah's strong covering without squares [Sh580] to both pairs $(V, V[G])$ and $(M(\alpha^*), V[G])$ for $[\tau]^{<\aleph_1}$. First notice that $V[G]$ is a c.c.c. extension of V, so it preserves cofinalities and $([\delta]^{<\mu})^V$ is unbounded (and even stationary) in $([\delta]^{<\mu})^{V[G]}$ for every regular μ and ordinal $\delta \geq \mu$. In particular, all the conditions of [Sh580] are satisfied by $(V, V[G])$.

Now let us turn to the pair $(M(\alpha^*), V[G])$. By the choice of α^*, $j(S_i) \restriction j(f)(\alpha^*)$ is unbounded in $([j(f)(\alpha^*)]^{<\aleph_i})^N$ for $i = 1, 2, 3$. In particular $\aleph_i^{M(\alpha^*)} = \aleph_i^N$ for $i = 1, 2, 3$. However, N and $V[G]$ have the same small sequence of ordinals. In particular, $\aleph_i^N = \aleph_i^{V[G]}$ and $([j(f)(\alpha^*)]^{<\aleph_i})^N = ([j(f)(\alpha^*)]^{<\aleph_i})^{V[G]}$ for every i, $1 \leq i \leq 3$.

Then the conditions of 6.7.2 of [Sh580] are satisfied by both pairs $(V, V[G])$ and $(M(\alpha^*), V[G])$. We like to use 3.3 of [Sh580] with cofinal in $[j(f)(\alpha^*)]^{\leq\aleph_0}$ families $\mathcal{P}^0 \in V$ and $\mathcal{P}^1 \in M(\alpha^*)$ so that $|\mathcal{P}^0|^V \leq j(f)(\alpha^*)$ and $|\mathcal{P}^1|^{M(\alpha^*)} \leq j(f)(\alpha^*)$. Take $\mathcal{P}^1 = j(S_1) \restriction j(f)(\alpha^*)$. Now, using c.c.c. of the forcing and working with names for elements of $\mathcal{P}^1$ it is easy to construct $\mathcal{P}^0 \in V$ such that each element of $\mathcal{P}^0$ contains an element of $\mathcal{P}^1$ and $|\mathcal{P}^0|^V \leq j(f)(\alpha^*)$. We are now ready to show unboundedness of $([j(f)(\alpha^*)]^{\leq\aleph_0})^V \cap ([j(f)(\alpha^*)]^{\leq\aleph_0})^{M(\alpha^*)}$ in $[j(f)(\alpha^*)]^{\leq\aleph_0}$

of the largest model $V[G]$. Work in $V[G]$. Let $Q \in [j(f)(\alpha^*)]^{\leq\aleph_0}$. Find $P_0^0 \in \mathcal{P}^0$, $P_0^0 \supseteq Q$. Then let $P_1^0 \in \mathcal{P}^1$, $P_1^0 \supseteq P_0^0$. Continue by induction and define an increasing sequence $\langle P_\nu^i \mid \nu < \omega_1, i < 2\rangle$ so that $P_0^0 \in \mathcal{P}^0$ and $P_\nu^0 \subseteq P_\nu^1 \in \mathcal{P}^1$. Then by 3.3 of [Sh580] sets $\{\nu < \omega_1 \mid \bigcup_{\nu'<\nu} P_{\nu'}^0 \in V\}$ and $\{\nu < \omega_1 \mid \bigcup_{\nu'<\nu} P_{\nu'}^1 \in M(\alpha^*)\}$ contain clubs. In particular there will be $\nu < \omega_1$ in both of them. Then $P = \bigcup_{\nu'<\nu} P_{\nu'}^0 = \bigcup_{\nu'<\nu} P_{\nu'}^1$ will be in $V \cap M(\alpha^*)$ and we are done. ∎of Subclaim 5. ∎of Claim 4. ∎

Let us now prove a stronger statement which relies on a different property.

THEOREM 1.2.1: (1) *Suppose that I is a κ-complete ideal over κ and the forcing with it (i.e., $\mathcal{P}(\kappa)/I$) is isomorphic to the forcing for adding λ-Cohen reals. Assume also*

(*) *some condition forces that "$j(\kappa) \geq (2^\kappa)^V$", where j is a generic embedding. Then $\lambda > 2^{<\kappa}$ implies $\lambda = 2^\kappa$.*

(2) *Similarly for Random.*

Proof: (1) Without loss of generality, let us assume the weakest condition, i.e., κ forces (*). Suppose that $\lambda < 2^\kappa$. Then $\lambda < j(\kappa)$ as $\mathcal{P}(\kappa)^V \subseteq N$. By [Sh430, 3.2(2)] in V we have $(\forall\theta < \kappa)(\mathrm{cov}(\theta, \aleph_1, \aleph_1, 2) < \kappa)$, hence in a generic ultrapower N, $\mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2) < j(\kappa)$. However, in V, $\lambda^+ \leq \mathrm{cov}(\lambda, \kappa, \aleph_1, 2) \leq \mathrm{cov}(\lambda, \aleph_1, \aleph_1, 2))^N < j(\kappa)$. The first inequality holds by [Gi-Sh2]. Hence $\lambda^+ < j(\kappa)$.

Now, by [Sh460, 2.6], there are regular $\delta < \mu < \kappa$ such that $\mathrm{cov}(\lambda, \mu, \mu, \delta) = \lambda$ (or see [Sh513, §1]).

Let us assume for simplification of the notation that $\mu = \aleph_2$, $\delta = \aleph_1$.

Let $\langle s_\alpha \mid \alpha < \lambda\rangle$ be generic for $\langle I, \subseteq\rangle$, i.e., a set of λ Cohens representing the generic G.

CLAIM 1: *There is a sequence of reals $\langle r_\alpha \mid \alpha < \lambda^+\rangle$ in a generic ultrapower such that for every $s \subseteq \omega_1$ and even $s \in {}^{\omega_1}V$ the final segment of $\langle r_\alpha \mid \alpha < \lambda^+\rangle$ is Cohen generic over $L[s]$.*

Proof: Let N be a generic ultrapower. Then ${}^\kappa N \subseteq N$ where ${}^\kappa N$ is in the sense of the generic extension. First note that $\langle s_\alpha \mid \alpha < \kappa\rangle$ is a sequence of κ Cohen reals over V and it belongs to N. Clearly every $s \subseteq \omega_1$ in N (or the same in $V[G]$) is a name in V interpreted using $\aleph_1$ Cohen reals only. Hence for every $\delta \leq \kappa$ of cofinality $> \aleph_1$, some final segment of $\langle s_\alpha \mid \alpha < \delta\rangle$ will be generic over $L[s]$. Then, in V, for I-almost every regular $\delta < \kappa$ there is a sequence $\vec{t}^\delta = \langle t_\alpha \mid \alpha < \delta\rangle$ such that for every $s \subseteq \omega_1$ and $\delta' \leq \delta$ of cofinality $> \aleph_1$, some final segment of

$\bar{t}^{\delta} \restriction \delta'$ is Cohen generic over $L[s]$. Back in N, we use this for some $\delta \geq \lambda^+$, for $\delta' = \lambda^+$ which is still below $j(\kappa)$. ∎of the claim.

Let us fix such a sequence $\langle r_\alpha \mid \alpha < \lambda^+\rangle$ in N. We split it into blocks each of length ω_1. Denote such a changed sequence by $\langle r_{\alpha i} \mid \alpha < \lambda^+, i < \omega_1\rangle$. Now back in V, let us use the fact that $\mathrm{cov}(\lambda, \aleph_2, \aleph_2, \aleph_1) = \lambda$. We know that for every $\alpha < \lambda^+$ the block $\langle \underset{\sim}{r}_{\alpha_i} \mid i < \omega_1\rangle$ is added by using only ω_1 Cohen reals from the λ Cohen reals, $\langle s_\beta \mid \beta < \lambda\rangle$. Work in V. For every $\alpha < \lambda^+$ and $i < \omega_1$ pick a condition $p_{\alpha i}$ in the Cohen forcing for adding λ-Cohen reals which decides the value of $\underset{\sim}{r}_{\alpha,i}(0)$. Let $\rho_{\alpha,i} \in 2$ be such a value, i.e., $p_{\alpha i} \Vdash$ "$\underset{\sim}{r}_{\alpha,i}(0) = \rho_{\alpha,i}$". Wlog,

$$p_{\alpha 0} \Vdash \left|\{i < \omega_1 \mid p_{\alpha,i} \in \underset{\sim}{G}\}\right| = \aleph_1.$$

For every $\alpha < \lambda^+$ and $i < \omega_1$ let $\mathrm{dom}\, p_{\alpha i} = \{\xi_{\alpha,i,\ell} \mid \ell < \ell_{\alpha i} < \omega\} \subseteq \lambda$ and $p_{\alpha i}(\xi_{\alpha,i,\ell}) = \eta_{\alpha,i,\ell} \in {}^{\omega>}2$. As $2^{<\kappa} < \lambda$ we can assume that $\rho_{\alpha,i}, \ell_{\alpha,i}$'s and $\eta_{\alpha,i,\ell}$'s do depend on α for $\alpha \in A_0$, $A_0 \subseteq \lambda^+$ unbounded. Thus, further, we shall drop the index α in these sets. The number of possibilities for $\langle \xi_{\alpha,0,\ell} \mid \ell < \ell_0\rangle$'s, $\alpha \in A_0$ is $\lambda^{\ell_0} = \lambda$. So we can assume that for some $\langle \xi_{0,\ell} \mid \ell < \ell_0\rangle$ for every $\alpha \in A_0$, $\langle \xi_{\alpha,0,\ell} \mid \ell < \ell_0\rangle = \langle \xi_{0,\ell} \mid \ell < \ell_0\rangle$. Hence also $p_{\alpha,0}$'s ($\alpha \in A_0$) will be the same. Drop the index α and denote $p_{\alpha,0}$ by p_0. Now, $\mathrm{cov}(\lambda, \aleph_2, \aleph_2, \aleph_1) = \lambda$, so there is a set $b \subseteq \lambda$ of cardinality $\aleph_1$ such that the following set is unbounded in λ^+:

$$A = \{\alpha \in A_0 \mid (\exists^{\aleph_1} i)(\forall \ell < \ell_i)(\xi_{\alpha,i,\ell} \in b) \quad \text{and} \quad (\forall \ell < \ell_0)(\xi_{0,\ell} \in b)\}.$$

We actually replace λ in cov by $\lambda^{<\omega} \times \omega_1$. Recall that $2^{<\kappa} < \lambda$. We can shrink A to an unbounded in λ^+ set A' and find uncountable $a \subseteq \omega_1$ with $0 \in a$ such that for every $\alpha, \beta \in A'$, $i \in a$ and $\ell < \ell_i$, $\xi_{\alpha,i,\ell} = \xi_{\beta,i,\ell}$. Again we drop the index α for α in A', $i \in a$, $\ell < \ell_i$ and denote $\xi_{\alpha,i,\ell}$ by $\xi_{i,\ell}$. Now, p_0 forces that in $L[\langle \underset{\sim}{s}_{\xi_{i,\ell}} \mid i \in a, \ell < \ell_i\rangle, \langle \ell_i \mid i \in a\rangle, \langle \eta_{i,\ell} \mid i \in a, \ell < \ell_i\rangle, \langle \rho_i \mid i \in a\rangle] = \underset{\sim}{M}$ we have information on $\langle \underset{\sim}{r}_{\alpha,i}(0) \mid i \in a\rangle$ for unboundedly many $\alpha < \lambda^+$. Thus for $i \in a$ and $\alpha \in A'$ the following holds:

if for every $\ell < \ell_i$, $s_{\xi_{i,\ell}}$ extends $\eta_{i,\ell}$, then $r_{\alpha,i}(0) = \rho_i$.

Apply the Claim 1 to M. We find some $\alpha^* < \lambda^+$ such that for every α, $\alpha^* \leq \alpha < \lambda^+$, $\langle r_{\alpha,i} \mid i < \omega_1\rangle$ is Cohen generic over M. Using c.c.c. of the forcing the value of α^* can be fixed already in V. Fix $\alpha^{**} \geq \alpha^*$ an element of A'. Recall that $\langle \eta_{i,\ell} \mid i \in a,\ \ell < \ell_i\rangle$ is a sequence of elements of ${}^{\omega>}2$ which belongs to V and $\langle s_{\xi_{i,\ell}} \mid i \in a, \ell < \ell_i\rangle$ are Cohen reals over V. So, for some $a^* \subseteq a$ of cardinality $\aleph_1$, $a^* \in M$ satisfies the following: for every $i \in a^*$ and $\ell < \ell_i$, $s_{\xi_{i,\ell}}$ extends

$\eta_{i,\ell}$. Then for every $i \in a^*$, $r_{\alpha^{**},i}(0) = \ell_i$. But $a^* \in M$, $\langle \ell_i \mid i \in a^* \rangle \in M$, so $\langle r_{\alpha^{**},i} \mid i < \omega_1 \rangle$ cannot be Cohen generic over M. Contradiction.

(2) Let us now deal with the Random reals case. Most of the proof repeats the Cohen reals case, but instead of choosing $p_{\alpha,i}, \xi_{\alpha,i,\ell}(\ell < \ell_{\alpha i})$, $\rho_{\alpha,i}$ we proceed as follows. Find, for every $\alpha < \lambda^+$ and $i < \omega_1$, $m_{\alpha,i}, n_{\alpha i} < \omega$, $\xi_{\alpha i,0} < \xi_{\alpha,i,1} < \cdots < \xi_{\alpha,i,m_{\alpha,i}-1}$ and a function $g_{\alpha,i}$ from ${}^{m_{\alpha,i}}({}^{n_{\alpha,i}}2)$ to $\{0,1\}$ such that

$$| \, g_{\alpha,i}(\underset{\sim}{s}_{\xi_{\alpha,i,0}}, \underset{\sim}{s}_{\xi_{\alpha,i,1}}, \ldots, \underset{\sim}{s}_{\xi_{\alpha,i,m_{\alpha,i}-1}}) \neq \underset{\sim}{r}_{\alpha,i}(0)| \, < \frac{1}{4}.$$

We view $g_{\alpha,i}$ as the continuous function on ${}^{m_{\alpha,i}}({}^{\omega}2)$ determined by its values on ${}^{m_{\alpha,i}}({}^{n_{\alpha,i}}2)$. For every $\alpha < \lambda^+$ pick an unbounded set $S_\alpha \subseteq \omega_1$, $m_\alpha, n_\alpha < \omega$ and g_α so that for every $i \in S_\alpha$, $m_{\alpha,i} = m_\alpha$, $n_{\alpha,i} = n_\alpha$ and $g_{\alpha,i} = g_\alpha$, $2^{<\kappa} < \lambda$, so we can assume wlog that there are S^*, m^*, n^* and g^* such that for some unbounded $A_0 \subseteq \lambda^+$ for every $\alpha \in A_0$, $S_\alpha = S^*$, $n_\alpha = n^*$, $m_\alpha = m^*$ and $g_\alpha = g^*$. Let $i_0 < i_1 < \cdots < i_n < \cdots (n < \omega)$ be any increasing sequence of elements of S^*. Consider for $\alpha \in A_0$, $\ell < \omega$,

$$\underset{\sim}{k}_{\alpha,\ell} = |\{\ell' < \ell | g^*(\langle \underset{\sim}{s}_{\xi_{\alpha,i_{\ell'},m}} | m < m^* \rangle) \neq \underset{\sim}{r}_{\alpha,i_{\ell'}}(0)\} \, | \, /\ell.$$

By basic probability some $q_{\alpha,\langle i_n | n<\omega \rangle}$ forces that

$$\text{“}\lim_{\ell\to\infty} \inf\{\underset{\sim}{k}_{\alpha,\ell'} \mid \ell' \geq \ell\} < \frac{1}{3}\text{”}.$$

Now, using $\mathrm{cov}(\lambda, \aleph_2, \aleph_2, \aleph_1) = \lambda$ for $[\lambda]^{m^*} \times S^*$ we find in V a set $b \subseteq \lambda$, $|b| = \aleph_1$ so that

$$A = \{\alpha \in A_0 \mid \text{ the cardinality of } \{i \in S^* \mid \forall \ell < m^* \xi_{\alpha,i,\ell} \in b\} \text{ is } \aleph_1\}$$

is unbounded in λ^+. Again, shrinking A to an unbounded $A' \subseteq \lambda^+$, using $2^{<\kappa} < \lambda$, we find $a \subseteq S^*$ of cardinality $\aleph_1$ so that for every $\alpha, \beta \in A'$, $i \in a$ and $\ell < m^*$, $\xi_{\alpha,i,\ell} = \xi_{\beta,i,\ell} = \xi_{i,\ell}$.

Consider now in $V[G]$

$$M = L[\langle s_{\xi_{i,\ell}} \mid i \in a, \ell < m^* \rangle, a, g^*].$$

Back in V, using Claim 1 for Random and c.c.c. find $\alpha^* < \lambda^+$ such that the weakest condition forces

“for every $\alpha, \alpha^* \leq \alpha < \lambda^+$ the sequence $\langle \underset{\sim}{r}_{\alpha,i} \mid i < \omega_1 \rangle$ is Random generic over $\underset{\sim}{M}$”.

Fix some $\alpha^{**} \geq \alpha^*$ in A'. Let $\vec{i} = \langle i_n \mid n < \omega \rangle$ be the sequence of the first ω elements of a. Choose a generic set G with $q_{\alpha^{**},\vec{i}} \in G$. Then,

$$\lim_{\ell \to \infty} \inf\{k_{\alpha^{**}\ell'} \mid \ell' \geq \ell\} < \frac{1}{3}.$$

But this is impossible since $\langle r_{\alpha^{**},i} \mid i < \omega_1 \rangle$ is a sequence random reals over M and $\vec{i} \in M$. Contradiction. ∎

2. Another construction of a model with a real-valued measurable cardinal

In this section we construct a model with a real-valued measurable cardinal which differs from the Solovay original. This answers negatively a question of D. Fremlin [Fr, (P1)]:

Let N be a model of ZFC and $\kappa \in N$ a real-valued measurable cardinal in N. Does it follow that there are inner models $M \subseteq N$ such that κ is a measurable in M and M-generic filter G for a random real p.o. set over M such that $G \in N$ and $N \cap \mathcal{P}(\kappa) \subseteq M[G]$?

Suppose that κ is a measurable and GCH holds. We define a forcing notion P as follows:

Definition 2.1: P consists of all triples $p = \langle p_0, p_1, p_2 \rangle$ so that

(1) $p_0 \subseteq \kappa$,
(2) p_1 is a function with domain contained in p_0,
(3) p_2 is a function defined over inaccessibles $\leq \kappa$,
(4) for every inaccessible δ, $|p_0 \cap \delta| < \delta$, $|\operatorname{dom} p_1 \cap \delta| < \delta$ and $|\operatorname{dom} p_2 \cap \delta| < \delta$,
(5) for every $\alpha \in \operatorname{dom} p_1$, $p_1(\alpha) \subseteq \alpha$,
(6) every element of p_0 is an ordinal of cofinality $\aleph_0$,
(7) for every limit ordinal β, if $\operatorname{cf} \beta > \aleph_0$, then $p_0 \cap \beta$ is not stationary in β, and if $\operatorname{cf} \beta = \aleph_0$, then $\beta \backslash (p \cap \beta)$ is unbounded in β,
(8) for every $\alpha \in \operatorname{dom} p_2$, $p_2(\alpha)$ is a closed subset of α disjoint with p_0.

Definition 2.2: Let $p, q \in P$, $p = \langle p_0, p_1, p_2 \rangle$ and $q = \langle q_0, q_1, q_2 \rangle$. Then $p \geq q$ iff

(1) $p_1 \subseteq q_1$,
(2) $\operatorname{dom} p_2 \supseteq \operatorname{dom} q_2$ and, for every $\alpha \in \operatorname{dom} q_2$, $p_2(\alpha)$ is an end extension of $q_2(\alpha)$,
(3) $p_0 \supseteq q_0$,
(4) for every $\delta < \kappa$, if δ is an inaccessible or a limit of inaccessibles and δ^* is the least inaccessible above δ, then $p_0 \cap [\delta, \delta^*)$ is an end extension of $q_0 \cap [\delta, \delta^*)$.

The forcing P is intended to add three objects. Thus, the first coordinates of P are producing a subset S of κ which is stationary in $V[S]$ and reflecting only in inaccessibles. The second coordinate is responsible for a kind of diamond sequence over S and the last coordinate adds clubs preventing reflection of S at inaccessibles and its stationarity.

The forcing P destroys the measurability of κ once used over $V = L[\mu]$. It is bad for our purpose. We are going to use a certain subforcing of P which will preserve measurability and contain the projection of P to the first two coordinates. But first let us study basic properties of P.

Let $P_0 = \{p_0 \mid \exists \langle p_1, p_2 \rangle \ \langle p_0, p_1, p_2 \rangle \in P\}$, $P_{01} = \langle \langle p_0, p_1 \rangle | \exists p_2 \ \langle p_0, p_1, p_2 \rangle \in P\}$. Let α be an inaccessible. We denote by $P \restriction \alpha$ the set

$$\{\langle p_0 \cap \alpha, \ p_1 \restriction \alpha, \ p_2 \restriction \alpha \rangle \mid \langle p_0, p_1, p_2 \rangle \in P\}$$

and by $P \backslash \alpha$ the set

$$\{\langle p_0 \backslash \alpha, \ p_2 \restriction [\alpha, \kappa), \ p_2 \restriction [\alpha + 1, \kappa) \rangle \mid \langle p_0, p_1, p_2 \rangle \in P\};$$

$P_0 \restriction \alpha$, $P_{01} \restriction \alpha$ and $P_0 \backslash \alpha$, $P_{01} \backslash \alpha$ are defined similarly.

The following is standard.

Claim 2.3: *Let α be an inaccessible. Then the following holds:*

(1) $P = P \restriction \alpha \times P \backslash \alpha$,
(2) $P_0 = P_0 \restriction \alpha \times P_0 \backslash \alpha$,
(3) $P_{01} = P_{01} \restriction \alpha \times P_{01} \backslash \alpha$.

Let $\alpha < \kappa$ be a limit ordinal and Q a forcing notion. Consider the following game Game (Q, α):

$$\begin{array}{lccccccc} \mathbf{I} & q_1 & & & q_3 & \cdots & & \cdots & & \\ & & \leq & \leq & & & & \geq & & \\ \mathbf{II} & & & q_2 & & & \cdots & & q_\beta & \cdots & q_\alpha \end{array}$$

where Players I and II are building an increasing sequence of elements of Q, I at even stages and II at odds. If at some stage $\beta < \alpha$, II cannot continue, i.e., there is no q above $\{q'_\beta | \beta' < \beta\}$, then I wins. Otherwise II wins.

Claim 2.4: *Player II has a winning strategy in the game* Game $(P \backslash \alpha, \alpha^+)$ *for every inaccessible α.*

Proof: Let α be an inaccessible. We define a winning strategy σ for Player II in the Game $(P \backslash \alpha, \alpha^+)$.

Let $\delta > \alpha$ be an inaccessible but not limit one. Denote by δ^- the supremum of inaccessibles below δ.

Let $p \in P\backslash\alpha$. We define $\overline{p}$ to be the condition obtained from $p = \langle p_0, p_1, p_2\rangle$ by adding $\sup(p_0 \cap [\delta^-, \delta)) + \sup(p_2(\delta))$ to $p_2(\delta)$ if $p_0 \cap [\delta^-, \delta) \neq \emptyset$ or $p_0 \cap [\delta^-, \delta) = \emptyset$ but $p_0 \cap \delta^-$ is unbounded in δ^-, where δ runs over inaccessibles above α which are not limit inaccessibles and $\sup(p_2(\delta)) = 0$ whenever $\delta \notin \operatorname{dom} p_2$.

Now we define σ to be dependent only on the last move of I at successive stages of the game. Set $\sigma(p_{\beta+1}) = \overline{p}_{\beta+1}$. Suppose $\beta \leq \alpha^+$ is limit and the game up to β

$$\begin{matrix} p_1 & & p_3 & \cdots & & \cdots \\ & p_2 & & \cdots & p_\gamma & \end{matrix}$$

was played according to σ. Then set $\sigma(\langle p_\gamma \mid \gamma < \beta\rangle) =$ the closure of $\bigcup_{\gamma<\beta} p_\gamma$. More precisely, let $\sigma(\langle p_\gamma \mid \gamma < \beta\rangle) = \langle p^0, p^1, p^2\rangle$ where $p^0 = \bigcup_{\gamma<\beta} p^0_\gamma$, $p^1 = \bigcup_{\gamma<\beta} p^1_\gamma$ and $\operatorname{dom}(p^2) = \bigcup_{\gamma<\beta} \operatorname{dom}(p^2_\gamma)$, $p^2(\xi) = \bigcup\{p^2_\gamma(\xi) \mid \gamma < \beta, \xi \in \operatorname{dom} p^2_\gamma\} \cup \{\sup(\cup\{p^2_\gamma(\xi) \mid \xi \in \operatorname{dom} p^\gamma_2, \gamma < \beta\})\}$, for $\xi \in \operatorname{dom} p^2$.

We need to check that such defined p is a condition. The only problem is to show that p^0 does not reflect at any τ, $\aleph_0 < \operatorname{cf}\tau < \tau$. So let τ be an ordinal such that $\aleph_0 < \operatorname{cf}\tau < \tau$ and $p^0 \cap \tau$ is unbounded in τ. Pick δ to be the first inaccessible above τ. Then $\delta^- \leq \tau$. If $\delta^- < \tau$, then starting with some $\gamma_0 < \beta$, $p^0_\gamma \cap [\delta^-, \delta) \neq \emptyset$. But then $p^2(\delta)$ will be a club of τ disjoint to $p^0 \cap \tau$. Suppose now that $\delta^- = \tau$. Then $C_0 = \{\sup\left(\bigcup_{\gamma'<\gamma}(p^0_{\gamma'} \cap \tau)\right) \mid \gamma < \beta,\ \gamma \text{ limit}\}$ is a club of τ. Also, $C_1 = \{\xi < \tau \mid \xi \text{ is a limit of inaccessibles}\}$ is a club of τ. Let $\xi \in C_0 \cap C_1$. Then for some inaccessible $\underline{\delta} < \tau$ $\xi = \underline{\delta}^-$. Let $\gamma_\xi < \beta$ be so that $\xi = \sup \bigcup_{\gamma'<\gamma_\xi}(p^0_{\gamma'} \cap \tau)$. Then $\xi \in p^2_{\gamma_\xi}(\underline{\delta})$, by the definition of σ and the procedure. Thus, this provides a club of τ disjoint to $p^0_\beta \cap \tau$. ∎of the claim

The following is now trivial.

Claim 2.5: *P preserves cofinalities and does not add new functions from ordinals less than the first inaccessible into V.*

Let U be a normal measure over κ and $j\colon V \to N$ the corresponding elementary embedding. Then, in N, $j(P) = j(P) \restriction \kappa \times j(P)\backslash\kappa$. Clearly, $(j(P) \restriction \kappa)^N = P$. Now let us produce inside V an N-generic subset of $j(P)\backslash\kappa$ with the set over the first coordinate nonstationary in V.

Claim 2.6: *There exists $\langle S, F_1, F_2\rangle$ such that*

(a) *$\langle S, F_1, F_2\rangle$ is $j(P)\backslash\kappa^+$ generic over N,*
(b) *S is not a stationary subset of $j(\kappa)$,*
(c) *S does not reflect.*

Proof: Let $\langle D_\alpha | \alpha < \kappa^+\rangle$ be the list of dense open subsets of $j(P)\backslash\kappa$ of N. Let $\sigma \in N$ be a winning strategy for Player II in Game $(j(P)\backslash\kappa, \kappa^+)$. It exists by Claim 2.4 applied in N to $j(P)$. Play the game from V so that I plays at stage $\beta+1$ an element $P_{\beta+1}$ of D_β which is above p_β, where $\beta < \kappa^+$. We will finish with a desired N-generic set. ∎

Force with P over V. Let G be a generic subset. We denote

$$\bigcup\{p_0 \mid \exists\langle p_1, p_2\rangle\ \langle p_0, p_1, p_2\rangle \in G\}$$

by S. For every $\alpha \in S$ let $A_\alpha = \bigcup\{p_1(\alpha) \mid \exists\langle p_0, p_1, p_2\rangle \in G \text{ and } \alpha \in \operatorname{dom} p_1\}$ and for inaccessible $\delta \leq \kappa$ let $C_\delta = \bigcup\{p_2(\delta) \mid \exists\langle p_0, p_1, p_2\rangle \in G \text{ and } \delta \in \operatorname{dom} p_2\}$. Then $S \subseteq \kappa$, and for every inaccessible $\delta \leq \kappa$, C_δ is a club of δ disjoint to S.

Claim 2.7: *S is a stationary nonreflecting subset of κ in $V[S, \langle A_\alpha \mid \alpha \in S\rangle, \langle C_\delta \mid \delta \text{ inaccessible and } \delta < \kappa\rangle]$.*

Proof: $\langle C_\delta \mid \delta < \kappa\rangle$ are witnessing the nonreflection. Suppose that S is nonstationary in $V[S, \langle A_\alpha \mid \alpha \in S\rangle, \langle C_\delta \mid \delta \text{ inaccessible and } \delta < \kappa\rangle]$. Return back to V and work with names. Suppose for simplicity that the empty condition forces the nonstationarity of S. Let $\underset{\sim}{C}$ be a name of witnessing club. Pick an elementary submodel N of $V_{2^{2^\kappa}}$ such that $P, \underset{\sim}{C} \in N$, $|N| < \kappa$ and $N \cap \kappa$ is an ordinal of cofinality $\aleph_0$. Let $\alpha = N \cap \kappa$ and $\langle \alpha_n \mid n < \omega\rangle$ be a cofinal in α sequence. Now by induction we construct an increasing sequence $\langle p_i \mid i < \omega\rangle$ of conditions of $P \restriction \kappa$ (i.e., P without the information on a club of κ disjoint to S) such that for every $i < \omega$

(a) $p_i \in N$,
(b) p_i decides the first element of $\underset{\sim}{C}$ above α_i,
(c) $\sup(p_i)_0 \geq \alpha_i$,

where $p_i = \langle (p_i)_0, (p_i)_1, (p_i)_2\rangle$.

Now, in V, let

$$p = \langle \bigcup_{i<\omega}(p_i)_0 \cup \{\alpha\}, \bigcup_{i<\omega}(p_i)_1,\ \{\langle \delta, \cup\{(p_i)_2(\delta) \mid i < \omega,\ \delta \in \operatorname{dom}(p_i)_2\}\rangle.$$

Then $p \Vdash \alpha \in \underset{\sim}{C} \cap \underset{\sim}{S}$. Contradiction. ∎

Claim 2.8: *κ is a measurable cardinal in $V[S, \langle A_\alpha \mid \alpha \in S\rangle]$.*

Proof: Just note that in N, $j(P_{01}) = P_{01} \times j(P_{01})\backslash\kappa^+$, since nothing is done in the interval $[\kappa, \kappa^+]$ by this forcing. By Claim 2.6, there is a $j(P_{01})\backslash\kappa^+$ generic over

N set in V. Thus it is easy to extend j to the embedding of $V[S, \langle A_\alpha \mid \alpha \in S\rangle]$. This insures the measurability of κ. ∎

Notice that κ is not measurable in

$$V[S, \langle A_\alpha \mid \alpha \in S\rangle, \langle C_\delta \mid \delta < \kappa,\ \delta \text{ inaccessible}\rangle]$$

since S is a stationary nonreflecting subset of κ.

Now, over a model $V[S, \langle A_\alpha \mid \alpha \in S\rangle]$ we are going to force a Boolean algebra B such that:

(a) κ is still measurable in $V[S, \langle A_\alpha \mid \in S\rangle, B]$,

(b) $j^*(B)/G(B)$ is isomorphic to the adding of $j(\kappa)$-Random reals, where j^* is the elementary embedding of $V[S, \langle A_\alpha \mid \alpha \in S\rangle, B]$ into its ultrapower and $G(B)$ is a generic subset of B.

First let us review some basics of product measure algebras. We refer to D. Fremlin [Fr2] for detailed presentation.

Suppose that B is a σ-algebra, i.e., a Boolean algebra all of whose countable suprema exist. A **measure on** B is a function μ: $B \to [0,1]$ so that: (a) $\mu(1_B) = 1$, and (b) whenever $\{b_n \mid n \in \omega\} \subseteq B$ with $b_n \wedge b_n = 0$ for $n \neq m$, then $\mu(V_n b_n) = \sum_n \mu(b_n)$. If in addition μ is **positive** (i.e., $\mu(b) = 0$ iff $b = 0$), then we say that $\langle B, \mu\rangle$ is a **measure algebra**. A measure algebra is always a complete Boolean algebra.

Suppose now that I is a set, and $\langle B_i, \mu_i\rangle$ for $i \in I$ are measure algebras. Call $C \in \prod_{i\in I} B_i$ a **cylinder** iff $C(i)$ is the unit element of B_i, except for a finite number of coordinates i. Let $B \supseteq \prod_{i\in I} B_i$ be the σ-algebra generated by the cylinders. It is known that there is a unique measure μ on B so that $\mu(C) = \prod_{i\in I} \mu_i(C(i))$ for any cylinder C. μ may not be positive, but there is a standard strategy: Let $I = \{b \in B \mid \mu(b) = 0\}$. Then I is an ideal, and $\overline{B} = B/I$ as usual is a σ-algebra consisting of equivalence classes $[b]$ for $b \in B$ (where $[b] = [c]$ **iff** the symmetric difference $(b - c) \vee (c - b) \in I$). We can define a positive measure $\overline{\mu}$ on $\overline{B}$ by: $\overline{\mu}([b]) = \mu(b)$. Thus, $\langle \overline{B}, \overline{\mu}\rangle$ is a measure algebra, called the product **measure algebra** of the $\langle B_i, \mu_i\rangle$'s.

Let $\mathbf{2}$ be the basic measure algebra $\langle P(2), \mu\rangle$ where μ is the measure: $\mu(\emptyset) = 0$, $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$, and $\mu(\{0,1\}) = 1$. For any set I, let $\mathbf{2}^I$ denote the product measure algebra of I copies of $\mathbf{2}$. We can then force with $\mathbf{2}^I$ with the natural proviso: b is a *stronger* condition than c iff $0 < b \le c$ in $\mathbf{2}^I$. This forcing obviously has the ω_1-c.c.

For $I = \omega$, $\mathbf{2}^I$ is just the usual random real forcing and, for $I = \lambda$, $\mathbf{2}^I$ is the λ-random real forcing. Let us denote them by Random and Random(λ), respectively.

We consider the σ-algebras $B_\alpha \subseteq \prod_{i<\alpha}(\mathcal{P}(2))_i$ generated by the cylinders, where $\alpha \le \kappa$ and $(\mathcal{P}(2))_i$ is just the i-th copy of $\mathcal{P}(2)$. The desired algebra B will be B_κ/I_κ, where the ideal I_κ of "null" sets is going to be added generically. More precisely, for α's of countable cofinality I_α's, will be added by forcing and then, for $\beta \le \kappa$ of uncountable cofinality I_β, will the union of I_α's, where $\alpha < \beta$, $\operatorname{cf}\alpha = \aleph_0$. The sequence of ideals $\langle I_\alpha \mid \alpha \le \kappa\rangle$ will be in $V_1 = V[S, \langle A_\alpha \mid \alpha \in S\rangle, \langle C_\delta \mid \delta < \kappa,\ \delta \text{ inaccessible}\rangle]$.

Let us work in V_1. We define by induction on $\alpha < \kappa$ a measure μ_α on B_α. Then I_α will be the ideal of μ_α-measure null sets. Actually there will be a lot of different measures over B_α's. We would like to prevent B_κ (and even its subalgebras of power κ) from carrying a measure. For this purpose, the "diamond" sequence $\langle A_\alpha \mid \alpha \in S\rangle$ will be used to destroy possible candidates.

If $\alpha < \min S$, then let μ_α be the usual product measure over B_α, i.e., one generated by attaching weight $1/2$ to $\{0\}$ and $\{1\}$, 0 to $\emptyset$ and 1 to $\{0,1\}$ in every component $(\mathcal{P}(2))_i$ $(i < \alpha)$ of the product $\prod_{i<\alpha}(\mathcal{P}(2))_i$. Set

$$I_\alpha = \{X \in B_\alpha \mid \mu_\alpha(X) = 0\}.$$

Suppose now that $\alpha < \kappa$ and for every $\beta < \alpha$ the measure μ_β over B_β was already defined. We need to define μ_α over B_α.

CASE 1: $\alpha \notin S$.

Pick an increasing continuous sequence $\langle \alpha_\tau \mid \tau < \operatorname{cf}\alpha\rangle$ witnessing nonstationarity of $S \cap \alpha$. In case $\operatorname{cf}\alpha = \aleph_0$ just use ω-sequence unbounded in α and disjoint with S. For every $\tau < \operatorname{cf}\alpha$ let $\mu(\tau)$ be $\mu_{\alpha_{\tau+1}} \restriction \big(B_{\alpha_{\tau+1}} \restriction [\alpha_\tau, \alpha_{\tau+1})\big)$, where $B_{\alpha_{\tau+1}} \restriction [\alpha_\tau, \alpha_{\tau+1})$ is the subalgebra of $\prod_{\alpha_\tau \le i < \alpha_{\tau+1}}(\mathcal{P}(2)))_i$ generated by the cylinders.

Let μ_α be the product measure of $\langle < B_{\alpha_{\tau+1}} \restriction [\alpha_\tau, \alpha_{\tau+1}), \mu(\tau) > \mid \tau < \operatorname{cf}\alpha\rangle$.

Notice that $\alpha_\tau \notin S$. Therefore, by induction, we can assume that for a limit τ the measure μ_{α_τ} over B_{α_τ} is the product measure of $\langle < B_{\alpha_\nu} \restriction [\alpha_\nu, \alpha_{\nu+1}), \mu(\nu) > \mid \nu < \tau\rangle$.

CASE 2: $\alpha \in S$.

Suppose that A_α codes in some reasonable fashion sequences $\langle \alpha_n \mid < \omega\rangle$, $\langle \varphi_n \mid n < \omega\rangle$, $\langle \mu(n) \mid n < \omega\rangle$ and $\langle a_n \mid n < \omega\rangle$ so that for every $n < \omega$

(a) $\langle \alpha_n \mid n < \omega\rangle$ is a cofinal in α sequence,

(b) a_n is a countable subset of $[\alpha_n, \alpha_{n+1})$,

(c) $\mu(n)$ is a measure over $B_{\alpha_{n+1}} \restriction a_n$ respecting the ideal $I_{\alpha_{n+1}} \restriction a_n$, i.e., for every $X \in B_{\alpha_{n+1}} \restriction a_n$, $\mu(n)(X) = 0$ iff $X \in I_{\alpha_{n+1}}$,

(d) φ_n: Random $\leftrightarrow B_{\alpha_{n+1}} \restriction a_n$ is a measure algebra isomorphism.

Denote $B_{\alpha_{n+1}} \restriction a_n$ by $B(n)$. Let us define a measure $\tilde{\mu}(n)$ over $B(n)$. Thus for every $n < \omega$ let us change the value $\mu(n)(\varphi_n(\{0\}))$ from $1/2$ to $1 - 1/\pi^2 n^2$ and those of $\varphi_n(\{1\})$ from $1/2$ to $1/\pi^2 n^2$. Let $\tilde{\mu}(n)$ be the measure obtained from $\mu(n)$ in such a fashion. Clearly, such local changes have no effect on the set of measure zero. Namely, for every $X \in B(n)$, $\mu(n)(X) = 0$ iff $\tilde{\mu}(n)(X) = 0$.

Define now the measure μ_α over B_α as the product measure of the measure algebras $\langle B(n), \tilde{\mu}(n)\rangle$ $(n < \omega)$ together with all the rest, i.e.,

$$\langle B_{\alpha_{n+1}} \restriction ([\alpha_n, \alpha_{n+1}) \backslash a_n),\ \mu_{\alpha_{n+1}} \restriction ([\alpha_n, \alpha_{n+1}) \backslash a_n)\rangle.$$

We claim that $\varphi = \bigcup_{n<\omega} \varphi_n$ cannot be extended to complete embedding into $\langle B_\alpha, \mu_\alpha\rangle$. The reason is that under φ the measure of the set $\bigcap_{n<\omega} \varphi_n(\{0\})$ should be zero, but $\mu_\alpha\big(\bigcap_{n<\omega} \varphi_n(\{0\})\big) = \prod_{n<\omega} \tilde{\mu}(n)\big(\varphi_n(\{0\})\big) = \prod_{n<\omega} \big(1 - 1/\pi^2 n^2\big)$, which equals $\sin(1) \neq 0$ by the Euler formula.

Notice that the ideal I_α of μ_α-measure zero sets will not be effected if for finitely many n's the measures $\mu(n)$ will be used in the product instead of $\tilde{\mu}(n)$'s. Also, if in the previous construction we do everything above some α_{n_0} for fixed $n_0 < \omega$, i.e., we define the measure over $B_\alpha \restriction [\alpha_{n_0}, \alpha)$ instead of all B_α, call it $\mu_\alpha \restriction [\alpha_{n_0}, \alpha)$ and its ideal $I_\alpha \restriction [\alpha_{n_0}, \alpha)$, then, for every $X \in B_\alpha$, $X \in I_\alpha$ iff $X \restriction \alpha_{n_0} \in I_{\alpha_{n_0}}$ and $X \restriction [\alpha_{n_0}, \alpha) \in I_\alpha \restriction [\alpha_{n_0}, \alpha)$. This means that once having I_{α_n}'s, initial segments of measures $\langle \tilde{\mu}(n) \mid n < \omega\rangle$ have no effect on I_α. This observation will be crucial further for showing measurability of κ.

If A_α does not guess the sequences as above, then we proceed as in Case 1.

This completes the definition of $\langle \mu_\alpha \mid \alpha < \kappa\rangle$ and hence also $\langle I_\alpha \mid \alpha \leq \kappa\rangle$.

We set $B = B_\kappa / I_\kappa$. Let $V_2 = V[S, \langle A_\alpha \mid \alpha \in S\rangle, \langle I_\alpha \mid \alpha < \kappa, \operatorname{cf}\alpha = \aleph_0\rangle]$. Then for every $\alpha < \kappa$, $I_\alpha \in V_2$. So $B \in V_2$. We will show the following claim, which has a proof similar to 2.7.

Claim 2.9: *Random(κ) does not embed into B in V_1 and also in V_2.*

Proof: Notice that V_2 and V_1 have the same reals. So if φ is an embedding of Random(κ) into B in V_2, then φ will also be such an embedding in V_1. Hence let us prove the claim for V_1.

Suppose otherwise. Let φ: Random(κ) $\longrightarrow B$ witnessing embedding. Back in V let us work with names. Let $\underset{\sim}{\varphi}$ be a name of φ and assume for simplicity that the empty condition forces this.

Pick N and $\langle \alpha_n \mid n < \omega\rangle$ to be as in Claim 2.7 with $\underset{\sim}{\varphi}$ replacing $\tilde{C}$.

We define sequences of conditions of $P \restriction \kappa$ of $\{p_n \mid n < \omega\} \subseteq N$ of ordinals $\langle \beta_n \mid n < \omega\rangle$, countable sets $\langle a_n \mid n < \omega\rangle$ and embedding $\langle \varphi_n \mid n < \omega\rangle$ so that

(a) $\sup(p_n)_0 \geq \alpha_n$,
(b) $\beta_n \geq \alpha_n$,
(c) $p_n \Vdash$ "$\underset{\sim}{\varphi}(\{0\}\beta_n)$ has nontrivial intersection with $\underset{\sim}{B} \restriction [\beta_n, \beta_{n+1})$", where $\{0\}_{\beta_n} \in (\mathcal{P}(2))_{\beta_n}$, i.e., the β_n-th copy of $\mathcal{P}(2)$,
(d) $a_n \subseteq \beta_n$,
(e) φ_n embeds $(\mathcal{P}(2))_{\beta_n}$ into $B_\kappa \restriction a_n$,
(f) $p_n \Vdash$ "$\check{\varphi}_n$ is equal to $\underset{\sim}{\varphi} \restriction (\mathcal{P}(2))_{\beta_n} \bmod \underset{\sim}{I}_\kappa$".

Since the forcing does not add new countable sequences of elements of V, there is no problem in carrying out the induction.

Denote by $\mu(n)$ the measure over $B_\kappa \restriction a_n$ induced by φ_n.

Now let $A_\alpha \subseteq \alpha$ be a code for such sequences

$$\langle \beta_n \mid n < \omega \rangle,\ \langle a_n \mid n < \omega \rangle,\ \langle \varphi_n \mid n < \omega \rangle$$

and $\langle \mu(n) \mid n < \omega \rangle$.

Set $p = \langle \bigcup_{n<\omega}(p_n)_0 \cup \{\alpha\}, \bigcup_{n<\omega}(p_n)_1 \cup \{\langle \alpha, A_\alpha \rangle\}, \{\langle \delta, \cup\{(p_n)_2(\delta) \mid n < \omega, \delta \in \operatorname{dom}(p_n)_2\}\rangle$.

Then $p \Vdash$ "$\sim \varphi$ does not embed $\prod_{n<\omega}(\mathcal{P}(2))_{\beta_n}$ into $B_\alpha / \underset{\sim}{I}_\alpha$", by the definition of I_α". Hence, also $p \Vdash$ "$\underset{\sim}{\varphi}$ does not embed Random(κ) into B". Contradiction.
∎

Claim 2.10: *κ is a measurable cardinal in V_1.*

Proof: Let $j\colon V \longrightarrow N$ be an elementary embedding witnessing the measurability of κ. We like to extend it to an embedding

$$\begin{aligned} j^* : V[S, \langle A_\alpha \mid \alpha \in S \rangle,\ \langle I_\alpha \mid \alpha < \kappa \text{ and } \operatorname{cf}\alpha = \aleph_0 \rangle] \\ \longrightarrow N[S^*,\ \langle A_\alpha \mid \alpha \in S^* \rangle,\ \langle I_\alpha \mid \alpha < j(\kappa) \text{ and } \operatorname{cf}\alpha = \aleph_0 \rangle]. \end{aligned}$$

By Claim 2.6, j extends to

$$j' : V[S, \langle A_\alpha \mid \alpha \in S \rangle] \longrightarrow N[S^*, \langle A_\alpha \mid \alpha \in S^* \rangle]$$

where $\langle S^* \backslash S,\ \langle A_\alpha \mid \alpha \in S^* \backslash S > \rangle \in V$ is $j(P_{01}) \backslash \kappa$ generic over N. We like to produce ideals $\langle I_\alpha \restriction (B_{j(\kappa)} \restriction [\kappa, j(\kappa)) \mid \kappa < \alpha < j(\kappa), \operatorname{cf}\alpha = \aleph_0 \rangle$ generic over N but in V. In order to define $\langle I_\alpha \mid \alpha < \kappa, \operatorname{cf}\alpha = \aleph_0 \rangle$ we used clubs witnessing nonreflection of S, i.e., $\langle C_\delta \mid \delta < \kappa,\ \delta \text{ inaccessible} \rangle$. By Claim 2.6, the only club which is needed in order to extend j but is missing in V is C_κ. But, we define generically I_α's only for α's of cofinality $\aleph_0$ and, moreover, initial segments have no influence on such I_α's. This means that the definition of

$\langle I_\alpha \restriction B_{j(\kappa)} \restriction [\kappa, j(\kappa)) \mid \kappa < \alpha < j(\kappa), \mathrm{cf}\,\alpha = \aleph_0\rangle$ can be carried out completely inside $N[S^*\backslash S, \langle A_\alpha \mid \alpha \in S^*\backslash S\rangle, \langle C_\delta \mid \kappa < \delta < j(\kappa), \delta \text{ inaccessible}\rangle]$. All the sets $S^*\backslash S, \langle A_\alpha \mid \alpha \in S^*\backslash S\rangle$ and $\langle C_\delta \mid \kappa < \delta < j(\kappa),\ \delta \text{ inaccessible of } N\rangle$ can be found inside V by Claim 2.6. Hence we have enough sets to extend j to j^*. Thus, the measurability of κ is preserved in $V_2 = V[S, \langle A_\alpha \mid \alpha \in S\rangle, \langle I_\alpha \mid \alpha < \kappa \text{ and } \mathrm{cf}\,\alpha = \aleph_0\rangle]$. ∎

Let $j^*\colon V_2 \longrightarrow N_2 = N[S^*, \langle A_\alpha \mid \alpha \in S^*\rangle\ , \langle I\alpha \mid \alpha < j(\kappa) \text{ and } \mathrm{cf}\,\alpha = \aleph_0\rangle]$ be the embedding of Claim 2.10.

CLAIM 2.11: *$j^*(B) \restriction [\kappa, j(\kappa))$ is isomorphic to* Random(κ^+) *in V_2.*

Proof: By Claim 2.6, there are in V, and hence in V_2, clubs

$$\langle C_\delta \mid \kappa < \delta \leq j(\kappa),\ \delta \text{ is an inaccessible in } N\rangle$$

witnessing nonreflection of $S^*\backslash S$ in every N-inaccessible $\delta \leq j(\kappa)$. Using them we define measures μ_α over $B_\alpha \restriction [\kappa, \alpha)$ agreeing with ideals I_α for every α, $\kappa < \alpha \leq j(\kappa)$ as was done for B_α's below κ in V_1. The final measure $\mu_{j(\kappa)}$ will turn $B_{j(\kappa)} \restriction [\kappa, j(\kappa))$ into measure algebra. Since $|j(\kappa)| = \kappa^+$, by the Maharam theorem, see [Fr2], we obtain the desired result. ∎

So we have the following:

THEOREM 2.12: *Let κ be the measurable cardinal of $L[\mu] = V$ (the minimal model with a measurable). Then κ is a real-valued measurable in V_2^B but, for every submodel V' of V_2^B, if κ is a measurable in V', then there is no $G \in V_2^B$ which is* Random(κ) *generic over V'.*

Proof: Suppose that $V' \subseteq V_2^B$, κ is a measurable in V', $G \subseteq$ Random(κ) generic over V' and $G \in V_2^B$. But then G is also Random(κ) generic over $V = L[\mu]$, since $L[\mu] \subseteq V'$ by its minimality. But V and V_2 have the same countable sequences of ordinals. So, G will be Random(κ)-generic also over V_2. This means that Random(κ) embeds B, which is impossible by Claim 2.9. ∎

3. The forcing with ideal cannot be isomorphic to Cohen×Random or Random×Cohen

The result for κ-complete ideals over κ was proved in [Gi-Sh1]. Max Bruke pointed out that the application of this in [Gi-Sh2] requires the result also for less than κ complete ideals as well. The purpose of this section is to close this gap.

THEOREM 3.1: *Suppose that I is a ω_1-complete ideal over some κ. Then the forcing with ideal (i.e., $\mathcal{P}(\kappa)/I$) cannot be isomorphic to Cohen$\times$Random or Random$\times$Cohen.*

Proof: Let us deal with the Random$\times$Cohen case. The Cohen$\times$Random case is similar.

Suppose otherwise. $\mathcal{P}(\kappa)/I \simeq$ Random$\times$Cohen. Without loss of generality, for some $\kappa_1 \leq \kappa$ and $f\colon \kappa \to \kappa_1$, $\kappa \Vdash_{\mathcal{P}(\kappa)/I}$ "κ_1 is the critical point of the generic embedding and $\tilde{f}$ represents κ_1 in the ultrapower". Define an ideal J over κ_1 to be the set of all $A \subseteq \kappa_1$ such that $f^{-1}{}''(A) \in I$. Denote $Q = \mathcal{P}(\kappa)/I$ and $Q_1 = \mathcal{P}(\kappa_1)/J$. Then Q_1 is a complete subordering of Q. By [Sh480], we can assume that Q_1 is ω^ω-bounding, since otherwise it adds a Cohen real which suffices for the argument of [Gi-Sh1]. We define a Q_1-name $\underset{\sim}{\tau} = \{\eta \in {}^{\omega>}2 \mid$ the condition $(\mathbf{1}_{\text{Random}}, \eta)$ is compatible with every element of $\underset{\sim}{G}(Q_1)\}$. For $\eta, \nu \in {}^{\omega>}2$ let us write $\eta \rhd \nu$ if the sequence η extends the sequence ν. The following two claims are obvious.

CLAIM 3.2: *$\underset{\sim}{\tau}$ is a Q_1-name of a nonempty subset of ${}^{\omega>}2$ closed under initial segments with no $\lhd$-maximal element and hence a tree.*

CLAIM 3.3: *$\Vdash_Q$ " the Cohen real is an ω-branch of $\underset{\sim}{\tau}$".*

CLAIM 3.4: *There is no $p \in Q_1$ and $\eta \in {}^{\omega>}2$ such that $p \Vdash_{Q_1}$ "for every $v \in {}^{\omega>}2$ $v \rhd \eta$ implies $v \in \underset{\sim}{\tau}$".*

Proof: Suppose otherwise. Let p, η be witnessing this. Then above p the forcing notion Q_1 is a complete subordering of Random. But it has to add a real. Hence it is isomorphic to Random, which is impossible by [Gi-Sh1]. ∎

Using ${}^\omega\omega$-boundness of Q_1, we can find $p_0 \in Q_1$ and a function $h\colon \omega \longrightarrow \omega$ such that $p_0 \Vdash_{Q_1}$ "for every $n < \omega$ and $\eta \in {}^n2$ there is v, $\eta \lhd v \in {}^{h(n)}2$ such that $v \notin \underset{\sim}{\tau}$". Let us assume for simplicity that this p_0 is just the weakest condition of Q_1.

Let $T^* = \{T \mid T \subseteq {}^{\omega>}2$ is a tree and for every $n < \omega$, $\eta \in {}^n2$ there is $v \rhd \eta$ such that $v \in {}^{h(n)}2$ and $v \notin T\}$. Consider also $T^*_m = \{T \cap {}^m2 \mid T \in T^*\}$ for $m < \omega$. T^* can be viewed as a tree if we identify it with $\bigcup_{m<\omega} T^*_m$ and define an order by setting $t_1 \lhd t_2$ iff for some $m < \omega$, $t_1 = t_2 \cap {}^{m>}2$. Then, clearly,

$$\Vdash_{Q_1} \text{"}\underset{\sim}{\tau} \text{ is an } \omega\text{-branch of } T^*\text{"}.$$

CLAIM 3.5: *Suppose that $n < \omega$, $q_0 \in$ Random, $\eta \in {}^n2$. Then there are $m < \omega$, q, v_0, v_1, t_0, t_1 such that*

(a) $q \in$ Random *and* $q \geq q_0$,
(b) $\eta \lhd v_0, v_1 \in {}^{\omega>}2$,
(c) $t_0, t_1 \subseteq {}^{m\geq}2$ *and* $t_0 \neq t_1$,
(d) $(q, v_i) \Vdash \underset{\sim}{\tau} \cap {}^{m\geq}2 = t_i$ *for* $i < 2$.

Proof: Find first some $q' \geq q_0$ and $v_0 \lhd \eta$ deciding $\underset{\sim}{\tau} \cap {}^{m\geq}2$. Let t_0 be the decided value, i.e., $(q', v_0) \Vdash \underset{\sim}{\tau} \cap {}^{m\geq}2 = t_0$. By Claim 3.4 there will be $m < \omega$ and $v \lhd \eta$, $v \in {}^m2 \backslash t_0$. Find some $(q, v_1) \geq (q', v)$ deciding $\underset{\sim}{\tau} \cap {}^m2$. Let t_1 be the forced value, i.e., $(q, v_1) \Vdash \underset{\sim}{\tau} \cap {}^m2 = t_1$. Since $(q, v_1) \Vdash v_1 \in \underset{\sim}{\tau}$, we have $(q, v_1) \Vdash v = v_1 \restriction m \in \underset{\sim}{\tau}$. But this means $t_0 \neq t_1$. ∎

CLAIM 3.6: *Suppose that $n, k < \omega$ and $q_0 \in$ Random. Then there are $q \in$ Random, $m < \omega$, $\langle v_{\eta,\ell} \mid \eta \in {}^n2,\ \ell < k\rangle$ and $\langle t_{\eta,\ell} \mid \eta \in {}^n2,\ \ell < k\rangle$ such that*

(a) $q \geq q_0$,
(b) $m \geq n$,
(c) *for every* $\eta_1, \eta_2 \in {}^n2$, *and* $\ell_1, \ell_2 < k$ $t_{\eta_1,\ell_1} = t_{\eta_2,\ell_2}$ *iff* $(\eta_1, \ell_1) = (\eta_2, \ell_2)$,
(d) *for every* $\eta \in {}^n2$, $\ell < k$,

$$\eta \lhd v_{\eta,\ell} \in {}^{\omega>}2, \quad t_{\eta,\ell} \in T^*_m \quad \text{and} \quad (q, v_{\eta,\ell}) \Vdash \underset{\sim}{\tau} \cap {}^m2 = t_{\eta,\ell}.$$

Proof: Just use the previous claim enough times. Thus, first, we generate a tree of $k \cdot (2^n + 1)$ possibilities for one $\eta \in {}^n2$ and then we repeat the argument of Claim 3.5 on all η's. ∎

CLAIM 3.7: *For every $n < \omega$, $k < \omega, q' \in$ Random and $\mathcal{E} > 0$ there are $m < \omega$, $q \geq q', \{q_\ell \mid \ell < \ell^*\} \subseteq$ Random pairwise disjoint, $\langle v_{\eta,\ell,j} \mid \eta \in {}^n2, \ell < \ell^*\ ,\ j < k\rangle$ and $\langle t_{\eta,\ell,j} \mid \eta \in {}^n2,\ \ell < \ell^*,\ j < k\rangle$ such that*

(a) $\mathrm{Lb}(q) \geq 1 - \mathcal{E}$ (Lb *denotes the Lebesgue measure*),
(b) $q = \bigcup_{\ell<\ell^*} q_\ell$,
(c) *if* $\eta \in {}^n2$, $\ell < \ell^*$ *and* $j < k$ *then* $v_{\eta,\ell,j} \in {}^{\omega>}2$ $v_{\eta,\ell,j} \rhd \eta$ *and* $(q_\ell, v_{\eta,\ell,j}) \Vdash \underset{\sim}{\tau} \cap {}^{m\geq}2 = t_{\eta,\ell,j}$,
(d) *for every* $\ell < \ell^*$,

$$t_{\eta_1 \ell,j_1} = t_{\eta_2,\ell,j_2} \quad \text{iff } (\eta_1, j_2) = (\eta_2, j_2).$$

Proof: We define by induction q_ℓ's using Claim 3.6. Thus if $\langle q_i \mid i \leq \ell\rangle$ is defined, then we apply Claim 3.6 to ${}^\omega2 \backslash \bigcup_{i\leq\ell} q_i$. The process stops after we reach ℓ^* s.t. $\mathrm{Lb}(\bigcup_{\ell<\ell^*} q_\ell) \geq 1 - \mathcal{E}$. ∎

CLAIM 3.8: *For every $n < \omega$ and $\mathcal{E} > 0$ there are $m, n \leq m < \omega$ and a function $H: T^*_m \longrightarrow 2$ such that for every $\eta \in {}^n2$ and $i \in 2$ we can find $q^{i,\eta}$, $\ell^{i,\eta} < \omega$, $\langle q^{i,\eta}_\ell \mid \ell < \ell^{i,\eta}\rangle$ and $\langle v^{i,\eta}_\ell \mid \ell < \ell^{i,\eta}\rangle$ such that for every $i < 2$ and $\ell < \ell^{i,\eta}$*

(a) $\eta \triangleleft v^{i,\eta}_\ell \in {}^{\omega>}2$,
(b) $q^{i,\eta}, q^{i,\eta}_\ell \in$ Random *and* $q^{i,\eta} = \bigcup_{\ell<\ell^i} q^{i,\eta}_\ell$,
(c) $\mathrm{Lb}(q^{i,\eta}) \geq 1 - \mathcal{E}$,
(d) $\langle q^{i,\eta}_\ell \mid \ell < \ell^{i,\eta}\rangle$ *are pairwise disjoint,*
(e) $\left(q^{i,\eta}_\ell, v^{i,\eta}_\ell\right) \Vdash$ "$H\left(\underset{\sim}{x} \cap {}^{m\geq}2\right) = i$".

Proof: For every $\eta \in {}^n2$ and $t \in T^*_m$ let

$$I_{\eta,t} = \{q \in \text{Random} \mid \text{ there is } v \in {}^{\omega>}2,\ v \triangleleft \eta \text{ such that } (q,v) \Vdash \text{"}\underset{\sim}{x} \cap {}^{m\geq}2 = t\text{"}\}.$$

Let $\{q_{\eta,t,\ell} \mid \ell < \ell_{\eta,t} \leq \omega\}$ be a maximal antichain subset of $I_{\eta,t}$. Let $q^*_{\eta,t} = \bigcup_{\ell<\ell_{\eta,t}} q_{\eta,t,\ell}$. Then $\bigcup_{t\in T^*_m} q^*_{\eta,t} = {}^\omega 2$ mod null set, since $\{q_{\eta,t,\ell} \mid t \in T^*_m \ell < \ell_{\eta,t}\}$ is a predense subset of Random (but not necessarily antichain). So $\mathrm{Lb}\left(\bigcup_{t\in T^*_m} q^*_{\eta,t}\right) = 1$.

It is enough to prove the following statement:

(*) There exists $H: T^*_m \longrightarrow 2$ so that for every $\eta \in {}^n2$ and $i < 2$

$$\mathrm{Lb}\left(\bigcup\{q^*_{\eta,t} \mid t \in T^*_m \text{ and } H(t) = i\}\right) \geq 1 - \mathcal{E}/2,$$

since then we will be able to find a maximal antichain $\langle q^i_\ell \mid \ell < \ell^* \leq \omega\rangle$ in Random above $\bigcup\{q^*_{\eta,t} \mid t \in T^*_m \text{ and } H(t) = i\}$ together with $\langle v^i_\ell \mid \ell < \ell^*\rangle$ and $\langle t^i_\ell \mid \ell < \ell^*\rangle$ so that

$$(q^i_\ell, v^i_\ell) \Vdash \text{"}\underset{\sim}{x} \cap {}^{m\geq}2 = t^i_\ell \text{ and } H(t_\ell) = i\text{"}.$$

In order to reduce ℓ^* to a finite ℓ^i we note that the precision here is $1 - \mathcal{E}/2$ but only $1 - \mathcal{E}$ is needed.

So let us prove (*). We consider the set $\mathcal{H}$ of all functions $H: T^*_m \longrightarrow 2$. It is finite, but it is better to regard it as a probability space. All $H \in \mathcal{H}$ with the same probability. So we choose $H(t) \in \{0,1\}$ independently for the $t \in T^*_m$ with probability $1/2$.

We use m given by Claim 3.7 for our n, $\mathcal{E}'$ much smaller than $\mathcal{E}$ and k large enough. Given $\eta \in {}^n2$ and $i < 2$, we consider the probability of

$$\left(\mathrm{Lb}\left(\bigcup\{q^*_{\eta,t} \mid t \in T^*_m,\ H(t) = i\}\right) \geq 1 - \mathcal{E}/2\right)$$

in $\mathcal{H}$. It is ≤ 1, as the value is always ≤ 1 and is $\geq 1-1/2^k$. In order to prove the last inequality, let us use $\{q_\ell \mid \ell<\ell^*\}$ of Claim 3.7. Thus

$$\begin{aligned}
&\mathrm{Lb}\Big(\bigcup\Big\{q^*_{\eta,t}\mid t\in T^*_m \text{ and } H(t)=i\Big\}\Big)\\
&=\sum_{\ell<\ell^*}\mathrm{Lb}\left(q_\ell\cap\bigcup\Big\{q^*_{\eta,t}\mid t\in T^*_m \text{ and } H(t)=i\Big\}\right)\\
&\quad+Lb\Big[\Big({}^{\omega}2\setminus\bigcup_{\ell<\ell^*}q_\ell\Big)\cap\Big(\bigcup\Big\{q^*_{\eta,t}\mid t\in T^*_m \text{ and } H(t)=i\Big\}\Big)\Big]\\
&\geq\sum_{\ell<\ell^*}\left[Lb(q_\ell)\times\mathrm{Lb}\left(q_\ell\cap\bigcup\Big\{q^*_{\eta,t}\mid t\in T^*_m \text{ and } H(t)=i\Big\}\right)\right]/\mathrm{Lb}(q_\ell)-\mathcal{E}'.
\end{aligned}$$

By (a) of 3.7,

$$\mathrm{Lb}\left({}^{\omega}2\setminus\bigcup_{\ell<\ell^*}q_\ell\right)<\mathcal{E}'.$$

Now it suffices to show that for each $\ell<\ell^*$

$$\mathrm{Lb}(q_\ell\cap\cup\{q^*_{\eta,t}\mid t\in T^*_m,\ H(t)=i\})/Lb(q_\ell)\geq 1-\mathcal{E}/4$$

holds for enough H's. But $v_{\eta,\ell,j}$, $t_{\eta,\ell,j}$ $(j<k)$ of 3.7 are witnessing that the probability in $\mathcal{H}$ of the failure is $\leq 1/2^k$. Just in order to fail, H should take the value $1-i$ on $t_{\eta,\ell,j}$ for every $j<k$ and the probabilities of $0,1$ are equal. The probability of the failure for some $\eta\in{}^{n}2$, $i\in 2$ is then $\leq 2^{n+1}/2^k$. So, picking k large enough comparatively to n we will insure that most $H\in\mathcal{H}$ are fine, whereas we need only one. ∎

Now using Claim 3.8 we define by induction on $j<\omega$, n_j, m_j, H_j, $\langle q^{i,\eta}_j \mid i<2, \eta\in^{n_j}2\rangle$, $\langle q^{i,\eta}_{j,\ell}\mid \ell<\ell_j i<2,\eta\in{}^{n_j}2\rangle$ and $\langle v^{i,\eta}_{j,\ell}\mid \ell<\ell_j, i<2,\eta\in{}^{n_j}2\rangle$ such that

(1) $n_j<m_j<n_{j+1}$,
(2) m_jH_j, $\langle q^{i,\eta}_j \mid i<2,\eta\in{}^{n_j}2\rangle, \langle q^{i,\eta}_{j,\ell}\mid \ell<\ell^{\eta,i}_j, i<2,\eta\in{}^{n_j}2\rangle$ and $\langle v^{i,\eta}_{j,\ell}\mid \ell<\ell^{i,\eta}_j, i<2,\eta\in{}^{n_j}2\rangle$ are given by Claim 3.8 for $n=n_j$ and $\mathcal{E}=1/2^{2^{n_j}}$,
(3) length $\left(v^{i,\eta}_{j,\ell}\right)<n_{j+1}$ for every $i<2$, $\eta\in{}^{n_j}2, \ell<\ell^{i,\eta}_j$.

Now define a Q_1-name $\underset{\sim}{\sigma}\in{}^{\omega}2$ by setting

$$\underset{\sim}{\sigma}(j)=H_j\left(\underset{\sim}{\tau}\cap{}^{m_j\geq}2\right).$$

Claim 3.9: $\Vdash_Q$ "$\underset{\sim}{\sigma}$ *is a Cohen real over* V".

Proof: It is enough to show the following:

for every $\mathcal{E} > 0$ and $\eta^* \in {}^{\omega>}2$ the following holds:

for every $j < \omega$ large enough and $\nu \in {}^{\omega>}2$ of length $> j$, there are $q, \langle q_\ell \mid \ell < \ell^* \rangle$ and $\langle v_\ell \mid \ell < \ell^* \rangle$ such that

(a) $q, \langle q_\ell \mid \ell < \ell^* \rangle$ are in Random,

(b) $q = \bigcup_{\ell<\ell^*} q_\ell$,

(c) $\mathrm{Lb}(q) \geq 1 - \mathcal{E}$,

(d) $v_\ell \trianglelefteq \eta^*$ for every $\ell < \ell^*$,

(e) $(q_\ell, v_\ell) \Vdash_Q$ "$\underset{\sim}{x} \restriction [j, \text{ length } \nu) = \nu \restriction [j, \text{ length } \nu)$".

Proof of $(*)$: Pick $j < \omega$ such that $n_j >$ length η^* and $2^{-j} < \mathcal{E}/2$. Let ν be given. We choose by induction on $k \in [j, \text{ length } \nu)$ a set a_k and $\langle q_\eta \mid \eta \in a_k \rangle$ such that

(a) $a_k \subseteq {}^{n_k}2$ is nonempty,

(b) a_j is a singleton extending η^*,

(c) $\forall \eta \in a_{k+1} (\eta \restriction n_k \in a_k)$ and $\forall \eta \in a_k \; \exists \eta' \in a_{k+1} \; (\eta \triangleleft \eta')$,

(d) for every $\eta \in a_k$

$$(q_\eta, \eta) \Vdash \text{“} \bigwedge_{\ell=j}^{k-1} H_\ell \left(\underset{\sim}{x} \cap {}^{m_\ell \geq}2 \right) = \nu(\ell) \text{”}$$

i.e. $(q_\eta, \eta) \Vdash$ "$\underset{\sim}{x} \restriction [j, k-1) = \nu \restriction [j, k-1)$"

(e) for $\eta \in a_j$, $q_\eta = {}^\omega 2$,

(f) for $\eta \in a_k$, $\langle q_\rho \mid \eta \triangleleft \rho \in a_{k+1} \rangle$ is an antichain of Random above q_η and $\sum \{\mathrm{Lb}(q_\rho) \mid \eta \triangleleft \rho \in a_{k+1}\} / \mathrm{Lb}(q_\eta) \geq 1 - 1/2^k$.

There is no problem in caring on this induction. This completes the proof of $(*)$ and hence also the theorem. ∎

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