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More on cardinal invariants of Boolean algebras

Andrzej Rosłanowski[a,b,*], Saharon Shelah[a,c,1]

[a] *Institute of Mathematics, The Hebrew University, 91 904 Jerusalem, Israel*

[b] *Mathematical Institute, Wrocław University, 50 384 Wrocław, Poland*

[c] *Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA*

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Abstract

We address several questions of Donald Monk related to irredundance and spread of Boolean algebras, gaining both some ZFC knowledge and consistency results. We show in ZFC that $\mathrm{irr}(\mathbb{B}_0\times\mathbb{B}_1)=\max\{\mathrm{irr}(\mathbb{B}_0),\mathrm{irr}(\mathbb{B}_1)\}$. We prove consistency of the statement "there is a Boolean algebra $\mathbb{B}$ such that $\mathrm{irr}(\mathbb{B})<s(\mathbb{B}\circledast\mathbb{B})$" and we force a superatomic Boolean algebra $\mathbb{B}_*$ such that $s(\mathbb{B}_*)=\mathrm{inc}(\mathbb{B}_*)=\kappa$, $\mathrm{irr}(\mathbb{B}_*)=\mathrm{Id}(\mathbb{B}_*)=\kappa^+$ and $\mathrm{Sub}(\mathbb{B}_*)=2^{\kappa^+}$. Next we force a superatomic algebra $\mathbb{B}_0$ such that $\mathrm{irr}(\mathbb{B}_0)<\mathrm{inc}(\mathbb{B}_0)$ and a superatomic algebra $\mathbb{B}_1$ such that $\mathrm{t}(\mathbb{B}_1)>\mathrm{Aut}(\mathbb{B}_1)$. Finally we show that consistently there is a Boolean algebra $\mathbb{B}$ of size λ such that there is no free sequence in $\mathbb{B}$ of length λ, there is an ultrafilter of tightness λ (so $\mathrm{t}(\mathbb{B})=\lambda$) and $\lambda\notin\mathrm{Depth}_{\mathrm{Hs}}(\mathbb{B})$.

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0. Introduction

In the present paper we answer (sometimes partially only) several questions of Donald Monk concerning cardinal invariants of Boolean algebras. Most of our results are consistency statements, but we get some ZFC knowledge too.

For a systematic study and presentation of current research on cardinal invariants of Boolean algebras (as well as for a long list of open problems) we refer the reader to Monk [11]. In Section 1 we recall most of the needed definitions giving some pointers to relevant results from the literature.

* Corresponding author. Department of Mathematics, University of Nebraska at Omaha, Omaha, NE 68182, USA.

E-mail address: roslanowski@unomaha.edu (A. Rosłanowski).

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In Section 2 we show that the difference between $s_n(\mathbb{B})$ and $s_N(\mathbb{B})$ (for $n<N$) can be reasonably large, with the only restriction coming from the inequality $s_n(\mathbb{B}) \geqslant 2^{s_N}(\mathbb{B})$ (a consistency result; for the definitions of the invariants see Section 1). It is relevant for the description of the behaviour of spread in ultraproducts: we may conclude that it is consistent that $s(\prod_{n\in\omega} \mathbb{B}_n/D)$ is much larger than $\prod_{n\in\omega} s(\mathbb{B}_n)/D$. In the following section we answer [11, Problem 24] showing that $\mathrm{irr}(\mathbb{B}_0 \times \mathbb{B}_1) = \max\{\mathrm{irr}(\mathbb{B}_0), \mathrm{irr}(\mathbb{B}_1)\}$ (a ZFC result). A partial answer to [11, Problem 27] is given in Section 4, where we show that, consistently, there is a Boolean algebra $\mathbb{B}$ such that $\mathrm{irr}(\mathbb{B}) < s(\mathbb{B} \circledast \mathbb{B})$. In particular, this shows that the parallel statement to the result of Section 3 for free product may fail. Note that proving the result of Section 4 in ZFC is a really difficult task, as so far we even do not know if (in ZFC) there are Boolean algebras $\mathbb{B}$ satisfying $\mathrm{irr}(\mathbb{B}) < |\mathbb{B}|$. In Section 5 we force a superatomic Boolean algebra $\mathbb{B}$ such that $s(\mathbb{B}) = \mathrm{inc}(\mathbb{B}) = \kappa$, $\mathrm{irr}(\mathbb{B}) = \mathrm{Id}(\mathbb{B}) = \kappa^+$ and $\mathrm{Sub}(\mathbb{B}) = 2^{\kappa^+}$. This gives answers to [11, Problems 73,77,78] as stated (though the problems in ZFC remain open). Next we present some modifications of this forcing notion and in Section 6 we answer [11, Problems 79,81] forcing superatomic Boolean algebras $\mathbb{B}_0, \mathbb{B}_1$ such that $\mathrm{irr}(\mathbb{B}_0) < \mathrm{inc}(\mathbb{B}_0)$ and $\mathrm{Aut}(\mathbb{B}_1) < \mathrm{t}(\mathbb{B}_1)$. Finally, in the last section, we show that (consistently) there is a Boolean algebra $\mathbb{B}$ of size λ such that there is no free sequence in $\mathbb{B}$ of length λ, there is an ultrafilter in $\mathrm{Ult}(\mathbb{B})$ of tightness λ (so $\mathrm{t}(\mathbb{B}) = \lambda$) and $\lambda \notin \mathrm{Depth}_{\mathrm{Hs}}(\mathbb{B})$. This gives answers to [11, Problems 13,41]. Lastly, we use one of the results of [17] to show that $2^{\mathrm{cf}(\mathrm{t}(\mathbb{B}))} < \mathrm{t}(\mathbb{B})$ implies $\mathrm{t}(\mathbb{B}) \in \mathrm{Depth}_{\mathrm{Hs}}(\mathbb{B})$.

Notation. Our notation is rather standard and compatible with that of classical textbooks on set theory (like [6]) and Boolean algebras (like [10, 11]). However in forcing considerations we keep the older tradition that

the stronger condition is the greater one

Let us list some of our notation and conventions.

Notation 0.1. (1) A name for an object in a forcing extension is denoted with a dot above (like $\dot{X}$) with one exception: the canonical name for a generic filter in a forcing notion $\mathbb{P}$ will be called $\Gamma_{\mathbb{P}}$.

(2) $\alpha, \beta, \gamma, \delta, \ldots$ will denote ordinals and $\kappa, \mu, \lambda, \theta$ will stand for (always infinite) cardinals.

(3) For a set X and a cardinal λ, $[X]^{<\lambda}$ stands for the family of all subsets of X of size less than λ. If X is a set of ordinals then its order type is denoted by $\mathrm{otp}(X)$.

(4) In Boolean algebras we use $\vee$ (and $\bigvee$), $\wedge$ (and $\bigwedge$) and $-$ for the Boolean operations. If $\mathbb{B}$ is a Boolean algebra, $x \in \mathbb{B}$ then $x^0 = x$, $x^1 = -x$. The Stone space of the algebra $\mathbb{B}$ is called $\mathrm{Ult}(\mathbb{B})$. All Boolean algebras under considerations are assumed to be infinite.

(5) The subalgebra of $\mathbb{B}$ generated by a set $Y \subseteq \mathbb{B}$ is denoted by $\langle Y\rangle_{\mathbb{B}}$.

(6) The sign $\circledast$ stands for the operation of the free product of Boolean algebras and the product is denoted by $\times$.

1. The invariants

Here we recall definitions of relevant cardinal functions on Boolean algebras and their basic properties. Let us start with introducing the formalism from [13, Section 1].

Definition 1.1 (*see Roslanowski and Shelah* [13, *Definition* 1.1]). (1) For a (not necessary first order) theory T in the language of Boolean algebras plus one distinguished unary predicate P_0 plus, possibly, some others $P_1, P_2, \dots$ we define cardinal invariants inv_T, inv_T^+ of Boolean algebras by (for a Boolean algebra $\mathbb{B}$):

$$\mathrm{inv}_T(\mathbb{B}) \stackrel{\mathrm{def}}{=} \sup\{|P_0|\colon (B, P_n)_n \text{ is a model of } T\},$$
$$\mathrm{inv}_T^+(\mathbb{B}) \stackrel{\mathrm{def}}{=} \sup\{|P_0|^+\colon (B, P_n)_n \text{ is a model of } T\}.$$

(2) A theory T is *universal in* (P_0, P_1) if all sentences $\phi \in T$ are of the form

$$(\forall x_1, \dots, x_n \in P_0)\psi(\bar{x}),$$

where all occurrences of $x_1, \dots, x_n$ in ψ are free and P_0 does not appear there and any appearance of P_1 in ψ is in the form $P_1(x_{i_0}, \dots, x_{i_k})$ with no more complicated terms.

(3) The invariant $\mathrm{inv}_T^{(+)}$ is called *def.f.o.car. invariant* (definable first order cardinal invariant) if T is first order. If T is universal in (P_0, P_1) and first order except the demand that P_1 is a well ordering of P_0, then we $\mathrm{inv}_T^{(+)}$ is called *def.u.w.o.car. invariant* (definable universal well-ordered cardinal invariant).

One of the reasons for introducing the formalism of Definition 1.1 was that it allows a uniform approach to the questions concerning the behaviour of the respective invariants when we consider ultraproducts of Boolean algebras. For a systematic study of that topic we refer the reader to [13, Section 1], here let us only recall the use of "finite" versions of $\mathrm{inv}_T^{(+)}$ for infinite theories T.

Proposition 1.2 (see Roslanowski and Shelah [13, Conclusion 1.11]). *Suppose that $T = \{\phi_n\colon n < \omega\}$ and if T is supposed to describe a def.u.w.o.car. invariant, then ϕ_0 already says that P_1 is a well ordering of P_0. Let $T^n = \{\phi_m\colon m < n\}$ for $n < \omega$. Assume that D is a uniform ultrafilter on κ, $f\colon \kappa \to \omega$ is such that* $\lim_D f = \omega$. *Let $\mathbb{B}_i$ (for $i < \kappa$) be Boolean algebras, $\mathbb{B} = \prod_{i<\kappa} \mathbb{B}_i / D$.*
1. *If T is first order then* $\prod_{i<\kappa} \mathrm{inv}^+_{T^{f(i)}}(\mathbb{B}_i)/D \leqslant \mathrm{inv}_T^+(\mathbb{B})$.
2. *If T is u.w.o. then* $\mathrm{Depth} \prod_{i<\kappa}(\mathrm{inv}^+_{T^{f(i)}}(\mathbb{B}_i), <)/D \leqslant \mathrm{inv}_T^+(\mathbb{B})$.

It should be pointed out that both the approaches presented in Definition 1.1 (so that of [13, Section 1]) and "finite" versions of cardinal functions on Boolean algebras were present (in some sense) in the literature some time ago already; see for example [3, p. 432; 9].

Many of the cardinal functions on Boolean algebras can be described as either def.f.o.car. invariants or def.u.w.o.car. invariants. Some of them originated in cardinal functions on topological spaces and have various equivalent definitions in the Boolean algebraic context. In these cases the main difficulty in giving the suitable description is in choosing the right definition of the function.

1.1. Variations of independence

We start reviewing cardinal functions on Boolean algebras with those which van Douwen called *variations of independence* (see [3, Section 7]). The first two members of this family, *the independence number* ind and *the incomparability number* inc, have clear representations as def.f.o.car. invariants.

Definition 1.3. (1) $\phi_n^{\rm ind}$ is the formula which says that any non-trivial Boolean combination of $n+1$ elements of P_0 is non-zero (i.e. $\phi_n^{\rm ind}$ says that if $x_0,\ldots,x_n$ are pairwise distinct elements of P_0 then $\bigwedge_{l\leqslant n} x_l^{t(l)} \neq 0$ for each $t \in {}^{n+1}2$).

(2) For $0<n\leqslant\omega$ let $T^n_{\rm ind}=\{\phi_k^{\rm ind}\colon k<n\}$.

(3) For a Boolean algebra $\mathbb{B}$, $0<n\leqslant\omega$ we define ${\rm ind}_n(\mathbb{B})={\rm inv}_{T^n_{\rm ind}}(\mathbb{B})$ and ${\rm ind}^+_n(\mathbb{B})={\rm inv}^+_{T^n_{\rm ind}}(\mathbb{B})$. We will denote ${\rm ind}^{(+)}_\omega$ by ${\rm ind}^{(+)}$ too.

(4) Let $\phi^{\rm inc}\equiv(\forall x,y\in P_0)(x\neq y\Rightarrow(x\nleq y\ \&\ y\nleq x))$ and let $T_{\rm inc}=\{\phi^{\rm inc}\}$. For a Boolean algebra $\mathbb{B}$ we define ${\rm inc}(\mathbb{B})={\rm inv}_{T_{\rm inc}}(\mathbb{B})$ and ${\rm inc}^+(\mathbb{B})={\rm inv}^+_{T_{\rm inc}}(\mathbb{B})$.

The behaviour of the n-independence number ${\rm ind}_n$ seems to depend on the parity of n, see [13, Section 4.1]. This suggests that similar phenomena could occur for "finite" versions of other cardinal invariants too.

Other cardinal functions from this family are s, hd, hL and t. They all have origins in the topological context and our understanding of them in the context of Boolean algebras is due to Arhangelskii, van Douwen, Monk, Shapirovskii and others (see [3, Sections 7–9] and the respective chapters in [11] for the exact references and results). We should note here that it is not clear if the finite versions of these invariants (all defined below) have any topological meaning.

The spread $s(\mathbb{B})$ of a Boolean algebra $\mathbb{B}$ is

$$\sup\{|X|\colon X\subseteq {\rm Ult}(\mathbb{B}) \text{ is discrete in the relative topology}\}.$$

However, because of Definition 1.1, we prefer to think of the spread as

$$s(\mathbb{B})=\sup\{|X|\colon X\subseteq B \text{ is ideal-independent}\}$$

(it is one of the equivalent definitions, see [11, Theorem 13.1]). Thus we can write $s(\mathbb{B})=s_\omega(\mathbb{B})$, where

Definition 1.4. (1) ϕ_n^s is the formula saying that no member of P_0 can be covered by union of $n+1$ other elements of P_0.

(2) For $0<n\leqslant\omega$ let $T^n_s=\{\phi^s_k\colon k<n\}$.

(3) For a Boolean algebra $\mathbb{B}$ and $0<n\leqslant\omega$: $s_n^{(+)}(\mathbb{B})={\rm inv}^{(+)}_{T^n_s}(\mathbb{B})$. (So these are def.f.o.car. invariants.)

The hereditary density $\mathrm{hd}(\mathbb{B})$ and *the hereditary Lindelöf degree* $\mathrm{hL}(\mathbb{B})$ of a Boolean algebra $\mathbb{B}$ are treated in a similar manner. We use [11, Theorem 16.1; 11, Theorem 15.1] to define them as

$$\mathrm{hd}(\mathbb{B}) = \sup\{|\kappa|: \text{there is a strictly decreasing } \kappa\text{-sequence of ideals (in } \mathbb{B})\},$$
$$\mathrm{hL}(\mathbb{B}) = \sup\{|\kappa|: \text{there is a strictly increasing } \kappa\text{-sequence of ideals (in } \mathbb{B})\}.$$

This leads us directly to the following definition.

Definition 1.5. (1) Let the formula ψ say that P_1 is a well ordering of P_0 (denoted by $<_1$). For $n<\omega$ let ϕ_n^{hd}, ϕ_n^{hL} be the following formulas:

$$\phi_n^{\mathrm{hd}} \equiv \psi \;\&\; (\forall x_0,\ldots,x_{n+1} \in P_0)(x_0 <_1 \cdots <_1 x_{n+1} \Rightarrow x_0 \nleq x_1 \vee \cdots \vee x_{n+1}),$$
$$\phi_n^{\mathrm{hL}} \equiv \psi \;\&\; (\forall x_0,\ldots,x_{n+1} \in P_0)(x_{n+1} <_1 \cdots <_1 x_0 \Rightarrow x_0 \nleq x_1 \vee \cdots \vee x_{n+1}).$$

(2) For $0<n \leqslant \omega$ we let $T^n_{\mathrm{hd}} = \{\phi_k^{\mathrm{hd}}: k<n\}$, $T^n_{\mathrm{hL}} = \{\phi_k^{\mathrm{hL}}: k<n\}$.
(3) For a Boolean algebra $\mathbb{B}$ and $0<n \leqslant \omega$:

$$\mathrm{hd}_n^{(+)}(\mathbb{B}) = \mathrm{inv}^{(+)}_{T^n_{\mathrm{hd}}}(\mathbb{B}), \qquad \mathrm{hL}_n^{(+)}(\mathbb{B}) = \mathrm{inv}^{(+)}_{T^n_{\mathrm{hL}}}(\mathbb{B}).$$

[So these are def.u.w.o.car. invariants, $\mathrm{hL}(\mathbb{B})$ is $\mathrm{hL}_\omega(\mathbb{B})$ and $\mathrm{hd}(\mathbb{B})$ is $\mathrm{hd}_\omega(\mathbb{B})$.]

We use the following characterization of *tightness* (see [11, Theorem 4.20, pp. 157–158; Section 12], also [3, Theorems 7.2, 8.7]):

$$\mathrm{t}(\mathbb{B}) = \sup\{|\alpha|: \text{there exists a free sequence of the length } \alpha \text{ in } \mathbb{B}\}.$$

Definition 1.6. (1) Let ψ be the sentence saying that P_1 is a well ordering of P_0 (we denote the respective order by $<_1$). For $k,l<\omega$ let $\phi^{\mathrm{t}}_{k,l}$ be the sentence asserting that for each $x_0,\ldots,x_k,y_0,\ldots,y_l \in P_0$ if $x_0 <_1 \cdots <_1 x_k <_1 y_0 <_1 \cdots <_1 y_l$ then $\bigwedge_{i\leqslant k} x_i \nleq \bigvee_{i\leqslant l} y_i$.
(2) For $n,m \leqslant \omega$ let $T^{n,m}_{\mathrm{t}} = \{\phi^{\mathrm{t}}_{k,l}: k<n, l<m\} \cup \{\psi\}$ and for a Boolean algebra $\mathbb{B}$ let $\mathrm{t}_{n,m}(\mathbb{B}) = \mathrm{inv}_{T^{n,m}_{\mathrm{t}}}(\mathbb{B})$.
[So these are def.u.w.o.car. invariants, $\mathrm{t}(\mathbb{B})$ is $\mathrm{t}_{\omega,\omega}(\mathbb{B})$.]

Note that

Proposition 1.7. *For a Boolean algebra* $\mathbb{B}$

$$\mathrm{t}^{(+)}_{1,N} = \mathrm{hd}^{(+)}_N(\mathbb{B}) \quad \textit{and} \quad \mathrm{t}^{(+)}_{N,1}(\mathbb{B}) = \mathrm{hL}^{(+)}_N(\mathbb{B})$$

and $\mathrm{ind}^{(+)}(\mathbb{B}) \leqslant s^{(+)}(\mathbb{B}) \leqslant \mathrm{hd}^{(+)}(\mathbb{B})$, $\mathrm{hL}^{(+)}(\mathbb{B})$.

1.2. Irredundance and other invariants

One of the most mysterious cardinal functions on Boolean algebras is *the irredundance*. For a Boolean algebra $\mathbb{B}$, $\mathrm{irr}(\mathbb{B})$ is the supremum of cardinalities of sets $X \subseteq \mathbb{B}$ such that $(\forall x \in X)(x \notin \langle X \setminus \{x\}\rangle_{\mathbb{B}})$. Thus we have the following definition.

Definition 1.8 (*Compare Monk* [11, *p.* 144]). Let $n \leqslant \omega$ and let $T^n_{\rm irr}$ be the theory of the language of Boolean algebras plus a predicate P_0, which says that for each $m < n$ and a Boolean term $\tau(y_0,\dots,y_m)$ we have

$$(\forall x \in P_0)(\forall x_0,\dots,x_m \in P_0 \setminus \{x\})(x \neq \tau(x_0,\dots,x_m)).$$

For Boolean algebra $\mathbb{B}$ we define $\mathrm{irr}^{(+)}_n(\mathbb{B}) = \mathrm{inv}^{(+)}_{T^n_{\rm irr}}(\mathbb{B})$. [So these are def.f.o.car. invariants; $\mathrm{irr}_\omega(\mathbb{B}) = \mathrm{irr}(\mathbb{B})$).]

There is a number of open problems concerning the irredundance number (see, e.g., [11, Ch. 8]). The most basic of them is if, in ZFC, there is a Boolean algebra $\mathbb{B}$ such that $\mathrm{irr}(\mathbb{B}) < |\mathbb{B}|$. A number of related consistency results has been known already. The first example of an algebra $\mathbb{B}$ such that $\mathrm{irr}(\mathbb{B}) < |\mathbb{B}|$ was constructed by Rubin [15] under the assumption of $\diamondsuit$. Todorčević [19] showed that the proper forcing axiom implies that every uncountable Boolean algebra contains an uncountable irredundant subset. We refer the reader to [11, Ch. 8] (specially p. 134 there) for more on the history and the relevant results.

The depth $\mathrm{Depth}(\mathbb{B})$ of a Boolean algebra $\mathbb{B}$ is

$$\sup\{|X|: X \subseteq \mathbb{B} \text{ is well-ordered by the Boolean ordering}\}.$$

It should be clear that the depth can be represented as a def.u.w.o.car. invariant.

The number of ideals in $\mathbb{B}$ is denoted by $\mathrm{Id}(\mathbb{B})$, $\mathrm{Aut}(\mathbb{B})$ stands for *the number of automorphisms* of the algebra $\mathbb{B}$, and *the number of subalgebras* of $\mathbb{B}$ is denoted by $\mathrm{Sub}(\mathbb{B})$.

2. Forcing for spread

The aim of this section is to show that for N much larger than n, the following inequalities seem to be the only restriction on the jumps between s_N and s_n:

$$s(\mathbb{B}) \leqslant s_N(\mathbb{B}) \leqslant s_n(\mathbb{B}) \leqslant |\mathbb{B}| \leqslant 2^{s(\mathbb{B})} \leqslant 2^{s_N(\mathbb{B})}$$

(remember that by a theorem of Shapirovskii, $|\mathbb{B}| \leqslant 2^{s(\mathbb{B})}$, see [3, Theorem 10.9], also [11, Theorem 13.6]). The forcing notion defined in 2.1(2) below is a modification of the one from [18, Section 2] and a relative of the forcing notion from [16, Section 15].

Definition 2.1. (1) For a set w and a family $F \subseteq 2^w$ we define

$$\mathrm{cl}(F) = \{g \in 2^w: (\forall u \in [w]^{<\omega})(\exists f \in F)(f \restriction u = g \restriction u),$$

$\mathbb{B}_{(w,F)}$ is the Boolean algebra generated freely by $\{x_\alpha: \alpha \in w\}$ except that if $u_0, u_1 \in [w]^{<\omega}$ and there is no $f \in F$ such that $f \restriction u_0 \equiv 0$, $f \restriction u_1 \equiv 1$ then $\bigwedge_{\alpha \in u_1} x_\alpha \wedge \bigwedge_{\alpha \in u_0} (-x_\alpha) = 0$.

2. Let $\mu \leqslant \lambda$ be cardinals, $0 < n < \omega$. We define forcing notion $\mathbb{Q}^{\mu,\lambda}_n$: *a condition* is a pair $p = (w^p, F^p)$ such that $w^p \in [\lambda]^{<\mu}$, $F^p \subseteq 2^{w^p}$, $|F^p| < \mu$ and for every $u \in [w^p]^{\leqslant n}$

there is $f^*: w^p\setminus u\to 2$ such that if $h: u\to 2$ then $f^*\cup h\in F^p$; *the order* is given by $p\leqslant q$ if and only if $w^p\subseteq w^q$ and

$$(\forall f\in F^q)(f\restriction w^p\in \mathrm{cl}(F^p))\quad\text{and}\quad(\forall f\in F^p)(\exists g\in F^q)(f\subseteq g).$$

Proposition 2.2 (see Shelah [18, 2.6]). *(1) If $p\in\mathbb{Q}_n^{\mu,\lambda}$, $f\in F^p$ then f extends to a homomorphism from $\mathbb{B}_p$ to $\{0,1\}$ (i.e. it preserves the equalities from the definition of $\mathbb{B}_p$).*

(2) If $p\in\mathbb{Q}_n^{\mu,\lambda}$, $\tau(y_0,\dots,y_k)$ is a Boolean term and $\alpha_0,\dots,\alpha_k\in w^p$ are distinct then

$$\mathbb{B}_p\models\tau(x_{\alpha_0},\dots,x_{\alpha_k})\neq 0\quad\textit{if and only if}$$

$$(\exists f\in F^p)(\{0,1\}\models\tau(f(\alpha_0),\dots,f(\alpha_k))=1).$$

(3) If $p,q\in\mathbb{Q}_n^{\mu,\lambda}$, $p\leqslant q$ then $\mathbb{B}_p$ is a subalgebra of $\mathbb{B}_q$.

Proposition 2.3. *Assume $\mu^{<\mu}=\mu\leqslant\lambda$, $0<n<\omega$. Then*
1. *$\mathbb{Q}_n^{\mu,\lambda}$ is a μ-complete forcing notion of size $\lambda^{<\mu}$,*
2. *$\mathbb{Q}_n^{\mu,\lambda}$ satisfies μ^+-cc.*

Proof. This is almost exactly like [18, 2.7]. For (1) no changes are required; for (2) one has to check that the condition defined as there is really in $\mathbb{Q}_n^{\mu,\lambda}$. So suppose that $\langle p_\alpha:\ \alpha<\mu^+\rangle\subseteq\mathbb{Q}_n^{\mu,\lambda}$. Applying standard "cleaning procedure" find $\alpha_0<\alpha_1<\mu^+$ such that
- $\mathrm{otp}(w^{p_{\alpha_0}})=\mathrm{otp}(w^{p_{\alpha_1}})$,
- if $H: w^{p_{\alpha_0}}\to w^{p_{\alpha_1}}$ is the order preserving mapping then $H\restriction(w^{p_{\alpha_0}}\cap w^{p_{\alpha_1}})$ is the identity on $w^{p_{\alpha_0}}\cap w^{p_{\alpha_1}}$ and $F^{p_{\alpha_0}}=\{f\circ H:\ f\in F^{p_{\alpha_1}}\}$

(remember $\mu^{<\mu}=\mu$; use the Δ-system lemma). Let $w^q=w^{p_{\alpha_0}}\cup w^{p_{\alpha_1}}$ and

$$F^q=\{f\cup g:\ f\in F^{p_{\alpha_0}}\ \&\ g\in F^{p_{\alpha_1}}\ \&\ f\restriction(w^{p_{\alpha_0}}\cap w^{p_{\alpha_1}})=g\restriction(w^{p_{\alpha_0}}\cap w^{p_{\alpha_1}})\}.$$

To check that $q=(w^q,F^q)$ is in $\mathbb{Q}_n^{\mu,\lambda}$ suppose that $u\in[w^q]^{\leqslant n}$ and let $u^*=H^{-1}[u\cap w^{p_{\alpha_1}}]\cup(u\cap w^{p_{\alpha_0}})\in[w^{p_{\alpha_0}}]^{\leqslant n}$. Let $f_0^*: w^{p_{\alpha_0}}\setminus u^*\to 2$ be such that if $h: u^*\to 2$ then $f_0^*\cup h\in F^{p_{\alpha_0}}$. Next, let $f^*: w^{p_{\alpha_0}}\setminus u\to 2$ be such that $f_0^*\subseteq f^*$ and if $\alpha\in u^*\setminus u$ then $f^*(\alpha)=0$, and let $g^*: w^{p_{\alpha_1}}\setminus u\to 2$ be such that $f_0^*\circ H^{-1}\subseteq g^*$ and if $\alpha\in H[u^*]\setminus u$ then $g^*(\alpha)=0$. Now it should be clear that

$$\text{if}\quad h: u\to 2\qquad\text{then }(f^*\cup g^*)\cup h\in F^q.$$

Verifying that both $p_{\alpha_0}\leqslant q$ and $p_{\alpha_1}\leqslant q$ is even easier. $\square$

Let $\dot{\mathbb{B}}$ be the $\mathbb{Q}_n^{\mu,\lambda}$-name for $\bigcup\{\mathbb{B}_p: p\in\Gamma_{\mathbb{Q}_n^{\mu,\lambda}}\}$. It follows from Proposition 2.2 that

$$\Vdash_{\mathbb{Q}_n^{\mu,\lambda}}\text{“}\dot{\mathbb{B}}\text{ is a Boolean algebra generated by }\{x_\alpha:\ \alpha<\lambda\}\text{”}$$

and, for a condition $p\in\mathbb{Q}_n^{\mu,\lambda}$,

$$p\Vdash_{\mathbb{Q}_n^{\mu,\lambda}}\text{“}\langle x_\alpha:\ \alpha\in w^p\rangle_{\dot{\mathbb{B}}}=\mathbb{B}_p\text{”}.$$

Theorem 2.4. *Assume $\mu^{<\mu}=\mu\leqslant\lambda$ and $1<n<N<\omega$ are such that $2^{n/2}+n\leqslant N$. Then*

$$\Vdash_{\mathbb{Q}_n^{\mu,\lambda}} \text{“}\mathrm{ind}_n^+(\dot{\mathbb{B}})=\lambda^+ \quad \textit{and}\ \mathrm{t}_{1,N}^+(\dot{\mathbb{B}})=\mathrm{t}_{N,1}^+(\dot{\mathbb{B}})=\mathrm{ind}^+(\dot{\mathbb{B}})=\mu^+\text{”},$$

and hence $\Vdash_{\mathbb{Q}_n^{\mu,\lambda}}$ *“$s_{n-1}^+(\dot{\mathbb{B}})=\lambda^+$ and $s_N^+(\dot{\mathbb{B}})=\mu^+$”.*

Proof. It follows immediately from the definition of $\mathbb{Q}_n^{\mu,\lambda}$ (by density arguments, remembering Proposition 2.2) that

$$\Vdash_{\mathbb{Q}_n^{\mu,\lambda}} \text{“the sequence } \langle x_\alpha\colon \alpha<\lambda\rangle \text{ is } n\text{-independent”.}$$

Suppose now that $\langle\dot{a}_\beta\colon \beta<\mu^+\rangle$ is a $\mathbb{Q}_n^{\mu,\lambda}$-name for a μ^+-sequence of elements of $\dot{\mathbb{B}}$, $p\in\mathbb{Q}_n^{\mu,\lambda}$. For each $\beta<\mu^+$ choose a condition $p_\beta\geqslant p$, a Boolean term τ_β and ordinals $\bar{\alpha}(\beta,0)<\cdots<\bar{\alpha}(\beta,\ell_\beta)<\lambda$ such that

$$p_\beta \Vdash_{\mathbb{Q}_n^{\mu,\lambda}} \dot{a}_\beta=\tau_\beta(x_{\bar{\alpha}(\beta,0)},\ldots,x_{\bar{\alpha}(\beta,\ell_\beta)}).$$

By Δ-system arguments, passing to a subsequence and increasing p_β's, we may assume that

(i) $\tau_\beta=\tau$, $\ell_\beta=\ell$ and $\bar{\alpha}(\beta,0),\ldots,\bar{\alpha}(\beta,\ell)\in w^{p_\beta}$,

(ii) $\mathrm{otp}(w^{p_{\beta_0}})=\mathrm{otp}(w^{p_{\beta_1}})$ and $\mathrm{otp}(w^{p_{\beta_0}}\cap\bar{\alpha}(\beta_0,j))=\mathrm{otp}(w^{p_{\beta_1}}\cap\bar{\alpha}(\beta_1,j))$ for $j\leqslant\ell$, $\beta_0,\beta_1<\mu^+$,

(iii) $\{w^{p_\beta}\colon \beta<\mu^+\}$ forms a Δ-system of sets with heart w^*,

(iv) if $H_{\beta_0,\beta_1}: w^{p_{\beta_0}}\longrightarrow w^{p_{\beta_1}}$ is the order preserving mapping then $H_{\beta_0,\beta_1}\restriction w^*$ is the identity on w^* and $F^{p_{\beta_0}}=\{f\circ H_{\beta_0,\beta_1}\colon f\in F^{p_{\beta_1}}\}$.

After this “cleaning procedure” look at the conditions $p_0,\ldots,p_N$. We want to show that they have a common upper bound $q\in\mathbb{Q}_n^{\mu,\lambda}$ such that $q\Vdash_{\mathbb{Q}_n^{\mu,\lambda}}$ “$\dot{a}_0\wedge\bigwedge_{j<N}(-\dot{a}_{1+j})=0$”. To this end define

$$w^q=w^{p_0}\cup\cdots\cup w^{p_N}$$

and

$$\begin{aligned} F^q=\ &\{f_0\cup\cdots\cup f_N\colon f_0\in F^{p_0},\ldots,f_N\in F^{p_N},\ f_0\restriction w^*=\cdots=f_N\restriction w^*,\\ &\quad \text{and if } \{0,1\}\models\tau(f_0(\bar{\alpha}(0,0)),\ldots,f_0(\bar{\alpha}(0,\ell)))=1\\ &\quad \text{then for some } j\in[1,N]\\ &\quad \{0,1\}\models\tau(f_j(\bar{\alpha}(j,0)),\ldots,f_j(\bar{\alpha}(j,\ell)))=1\}. \end{aligned}$$

Let us check that $q=(w^q,F^q)$ is in $\mathbb{Q}_n^{\mu,\lambda}$. Clearly each $f\in F^q$ is a function from w^q to 2 and $|F^q|<\mu$. Suppose now that $u\in[w^q]^{\leqslant n}$. Let $u^*=u\cap w^*$ and $u^+=\bigcup_{i\leqslant N}H_{i,0}[u\cap w^{p_i}]\in[w^{p_0}]^{\leqslant n}$. One of the sets u^*, $u^+\setminus u^*$ has size at most $n/2$, and first we deal with the case $|u^*|\leqslant n/2$. Choose $f^*: w^{p_0}\setminus u^+\longrightarrow 2$ such that

$(\forall h: u^+ \to 2)(f^* \cup h \in F^{p_0})$. For each $v \subseteq u^*$ choose $h_v : u^+ \to 2$ such that $h_v \restriction v \equiv 1$, $h_v \restriction (u^* \setminus v) \equiv 0$ and

> if there is $h: u^+ \to 2$ satisfying the above demands and such that
> $\{0,1\} \models \tau((f^* \cup h)(\bar{\alpha}(0,0)), \ldots, (f^* \cup h)(\bar{\alpha}(0,\ell))) = 1$
> then h_v has this property.

Since $2^{|u^*|} + n \leqslant N$ we may choose distinct $i_v \in [1,N]$ for $v \subseteq u^*$ such that $w^{p_{i_v}} \cap u = u^*$. Now we define functions $f_i^* : w^{p_i} \setminus u \to 2$ (for $i \leqslant N$) as follows:

- if $i = i_v$, $v \subseteq u^*$ then $f_i^* = (f^* \cup h_v) \circ H_{i,0}(w^{p_i} \setminus u)$,
- if $i \notin \{i_v : v \subseteq u^*\}$ then $f_i^* \supseteq f^* \circ H_{i,0}$ is such that $f_i^*(\alpha) = 0$ for all $\alpha \in H_{i,0}^{-1}[u^+] \setminus u$.

Suppose that $h: u \to 2$ and let $f_i = f_i^* \cup (h \restriction (u \cap w^{p_i}))$. It should be clear that for each $i \leqslant N$ we have $f_i \in F^{p_i}$ and $f_i \restriction w^* = f_0 \restriction w^*$ (remember the choice of f^*). Assume that $\{0,1\} \models \tau(f_0(\bar{\alpha}(0,0)), \ldots, f_0(\bar{\alpha}(0,\ell))) = 1$. Look at $v = h^{-1}[\{1\}] \cap u^*$ and the corresponding i_v. By the above assumption and the choice of $h_v, f_{i_v}^*$ we have

$$\{0,1\} \models \tau(f_{i_v}(\bar{\alpha}(i_v,0)), \ldots, f_{i_v}(\bar{\alpha}(i_v,\ell))) = 1.$$

This shows that $\bigcup_{i \leqslant N} f_i \in F^q$ and hence we conclude $q \in \mathbb{Q}_n^{\mu,\lambda}$. If $|u^+ \setminus u^*| \leqslant n/2$ then we proceed similarly: for $v \subseteq u^+ \setminus u^*$ we choose distinct $i_v \in [1,N]$ such that $w^{p_{i_v}} \cap u = u^*$. We pick f^* as in the previous case and we define $f_i^* : w^{p_i} \setminus u \to 2$ (for $i \leqslant N$) as follows

- if $i = i_v$, $v \subseteq u^+ \setminus u^*$ then $f_i^* \supseteq f^* \circ H_{i,0}$ and $(\forall \alpha \in u^+ \setminus u^*)(f_i^*(H_{0,i}(\alpha)) = 1 \Leftrightarrow \alpha \in v)$,
- if $i \notin \{i_v : v \subseteq u^+ \setminus u^*\}$ then $f_i^* \supseteq f^* \circ H_{i,0}$ is such that $f_i^*(\alpha) = 0$ for all $\alpha \in H_{i,0}^{-1}[u^+] \setminus u$.

Next we argue like before to show that $q \in \mathbb{Q}_n^{\mu,\lambda}$.

Checking that q is a common upper bound of $p_0, \ldots, p_N$ is straightforward. Finally, by the definition of F^q and by Proposition 2.2(2) we see that

$$q \Vdash_{\mathbb{Q}_n^{\mu,\lambda}} \text{“}\dot{a}_0 \wedge \bigwedge_{j=1}^{N} (-\dot{a}_j) = 0\text{”}.$$

Thus we have proved that $\Vdash_{\mathbb{Q}_n^{\mu,\lambda}}$ “$\mathrm{t}_{1,N}^+(\dot{\mathbb{B}}) \leqslant \mu^+$”. The same arguments show that $\Vdash_{\mathbb{Q}_n^{\mu,\lambda}}$ “$\mathrm{t}_{N,1}^+(\dot{\mathbb{B}}) \leqslant \mu^+$” (just considering $-\dot{a}_\alpha$ instead of $\dot{a}_\alpha$ and $\{0,\ldots,N-1\}$, $\{N\}$ as the two groups of indexes there).

To show that the equalities hold one can prove even more: in $\mathbf{V}^{\mathbb{Q}_n^{\mu,\lambda}}$, there is an independent subset of $\dot{\mathbb{B}}$ of size μ. The construction of the set is easy once you note that if $p \in \mathbb{Q}_n^{\mu,\lambda}$, $\alpha \in \lambda \setminus w^p$ and $w^q = w^p \cup \{\alpha\}$, $F^q = \{f \in 2^{w^q} : f \restriction w^p \in F^p\}$ then $q = (w^q, F^q)$ is a condition in $\mathbb{Q}_n^{\mu,\lambda}$ stronger than p. $\square$

Conclusion 2.5. *Assume that $\mu^{<\mu} = \mu < \lambda \leqslant \chi$. Then there is a forcing notion $\mathbb{P}$ which does not change cardinalities and cofinalities and such that in $\mathbf{V}^{\mathbb{P}}$: $2^\mu \geqslant \chi$ and there are Boolean algebras $\mathbb{B}_0, \mathbb{B}_1, \mathbb{B}_2, \ldots$ of size λ satisfying*

$$\mathrm{ind}_{n+1}^+(\mathbb{B}_n) = \lambda^+ \quad \textit{and} \quad \mathrm{hd}^+(\mathbb{B}_n) = \mathrm{hL}^+(\mathbb{B}_n) = \mathrm{ind}^+(\mathbb{B}_n) = \mu^+.$$

Consequently, in $\mathbf{V}^{\mathbb{P}}$, *for every non-principal ultrafilter* $\mathscr{D}$ *on* ω *we have*

$$\mathrm{inv}\left(\prod_{n<\omega} \mathbb{B}_n/\mathscr{D}\right) = \lambda^{\omega} \quad \text{and} \quad \prod_{n\in\omega} \mathrm{inv}(\mathbb{B}_n)/\mathscr{D} = \mu^{\omega},$$

for $\mathrm{inv} \in \{\mathrm{ind}, \mathrm{t}, \mathrm{hd}, \mathrm{hL}, s\}$.

Proof. Let $\mathbb{P}_0$ be the forcing notion adding χ many Cohen subsets of μ (with conditions of size $<\mu$) and for $n>0$ let $\mathbb{P}_n$ be $\mathbb{Q}_n^{\mu,\lambda}$. Let $\mathbb{P}$ be the $<\mu$-support product of the $\mathbb{P}_n$'s (so if $\mu=\omega$ then $\mathbb{P}$ is the finite support product of the $\mathbb{P}_n$'s and otherwise it is the full product).

Claim 2.5.1. $\mathbb{P}$ *is a* μ*-closed* μ^+*-cc forcing notion of size* $\chi^{<\mu}$.

Proof of the Claim. Modify the Proof of Proposition 2.3. □

Let $\dot{\mathbb{B}}_n$ be the $\mathbb{P}_{n+1}$-name (and so $\mathbb{P}$-name) for the Boolean algebra added by forcing with $\mathbb{P}_{n+1}$.

Claim 2.5.2. *For* $n\in\omega$, $\mathrm{inv} \in \{\mathrm{ind}, \mathrm{t}, \mathrm{hd}, \mathrm{hL}, s\}$ *we have*

$$\Vdash_{\mathbb{P}} \text{“}\mathrm{ind}^+_{n+1}(\dot{\mathbb{B}}_n) = \lambda^+ \quad \text{and} \quad \mathrm{inv}^+(\dot{\mathbb{B}}_n) = \mu^+\text{”}.$$

Proof of the Claim. Repeat the proof of Theorem 2.4 with suitable changes to show that in $\mathbf{V}^{\mathbb{P}}$, for each n, we have

$$\mathrm{ind}^+_{n+1}(\dot{\mathbb{B}}_n) = \lambda^+ \quad \text{and} \quad \mathrm{t}^+_{1,2^n+n}(\dot{\mathbb{B}}_n) = \mathrm{t}_{2^n+n,1}(\dot{\mathbb{B}}_n) = \mathrm{ind}^+(\dot{\mathbb{B}}_n) = \mu^+.$$

Alternatively, first note that if $\mathbb{Q}$ is a μ-closed μ^+-cc forcing notion then $(\mathbb{Q}_n^{\mu,\lambda})^{\mathbf{V}^{\mathbb{Q}}} = (\mathbb{Q}_n^{\mu,\lambda})^{\mathbf{V}}$, so $\mathbb{Q}\times\mathbb{Q}_n^{\mu,\lambda} = \mathbb{Q} * \mathbb{Q}_n^{\mu,\lambda}$. Moreover, if we start with $\mu^{<\mu}=\mu$ then $\mathbf{V}^{\mathbb{Q}} \models \mu^{<\mu} = \mu$ and we may use Theorem 2.4 for $\mathbb{Q}_n^{\mu,\lambda}$ in $\mathbf{V}^{\mathbb{Q}}$.

Now apply Proposition 1.7. □

The “consequently” part of the conclusion follows from Proposition 1.2. □

Remark 2.6. Note that the examples when the spread of ultraproduct is larger than the ultraproduct of the spreads which were known before provided “a successor” difference only. Conclusion 2.5 shows that the jump can be larger, but we do not know if one can get it in ZFC (i.e. assuming suitable cardinal arithmetic only).

Problem 2.7. Can one improve Theorem 2.4 getting it for $N=n+1$?

3. Irredundance of products

This and the next sections are motivated by a result of Heindorf [4] who showed that $\mathrm{irr}(\mathbb{B}) \leqslant s(\mathrm{Ult}(\mathbb{B}) \times \mathrm{Ult}(\mathbb{B}))$. In Theorem 3.1 below we show that $\mathrm{irr}(\mathbb{B}_0 \times \mathbb{B}_1) = \max\{\mathrm{irr}(\mathbb{B}_0), \mathrm{irr}(\mathbb{B}_1)\}$ thus answering [11, Problem 24]. A parallel question for free products of Boolean algebras will be addressed in the next section. It should be noted here that the proof of the ZFC result was written as a result of an analysis why a forcing proof of consistency of an inequality (similar to the one from the next section) failed.

Theorem 3.1. *For Boolean algebras* $\mathbb{B}_0, \mathbb{B}_1$:

$$\mathrm{irr}(\mathbb{B}_0 \times \mathbb{B}_1) = \max\{\mathrm{irr}(\mathbb{B}_0), \mathrm{irr}(\mathbb{B}_1)\}.$$

Proof. Clearly $\mathrm{irr}(\mathbb{B}_0 \times \mathbb{B}_1) \geqslant \max\{\mathrm{irr}(\mathbb{B}_0), \mathrm{irr}(\mathbb{B}_1)\}$, so we have to deal with the converse inequality only. Assume that a sequence $\bar{x} = \langle (x^0_\alpha, x^1_\alpha) : \alpha < \lambda \rangle \subseteq \mathbb{B}_0 \times \mathbb{B}_1$ is irredundant. Thus, for each $\alpha < \lambda$, we have homomorphisms $f^0_\alpha, f^1_\alpha : \mathbb{B}_0 \times \mathbb{B}_1 \to \{0,1\}$ such that $f^0_\alpha(x^0_\alpha, x^1_\alpha) = 0$, $f^1_\alpha(x^0_\alpha, x^1_\alpha) = 1$ and

$$(\forall \beta \in \lambda \setminus \{\alpha\})(f^0_\alpha(x^0_\beta, x^1_\beta) = f^1_\alpha(x^0_\beta, x^1_\beta)).$$

By shrinking the sequence $\bar{x}$ if necessary, we may assume that one of the following occurs:

(i) $(\forall \alpha < \lambda)(f^0_\alpha(1,0) = f^1_\alpha(1,0) = 0)$,
(ii) $(\forall \alpha < \lambda)(f^0_\alpha(1,0) = f^1_\alpha(1,0) = 1)$,
(iii) $(\forall \alpha < \lambda)(f^0_\alpha(1,0) = 0 \;\&\; f^1_\alpha(1,0) = 1)$,
(iv) $(\forall \alpha < \lambda)(f^0_\alpha(1,0) = 1 \;\&\; f^1_\alpha(1,0) = 0)$.

If the first clause occurs then we may define (for $\alpha < \lambda$) homomorphisms $h^0_\alpha, h^1_\alpha : \mathbb{B}_1 \to \{0,1\}$ by $h^\ell_\alpha(x) = f^\ell_\alpha(1,x)$ (remember that in this case we have $f^\ell_\alpha(0,1) = 1$). Clearly these homomorphisms witness that the sequence $\langle x^1_\alpha : \alpha < \lambda \rangle \subseteq \mathbb{B}_1$ is irredundant (and thus $\mathrm{irr}^+(\mathbb{B}_1) > \lambda$). Similarly, if (ii) holds then the sequence $\langle x^0_\alpha : \alpha < \lambda \rangle \subseteq \mathbb{B}_0$ is irredundant and $\mathrm{irr}^+(\mathbb{B}_0) > \lambda$.

Since $f^\ell_\alpha(1,0) = 0 \Leftrightarrow f^\ell_\alpha(0,1) = 1$ and the algebras $\mathbb{B}_0, \mathbb{B}_1$ are in symmetric positions, we may assume that clause (iv) holds, so $f^\ell_\alpha(0,1) = \ell$ (for $\ell < 2$, $\alpha < \lambda$).

For $\alpha < \lambda$ and $\ell < 2$ let $g^\ell_\alpha : \lambda \to 2$ be given by $g^\ell_\alpha(\beta) = f^\ell_\alpha(x^0_\beta, x^1_\beta)$ for $\beta < \lambda$. Note that $\beta \neq \alpha$ implies $g^0_\alpha(\beta) = g^1_\alpha(\beta)$ (remember the choice of the f^ℓ_α's). Next, for $\ell < 2$ let $F_\ell = \{g^\ell_\alpha : \alpha < \lambda\}$ and let $\mathbb{B}^*_\ell$ be the algebra $\mathbb{B}_{(\lambda, F_\ell)}$ (see Definition 2.1(1)).

Claim 3.1.1. *Assume that* $A \subseteq \lambda$ *and* $\ell < 2$ *are such that*

$(\boxtimes^\ell_A)$ *the mappings* $\{x_\beta : \beta \in A\} \to \{0,1\} : x_\beta \mapsto g^k_\alpha(\beta)$ (*for* $k = 0,1$ *and* $\alpha \in A$) *extend to homomorphisms from* $\langle x_\beta : \beta \in A \rangle_{\mathbb{B}^*_\ell}$ *onto* $\{0,1\}$.

Then the sequence $\langle x^\ell_\alpha : \alpha \in A \rangle \subseteq \mathbb{B}_\ell$ *is irredundant.*

Proof of the Claim. First note that the assumption $(\boxtimes^\ell_A)$ implies that the sequence $\langle x_\beta\colon \beta\in A\rangle\subseteq \mathbb{B}^*_\ell$ is irredundant. Now, the mapping $x^\ell_\beta\mapsto x_\beta$ extends to a homomorphism from the algebra $\langle x^\ell_\beta:\beta<\lambda\rangle_{\mathbb{B}_\ell}$ onto $\mathbb{B}^*_\ell$. [Why? Note that, since $f^0_\alpha(1,0)=1=f^1_\alpha(0,1)$, the mappings $x^\ell_\beta\mapsto f^\ell_\alpha(x^0_\beta,x^1_\beta)=g^\ell_\alpha(\beta)$ extend to homomorphisms from $\mathbb{B}_\ell$ onto $\{0,1\}$. Now look at the definition of the algebra $\mathbb{B}^*_\ell$; remember Proposition 2.2(2).] Consequently we get that the sequence $\langle x^\ell_\beta\colon \beta\in A\rangle\subseteq\mathbb{B}_\ell$ is irredundant. $\square$

It follows from Claim 3.1.1 that if there are $A\in[\lambda]^\lambda$ and $\ell<2$ such that $(\boxtimes^\ell_A)$ holds true then the algebra $\mathbb{B}_\ell$ has an irredundant sequence of length λ (i.e. $\mathrm{irr}^+(\mathbb{B}_\ell)>\lambda$). So the proof of the theorem will be concluded when we show the following claim.

Claim 3.1.2. *Let $\ell<2$. Assume that there is no $A\in[\lambda]^\lambda$ such that $(\boxtimes^\ell_A)$ holds. Then $s^+(\mathbb{B}_{1-\ell})>\lambda$ (so $\mathrm{irr}^+(\mathbb{B}_{1-\ell})>\lambda$ too).*

Proof of the Claim. By induction on $\xi<\lambda$ we build a sequence $\langle(u_\xi,v_\xi)\colon \xi<\lambda\rangle$ such that for each $\xi<\lambda$:
(a) $u_\xi,v_\xi\in[\lambda]^{<\omega}$ are disjoint,
(b) $(u_\xi\cup v_\xi)\cap\bigcup_{\zeta<\xi}(u_\zeta\cup v_\zeta)=\emptyset$,
(c) $\mathbb{B}^*_\ell\models\bigwedge_{\gamma\in u_\xi}x_\gamma\wedge\bigwedge_{\gamma\in v_\xi}(-x_\gamma)=0$,
(d) $\mathbb{B}^*_{1-\ell}\models\bigwedge_{\gamma\in u_\xi}x_\gamma\wedge\bigwedge_{\gamma\in v_\xi}(-x_\gamma)\neq 0$.
Suppose we have defined u_ζ,v_ζ for $\zeta<\xi$. The set $A=\lambda\setminus\bigcup_{\zeta<\xi}(u_\zeta\cup v_\zeta)$ is of size λ, so (by our assumptions) $(\boxtimes^\ell_A)$ fails. This means that one of the mappings

$$\{x_\beta\colon \beta\in A\}\to\{0,1\}:x_\beta\mapsto g^k_\alpha(\beta)\quad (k=0,1,\ \alpha\in A)$$

does not extend to a homomorphism from $\langle x_\beta\colon \beta\in A\rangle_{\mathbb{B}^*_\ell}$. But, by the definition of $\mathbb{B}^*_\ell$, the mappings $x_\beta\mapsto g^\ell_\alpha(\beta)$ do extend (see Proposition 2.2(1)). So we find finite disjoint sets $u_\xi,v_\xi\subseteq A$ such that $\mathbb{B}^*_\ell\models\bigwedge_{\gamma\in u_\xi}x_\gamma\wedge\bigwedge_{\gamma\in v_\xi}(-x_\gamma)=0$, but for some $\alpha<\lambda$, $g^{1-\ell}_\alpha\restriction u_\xi\equiv 1$ and $g^{1-\ell}_\alpha\restriction v_\xi\equiv 0$. The latter implies that $\mathbb{B}^*_{1-\ell}\models\bigwedge_{\gamma\in u_\xi}x_\gamma\wedge\bigwedge_{\gamma\in v_\xi}(-x_\gamma)\neq 0$. This finishes the construction.

The demand (d) means that (by Proposition 2.2) for each $\xi<\lambda$ we find $\alpha_\xi<\lambda$ such that $g^{1-\ell}_{\alpha_\xi}\restriction u_\xi\equiv 1$ and $g^{1-\ell}_{\alpha_\xi}\restriction v_\xi\equiv 0$. On the other hand, by (c), there is no $\alpha<\lambda$ such that $g^\ell_\alpha\restriction u_\xi\equiv 1$ and $g^\ell_\alpha\restriction v_\xi\equiv 0$. But now, if $\alpha\notin u_\xi\cup v_\xi$ then $g^{1-\ell}_\alpha\restriction(u_\xi\cup v_\xi)=g^\ell_\alpha\restriction(u_\xi\cup v_\xi)$, so necessarily $\alpha_\xi\in u_\xi\cup v_\xi$. Let $y_\xi=\bigwedge_{\gamma\in u_\xi}x^{1-\ell}_\gamma\wedge\bigwedge_{\gamma\in v_\xi}(-x^{1-\ell}_\gamma)\in\mathbb{B}_{1-\ell}$ and $h_\xi:\langle x^{1-\ell}_\beta\colon \beta<\lambda\rangle_{\mathbb{B}_{1-\ell}}\to\{0,1\}$ be a homomorphism defined by $h_\xi(x^{1-\ell}_\beta)=f^{1-\ell}_{\alpha_\xi}(x^0_\beta,x^1_\beta)=g^{1-\ell}_{\alpha_\xi}(\beta)$. It follows from the above discussion that (h_ξ is well defined and)

$$h_\xi(y_\zeta)=1\quad\text{if and only if}\quad\xi=\zeta,$$

showing that the sequence $\langle y_\xi\colon 0<\xi<\lambda\rangle$ is ideal independent and irredundant. This finishes the proof of the claim and that of the theorem. $\square$

4. Forcing for spread and irredundance

In this section we show that, consistently, there is a Boolean algebra $\mathbb{B}$ such that $\mathrm{irr}(\mathbb{B}) < s(\mathbb{B} \circledast \mathbb{B})$. This gives a partial answer to [11, Problem 27]. Moreover, it shows that a statement parallel to Theorem 3.1 for the free product (instead of product) is not provable in ZFC. Note that before trying to answer [11, Problem 27] in ZFC one should first construct a ZFC example of a Boolean algebra $\mathbb{B}$ such that $\mathrm{irr}(\mathbb{B}) < |\mathbb{B}|$ – so far no such example is known.

Definition 4.1. (1) We define a forcing notion $\mathbb{Q}^*$ by:
a condition is a tuple $p = \langle u^p, \langle f^p_{0,\alpha}, f^p_{1,\alpha}, f^p_{2,\alpha} \colon \alpha \in u^p \rangle\rangle$ such that
(a) $u^p \subseteq \omega_1$ is finite,
(b) $f^p_{\ell,\alpha} : u^p \times 2 \to \{0,1\}$ for $\ell < 3, \alpha \in u^p$,
(c) $f^p_{0,\alpha} \restriction (u^p \cap \alpha) \times 2 = f^p_{1,\alpha} \restriction (u^p \cap \alpha) \times 2 = f^p_{2,\alpha} \restriction (u^p \cap \alpha) \times 2$ for $\alpha \in u^p$,
(d) $f^p_{0,\alpha}(\alpha, 0) = 1$, $f^p_{0,\alpha}(\alpha, 1) = 0$ (for $\alpha \in u^p$),
(e) $f^p_{1,\alpha}(\alpha, 0) = 0$, $f^p_{1,\alpha}(\alpha, 1) = 1$ (for $\alpha \in u^p$),
(f) $f^p_{0,\alpha}(\beta, 0) = 0$ or $f^p_{1,\alpha}(\beta, 1) = 0$ (for distinct $\alpha, \beta \in u^p$),
(g) $f^p_{0,\alpha}(\beta, 0) = 0$ or $f^p_{2,\alpha}(\beta, 1) = 0$ (for $\alpha, \beta \in u^p$),
(h) $f^p_{1,\alpha}(\beta, 1) = 0$ or $f^p_{2,\alpha}(\beta, 0) = 0$ (for $\alpha, \beta \in u^p$),
(i) $f^p_{2,\alpha}(\beta, 0) = 0$ or $f^p_{2,\alpha}(\beta, 1) = 0$ (for $\alpha, \beta \in u^p$);
the order is defined by $p \leqslant q$ if and only if $u^p \subseteq u^q$, and $f^q_{\ell,\alpha} \restriction (u^p \times 2) = f^p_{\ell,\alpha}$ for $\alpha \in u^p$, $\ell < 3$ and for each $\alpha \in u^q$, $\ell < 3$:

$$f^q_{\ell,\alpha} \restriction (u^p \times 2) \in \{f^p_{k,\beta} \colon \beta \in u^p,\ k < 3\}.$$

(2) For a condition $p \in \mathbb{Q}^*$ let $\mathbb{B}^*_p$ be the algebra $\mathbb{B}_{(w,F)}$, where $w = u^p \times 2$ and $F = \{f^p_{\ell,\alpha} \colon \alpha \in u^p, \ell < 3\}$ (see Definition 2.1(1)).
(3) Let $\dot{\mathbb{B}}^*, \dot{f}_{\ell,\alpha}$ (for $\ell < 3$, $\alpha < \omega_1$) be $\mathbb{Q}^*$-names such that

$$\Vdash_{\mathbb{Q}^*} \text{“}\dot{\mathbb{B}}^* = \bigcup\{\mathbb{B}^*_p \colon p \in \Gamma_{\mathbb{Q}^*}\} \quad \text{and} \quad \dot{f}_{\ell,\alpha} = \bigcup\{f^p_{\ell,\alpha} \colon p \in \Gamma_{\mathbb{Q}^*}, \alpha \in u^p\}\text{”}.$$

Remark 4.2. The forcing with $\mathbb{Q}^*$ is supposed to add a Boolean algebra $\dot{\mathbb{B}}^*$ with two sequences $\langle x_{\alpha,0} \colon \alpha < \omega_1\rangle, \langle x_{\alpha,1} \colon \alpha < \omega_1 \rangle \subseteq \dot{\mathbb{B}}^*$. These sequences will build the example for $s(\dot{\mathbb{B}}^* \circledast \dot{\mathbb{B}}^*) = \omega_1$: the sequence $\langle x_{\alpha,0} \wedge x_{\alpha,1} \colon \alpha < \omega_1 \rangle \subseteq \dot{\mathbb{B}}^* \circledast \dot{\mathbb{B}}^*$ will be ideal independent and functions $\dot{f}_{\alpha,0}, \dot{f}_{\alpha,1}$ will witness it (by Definition 4.1(d)–(f); see Proposition 4.4 below). But we want to have $\mathrm{irr}(\dot{\mathbb{B}}^*) = \omega_0$, and we will have to amalgamate conditions from $\mathbb{Q}^*$ (in a suitable way). For this to work we have additional functions $f^p_{3,\alpha}$ and requirements (Definition 4.1(g), (h), (i)), which allow us to interchange (in some sense) functions $f^p_{0,\alpha}, f^p_{2,\alpha}$ and $f^p_{1,\alpha}, f^p_{2,\alpha}$ in the amalgamations (see Theorem 4.5 and Proposition 4.3).

Proposition 4.3. (1) $\mathbb{Q}^*$ *is a ccc forcing notion.*
(2) *If* $p, q \in \mathbb{Q}^*$, $p \leqslant q$ *then* $\mathbb{B}_p$ *is a subalgebra of* $\mathbb{B}_q$.

(3) *In* $\mathbf{V}^{\mathbb{Q}^*}$, $\dot{f}_{\ell,\alpha}:\omega_1\times 2\to 2$ (*for* $\alpha<\omega_1$ *and* $\ell<3$) *and* $\dot{\mathbb{B}}^*$ *is the Boolean algebra* $\mathbb{B}_{(w,F)}$, *where* $w=\omega_1\times 2$ *and* $F=\{\dot{f}_{\ell,\alpha}\colon \alpha<\omega_1,\ell<3\}$.

Proof. (1) Suppose that $\mathscr{A}\subseteq\mathbb{Q}^*$ is uncountable. Applying Δ-system arguments find $p,q\in\mathscr{A}$ such that letting $u^*=u^p\cap u^q$ we have
(i) $\max(u^*)<\min(u^p\setminus u^*)\leqslant\max(u^p\setminus u^*)<\min(u^q\setminus u^*)$,
(ii) $|u^p|=|u^q|$ and if $H:u^p\to u^q$ is the order isomorphism, $\alpha\in u^p$ and $\ell<3$ then $f^p_{\ell,\alpha}=f^q_{\ell,H(\alpha)}\circ(H\times\mathrm{id})$.
Now let $u^r=u^p\cup u^q$ and for $\ell<3$ and $\alpha\in u^r$ let:

$$f^r_{\ell,\alpha}=\begin{cases} f^p_{\ell,\alpha}\cup f^q_{\ell,\alpha} & \text{if } \alpha\in u^p\cap u^q,\\ f^p_{\ell,\alpha}\cup f^q_{2,H(\alpha)}\restriction(u^q\setminus u^p) & \text{if } \alpha\in u^p\setminus u^q,\\ f^q_{\ell,\alpha}\cup f^p_{2,H^{-1}(\alpha)}\restriction(u^p\setminus u^q) & \text{if } \alpha\in u^q\setminus u^p.\end{cases}$$

It is a routine to check that this defines a condition in $\mathbb{Q}^*$ stronger than both p and q.
(2) Should be clear.
(3) Note that if $p\in\mathbb{Q}^*$, $\alpha_0\in u^p$ and $\beta\in\omega_1\setminus u^p$ then letting $u^q=u^p\cup\{\beta\}$ and

$$f^q_{\ell,\alpha}=\begin{cases} f^p_{\ell,\alpha}\cup\{((\beta,0),0),((\beta,1),0)\} & \text{if } \alpha\in u^p,\ \ell<3,\\ f^p_{2,\alpha_0}\cup\{((\beta,0),1-\ell),((\beta,1),\ell)\} & \text{if } \alpha=\beta,\ \ell<2,\\ f^p_{2,\alpha_0}\cup\{((\beta,0),0),((\beta,1),0)\} & \text{if } \alpha=\beta,\ \ell=2,\end{cases}$$

we get a condition $q\in\mathbb{Q}^*$ stronger than p and such that $\beta\in w^q$. Now, the rest should be clear. $\square$

Proposition 4.4. $\Vdash_{\mathbb{Q}^*}$"$s(\dot{\mathbb{B}}^*\circledast\dot{\mathbb{B}}^*)=\omega_1$".

Proof. To avoid confusion between the two copies of $\dot{\mathbb{B}}^*$ in $\dot{\mathbb{B}}^*\circledast\dot{\mathbb{B}}^*$, let us denote an element $a\wedge b\in\dot{\mathbb{B}}^*\circledast\dot{\mathbb{B}}^*$ such that a is from the first copy of $\dot{\mathbb{B}}^*$ and b is from the second one, by $\langle a,b\rangle$. With this convention, for each $\alpha<\omega_1$ let $\dot{y}_\alpha=\langle x_{\alpha,0},x_{\alpha,1}\rangle$ and let $\dot{f}_\alpha:\dot{\mathbb{B}}^*\circledast\dot{\mathbb{B}}^*\to\{0,1\}$ be a homomorphism such that (for $\beta<\omega_1$, $i<2$)

$$\dot{f}_\alpha(\langle x_{\beta,i},1\rangle)=\dot{f}_{0,\alpha}(\beta,i)\quad\text{and}\quad\dot{f}_\alpha(\langle 1,x_{\beta,i}\rangle)=\dot{f}_{1,\alpha}(\beta,i).$$

Note that, by Definition 4.1(d), (e), for each $\alpha<\omega_1$,

$$\dot{f}_\alpha(\dot{y}_\alpha)=\dot{f}_{0,\alpha}(\alpha,0)\wedge\dot{f}_{1,\alpha}(\alpha,1)=1,$$

and if $\beta\in\omega_1\setminus\{\alpha\}$ then (by Definition 4.1(f))

$$\dot{f}_\alpha(\dot{y}_\beta)=\dot{f}_{0,\alpha}(\beta,0)\wedge\dot{f}_{1,\alpha}(\beta,1)=0.$$

Hence we conclude that

$$\Vdash_{\mathbb{Q}^*}\ \text{“}\langle\dot{y}_\alpha\colon\alpha<\omega_1\rangle\text{ is ideal-independent”},$$

finishing the proof. $\square$

Theorem 4.5. $\Vdash_{\mathbb{Q}^*}$ "$\mathrm{irr}_5^+(\dot{\mathbb{B}}^*) = \omega_1$".

Proof. Let $\langle \dot{a}_\beta\colon \beta<\omega_1\rangle$ be a $\mathbb{Q}^*$-name for an ω_1-sequence of elements of $\dot{\mathbb{B}}^*$, $p\in\mathbb{Q}^*$. For $\beta<\omega_1$ choose a condition $p_\beta \geqslant p$, a Boolean term τ_β, ordinals $\bar{\alpha}(\beta,0)\leqslant\cdots\leqslant\bar{\alpha}(\beta,\ell_\beta)<\omega_1$ and $\bar{\imath}(\beta,0),\ldots,\bar{\imath}(\beta,\ell_\beta)\in\{0,1\}$ such that

$$p_\beta \Vdash_{\mathbb{Q}^*} \dot{a}_\beta = \tau_\beta(x_{\bar{\alpha}(\beta,0),\bar{\imath}(\beta,0)},\ldots,x_{\bar{\alpha}(\beta,\ell_\beta),\bar{\imath}(\beta,\ell_\beta)}).$$

Applying standard "cleaning procedure" we may assume that for $\beta,\beta_0,\beta_1<\omega_1$:

(i) $\tau_\beta=\tau$, $\ell_\beta=\ell$,

(ii) $\{(\bar{\alpha}(\beta,j),\bar{\imath}(\beta,j))\colon j\leqslant\ell\} = u^{p_\beta}\times 2$ is an enumeration which does not depend on β if we treat it modulo otp (so $2\cdot|u^{p_\beta}|=\ell+1$ and we may write $\tau(x_{\gamma,i}\colon \gamma\in u^{p_\beta}, i<2)$),

(iii) $\{u^{p_\beta}\colon \beta<\omega_1\}$ forms a Δ-system of sets with the heart u^*, and if $\beta_0<\beta_1<\omega_1$ then

$$\max(u^*) < \min(u^{p_{\beta_0}}\setminus u^*) \leqslant \max(u^{p_{\beta_0}}\setminus u^*) < \min(u^{p_{\beta_1}}\setminus u^*),$$

(iv) $|u^{p_{\beta_0}}| = |u^{p_{\beta_1}}|$ and if $H_{\beta_0,\beta_1}: u^{p_{\beta_0}} \to u^{p_{\beta_1}}$ is the order preserving mapping then $f^{p_{\beta_0}}_{k,\alpha} = f^{p_{\beta_1}}_{k,H_{\beta_0,\beta_1}(\alpha)} \circ (H_{\beta_0,\beta_1}\times \mathrm{id})$ (for $\alpha\in u^{p_{\beta_0}}$, $k<3$).

Now we are going to define a condition q stronger than $p_0,\ldots,p_5$. We put $u^q = \bigcup_{i<6} u^{p_i}$ and we define functions $f^q_{\ell,\alpha}: u^q\times 2\to 2$ (for $\alpha\in u^q$ and $\ell<3$) as follows.

($\boxtimes$) If $\alpha\in u^*$, $\ell<3$ then $f^q_{\ell,\alpha} = \bigcup_{i<6} f^{p_i}_{\ell,\alpha}$.

($\boxplus_0$) If $\alpha\in u^{p_0}\setminus u^*$ then

$$f^q_{0,\alpha} = f^{p_0}_{0,\alpha} \cup f^{p_1}_{0,H_{0,1}(\alpha)} \cup f^{p_2}_{0,H_{0,2}(\alpha)} \cup f^{p_3}_{2,H_{0,3}(\alpha)} \cup f^{p_4}_{2,H_{0,4}(\alpha)} \cup f^{p_5}_{2,H_{0,5}(\alpha)},$$

$$f^q_{1,\alpha} = f^{p_0}_{1,\alpha} \cup f^{p_1}_{2,H_{0,1}(\alpha)} \cup f^{p_2}_{2,H_{0,2}(\alpha)} \cup f^{p_3}_{1,H_{0,3}(\alpha)} \cup f^{p_4}_{1,H_{0,4}(\alpha)} \cup f^{p_5}_{2,H_{0,5}(\alpha)},$$

$$f^q_{2,\alpha} = \bigcup_{i<6} f^{p_i}_{2,H_{0,i}(\alpha)}.$$

($\boxplus_i$) If $\alpha\in u^{p_i}\setminus u^*$, $0<i<6$ and $\ell<3$ then $f^q_{\ell,\alpha} = f^{p_i}_{\ell,\alpha}\cup\bigcup_{j\neq i} f^{p_j}_{2,H_{i,j}(\alpha)}$.

It follows from (iv) and Definition 4.1(c) that the functions $f^q_{\ell,\alpha}$ are well defined.

Claim 4.5.1. *The tuple $q=\langle u^q, \langle f^q_{\ell,\alpha}\colon \ell<3, \alpha\in u^q\rangle\rangle$ is a condition in $\mathbb{Q}^*$ stronger than $p_0,\ldots,p_5$.*

Proof of the Claim. To show that $q\in\mathbb{Q}^*$ one has to check the demands (a)–(i) of Definition 4.1. The only possible problems could be caused by clauses (f)–(i). If functions $f^q_{\ell,\alpha}$ were defined in clauses ($\boxtimes$), ($\boxplus_i$) then easily these demands are met. To deal with instances of ($\boxplus_0$) (i.e. when $\alpha\in u^{p_0}\setminus u^*$) note that in the definition of $f^q_{\ell,\alpha}$ ($\ell<2$, $\alpha\in u^{p_0}\setminus u^*$) a part of the form $f^{p_j}_{\ell,H_{i,j}(\alpha)}$ "meets" $f^{p_j}_{2,H_{i,j}(\alpha)}$ on the side of $f^q_{1-\ell,\alpha}$. Therefore, by (g), (h) of Definition 4.1, we have no problems with checking demand (f). Clause 4.1(i) is immediate and (g), (h) should be clear too. □

Claim 4.5.2.

$$\tau(x_{\gamma,i}\colon \gamma\in u^{p_0}, i<2)\in\langle\tau(x_{\gamma,i}\colon \gamma\in u^{p_j}, i<2)\colon 0<j<6\rangle_{\mathbb{B}^*_q}.$$

Consequently $q\Vdash_{\mathbb{Q}^*}$ "$\dot{a}_0\in\langle\dot{a}_j\colon 0<j<6\rangle_{\dot{\mathbb{B}}^*}$".

Proof of the Claim. Suppose that

$$\tau(x_{\gamma,i}\colon \gamma\in u^{p_0}, i<2)\notin\langle\tau(x_{\gamma,i}\colon \gamma\in u^{p_j}, i<2)\colon 0<j<6\rangle_{\mathbb{B}^*_q}.$$

Then we find two homomorphisms $h_0,h_1:\mathbb{B}^*_q\to\{0,1\}$ such that

$$h_0(\tau(x_{\gamma,i}\colon \gamma\in u^{p_0}, i<2))\neq h_1(\tau(x_{\gamma,i}\colon \gamma\in u^{p_0}, i<2))$$

but

$$h_0(\tau(x_{\gamma,i}\colon \gamma\in u^{p_j}, i<2))=h_1(\tau(x_{\gamma,i}\colon \gamma\in u^{p_j}, i<2))\quad\text{for } 0<j<6.$$

By the definition of the algebra $\mathbb{B}^*_q$ each its homomorphism into $\{0,1\}$ is generated by one of the functions $f^q_{\ell,\alpha}$ (for $\ell<3$, $\alpha\in u^q$). So we find $\ell_0,\ell_1<3$ and $\alpha_0,\alpha_1\in u^q$ such that $h_k\supseteq f^q_{\ell_k,\alpha_k}$. Now we have to consider several cases corresponding to the way the $f^q_{\ell_k,\alpha_k}$ were defined.

Case A: $\alpha_k\in u^*$, $\alpha_{1-k}\in u^{p_i}$, $i<6$. Then look at the definition ($\boxtimes$) of $f^q_{\ell_k,\alpha_k}$ – it copies $f^{p_0}_{\ell_k,\alpha_k}$ everywhere (remember (iv)). On the other hand, whatever clause was used to define $f^q_{\ell_{1-k},\alpha_{1-k}}$, there is $j\in(0,6)$ such that $f^q_{\ell_{1-k},\alpha_{1-k}}\restriction(u^{p_j}\times 2)$ is a copy of $f^q_{\ell_{1-k},\alpha_{1-k}}\restriction(u^{p_0}\times 2)$. Hence we may conclude that (for this j)

$$h_{1-k}(\tau(x_{\gamma,i}\colon \gamma\in u^{p_j}, i<2))\neq h_k(\tau(x_{\gamma,i}\colon \gamma\in u^{p_j}, i<2)),$$

a contradiction.

Case B: $\alpha_k\in u^{p_0}\setminus u^*$, $\alpha_{1-k}\in u^{p_i}\setminus u^*$, $0<i<6$. Then we repeat the argument of the previous Case, choosing j in such a way that $j\neq i$ and: if $\ell_k=0$ then $j\in\{1,2\}$, if $\ell_k=1$ then $j\in\{3,4\}$.

Case C: $\alpha_k\in u^{p_{i'}}\setminus u^*$, $\alpha_{1-k}\in u^{p_{i''}}\setminus u^*$, $0<i',i''<6$. Like above, but now take $j\in\{1,\dots,5\}\setminus\{i',i''\}$.

Case D: $\alpha_0,\alpha_1\in u^{p_0}\setminus u^*$. This is the most complicated case. We may repeat the previous argument in some cases letting:

$$j=\begin{cases}1 & \text{if } (\ell_0,\ell_1)\in\{(0,0),(0,2),(2,0),(2,2)\},\\ 3 & \text{if } (\ell_0,\ell_1)\in\{(1,1),(1,2),(2,1)\}.\end{cases}$$

This leaves us with two symmetrical cases: $(\ell_0,\ell_1)=(0,1)$ or $(\ell_0,\ell_1)=(1,0)$. So suppose that $\ell_0=0$, $\ell_1=1$ and let

$$x\stackrel{\text{def}}{=}h_0(\tau(x_{\gamma,i}\colon \gamma\in u^{p_5}, i<2))=h_1(\tau(x_{\gamma,i}\colon \gamma\in u^{p_5}, i<2)).$$

Since $f^q_{0,\alpha_0}\restriction(u^{p_4}\times 2)$ is a copy of $f^q_{0,\alpha_0}\restriction(u^{p_5}\times 2)$ we conclude that

$$x=h_0(\tau(x_{\gamma,i}\colon\ \gamma\in u^{p_4},i<2))=h_1(\tau(x_{\gamma,i}\colon\ \gamma\in u^{p_4},i<2)),$$

and, since $f^q_{1,\alpha_1}\restriction(u^{p_4}\times 2)$ is a copy of $f^q_{1,\alpha_1}\restriction(u^{p_0}\times 2)$ we get

($\square$) $x=h_1(\tau(x_{\gamma,i}\colon\ \gamma\in u^{p_0},i<2))$.

Next, $f^q_{1,\alpha_1}\restriction(u^{p_2}\times 2)$ is a copy of $f^q_{1,\alpha_1}\restriction(u^{p_5}\times 2)$ and therefore

$$x=h_1(\tau(x_{\gamma,i}\colon\ \gamma\in u^{p_2},i<2))=h_0(\tau(x_{\gamma,i}\colon\ \gamma\in u^{p_2},i<2)).$$

But $f^q_{0,\alpha_0}\restriction(u^{p_2}\times 2)$ is a copy of $f^q_{0,\alpha_0}\restriction(u^{p_0}\times 2)$, so we conclude that

($\odot$) $x=h_0(\tau(x_{\gamma,i}\colon\ \gamma\in u^{p_0},i<2))$.

But now ($\square$) + ($\odot$) contradict the choice of h_0,h_1. The other case is similar. This finishes the proof of the claim and of the theorem. $\square$

Conclusion 4.6. *It is consistent that there exists a Boolean algebra $\mathbb{B}$ such that*

$$\omega_0=\mathrm{irr}(\mathbb{B})\quad \textit{and}\quad s(\mathbb{B}\circledast\mathbb{B})=\mathrm{irr}(\mathbb{B}\circledast\mathbb{B})=\omega_1.$$

Remark 4.7. We may use any cardinal $\mu=\mu^{<\mu}$ instead of ω and μ^+ instead of ω_1 in Definition 4.1 and then Propositions 4.3 and 4.4. But we do not know if the difference between the respective cardinal invariants can be larger.

Problem 4.8. Is it consistent that there is a Boolean algebra $\mathbb{B}$ such that

$$(\mathrm{irr}(\mathbb{B}))^+<|\mathbb{B}|?\quad (\mathrm{irr}(\mathbb{B}))^+<s(\mathbb{B}\circledast\mathbb{B})?$$

5. Forcing a superatomic Boolean algebra

In this section we give partial answers to [11, Problems 73, 77, 78] showing that, consistently, there is a superatomic Boolean algebra $\mathbb{B}$ such that $s(\mathbb{B})=\mathrm{inc}(\mathbb{B})<\mathrm{irr}(\mathbb{B})=\mathrm{Id}(\mathbb{B})<\mathrm{Sub}(\mathbb{B})$. The forcing notion we use is a variant of the one of Martinez [8], which in turn was a modification of the forcing notion used in Baumgartner–Shelah [2]. For more information on superatomic Boolean algebras we refer the reader to Koppelberg [7], Roitman [12] and Monk [11].

Definition 5.1. Let κ be a cardinal. For a pair $s=(\alpha,\xi)\in\kappa^+\times\kappa$ we will write $\alpha(s)=\alpha$ and $\xi(s)=\xi$. We define a forcing notion $\mathbb{P}_\kappa$ as follows:

a condition is a tuple

$$p=\langle w^p,u^p,a^p,\langle f^p_s\colon\ s\in u^p\rangle,\langle y^p_{s_0,s_1}\colon\ s_0,s_1\in u^p,\ \ s_0\neq s_1,\ \ \alpha(s_0)\leqslant\alpha(s_1)\rangle\rangle$$

such that

(a) $a^p\subseteq w^p\in[\kappa^+]^{<\kappa}$, $u^p\in[w^p\times\kappa]^{<\kappa}$, and $\alpha\in w^p\Rightarrow(\alpha,0),(\alpha,1)\in u^p$,

(b) for $s\in u^p$, $f^p_s:u^p\to\{0,1\}$ is such that $f^p_s(s)=1$ and

$$(\forall t\in u^p)(\alpha(t)\leqslant\alpha(s)\ \&\ t\neq s\Rightarrow f^p_s(t)=0),$$

(c) if $\alpha<\beta$, $\alpha,\beta\in a^p$ then $f^p_{\alpha,0}(\beta,0)=f^p_{\alpha,1}(\beta,0)$,
(d) if $s_0,s_1\in u^p$ are distinct, $\alpha(s_0)\leqslant\alpha(s_1)$ then $y^p_{s_0,s_1}\in[u^p\cap(\alpha(s_0)\times\kappa)]^{<\omega}$ and for every $t\in u^p$

$$f^p_t(s_0)=1\ \&\ f^p_t(s_1)\neq f^p_{s_0}(s_1)\quad\Rightarrow\quad(\exists s\in y^p_{s_0,s_1})(f^p_t(s)=1);$$

the order is given by $p\leqslant q$ if and only if $w^p\subseteq w^q$, $u^p\subseteq u^q$, $a^p=a^q\cap w^p$, $y^q_{s_0,s_1}=y^p_{s_0,s_1}$ (for distinct $s_0,s_1\in u^p$ such that $\alpha(s_0)\leqslant\alpha(s_1)$), $f^p_s\subseteq f^q_s$ (for $s\in u^p$) and

$$(\forall s\in u^q)(\exists t\in u^p)(f^q_s\restriction u^p=f^p_t \text{ or } f^q_s\restriction u^p=\mathbf{0}_{u^p}).$$

Definition 5.2. We say that conditions $p,q\in\mathbb{P}_\kappa$ are *isomorphic* if there is a bijection $H:u^p\to u^q$ (called *the isomorphism from p to q*) such that
1. $(\forall s\in u^p)(\mathrm{otp}(\alpha(s)\cap w^p)=\mathrm{otp}(\alpha(H(s))\cap w^q)\ \ \&\ \xi(s)=\xi(H(s)))$,
2. $(\forall\beta\in w^p)(\alpha(H(\beta,0))\in a^q\ \Leftrightarrow\ \beta\in a^p)$,
3. $(\forall s\in u^p)(f^p_s=f^q_{H(s)}\circ H)$,
4. $(\forall s_0,\,s_1\in u^p)(\alpha(s_0)\leqslant\alpha(s_1)\ \Rightarrow\ y^p_{s_0,s_1}=\{s\in u^p\colon H(s)\in y^q_{H(s_0),H(s_1)}\})$.

Proposition 5.3. *Assume $\kappa^{<\kappa}=\kappa$. Then $\mathbb{P}_\kappa$ is a κ-complete κ^+-cc forcing notion of size κ^+.*

Proof. It should be clear that $\mathbb{P}_\kappa$ is κ-complete and $|\mathbb{P}_\kappa|=\kappa^+$. Moreover, there is κ many isomorphism types of conditions in $\mathbb{P}_\kappa$ (and a condition in $\mathbb{P}_\kappa$ is determined by its isomorphism type and the set w^p). Now, to show the chain condition assume that $\mathscr{A}\subseteq\mathbb{P}_\kappa$ is of size κ^+. Applying the Δ-system lemma choose pairwise isomorphic conditions $p_0,p_1,p_2\in\mathscr{A}$ such that $\{w^{p_0},w^{p_1},w^{p_2}\}$ forms a Δ-system with heart w^* and such that for $i<j<3$

$$\sup(w^*)<\min(w^{p_i}\setminus w^*)\leqslant\sup(w^{p_i})<\min(w^{p_j}\setminus w^*)$$

(remember $\kappa^{<\kappa}=\kappa$). For $i,j<3$ let $H_{i,j}:u^{p_i}\to u^{p_j}$ be the isomorphism from p_i to p_j. We are going to define a condition $q\in\mathbb{P}_\kappa$ which will be an upper bound to p_1,p_2 (note: not p_0!). To this end we first let

$$w^q=w^{p_0}\cup w^{p_1}\cup w^{p_2},\quad u^q=u^{p_0}\cup u^{p_1}\cup u^{p_2},\quad a^q=a^{p_1}\cup a^{p_2}.$$

To define functions f^q_s we use the approach which can be described as "put zero whenever possible". Thus we let
- if $s\in u^{p_1}\setminus u^{p_0}$ then $f^q_s=\mathbf{0}_{u^{p_0}}\cup f^{p_1}_s\cup\mathbf{0}_{u^{p_2}}$,
- if $s\in u^{p_2}\setminus u^{p_0}$ then $f^q_s=\mathbf{0}_{u^p_0}\cup\mathbf{0}_{u^p_1}\cup f^{p_2}_s$,
- if $s\in u^{p_0}$ then $f^q_s=f^{p_0}_s\cup f^{p_1}_{H_{0,1}(s)}\cup f^{p_2}_{H_{0,2}(s)}$.

It should be clear that the functions f^q_s are well defined. Now we are going to define the sets $y^q_{s_0,s_1}$ for distinct $s_0,s_1\in u^q$ such that $\alpha(s_0)\leqslant\alpha(s_1)$. It is done by cases considering all possible configurations. Thus we put
- if $s_0,s_1\in u^{p_i}$, $i<3$ then $y^q_{s_0,s_1}=y^{p_i}_{s_0,s_1}$,
- if $s_0\in u^{p_1}\setminus u^{p_0}$, $s_1\in u^{p_2}\setminus u^{p_0}$ then $y^q_{s_0,s_1}=\{H_{2,0}(s_1)\}$,

- if $s_0 \in u^{p_0}$, $s_1 \in u^{p_i}$, $i \in \{1,2\}$ then

$$y^q_{s_0,s_1} = \begin{cases} \emptyset & \text{if } H_{i,0}(s_1) = s_0, \\ \{H_{i,0}(s_1)\} & \text{if } \alpha(H_{i,0}(s_1)) < \alpha(s_0), \\ y^{p_0}_{s_0,H_{i,0}(s_1)} & \text{if } \alpha(s_0) \leqslant \alpha(H_{i,0}(s_1)),\ s_0 \neq H_{i,0}(s_1). \end{cases}$$

We claim that

$$q = \langle w^q, u^q, a^q, \langle f^q_s \colon s \in u^q \rangle, \langle y^q_{s_0,s_1} \colon s_0, s_1 \in u^q,\ s_0 \neq s_1, \alpha(s_0) \leqslant \alpha(s_1) \rangle \rangle$$

is a condition in $\mathbb{P}_\kappa$ and for this we have to check the demands of Definition 5.1. Clauses (a) and (b) should be obvious. To check Definition 5.1(c) note that $a^q \cap w^{p_0} = a^q \cap w^*$ and therefore there are no problems when $\alpha \in a^q \cap w^{p_0}$. If $\alpha \in a^q \cap (w^{p_1} \setminus w^{p_0})$ and $\alpha < \beta \in a^q \cap (w^{p_2} \setminus w^{p_0})$ then $f^q_{\alpha,0}(\beta,0) = f^q_{\alpha,1}(\beta,0) = 0$. In all other instances we use the clause (c) of Definition 5.1 for p_1, p_2.

Now we have to verify the demand 5.1(d). Suppose that s_0, s_1 are distinct members of u^q and $\alpha(s_0) \leqslant \alpha(s_1)$. If $s_0, s_1 \in u^{p_i}$ for some $i < 3$ then easily the set $y^q_{s_0,s_1}$ has the required property. So suppose now that $s_0 \in u^{p_1} \setminus u^{p_0}$, $s_1 \in u^{p_2} \setminus u^{p_0}$ (so then $f^q_{s_0}(s_1) = 0$) and let $t \in u^q$ be such that $f^q_t(s_1) = 1 = f^q_t(s_0)$. Then necessarily $t \in u^{p_0}$ and $f^q_t(H_{2,0}(s_1)) = f^q_t(s_1) = 1$, so we are done in this case. Finally, let us assume that $s_0 \in u^{p_0}$ and $s_1 \in u^{p_i}$, $0 < i < 3$. Note that if $f^q_t(s_0) = 1$ then $t \in u^{p_0}$. Now, if $H_{i,0}(s_1) = s_0$ then $f^q_t(s_0) = f^q_t(s_1)$ for every $t \in u^{p_0}$ and there are no problems (i.e. no f^q_t has to be taken care of). If $\alpha(H_{i,0}(s_1)) < \alpha(s_0)$ then the set $y^q_{s_0,s_1} = \{H_{i,0}(s_1)\}$ will work as for every $t \in u^{p_0}$ we have $f^q_t(H_{i,0}(s_1)) = f^q_t(s_1)$ (and $f^q_{s_0}(s_1) = 0$). For the same reason the set $y^q_{s_0,s_1}$ has the required property in the remaining case too.

Checking that the condition q is stronger than both p_1 and p_2 is straightforward (note: we do not claim that q is stronger than p_0). $\square$

Lemma 5.4. *If $p \in \mathbb{P}_\kappa$, $t \in \kappa^+ \times \kappa$ then there is $q \in \mathbb{P}_\kappa$ such that $p \leqslant q$ and $t \in u^q$.*

Proof. Suppose $t = (\alpha, \xi) \in (\kappa^+ \times \kappa) \setminus u^p$. Put $w^q = w^p \cup \{\alpha\}$, $a^q = a^p$ and $u^q = u^p \cup \{(\alpha,0), (\alpha,1), (\alpha,\xi)\}$. For $s \in u^p$ let $f^q_s = f^p_s \cup \mathbf{0}_{u^q \setminus u^p}$ and for $s \in u^q \setminus u^p$ let f^q_s be such that $f^q_s(s) = 1$ and $f^q_s \restriction u^q \setminus \{s\} \equiv 0$. Finally, for distinct $s_0, s_1 \in u^q$ such that $\alpha(s_0) \leqslant \alpha(s_1)$ let

$$y^q_{s_0,s_1} = \begin{cases} y^p_{s_0,s_1} & \text{if } s_0, s_1 \in u^p, \\ \emptyset & \text{otherwise.} \end{cases}$$

Check that $q = \langle w^q, u^q, a^q, \langle f^q_s \colon s \in u^q \rangle, \langle y^q_{s_0,s_1} \colon s_0, s_1 \in u^q \rangle \rangle \in \mathbb{P}_\kappa$ is as required. $\square$

For $p \in \mathbb{P}_\kappa$ let $\mathbb{B}_p$ be the algebra $\mathbb{B}_{(u^p, F^p)}$ (see Definition 2.1(1)), where $F^p = \{f^p_s \colon s \in u^p\} \cup \{\mathbf{0}_{u^p}\}$, and let $\dot{\mathbb{B}}_*$ be a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa} \text{“}\dot{\mathbb{B}}_* = \bigcup \{\mathbb{B}_p \colon p \in \Gamma_{\mathbb{P}_\kappa}\}\text{”}.$$

Furthermore, for $s\in\kappa^+\times\kappa$ let $\dot{f}_s$ be a $\mathbb{P}_\kappa$-name such that

$$\Vdash_{\mathbb{P}_\kappa} \text{“}\dot{f}_s=\bigcup\{f^p_s\colon\ p\in\Gamma_{\mathbb{P}_\kappa}\ \&\ s\in u^p\}\text{”}.$$

Proposition 5.5. *Assume $\kappa^{<\kappa}=\kappa$. Then in $\mathbf{V}^{\mathbb{P}_\kappa}$:*
(1) *$\dot{\mathbb{B}}_*$ is the algebra $\mathbb{B}_{(W,\dot{F})}$, where $W=\kappa^+\times\kappa$ and $\dot{F}=\{\dot{f}_s\colon s\in\kappa^+\times\kappa\}\cup\{\mathbf{0}_{\kappa^+\times\kappa}\}$,*
(2) *the algebra $\dot{\mathbb{B}}_*$ is superatomic,*
(3) *if $s\in\kappa^+\times\kappa$ and $b\in\dot{\mathbb{B}}_*$ then there are finite $v_0\subseteq v_1\subseteq\alpha(s)\times\kappa$ such that either $x_s\wedge b$ or $x_s\wedge(-b)$ equals to*

$$\bigvee\left\{x_t\wedge\bigwedge_{\substack{t'\in v_1\\ \alpha(t')<\alpha(t)}}(-x_{t'})\colon\ t\in v_0\right\},$$

(4) *the height of $\dot{\mathbb{B}}_*$ is κ^+ and $\{x_{\alpha,\xi}\colon\ \xi\in\kappa\}$ are representatives of atoms of rank $\alpha+1$,*
(5) $\mathrm{irr}(\dot{\mathbb{B}}_*)=\kappa^+$.

Proof. (1) First note that if $p\leqslant q$ then $\mathbb{B}_p$ is a subalgebra of $\mathbb{B}_q$. Next, remembering Lemma 5.4, conclude that

$$\Vdash_{\mathbb{P}^*} \text{“}\dot{\mathbb{B}}_* \text{ is a Boolean algebra generated by } \langle x_s\colon\ s\in\kappa^+\times\kappa\rangle\text{”}.$$

Clearly, by Lemma 5.4, $\Vdash$ “$\dot{f}_s:\kappa^+\times\kappa\longrightarrow\{0,1\}$” and $p\Vdash$ “$\dot{f}_s\restriction u^p=f^p_s$” (for $s\in u^p$, $p\in\mathbb{P}_\kappa$). So it should be clear that $\Vdash_{\mathbb{P}_\kappa}\dot{\mathbb{B}}_*=\mathbb{B}_{(W,\dot{F})}$, where $W=\kappa^+\times\kappa$ and $\dot{F}=\{\dot{f}_s\colon s\in\kappa^+\times\kappa\}\cup\{\mathbf{0}_{\kappa^+\times\kappa}\}$.

(2) It follows from Definition 5.1(b) that for each $s\in\kappa^+\times\kappa$

$$\Vdash_{\mathbb{P}_\kappa} \text{“}\dot{f}_s(s)=1 \quad\text{and}\quad (\forall t\in\kappa^+\times\kappa)(\alpha(t)\leqslant\alpha(s)\ \&\ t\neq s\Rightarrow \dot{f}_s(t)=0)\text{”}.$$

Now work in $\mathbf{V}^{\mathbb{P}_\kappa}$. Let $\dot{J}_\alpha$ be the ideal in $\dot{\mathbb{B}}_*$ generated by $\{x_{\beta,\xi}\colon\ \beta<\alpha,\xi\in\kappa\}$ (for $\alpha\leqslant\kappa^+$; if $\alpha=0$ then $\dot{J}_\alpha=\{0\}$). It follows from the previous remark that $x_{\alpha,\xi}\notin\dot{J}_\alpha$ (for all $\xi\in\kappa$; remember Proposition 2.2).

Suppose that s_0, s_1 are distinct, $\alpha(s_0)=\alpha(s_1)=\alpha<\kappa^+$ and suppose that $t\in\kappa^+\times\kappa$ is such that $\dot{f}_t(s_0)=\dot{f}_t(s_1)=1$. Let $p\in\Gamma_{\mathbb{P}^*}$ be such that $t,s_0,s_1\in u^p$. It follows from Definition 5.1(d) that there is $s\in y^p_{s_0,s_1}$ such that $f^p_t(s)=1$. Hence (applying Proposition 2.2) we may conclude that

$$\dot{\mathbb{B}}_*\models x_{s_0}\wedge x_{s_1}\leqslant\bigvee\{x_s\colon\ s\in y^p_{s_0,s_1}\},$$

and therefore $x_{s_0}\wedge x_{s_1}\in\dot{J}_\alpha$.

Now suppose that $s_0,s_1\in\kappa^+\times\kappa$ are such that $\alpha(s_0)<\alpha(s_1)$ and let $p\in\Gamma_{\mathbb{P}_\kappa}$ be such that $s_0,s_1\in u^p$. If $f^p_{s_0}(s_1)=0$ then, by similar considerations as above, we have $x_{s_0}\wedge x_{s_1}\in\dot{J}_\alpha$. Similarly, if $f^p_{s_0}(s_1)=1$ then $x_{s_0}\wedge(-x_{s_1})\in\dot{J}_\alpha$. Hence we may conclude that $x_{s_0}/\dot{J}_\alpha$ is an atom in $\dot{\mathbb{B}}_*/\dot{J}_\alpha$.

Finally, note that the ideal $\dot{J}_{\kappa^+}$ is maximal (as $\{x_s\colon s\in\kappa^+\times\kappa\}$ are generators of the algebra $\dot{\mathbb{B}}_*$) and hence the algebra $\dot{\mathbb{B}}$ is superatomic.

(3) For $\alpha\leqslant\kappa^+$ let $\dot{J}_\alpha$ be the ideal of $\dot{\mathbb{B}}_*$ defined as above. Note that if $a\in\dot{J}_\alpha\setminus\{0\}$ then there is a finite set $v\subseteq\alpha\times\kappa$ such that

$$a\leqslant\bigvee_{t\in v}x_t\quad\text{and}\quad(\forall t\in v)(x_t\wedge a\notin\dot{J}_{\alpha(t)}).$$

A set v with these properties will be called *a good α-cover for a*.

We know already that $x_s/\dot{J}_{\alpha(s)}$ is an atom in $\dot{\mathbb{B}}_*/\dot{J}_{\alpha(s)}$ and therefore either $x_s\wedge b\in\dot{J}_{\alpha(s)}$ or $x_s\wedge(-b)\in\dot{J}_{\alpha(s)}$. We may assume that the first takes place. Applying repeatedly the previous remark find a finite set $v_1\subseteq\alpha(s)\times\kappa$ such that for every $t\in v_1\cup\{s\}$:
1. if $x_t\wedge(x_s\wedge b)\in\dot{J}_{\alpha(t)}\setminus\{0\}$ then there is a good $\alpha(t)$-cover $v\subseteq v_1$ for $x_t\wedge(x_s\wedge b)$,
2. if $x_t\wedge(-x_s\vee -b)\in\dot{J}_{\alpha(t)}\setminus\{0\}$ then there is a good $\alpha(t)$-cover $v\subseteq v_1$ for $x_t\wedge(-x_s\vee -b)$.

Now let $v_0=\{t\in v_1\colon x_t\wedge(x_s\wedge b)\notin\dot{J}_{\alpha(t)}\}$ and check that

$$x_s\wedge b=\bigvee\left\{x_t\wedge\bigwedge_{\substack{t'\in v_1\\ \alpha(t')<\alpha(t)}}(-x_{t'})\colon t\in v_0\right\},$$

as required.

(4) Almost everything what we need for this conclusion was done in clause (2) above except that we have to check that, for each $\alpha<\kappa^+$, $\{x_{\alpha,\xi}/\dot{J}_\alpha\colon \xi<\kappa\}$ lists *all* atoms of the algebra $\dot{\mathbb{B}}_*/\dot{J}_\alpha$. So suppose that $b/\dot{J}_\alpha$ is an atom in $\dot{\mathbb{B}}_*/\dot{J}_\alpha$. We may assume that $b=\bigwedge_{t\in w}x_t\wedge\bigwedge_{t\in u}(-x_t)$ and that $\alpha(t)>\alpha$ for $t\in w$ (otherwise either $b\in\dot{J}_\alpha$ or $b/\dot{J}_\alpha=x_s/\dot{J}_\alpha$ for some s with $\alpha(s)=\alpha$).

Suppose that $w=\emptyset$. Let $p\in\mathbb{P}_\kappa$. We may find a condition $q\geqslant p$ such that $u\subseteq u^q$ and then take $t\in(\{\alpha\}\times\kappa)\setminus u^q$. Exactly as in the proof of Lemma 5.4 we define a condition $r\in\mathbb{P}_\kappa$ stronger than q and such that $t\in u^r$. Note that for this condition we have $r\Vdash x_t\leqslant b$ and we easily finish.

Let $s\in w$ (so $\alpha(s)>\alpha$) and $b^*=\bigwedge_{t\in w\setminus\{s\}}x_t\wedge\bigwedge_{t\in u}(-x_t)$ (so $b=b^*\wedge x_s$). It follows from the third clause that we find finite sets $v_0\subseteq v_1\subseteq\alpha(s)\times\kappa$ such that

$$c\stackrel{\text{def}}{=}\bigvee\left\{x_t\wedge\bigwedge_{\substack{t'\in v_1\\ \alpha(t')<\alpha(t)}}(-x_{t'})\colon t\in v_0\right\}\in\{x_s\wedge b^*,x_s\wedge(-b^*)\}.$$

Now we want to show that there is an element $x_{\alpha,\zeta}$ which is $\dot{J}_\alpha$-smaller than b (which will finish the proof). Let $q\in\mathbb{P}_\kappa$ be such that $w\cup u\cup v_1\subseteq u^q$.

If $c=x_s\wedge(-b^*)$ then we repeat arguments similar to those from the previous paragraph but with a modified version of Lemma 5.4: defining the condition r with the property that $t\in u^r$, we use the function $f^q_s\cup\{(t,1)\}$ as f^r_t (check that no changes are needed in the definition of $y^r_{s_0,s_1}$). Then easily $r\Vdash x_t\leqslant x_s\wedge(-c)=b$.

Finally, if $c = x_s \wedge b^*$ then we take $s' \in v_0$ such that $\alpha(s')$ is maximal possible. If $\alpha(s') > \alpha$ then similarly as in the previous case we find a condition r which forces that $x_t \leqslant x_s \wedge b^* = b$ (just use $f^q_{s'} \cup \{(t,1)\}$). If $\alpha(s') \leqslant \alpha$ it is even easier, as then necessarily $\alpha(s') = \alpha$; now look at $x_{s'}$ and note that $q \Vdash$ "$x_{s'} \wedge -b \leqslant \bigvee_{t' \in v_1, \alpha(t') < \alpha} x_{t'}$".

(5) Look at the demand 5.1(c): it means that if $\alpha, \beta \in \dot{a} \stackrel{\text{def}}{=} \bigcup \{a^p \colon p \in \Gamma_{\mathbb{P}_\kappa}\}$ are distinct then $\dot{f}_{\alpha,0}(\beta,0) = \dot{f}_{\alpha,1}(\beta,0)$. As $\dot{f}_{\alpha,0}(\alpha,0) = 1$, $\dot{f}_{\alpha,1}(\alpha,0) = 0$ we conclude that $\dot{f}_{\alpha,0}, \dot{f}_{\alpha,1}$ determine homomorphisms from $\dot{\mathbb{B}}_*$ to $\{0,1\}$ witnessing $x_{\alpha,0} \notin \langle x_{\beta,0} : \beta \in \dot{a} \setminus \{\alpha\} \rangle_{\dot{\mathbb{B}}_*}$. Since clearly $\Vdash |\dot{a}| = \kappa^+$ the proof is finished. $\square$

Proposition 5.6. *Assume $\kappa^{<\kappa} = \kappa$. Then*

$$\Vdash_{\mathbb{P}_\kappa} \operatorname{inc}(\dot{\mathbb{B}}_*) = s(\dot{\mathbb{B}}_*) = \kappa.$$

Proof. Suppose that $\langle \dot{b}_\alpha \colon \alpha < \kappa^+ \rangle$ is a $\mathbb{P}_\kappa$-name for a κ^+-sequence of elements of $\dot{\mathbb{B}}_*$ and

$$p \Vdash_{\mathbb{P}_\kappa} \text{“}\langle \dot{b}_\alpha \colon \alpha < \kappa^+ \rangle \text{ are pairwise incomparable”.}$$

Applying Δ-lemma and "standard cleaning" choose pairwise isomorphic conditions p_0, p_1, p_2 stronger than p, sets v_1, v_2, a Boolean term τ and $\alpha_1 < \alpha_2 < \kappa^+$ such that

- $\{w^{p_0}, w^{p_1}, w^{p_2}\}$ forms a Δ-system with heart w^*,
- $\sup(w^*) < \min(w^{p_i} \setminus w^*) \leqslant \sup(w^{p_i}) < \min(w^{p_j} \setminus w^*)$ for $i < j < 3$,
- $v_i \in [u^{p_i}]^{<\omega}$ for $i = 1, 2$,
- if $H_{i,j}$ is the isomorphism from p_i to p_j then $v_2 = H_{2,1}[v_1]$,
- $p_i \Vdash$ "$\dot{b}_{\alpha_i} = \tau(x_s : s \in v_i)$" for $i = 1, 2$.

Considering two cases, we are going to define a condition r stronger than p_1, p_2. The condition r will be defined in a similar manner as the condition q in the proof of Proposition 5.3.

Case A: $\{0,1\} \models \tau(0 : t \in v_1) = 0$. First choose $s^* \in u^{p_2} \setminus u^{p_0}$ such that if there is $s \in u^{p_2} \setminus u^{p_0}$ with the property that

$$\{0,1\} \models \tau(f^{p_2}_s(t) : t \in v_2) = 1$$

then s^* is like that.

Now we proceed as in Proposition 5.3 using $f^{p_2}_{s^*}$ instead of $\mathbf{0}_{u^p_2}$. So we let

$$w^r = w^{p_0} \cup w^{p_1} \cup w^{p_2}, \qquad u^r = u^{p_0} \cup u^{p_1} \cup u^{p_2}, \qquad a^r = a^{p_1} \cup a^{p_2},$$

and we define functions f^r_s as follows:

- if $s \in u^{p_0}$ then $f^r_s = f^{p_0}_s \cup f^{p_1}_{H_{0,1}(s)} \cup f^{p_2}_{H_{0,2}(s)}$,
- if $s \in u^{p_1} \setminus u^{p_0}$ then $f^r_s = \mathbf{0}_{u^{p_0}} \cup f^{p_1}_s \cup f^{p_2}_{s^*}$,
- if $s \in u^{p_2} \setminus u^{p_0}$ then $f^r_s = \mathbf{0}_{u^{p_0}} \cup \mathbf{0}_{u^{p_1}} \cup f^{p_2}_s$

(check that the functions f^r_s are well defined). Next, for distinct $s_0, s_1 \in u^r$ such that $\alpha(s_0) \leqslant \alpha(s_1)$, we define the sets $y^r_{s_0,s_1}$:

- if $s_0, s_1 \in u^{p_i}$, $i < 3$ then $y^r_{s_0,s_1} = y^{p_i}_{s_0,s_1}$,
- if $s_0 \in u^{p_1} \setminus u^{p_0}$, $s_1 \in u^{p_2} \setminus u^{p_0}$ then $y^r_{s_0,s_1} = \{H_{1,0}(s_0)\}$,

- if $s_0 \in u^{p_0}$, $s_1 \in u^{p_i}$, $i \in \{1,2\}$ then

$$y^r_{s_0,s_1} = \begin{cases} \emptyset & \text{if } H_{i,0}(s_1) = s_0, \\ \{H_{i,0}(s_1)\} & \text{if } \alpha(H_{i,0}(s_1)) < \alpha(s_0), \\ y^{p_0}_{s_0,H_{i,0}(s_1)} & \text{if } \alpha(s_0) \leqslant \alpha(H_{i,0}(s_1)),\ s_0 \neq H_{i,0}(s_1). \end{cases}$$

Similarly as in Proposition 5.3 one checks that

$$r = \langle w^r, u^r, a^r, \langle f^r_s\colon s \in u^r\rangle, \langle y^r_{s_0,s_1}\colon s_0, s_1 \in u^r,\ s_0 \neq s_1,\ \alpha(s_0) \leqslant \alpha(s_1)\rangle\rangle$$

is a condition in $\mathbb{P}_\kappa$ stronger than both p_1 and p_2. Moreover, it follows from the definition of f^r_s's that

$$\mathbb{B}_r \models \tau(x_t\colon t \in v_1) \leqslant \tau(x_t\colon t \in v_2)$$

(see Proposition 2.2). Consequently $r \Vdash \dot{b}_{\alpha_1} \leqslant \dot{b}_{\alpha_2}$, a contradiction.

Case B: $\{0,1\} \models \tau(0\colon t \in v_1) = 1$. Define r almost exactly like in Case A, except that when choosing $s^* \in u^{p_2} \setminus u^{p_0}$ ask if there is $s \in u^{p_2} \setminus u^{p_0}$ such that

$$\{0,1\} \models \tau(f^{p_2}_s(t)\colon t \in v_2) = 0$$

(and if so then s^* has this property). Continue like before getting a condition r stronger than p_1, p_2 and such that

$$\mathbb{B}_r \models \tau(x_t\colon t \in v_1) \geqslant \tau(x_t\colon t \in v_2)$$

and therefore $r \Vdash \dot{b}_{\alpha_1} \geqslant \dot{b}_{\alpha_2}$, a contradiction finishing the proof. □

Theorem 5.7. *Assume* $\kappa^{<\kappa} = \kappa$. *Then*

$$\Vdash_{\mathbb{P}_\kappa} \mathrm{Id}(\dot{\mathbb{B}}) = 2^\kappa = (2^\kappa)^{\mathbf{V}}.$$

Proof. Let $\mathscr{K}$ be the family of all pairs (p, τ) such that $p \in \mathbb{P}_\kappa$ and $\tau = \tau(x_s : s \in v)$ is a Boolean term, $v \subseteq u^p$. For each ordinal $\alpha < \kappa^+$ we define a relation E^-_α on $\mathscr{K}$ as follows: $(p_0, \tau_0)\, E^-_\alpha\, (p_1, \tau_1)$ if and only if

(i) the conditions p_0, p_1 are isomorphic,
(ii) $w^{p_0} \cap \alpha = w^{p_1} \cap \alpha$,
(iii) if $H : u^{p_0} \to u^{p_1}$ is the isomorphism from p_0 to p_1 then $\tau_1 = H(\tau_0)$ (i.e. $\tau_0 = \tau(x_s\colon s \in v)$, $\tau_1 = \tau(x_{H(s)}\colon s \in v)$).

A relation E_α on $\mathscr{K}$ is defined by

$$(p_0, \tau_0)\, E_\alpha\, (p_1, \tau_1) \quad \text{if and only if} \quad (p_0, \tau_0)\, E^-_\alpha\, (p_1, \tau_1) \text{ and}$$

(iv) if $\beta \in w^{p_0}$ then

$$\beta - \sup(w^{p_0} \cap \beta) = H(\beta) - \sup(w^{p_1} \cap H(\beta)) \bmod \kappa$$

and

$$\beta \geqslant \sup(w^{p_0} \cap \beta) + \kappa \quad \text{if and only if } H(\beta) \geqslant \sup(w^{p_1} \cap H(\beta)) + \kappa.$$

Claim 5.7.1. *For each $\alpha<\kappa^+$, E_α, E^-_α are equivalence relations on $\mathscr{K}$ with κ many equivalence classes.*

Claim 5.7.2. *Suppose that $\alpha<\kappa^+$, $(p_0,\tau_0)\ E_\alpha\ (p_1,\tau_1)$ and $p_0\leqslant q_0$. Then there is $q_1\in\mathbb{P}_\kappa$ such that $p_1\leqslant q_1$ and $(q_0,\tau_0)\ E^-_\alpha\ (q_1,\tau_1)$.*

Claim 5.7.3. *Suppose that $\dot{I}$ is a $\mathbb{P}_\kappa$-name for an ideal in the algebra $\dot{\mathbb{B}}_*$ and let $\mathscr{K}(\dot{I})=\{(p,\tau)\in\mathscr{K}\colon\ p\Vdash\tau\in\dot{I}\}$. Then there is $\alpha=\alpha(\dot{I})<\kappa^+$ such that*

$$\mathscr{K}(\dot{I})=\bigcup\{(p,\tau)/E_\alpha\colon\ (p,\tau)\in\mathscr{K}(\dot{I})\}.$$

Proof of the Claim. Assume not. Then for each $\alpha<\kappa^+$ we find $(p^\alpha_0,\tau^\alpha_0)\in\mathscr{K}(\dot{I})$ and $(p^\alpha_1,\tau^\alpha_1)\notin\mathscr{K}(\dot{I})$ such that $(p^\alpha_0,\tau^\alpha_0)\ E_\alpha\ (p^\alpha_1,\tau^\alpha_1)$. Take $q^\alpha_1\geqslant p^\alpha_1$ such that $q^\alpha_1\Vdash\tau^\alpha_1\notin\dot{I}$ and use Claim 5.7.2 to find $q^\alpha_0\geqslant p^\alpha_0$ such that $(q^\alpha_0,\tau^\alpha_0)\ E^-_\alpha\ (q^\alpha_1,\tau^\alpha_1)$. Now use the Δ-system lemma and clause (i) of the definition of E^-_α to find $\alpha_0<\alpha_1<\alpha_2<\alpha_3<\kappa^+$ such that letting $q_2=q^{\alpha_2}_1$, $\tau_2=\tau^{\alpha_2}_1$ and $q_i=q^{\alpha_i}_0$, $\tau_i=\tau^{\alpha_i}_0$ for $i\neq 2$ we have

- the conditions $q_0,\dots,q_3$ are pairwise isomorphic (and for $i,j<4$ let $H_{i,j}:u^{q_i}\to u^{q_j}$ be the isomorphism from q_i to q_j),
- $\{w^{q_0},w^{q_1},w^{q_2},w^{q_3}\}$ forms a Δ-system with heart w^*,
- $\sup(w^*)<\min(w^{q_i}\setminus w^*)\leqslant\sup(w^{q_i}\setminus w^*)<\min(w^{q_j}\setminus w^*)$ when $i<j<4$,
- $\tau_i=H_{i,j}(\tau_j)$ (i.e. we have the same term).

Now we define a condition $q\in\mathbb{P}_\kappa$ in a similar manner as in Propisitions 5.3 and 5.6. First we fix $s^*\in u^{q_3}\setminus u^{q_0}$ such that

if there is $s\in u^{q_3}\setminus u^{q_0}$ with the property that $f^{q_3}_s(\tau_3)=1$
then s^* is like that.

We put

$$w^q=w^{q_0}\cup\dots\cup w^{q_3},\qquad u^q=u^{q_0}\cup\dots\cup u^{q_3},\qquad a^q=a^{q_1}\cup a^{q_2}\cup a^{q_3},$$

and we define f^q_s as follows:

$$f^q_s=\begin{cases}\bigcup_{i<4}f^{q_i}_{H_{0,i}}(s) & \text{if } s\in u^{q_0},\\ \mathbf{0}_{u^{q_0}}\cup f^{q_1}_s\cup f^{q_2}_{H_{3,2}(s^*)}\cup f^{q_3}_{s^*} & \text{if } s\in u^{q_1}\setminus u^{q_0},\\ \mathbf{0}_{u^{q_0}}\cup\mathbf{0}_{u^{q_1}}\cup f^{q_2}_s\cup f^{q_3}_{s^*} & \text{if } s\in u^{q_2}\setminus u^{q_0},\\ \mathbf{0}_{u^{q_0}}\cup\mathbf{0}_{u^{q_1}}\cup\mathbf{0}_{u^{q_2}}\cup f^{q_3}_s & \text{if } s\in u^{q_3}\setminus u^{q_0}.\end{cases}$$

Finally, for distinct $s_0,s_1\in u^q$ such that $\alpha(s_0)\leqslant\alpha(s_1)$, we define

$$y^q_{s_0,s_1}=\begin{cases}y^{q_i}_{s_0,s_1} & \text{if } s_0,s_1\in u^{q_i},\ i<4,\\ \{H_{i,0}(s_0)\} & \text{if } s_0\in u^{q_i}\setminus u^{q_0},\ s_1\in u^{q_j}\setminus u^{q_0},\ 0<i<j<4,\\ \emptyset & \text{if } s_0\in u^{q_0},\ s_1\in u^{q_i},\ 0<i<4,\ H_{i,0}(s_1)=s_0,\\ \{H_{i,0}(s_1)\} & \text{if } s_0\in u^{q_0},\ s_1\in u^{q_i},\ 0<i<4,\ \alpha(H_{i,0}(s_1))<\alpha(s_0),\\ y^{q_0}_{s_0,H_{i,0}(s_1)} & \text{otherwise}.\end{cases}$$

It should be a routine to check that this defines a condition $q\in\mathbb{P}_\kappa$ stronger than q_1, q_2, q_3 and that (by Proposition 2.2) $\mathbb{B}_q\models\tau_2\leqslant\tau_1\vee\tau_3$ (remember that the terms are isomorphic). But this means that

$$q \Vdash_{\mathbb{P}_\kappa} \text{“}\tau_1^{\alpha_2} \leqslant \tau_0^{\alpha_1} \vee \tau_0^{\alpha_3} \ \&\ \tau_1^{\alpha_2} \notin \dot{I} \ \&\ \tau_0^{\alpha_1}, \tau_0^{\alpha_3} \in \dot{I}\text{”},$$

a contradiction finishing the proof of the claim. □

Now, using Claim 5.7.3, we may easily finish: if $\dot{I}_0$, $\dot{I}_1$ are $\mathbb{P}_\kappa$-names for ideals in $\dot{\mathbb{B}}_*$ such that $\mathscr{K}(\dot{I}_0) = \mathscr{K}(\dot{I}_1)$ then $\Vdash \dot{I}_0 = \dot{I}_1$. But Claim 5.7.3 says that $\mathscr{K}(\dot{I})$ is determined by $\alpha(\dot{I})$ and a family of equivalence classes of $E_{\alpha(\dot{I})}$. So we have at most $\kappa^+ \cdot 2^\kappa = 2^\kappa$ possibilities for $\mathscr{K}(\dot{I})$. Finally note that $|\mathbb{P}_\kappa| = \kappa^+$ and $\mathbb{P}_\kappa$ satisfies the κ^+-cc, so $\Vdash_{\mathbb{P}_\kappa} 2^\kappa = (2^\kappa)^{\mathbf{V}}$. □

Conclusion 5.8. *It is consistent that there is a superatomic Boolean algebra* $\mathbb{B}$ *such that*

$$s(\mathbb{B}) = \mathrm{inc}(\mathbb{B}) = \kappa, \qquad \mathrm{irr}(\mathbb{B}) = \mathrm{Id}(\mathbb{B}) = \kappa^+ \quad \textit{and} \quad \mathrm{Sub}(\mathbb{B}) = 2^{\kappa^+}.$$

6. Modifications of $\mathbb{P}_\kappa$

In this section we modify the forcing notion $\mathbb{P}_\kappa$ of Definition 5.1 and we get two new models. The first model shows the consistency of “there is a superatomic Boolean algebra $\mathbb{B}$ such that $\mathrm{irr}(\mathbb{B}) < \mathrm{inc}(\mathbb{B})$” answering [11, Problem 79]. Next we solve [11, Problem 81] showing that possibly there is a superatomic Boolean algebra $\mathbb{B}$ with $\mathrm{Aut}(\mathbb{B}) < \mathrm{t}(\mathbb{B})$.

Definition 6.1. Let κ be a cardinal. A forcing notion $\mathbb{P}^0_\kappa$ is defined like $\mathbb{P}_\kappa$ of Definition 5.1 but the demand 5.1(c) is replaced by
(c^0) if $\alpha < \beta$, $\alpha, \beta \in a^p$ then $(\exists s \in u^p)(f^p_s(\alpha, 0) = 1 \ \&\ f^p_s(\beta, 0) = 0)$.

Naturally we have a variant of Definition 5.2 of isomorphic conditions for the forcing notion $\mathbb{P}^0_\kappa$ (with no changes) and similarly as for the case of $\mathbb{P}_\kappa$ we define algebras $\mathbb{B}_p$ (for $p \in \mathbb{P}^0_\kappa$) and $\mathbb{P}^0_\kappa$-names $\dot{\mathbb{B}}^0_*$, $\dot{f}^0_s$ (for $s \in \kappa^+ \times \kappa$).

Proposition 6.2. *Assume* $\kappa^{<\kappa} = \kappa$. *Then* $\mathbb{P}_\kappa$ *is a* κ*-complete* κ^+*-cc forcing notion of size* κ^+.

Proof. Repeat the proof of Proposition 5.3 (with no changes). □

Proposition 6.3. *Assume* $\kappa^{<\kappa} = \kappa$. *Then in* $\mathbf{V}^{\mathbb{P}_\kappa}$:
1. $\dot{\mathbb{B}}^0_*$ *is the algebra* $\mathbb{B}_{(W, \dot{F})}$, *where* $W = \kappa^+ \times \kappa$ *and* $\dot{F} = \{\dot{f}^0_s\colon s \in \kappa^+ \times \kappa\} \cup \{\mathbf{0}_{\kappa^+ \times \kappa}\}$,
2. *the algebra* $\dot{\mathbb{B}}^0_*$ *is superatomic* (*of height* κ^+) *and* $\{x_{\alpha,\xi}\colon \xi \in \kappa\}$ *are representatives of atoms of rank* $\alpha + 1$,
3. $\mathrm{inc}(\dot{\mathbb{B}}^0_*) = \kappa^+$.

Proof. The proofs of the first two clauses are repetitions of that of Proposition 5.5(1)–(4) (so we have the respective version of Proposition 5.5(3) too).

To show the third clause let $\dot{a} \stackrel{\rm def}{=} \bigcup\{a^p\colon p\in\Gamma_{\mathbb{P}^0_\kappa}\}$. It should be clear that $\Vdash |\dot{a}|=\kappa^+$. Note that if $\alpha,\beta\in a^p$, $\alpha<\beta$ then, by Definition 6.1(c^0), $\mathbb{B}_p\models x_{\alpha,0}\not\leqslant x_{\beta,0}$ and by the respective variant of Definition 5.1(b) we have $\mathbb{B}_p\models x_{\beta,0}\not\leqslant x_{\alpha,0}$. Consequently the sequence $\langle x_{\alpha,0}\colon \alpha\in\dot{a}\rangle$ witnesses $\mathrm{inc}(\dot{\mathbb{B}}^0_*)=\kappa^+$. □

Proposition 6.4. *Assume* $\kappa^{<\kappa}=\kappa$. *Then* $\Vdash_{\mathbb{P}^0_\kappa} \mathrm{irr}_3^+(\dot{\mathbb{B}}^0_*)=\kappa^+$.

Proof. Let $\langle \dot{b}_\alpha\colon \alpha<\kappa^+\rangle$ be a $\mathbb{P}^0_\kappa$-name for a κ^+-sequence of elements of $\dot{\mathbb{B}}^0_*$ and let $p\in\mathbb{P}^0_\kappa$. Find pairwise isomorphic conditions p_i, sets v_i, ordinals α_i (for $i<7$) and a Boolean term τ such that

- $p\leqslant p_0,\dots,p_7$, $\alpha_0<\alpha_1<\dots<\alpha_6<\kappa^+$, $v_i\in[u^{p_i}]^{<\omega}$ for $i<7$,
- $\{w^{p_0},\dots,w^{p_6}\}$ forms a Δ-system with heart w^*,
- $\sup(w^*)<\min(w^{p_i}\setminus w^*)\leqslant\sup(w^{p_i})<\min(w^{p_j}\setminus w^*)$ for $i<j<7$,
- if $H_{i,j}$ is the isomorphism from p_i to p_j then $v_j=H_{i,j}[v_i]$ (for $i,j<7$),
- $p_i\Vdash$ "$\dot{b}_{\alpha_i}=\tau(x_s\colon s\in v_i)$" for $i<7$.

Now we are going to define an upper bound q to the conditions $p_3,\dots,p_6$. For this we let

$$w^q=\bigcup_{i<7}w^{p_i},\qquad u^q=\bigcup_{i<7}u^{p_i},\qquad a^q=\bigcup_{2<i<7}a^{p_i}$$

and for $s\in w^q$ we define

$$f^q_s=\begin{cases}\bigcup_{j<7}f^{p_j}_{H_{0,j}(s)} & \text{if } s\in u^{p_0},\\ \mathbf{0}_{u^{p_0}\cup u^{p_2}\cup u^{p_4}}\cup f^{p_1}_s\cup f^{p_3}_{H_{1,3}(s)}\cup f^{p_5}_{H_{1,5}(s)}\cup f^{p_6}_{H_{1,6}(s)} & \text{if } s\in u^{p_1}\setminus u^{p_0},\\ \mathbf{0}_{u^{p_0}\cup u^{p_1}\cup u^{p_5}}\cup f^{p_2}_s\cup f^{p_3}_{H_{2,3}(s)}\cup f^{p_4}_{H_{2,4}(s)}\cup f^{p_6}_{H_{2,6}(s)} & \text{if } s\in u^{p_2}\setminus u^{p_0},\\ \mathbf{0}_{u^{p_0}\cup u^{p_1}\cup u^{p_2}\cup u^{p_6}}\cup f^{p_3}_s\cup f^{p_4}_{H_{3,4}(s)}\cup f^{p_5}_{H_{3,5}(s)} & \text{if } s\in u^{p_3}\setminus u^{p_0},\\ \mathbf{0}_{u^q\setminus u^{p_i}}\cup f^{p_i}_s & \text{if } s\in u^{p_i}\setminus u^{p_0},\ 3<i.\end{cases}$$

Next, for distinct $s_0,s_1\in u^q$ such that $\alpha(s_0)\leqslant\alpha(s_1)$, we define $y^q_{s_0,s_1}$ considering all possible configurations separately. Thus we put

- if $s_0,s_1\in u^{p_i}$, $i<7$ then $y^q_{s_0,s_1}=y^{p_i}_{s_0,s_1}$,
- if $s_0\in u^{p_0}\setminus u^{p_1}$, $s_1\in u^{p_i}\setminus u^{p_0}$, $0<i<7$ then

$$y^q_{s_0,s_1}=\begin{cases}\emptyset & \text{if } H_{i,0}(s_1)=s_0,\\ \{H_{i,0}(s_1)\} & \text{if } \alpha(H_{i,0}(s_1))<\alpha(s_0),\\ y^{p_0}_{s_0,H_{i,0}(s_1)} & \text{otherwise.}\end{cases}$$

- if $s_0\in u^{p_i}\setminus u^{p_0}$, $s_1\in u^{p_j}\setminus u^{p_0}$, $0<i<j<7$ then

$$y^q_{s_0,s_1}=\begin{cases}\{H_{i,k}(s_0)\colon k<i\} & \text{if } H_{j,i}(s_1)=s_0,\\ \{H_{i,k}(s_0)\colon k<i\}\cup\{H_{j,i}(s_1)\} & \text{if } \alpha(H_{j,i}(s_1))<\alpha(s_0),\\ \{H_{i,k}(s_0)\colon k<i\}\cup y^{p_i}_{s_0,H_{j,i}(s_1)} & \text{otherwise.}\end{cases}$$

It is not difficult to check that the above formulas define a condition $q\in\mathbb{P}^0_\kappa$ stronger than p_3, p_4, p_5, p_6 (just check all possible cases). Moreover, applying Proposition 2.2, one sees that

$$\begin{aligned}\mathbb{B}_q\models\tau(x_s\colon s\in v_3)=&\ (\tau(x_s\colon s\in v_4)\wedge\tau(x_s\colon s\in v_5))\\ &\vee(\tau(x_s\colon s\in v_4)\wedge\tau(x_s\colon s\in v_6))\\ &\vee(\tau(x_s\colon s\in v_5)\wedge\tau(x_s\colon s\in v_6)).\end{aligned}$$

Hence

$$q\Vdash_{\mathbb{P}^0_\kappa}\text{“}\dot b_{\alpha_3}\in\langle\dot b_{\alpha_4},\dot b_{\alpha_5},\dot b_{\alpha_6}\rangle_{\dot{\mathbb{B}}^0_*}\text{”},$$

finishing the proof. □

Conclusion 6.5. *It is consistent that there is a superatomic Boolean algebra* $\mathbb{B}$ *such that* $\mathrm{inc}(\mathbb{B})=\kappa^+$ *and* $\mathrm{irr}(\mathbb{B})=\kappa$.

For the next model we need a more serious modification of $\mathbb{P}_\kappa$ involving a change in the definition of the order.

Definition 6.6. For an uncountable cardinal κ we define a forcing notion $\mathbb{P}^1_\kappa$ like $\mathbb{P}_\kappa$ of Definition 5.1 except that the clause 5.1(c) is replaced by
(c^1) if $\alpha<\beta$, $\alpha,\beta\in a^p$ then $f^p_{\alpha,0}(\beta,0)=1$
and we add the following requirement
(e) if $(1,\xi)\in u^p$ then the set $\{\zeta<\kappa\colon (0,\zeta)\in u^p\ \&\ f^p_{0,\zeta}(1,\xi)=1\}$ is infinite.
Moreover, we change the definition of the order demanding additionally that, if $p\leqslant q$,
(α) if $(1,\xi)\in u^p$, $(0,\zeta)\in u^q\setminus u^p$ then $f^q_{0,\zeta}(1,\xi)=0$, and
(β) if $(1,\xi)\in u^q\setminus u^p$ then the set $\{(0,\zeta)\in u^p\colon f^q_{0,\zeta}(1,\xi)=1\}$ is finite.

Like before we have the respective variants of Proposition 5.3, Lemma 5.4 and Proposition 5.5 for $\mathbb{P}^1_\kappa$ which we formulate below. The $\mathbb{P}^1_\kappa$-names $\dot{\mathbb{B}}^1_*$ and $\dot f^1_s$ are defined like $\dot{\mathbb{B}}_*$ and $\dot f_s$.

Proposition 6.7. *Assume* $\omega_0<\kappa=\kappa^{<\kappa}$. *Then* $\mathbb{P}^1_\kappa$ *is a* κ*-complete* κ^+*-cc forcing notion of size* κ^+.

Proof. Repeat the arguments of Proposition 5.3 with the following small adjustments. First note that we may assume $|w^*|>2$. Next, if $a^{p_2}\setminus w^*\neq\emptyset$ then we let $\alpha=\min(a^{p_2}\setminus w^*)$ and defining f^q_s for $s\in u^{p_1}\setminus u^{p_0}$ we put $f^q_s=\mathbf{0}_{u^{p_0}}\cup f^{p_1}_s\cup f^{p_2}_{\alpha,0}$. (No other changes needed.) □

Proposition 6.8. *Assume* $\omega_0<\kappa=\kappa^{<\kappa}$. *Then in* $\mathbf{V}^{\mathbb{P}^1_\kappa}$:
(1) $\dot{\mathbb{B}}^1_*$ *is the algebra* $\mathbb{B}_{(W,\dot F)}$, *where* $W=\kappa^+\times\kappa$ *and* $\dot F=\{\dot f^1_s\colon s\in\kappa^+\times\kappa\}\cup\{\mathbf{0}_{\kappa^+\times\kappa}\}$,
(2) *the algebra* $\dot{\mathbb{B}}^1_*$ *is superatomic*,

(3) *if* $s\in\kappa^+\times\kappa$ *and* $b\in\dot{\mathbb{B}}^1_*$ *then there are finite* $v_0\subseteq v_1\subseteq\alpha(s)\times\kappa$ *such that either* $x_s\wedge b$ *or* $x_s\wedge(-b)$ *equals to*

$$\bigvee\left\{x_t\wedge\bigwedge_{\substack{t'\in v_1\\ \alpha(t')<\alpha(t)}}(-x_{t'})\colon t\in v_0\right\},$$

(4) *the height of* $\dot{\mathbb{B}}^1_*$ *is* κ^+ *and* $\{x_{\alpha,\xi}\colon\xi\in\kappa\}$ *are representatives of atoms of rank* $\alpha+1$,
(5) $\mathrm{t}(\dot{\mathbb{B}}^1_*)=\kappa^+$.

Proof. (1)–(3) Repeat the arguments of Proposition 5.5(1)–(3) with no changes.

(4) Like Proposition 5.5(4), but the cases $\alpha=0$ and $\alpha=1$ are considered separately (for $\alpha>1$ no changes are required).

(5) Let $\dot{a}\stackrel{\mathrm{def}}{=}\bigcup\{a^p\colon p\in\Gamma_{\mathbb{P}_\kappa}\}$ and look at the sequence $\langle -x_{\alpha,0}\colon\alpha\in\dot{a}\rangle$. It easily follows from Definition 6.6(c[1]) that it is a free sequence (so it witnesses $\mathrm{t}(\dot{\mathbb{B}}^1_*)=\kappa^+$). $\square$

Theorem 6.9. *Assume* $\omega_0<\kappa=\kappa^{<\kappa}$. *Then* $\Vdash_{\mathbb{P}^1_\kappa}$ "$\mathrm{Aut}(\dot{\mathbb{B}}^1_*)=\kappa$".

Proof. It follows from Proposition 6.7 that, in $\mathbf{V}^{\mathbb{P}^1_\kappa}$, $\kappa=\kappa^{<\kappa}$. By Proposition 6.8(2), (4) we have that each automorphism of $\dot{\mathbb{B}}^1_*$ is determined by its values on atoms of $\dot{\mathbb{B}}^1_*$ and $\{x_{0,\xi}\colon\xi<\kappa\}$ is the list of the atoms of $\dot{\mathbb{B}}^1_*$. Therefore it is enough to show that in $\mathbf{V}^{\mathbb{P}^1_\kappa}$:

$$\text{if }\ \dot{h}\colon\dot{\mathbb{B}}^1_*\to\dot{\mathbb{B}}^1_*\text{ is an automorphism then }|\{\xi<\kappa\colon\ \dot{h}(x_{0,\xi})\neq x_{0,\xi}\}|<\kappa.$$

So assume that $\dot{h}$ is a $\mathbb{P}^1_\kappa$-name for an automorphism of the algebra $\dot{\mathbb{B}}^1_*$ and $p\in\mathbb{P}^1_\kappa$ is such that $0,1\in w^p$. Now we consider three cases.

Case A: For each $q\geqslant p$ there are $r\in\mathbb{P}^1_\kappa$ and distinct $\xi,\zeta<\kappa$ such that

$$q\leqslant r,\quad (0,\xi),(0,\zeta)\in u^r\setminus u^q,\quad f^r_{0,\xi}\restriction u^q\equiv\mathbf{0}\quad\text{and}\quad r\Vdash_{\mathbb{P}^1_\kappa}\text{“ }\dot{h}(x_{0,\xi})=x_{0,\zeta}\text{”}.$$

Construct inductively a sequence $\langle q_n,\xi_n,\zeta_n\colon n<\omega\rangle$ such that
- $q_n\in\mathbb{P}^1_\kappa,\ \xi_n,\zeta_n<\kappa,\ \xi_n\neq\zeta_n,\ p=q_0\leqslant q_1\leqslant q_2\leqslant\cdots,$
- $(0,\xi_n),(0,\zeta_n)\in u^{q_{n+1}}\setminus u^{q_n}$ and $f^{q_{n+1}}_{0,\xi_n}\restriction u^{q_n}\equiv\mathbf{0}$,
- $q_{n+1}\Vdash$ “ $\dot{h}(x_{0,\xi_n})=x_{0,\zeta_n}$”.

Choose $\xi<\kappa$ such that $(1,\xi)\notin\bigcup_{n<\omega}u^{q_n}$. Now we are defining a condition $r\in\mathbb{P}^1_\kappa$. First we put

$$w^r=\bigcup_{n<\omega}w^{q_n},\qquad u^r=\{(1,\xi)\}\cup\bigcup_{n<\omega}u^{q_n}\quad\text{and}\quad a^r=\bigcup_{n<\omega}a^{q_n}.$$

Next for $s\in u^r$ we put

$$f^r_s=\begin{cases}\{\langle(1,\xi),1\rangle\}\cup\bigcup_{m>n}f^{q_m}_s & \text{if }\ s=(0,\xi_n),\ n\in\omega,\\ \{\langle(1,\xi),0\rangle\}\cup\bigcup_{m>n}f^{q_m}_s & \text{if }\ s\in u^{q_n}\setminus\{(0,\xi_\ell):\ell\leqslant n\},\ n\in\omega,\\ \mathbf{0}_{u^r\setminus\{s\}}\cup\{\langle s,1\rangle\} & \text{if }\ s=(1,\xi).\end{cases}$$

Furthermore, if $s_0, s_1 \in u^r$ are distinct and such that $\alpha(s_0) \leqslant \alpha(s_1)$ then we define $y^r_{s_0,s_1}$ as follows:
- if $(1,\xi) \notin \{s_0, s_1\}$ then $y^r_{s_0,s_1} = y^{q_n}_{s_0,s_1}$, where $n<\omega$ is such that $s_0, s_1 \in u^{q_n}$,
- if $s_0 = (1,\xi)$, $s_1 \in u^{q_n}$, $n<\omega$ then $y^r_{s_0,s_1} = \{(0,\ \xi_m)\colon\ m \leqslant n\}$,
- if $s_1 = (1,\xi)$, $s_0 \in u^{q_n}$, $\alpha(s_0)=1$, $n<\omega$ then $y^r_{s_0,s_1} = \{(0,\ \xi_m)\colon\ m \leqslant n\}$.

It is not difficult to check that the above formulas define a condition $r \in \mathbb{P}^1_\kappa$ stronger than all q_n (verifying Definition 5.1(d) remember that $f^{q_{n+1}}_{0,\xi_n} \restriction u^{q_n} \equiv \mathbf{0}$). Note that $r \Vdash (\forall n<\omega)(x_{0,\xi_n} \leqslant x_{1,\xi})$ and hence $r \Vdash (\forall n<\omega)(x_{0,\zeta_n} \leqslant \dot{h}(x_{1,\xi}))$. Take a condition r^* stronger than r and such that for some $\zeta<\kappa$ we have $(1,\zeta) \in u^{r^*}$ and $r^* \Vdash \dot{h}(x_{1,\xi})/\dot{J}_1 = x_{1,\zeta}/\dot{J}_1$, where $\dot{J}_1$ is the ideal of $\dot{\mathbb{B}}^1_*$ generated by atoms (remember Proposition 6.8(4)). Then for some N we have $r^* \Vdash (\forall n \geqslant N)(x_{0,\zeta_n} \leqslant x_{1,\zeta})$. Now look at the definition of the order in $\mathbb{P}^1_\kappa$: by Definition 6.6(β) we have $(1,\zeta) \in u^r$. If $(1,\zeta) \in u^{q_n}$ for some $n<\omega$ then we get immediate contradiction with Definition 6.6(α), so the only possibility is that $\xi=\zeta$. But then look at the definition of the functions f^r_{0,ζ_n} – they all take value 0 at $(1,\xi)$ so $r \Vdash x_{0,\zeta_n} \not\leqslant x_{1,\xi}$, a contradiction. Thus necessarily Case A does not hold.

Case B: There are $p^* \geqslant p$ and $t \in u^{p^*}$ such that for each $q \geqslant p^*$ there are $r \in \mathbb{P}^1_\kappa$ and distinct $\xi, \zeta < \kappa$ with:

$$q \leqslant r,\ (0,\ \xi), (0,\zeta) \in u^r \setminus u^q,\ r \Vdash_{\mathbb{P}^1_\kappa} \text{“}\dot{h}(x_{0,\xi}) = x_{0,\zeta}\text{”},\ f^r_{0,\xi}(t)=1 \text{ and}$$
$$(\forall s \in u^q)(\alpha(s)<\alpha(t) \Rightarrow f^r_{0,\xi}(s)=0).$$

First note that (by Definition 6.6(α)) necessarily $\alpha(t)>1$. Now apply the procedure of Case A with the following modifications. Choosing q_n, ξ_n, ζ_n we demand that $q_0 = p^*$, $f^{q_{n+1}}_{0,\xi_n}(t)=1$ and $(\forall s \in u^{q_n})(\alpha(s)<\alpha(t) \Rightarrow f^{q_{n+1}}_{0,\xi_n}(s)=0)$. Next, defining the condition r we declare that $f^r_{1,\xi} = \bigcup_{n<\omega} f^{q_n}_t \cup \{\langle(1,\xi),1\rangle\}$ and in the definition of $y^r_{s_0,s_1}$ we let
- if $s_0=(1,\xi)$ and either $s_1=t$ or $\alpha(s_1)<\alpha(t)$ then $y^r_{s_0,s_1}=\emptyset$,
- if $s_0=(1,\xi)$ and $\alpha(s_1) \geqslant \alpha(t)$, $s_1 \neq t$ then $y^r_{s_0,s_1} = y^{q_n}_{t,s_1}$, where $n<\omega$ is such that $s_1 \in u^{q_n}$.

Continuing as in the Case A we get a contradiction.

Case C: Neither Case A nor Case B hold.

Let $q_0 \geqslant p$ witness that Case A fails. So for each $r \geqslant q_0$ and distinct $\xi,\zeta<\kappa$ such that $(0,\ \xi), (0,\zeta) \in u^r \setminus u^{q_0}$ if $r \Vdash \dot{h}(x_{0,\xi}) = x_{0,\zeta}$ then $(\exists t \in u^{q_0})(f^r_{0,\xi}(t)=1)$.

Now, since Case B fails and $\mathbb{P}^1_\kappa$ is κ-complete (and $\kappa>\omega$) we may build a condition $q_1 \geqslant q_0$ such that if

$$t \in u^{q_1},\ r \geqslant q_1,\ (0,\ \xi), (0,\zeta) \in u^r \setminus u^{q_1},\ r \Vdash \dot{h}(x_{0,\xi}) = x_{0,\zeta},\ f^r_{0,\xi}(t)=1$$

and

$$(\forall s \in u^{q_1})(\alpha(s)<\alpha(t) \Rightarrow f^r_{0,\xi}(s)=0)$$

then $\xi=\zeta$.

Next choose a condition $q_2 \geqslant q_1$ such that

$$q_2 \Vdash_{\mathbb{P}^1_\kappa} \text{“}(\forall (0,\ \xi) \in u^{q_1})(\exists (0,\zeta) \in u^{q_2})(\ \dot{h}(x_{0,\zeta}) = x_{0,\xi})\text{”}.$$

It follows from the choices of q_0 and q_1 that

$$q_2 \Vdash_{\mathbb{P}^1_\kappa} \text{“}(\forall \xi < \kappa)(\dot{h}(x_{0,\xi}) \neq x_{0,\xi} \Rightarrow (0,\xi) \in u^{q_2})\text{”},$$

finishing the proof. □

Conclusion 6.10. *It is consistent that there is a superatomic Boolean algebra* $\mathbb{B}$ *such that* $\mathrm{t}(\mathbb{B}) = \kappa^+$ *and* $\mathrm{Aut}(\mathbb{B}) = \kappa$.

7. When tightness is singular

In this section we will show that, consistently, there is a Boolean algebra with tightness λ and such that there is an ultrafilter with this tightness but there is no free sequence of length λ and no homomorphic image of the algebra has depth λ. This gives partial answers to [11, Problems 13, 41]. Next we show some bounds on possible consistency results here showing that sometimes we may find quotients with depth equal to the tightness of the original algebra.

Let us recall that a sequence $\langle b_\alpha : \alpha < \xi \rangle$ of elements of a Boolean algebra $\mathbb{B}$ is (algebraically) free if for each finite sets $F, G \subseteq \xi$ such that $\max(F) < \min(G)$ we have

$$\mathbb{B} \models \bigwedge_{\alpha \in F} b_\alpha \wedge \bigwedge_{\alpha \in G} (-b_\alpha) \neq 0.$$

Existence of algebraically free sequences of length α is equivalent to the existence of free sequences of length α in the space of ultrafilters $\mathrm{Ult}(\mathbb{B})$.

Before we formulate our main Definition 7.1, let us explain why the cardinal λ there is singular of uncountable cofinality. We want to force a Boolean algebra $\mathbb{B}$ such that $\mathrm{t}(\mathbb{B}) = \lambda$ (and there is an ultrafilter on $\mathbb{B}$ with tightness λ) but there is no free sequence of length λ. It follows from Arhangelskii [1] that we should demand that λ is singular (see [5, Corollary 7.11]). On the other hand, by [11, Theorem 12.2], if λ is singular of countable cofinality and $\mathrm{t}(\mathbb{B}) = \lambda$ then $\mathbb{B}$ has a free sequence of length λ.

Definition 7.1. (1) *A good parameter* is a tuple $S = (\mu, \lambda, \bar{\chi})$ such that μ, λ are cardinals satisfying

$$\mu = \mu^{<\mu} < \mathrm{cf}(\lambda) < \lambda \quad \text{and} \quad (\forall \alpha < \mathrm{cf}(\lambda))(\forall \xi < \mu)(\alpha^{\xi} < \mathrm{cf}(\lambda))$$

and $\bar{\chi} = \langle \chi_i : i < \mathrm{cf}(\lambda) \rangle$ is a strictly increasing sequence of regular cardinals such that $\mathrm{cf}(\lambda) < \chi_0$, $(\forall i < \mathrm{cf}(\lambda))(\chi_i^{<\mu} = \chi_i)$ and $\lambda = \sup_{i < \mathrm{cf}(\lambda)} \chi_i$.

(2) Let $S = (\mu, \lambda, \bar{\chi})$ be a good parameter. Put

$$\mathscr{X}_S = \{(i, \xi) : i < \mathrm{cf}(\lambda) \ \&\ 0 \leqslant \xi \leqslant \chi_i^+\}$$

and define a forcing notion $\mathbb{Q}_S$ as follows.

A condition is a tuple $p=\langle\gamma^p,w^p,u^p,\langle f^p_{i,\xi,\alpha}\colon (i,\xi)\in u^p,\ \alpha<\gamma^p\rangle\rangle$ such that
(a) $\gamma^p<\mu$, $w^p\in[\mathrm{cf}(\lambda)]^{<\mu}$, $u^p\in[\mathscr{X}_S]^{<\mu}$,
(b) $(\forall i\in w^p)((i,0),(i,\chi_i^+)\in u^p)$ and if $(i,\xi)\in u^p$ then $i\in w^p$,
(c) for $(i,\xi)\in u^p$ and $\alpha<\gamma^p$, $f^p_{i,\xi,\alpha}:u^p\to 2$ is a function such that if $\zeta<\xi$, $(i,\zeta)\in u^p$ then $f^p_{i,\xi,\alpha}(i,\zeta)=0$, if $\xi\leqslant\zeta\leqslant\chi_i^+$, $(i,\zeta)\in u^p$ then $f^p_{i,\xi,\alpha}(i,\zeta)=1$, and $f^p_{i,\xi,\alpha}\restriction(u^p\setminus(\{i\}\times\chi_i^+))=f^p_{i,0,\alpha}\restriction(u^p\setminus(\{i\}\times\chi_i^+))$;
the order is given by $p\leqslant q$ if and only if $\gamma^p\leqslant\gamma^q, w^p\subseteq w^q$, $u^p\subseteq u^q$, $f^p_{i,\xi,\alpha}\subseteq f^q_{i,\xi,\alpha}$ (for $(i,\xi)\in u^p$, $\alpha<\gamma^p$) and

$$(\forall(i,\xi,\alpha)\in u^q\times\gamma^q)(f^q_{i,\xi,\alpha}\restriction u^p\in\{f^p_{j,\zeta,\beta}\colon (j,\zeta,\beta)\in u^p\times\gamma^p\}\cup\{\mathbf{0}_{u^p}\}).$$

(3) We say that conditions $p,q\in\mathbb{Q}_S$ are isomorphic if $\gamma^p=\gamma^q$, $\mathrm{otp}(w^p)=\mathrm{otp}(w^q)$ and there is a bijection $H:u^p\to u^q$ (called *the isomorphism from p to q*) such that if $H_0:w^p\to w^q$ is the order preserving mapping then:
(α) $H(i,\xi)=(H_0(i),\zeta)$ for some ζ,
(β) for each $i\in w^p$, the mapping

$$H^i\colon \{\xi\leqslant\chi_i^+\colon (i,\xi)\in u^p\}\to\{\zeta\leqslant\chi^+_{H_0(i)}\colon (H_0(i),\zeta)\in u^q\}$$

given by $H(i,\xi)=(H_0(i),H^i(\xi))$ is the order preserving isomorphism,
(γ) $(\forall\alpha<\gamma^p)(\forall(i,\xi)\in u^p)(f^p_{i,\xi,\alpha}=f^q_{H(i,\xi),\alpha}\circ H)$.

Remark 7.2. (1) Note that there are only μ isomorphism types of conditions in $\mathbb{Q}_S$.
(2) Variants of the forcing notion $\mathbb{Q}_S$ are used in [14] to deal with attainment problems for equivalent definitions of hd, hL.

Proposition 7.3. *Let $S=(\mu,\lambda,\bar\chi)$ be a good parameter. Then $\mathbb{Q}_S$ is a μ-complete μ^+-cc forcing notion.*

Proof. Easily $\mathbb{Q}_S$ is μ-closed. To show the chain condition suppose that $\mathscr{A}\subseteq\mathbb{Q}_S$ is of size μ^+. Since $\mu^{<\mu}=\mu$ we may apply standard cleaning procedure and find isomorphic conditions $p,q\in\mathscr{A}$ such that if $H:u^p\to u^q$ is the isomorphism from p to q and $H_0:w^p\to w^q$ is the order preserving mapping then
- $H_0\restriction w^p\cap w^q$ is the identity on $w^p\cap w^q$, and
- $H\restriction u^p\cap u^q$ is the identity on $u^p\cap u^q$.

Next put $\gamma^r=\gamma^p=\gamma^q$, $w^r=w^p\cup w^q$, $u^r=u^p\cup u^q$. For $(i,\xi)\in u^r$ and $\alpha<\gamma^r$ we define $f^r_{i,\xi,\alpha}$ as follows:
1. if $(i,\xi)\in u^p$, $i\in w^p\setminus w^q$ then $f^r_{i,\xi,\alpha}=f^p_{i,\xi,\alpha}\cup f^q_{H_0(i),0,\alpha}$,
2. if $(i,\xi)\in u^q$, $i\in w^q\setminus w^p$ then $f^r_{i,\xi,\alpha}=f^p_{H_0^{-1}(i),0,\alpha}\cup f^q_{i,\xi,\alpha}$,
3. if $i\in w^p\cap w^q$ then

$$f^r_{i,\xi,\alpha}=(f^p_{i,0,\alpha}\cup f^q_{i,0,\alpha})\restriction(u^r\setminus\{i\}\times\chi_i^+)\cup\mathbf{0}_{(\{i\}\times[0,\xi))\cap u^r}\cup\mathbf{1}_{(\{i\}\times[\xi,\chi_i^+])\cap u^r}.$$

Checking that $r\stackrel{\mathrm{def}}{=}\langle\gamma^r,w^r,u^r,\langle f^r_{i,\xi}\colon (i,\xi)\in u^r\rangle\rangle\in\mathbb{Q}_S$ is a condition stronger than both p and q is straightforward. $\square$

For a condition $p\in\mathbb{Q}_S$ let $\mathbb{B}_p$ be the Boolean algebra $\mathbb{B}_{(u^p,F^p)}$ for

$$F^p \stackrel{\rm def}{=} \{f^p_{i,\xi,\alpha}\colon (i,\xi)\in u^p,\ \alpha<\gamma^p\}\cup\{\mathbf{0}_{u^p}\}$$

(see Definition 2.1). Naturally we define $\mathbb{Q}_S$-names $\dot{\mathbb{B}}^*_S$ and $\dot{f}_{i,\xi,\alpha}$ (for $i<\mathrm{cf}(\lambda)$, $\xi<\chi_i^+$, $\alpha<\mu$) by

$$\Vdash_{\mathbb{Q}_S} \text{“}\dot{\mathbb{B}}^*_S=\bigcup\{\mathbb{B}_p\colon\ p\in\Gamma_{\mathbb{Q}_S}\},\quad \dot{f}_{i,\xi,\alpha}=\bigcup\{f^p_{i,\xi,\alpha}\colon\ (i,\xi,\alpha)\in u^p\times\gamma^p,\ p\in\Gamma_{\mathbb{Q}_S}\}\text{”}.$$

Further, let $\dot{\mathbb{B}}_S$ be the $\mathbb{Q}_S$-name for the subalgebra $\langle x_{i,\xi}\colon\ i<\mathrm{cf}(\lambda),\xi<\chi_i^+\rangle_{\dot{\mathbb{B}}^*_S}$ of $\dot{\mathbb{B}}^*_S$.

Proposition 7.4. *Assume $S=(\mu,\lambda,\bar\chi)$ is a good parameter. Then in $\mathbf{V}^{\mathbb{Q}_S}$:*
(1) $\dot{f}_{i,\xi,\alpha}:\mathscr{X}_S\to 2$ *(for $\alpha<\mu$, $i<\mathrm{cf}(\lambda)$ and $\xi\leqslant\chi_i^+$),*
(2) $\dot{\mathbb{B}}^*_S$ *is the Boolean algebra $\mathbb{B}_{(\mathscr{X}_S,\dot F)}$, where $\dot F=\{\dot f_{i,\xi,\alpha}\colon\ (i,\xi)\in\mathscr{X}_S,\ \alpha<\mu\}$,*
(3) *for each $i<\mathrm{cf}(\lambda)$, the sequence $\langle -x_{i,\xi}\colon\ \xi<\chi_i^+\rangle$ is (algebraically) free in the algebra $\dot{\mathbb{B}}_S$,*
(4) $\mathbf{0}_{\mathscr{X}_S}\in\mathrm{cl}(\dot F)$, *so it determines a homomorphism from $\dot{\mathbb{B}}^*_S$ to 2 (so an ultrafilter). Its restriction $\mathbf{0}_{\mathscr{X}_S}\restriction\dot{\mathbb{B}}_S$ has tightness λ.*

Proof. (1)–(3) Should be clear.

(4) First note that if $p\in\mathbb{Q}_S$ and $i<\mathrm{cf}(\lambda)$ then there is a condition $q\in\mathbb{Q}_S$ stronger than p and such that $i\in w^q$ and

$$(\exists\alpha<\gamma^q)(f^q_{i,0,\alpha}\restriction(u^p\setminus\{i\}\times(\chi_i^++1))\equiv 0).$$

Hence we immediately conclude that $\mathbf{0}_{\mathscr{X}_S}\in\mathrm{cl}(\dot F)$. Now we look at the restriction $\mathbf{0}_{\mathscr{X}_S}\restriction\dot{\mathbb{B}}_S$. First fix $i<\mathrm{cf}(\lambda)$ and let $\dot Y_i=\{\dot f_{i,\xi,\alpha}\restriction\dot{\mathbb{B}}_S\colon\ \xi<\chi_i^+,\ \alpha<\mu\}$ (so $\dot Y_i$ is a family of homomorphisms from $\dot{\mathbb{B}}_S$ to 2 and it can be viewed as a family of ultrafilters on $\dot{\mathbb{B}}_S$). It follows from the previous remark (and Definition 7.1(2c)) that $\mathbf{0}_{\mathscr{X}_S}\restriction\dot{\mathbb{B}}_S\in\mathrm{cl}(\dot Y_i)$. We claim that $\mathbf{0}_{\mathscr{X}_S}\restriction\dot{\mathbb{B}}_S$ is not in the closure of any subset of $\dot Y_i$ of size less than χ_i^+. So assume that $\dot X$ is a $\mathbb{Q}_S$-name for a subset of $\dot Y_i$ such that $\Vdash |\dot X|\leqslant\chi_i$ (and we will think that $\Vdash \dot X\subseteq\chi_i^+\times\mu$). Since $\mathbb{Q}_S$ satisfies the μ^+-cc we find $\xi<\chi_i^+$ such that $\Vdash$ $\dot X\subseteq\xi\times\mu$. Now note that Definition 7.1(2c) implies that $\Vdash (\forall(\zeta,\alpha)\in\dot X)(\dot f_{i,\zeta,\alpha}(i,\xi)=1)$, so $\Vdash \mathbf{0}_{\mathscr{X}_S}\restriction\dot{\mathbb{B}}_S\notin\mathrm{cl}(\dot X)$. Hence the tightness of the ultrafilter $\mathbf{0}_{\mathscr{X}_S}\restriction\dot{\mathbb{B}}_S$ is λ. $\square$

Theorem 7.5. *Assume that $S=(\mu,\lambda,\bar\chi)$ is a good parameter. Then in $\mathbf{V}^{\mathbb{Q}_S}$:*
(1) *there is no algebraically free sequence of length λ in $\dot{\mathbb{B}}_S$,*
(2) *if $\dot I$ is an ideal in $\dot{\mathbb{B}}_S$ then* $\mathrm{Depth}(\dot{\mathbb{B}}_S/\dot I)<\lambda$.

Proof. (1) Assume that $\langle\dot b_\alpha\colon\ \alpha<\lambda\rangle$ is a $\mathbb{Q}_S$-name for a λ-sequence of elements of $\dot{\mathbb{B}}_S$ and $p\in\mathbb{Q}_S$. For each $i<\mathrm{cf}(\lambda)$ and $\xi<\chi_i^+$ choose a condition $p_{i,\xi}\in\mathbb{Q}_S$ stronger than p, a finite set $v_{i,\xi}\subseteq u^{p_{i,\xi}}$ and a Boolean term $\tau_{i,\xi}$ such that

$$p_{i,\xi}\Vdash_{\mathbb{Q}_S}\dot b_{\chi_i+\xi}=\tau_{i,\xi}(x_{j,\zeta}:(j,\zeta)\in v_{i,\xi}).$$

Let us fix $i<\mathrm{cf}(\lambda)$ for a moment. Applying the Δ-system lemma and standard cleaning (and using the assumption that $\chi_i^{<\mu}=\chi_i=\mathrm{cf}(\chi_i)$) we may find a set $Z_i\in[\chi_i^+]^{\chi_i^+}$ such that

$(\alpha)_i$ all conditions $p_{i,\xi}$ for $\xi\in Z_i$ are isomorphic,

$(\beta)_i$ $\{u^{p_{i,\xi}}\colon \xi\in Z_i\}$ forms a Δ-system with heart u_i,

$(\gamma)_i$ if $\xi_0,\xi\in Z_i$ and $H:u^{p_{i,\xi_0}}\to u^{p_{i,\xi_1}}$ is the isomorphism from p_{i,ξ_0} to p_{i,ξ_1} then $H[v_{i,\xi_0}]=v_{i,\xi_1}$ and $H\restriction u_i$ is the identity on u_i,

$(\delta)_i$ $\tau_{i,\xi}=\tau_i$ (for each $\xi\in Z_i$),

$(\varepsilon)_i$ $u^{p_{i,\xi_0}}\cap\{(j,\zeta)\colon j<i\,\&\,\zeta<\chi_j^+\}=u^{p_{i,\xi_1}}\cap\{(j,\zeta)\colon j<i\,\&\,\zeta<\chi_j^+\}$ whenever $\xi_0,\xi_1\in Z_i$.

Apply the cleaning procedure and the Δ-system lemma again to get a set $J\in[\mathrm{cf}(\lambda)]^{\mathrm{cf}(\lambda)}$ such that

$(\alpha)^*$ if $i_0,i_1\in J$, $\xi_0\in Z_{i_0}$, $\xi_1\in Z_{i_1}$ then the conditions $p_{i_0,\xi_0}, p_{i_1,\xi_1}$ are isomorphic,

$(\beta)^*$ $\{u_i:i\in J\}$ forms a Δ-system with heart u^*,

$(\gamma)^*$ if $i_0,i_1\in J$, $\xi_0\in Z_{i_0}$, $\xi_1\in Z_{i_1}$ and $H:u^{p_{i_0,\xi_0}}\to u^{p_{i_1,\xi_1}}$ is the isomorphism from p_{i_0,ξ_0} to p_{i_1,ξ_1} then $H[v_{i_0,\xi_0}]=v_{i_1,\xi_1}$, $H[u_{i_0}]=u_{i_1}$ and $H\restriction u^*$ is the identity on u^*,

$(\delta)^*$ $\tau_i=\tau$ (for $i\in J$)

(remember the assumptions on $\mathrm{cf}(\lambda)$ in Definition 7.1(1)). Now choose $i_0\in J$ such that $\sup\{i<\mathrm{cf}(\lambda)\colon (i,0)\in u^*\}<i_0$ and pick $\xi_0^0,\xi_1^0\in Z_{i_0}$, $\xi_0^0<\xi_1^0$. Next take $i_1\in J$ such that

$$i_1>i_0+\sup\{i<\mathrm{cf}(\lambda):(i,0)\in u^{p_{i_0,\xi_0^0}}\cup u^{p_{i_0,\xi_1^0}}\}$$

and $u_{i_1}\cap(u^{p_{i_0,\xi_0^0}}\cup u^{p_{i_0,\xi_1^0}})=u^*$. Finally pick $\xi_0^1,\xi_1^1\in Z_{i_1}$ such that $\xi_0^1<\xi_1^1$ and, for $\ell<2$,

$$u^{p_{i_1,\xi_\ell^1}}\cap(u^{p_{i_0,\xi_0^0}}\cup u^{p_{i_0,\xi_1^0}})=u^*.$$

To make our notation somewhat simpler let $p_\ell^k=p_{i_k,\xi_\ell^k}$, $\tau_\ell^k=\tau(x_{j,\zeta}\colon (j,\zeta)\in v_{i_k,\xi_\ell^k})$ (for $k,\ell<2$) and let $H^{k_0,\ell_0}_{k_1,\ell_1}:u^{p^{k_0}_{\ell_0}}\to u^{p^{k_1}_{\ell_1}}$ be the isomorphism from $p^{k_0}_{\ell_0}$ to $p^{k_1}_{\ell_1}$ (for $k_0,k_1,\ell_0,\ell_1<2$).

It follows from the choice of i_k,ξ_ℓ^k that:

(i) if $(i,0)\in u^*$, $k<2$, $\xi<\chi_i^+$ then $(i,\xi)\in u^{p_0^k}\Leftrightarrow(i,\xi)\in u^{p_1^k}$,

(ii) if $i\in(w^{p_0^0}\cup w^{p_1^0})\cap(w^{p_0^1}\cup w^{p_1^1})$ then $(i,0)\in u^*$.

Now we are defining a condition q stronger than all p_ℓ^k. So we put $\gamma^q=\gamma^{p_0^0}$, $w^q=w^{p_0^0}\cup w^{p_1^0}\cup w^{p_0^1}\cup w^{p_1^1}$, $u^q=u^{p_0^0}\cup u^{p_1^0}\cup u^{p_0^1}\cup u^{p_1^1}$, and for $(j,\zeta)\in u^q$ and $\alpha<\gamma^q$ we define $f^q_{j,\zeta,\alpha}:u^q\to 2$ in the following manner. We declare that

$$f^q_{j,\zeta,\alpha}\restriction(\{j\}\times[0,\zeta))\cap u^q\equiv 0\quad\text{and}\quad f^q_{j,\zeta,\alpha}\restriction(\{j\}\times[\zeta,\chi_j^+])\cap u^q\equiv 1,$$

and now we define $f^q_{j,\zeta,\alpha}$ on $u^q\setminus(\{j\}\times[0,\chi_j^+])$ letting:

– if $(j,0)\in u^*$ then

$$f^q_{j,\zeta,\alpha}\supseteq\bigcup_{\ell,k<2}f^{p_\ell^k}_{j,0,\alpha}\restriction(u^{p_\ell^k}\setminus\{j\}\times\chi_j^+),$$

[note that in this case we have: $f^q_{j,\zeta,\alpha}(\tau_0^k)=f^q_{j,\zeta,\alpha}(\tau_1^k)$ for $k=0,1$]

– if $(j,0)\in u_{i_0}\setminus u^*$ then

$$f^q_{j,\zeta,\alpha}\supseteq\bigcup_{\ell<2} f^{p^0_\ell}_{j,0,\alpha}\restriction(u^{p^0_\ell}\setminus\{j\}\times\chi^+_j)\cup\bigcup_{\ell<2} f^{p^1_\ell}_{H^{0,0}_{1,0}(j,0),\alpha},$$

[note that then $f^q_{j,\zeta,\alpha}(\tau^1_0)=f^q_{j,\zeta,\alpha}(\tau^1_1)$]

– if $(j,0)\in u^{p^k_\ell}\setminus\bigcup\{u^{p^{k'}_{\ell'}}\colon (k',\ell')\neq(k,\ell),\ k',\ell'<2\}$ then

$$f^q_{j,\zeta,\alpha}=\bigcup_{k',\ell'<2} f^{p^{k'}_{\ell'}}_{H^{k,\ell}_{k',\ell'}(j,\zeta),\alpha},$$

[again, $f^q_{j,\zeta,\alpha}(\tau^1_0)=f^q_{j,\zeta,\alpha}(\tau^1_1)$]

– if $(j,0)\in u_{i_1}\setminus u^*$ and, say, $(j,\zeta)\in u^{p^1_0}$ then let $j^*\in w^{p^0_0}$ be the isomorphic image of j (in the isomorphism from p^1_0 to p^0_0). Choose $\zeta^*<\chi^+_{j^*}$ such that, if possible then, $f^{p^0_0}_{j^*,\zeta^*,\alpha}(\tau^0_0)=0$ (if there is no such ζ^* take $\zeta^*=0$). Let $\zeta'=\min\{\xi\colon (j^*,\xi)\in u^{p^0_1}\ \&\ \xi\geqslant\zeta^*\}$ and

$$f^q_{j,\zeta,\alpha}\supseteq f^{p^0_0}_{j^*,\zeta^*,\alpha}\cup f^{p^0_1}_{j^*,\zeta',\alpha}\cup\bigcup_{\ell<2} f^{p^1_\ell}_{j,0,\alpha}\restriction(u^{p^1_\ell}\setminus\{j\}\times\chi^+_j)$$

[note that $f^q_{j,\zeta,\alpha}(\tau^0_0)\leqslant f^q_{j,\zeta,\alpha}(\tau^1_1)$].

It is a routine to check that $q=\langle\gamma^q,w^q,u^q,\langle f^q_{j,\zeta,\alpha}\colon (j,\zeta)\in u^q,\ \alpha<\gamma^q\rangle\rangle\in\mathbb{Q}_S$ is a condition stronger than all p^k_ℓ. It follows from the remarks on $f^q_{j,\zeta,\alpha}(\tau^1_1)$ we made when we defined $f^q_{j,\zeta,\alpha}$ that, by Proposition 2.2, $\mathbb{B}_q\models\tau^0_0\wedge\tau^0_1\wedge\tau^1_0\leqslant\tau^1_1$. Hence we conclude that q forces that the sequence $\langle\dot b_\alpha\colon\alpha<\lambda\rangle$ is not free as witnessed by $\{\chi_{i_0}+\xi^0_0,\chi_{i_0}+\xi^0_1,\chi_{i_1}+\xi^1_0\}$ and $\{\chi_{i_1}+\xi^1_1\}$.

(2) Suppose that $\dot I$ is a $\mathbb{Q}_S$-name for an ideal in $\dot{\mathbb{B}}_S$ and a condition $p\in\mathbb{Q}_S$ is such that $p\Vdash_{\mathbb{Q}_S}$"Depth$(\dot{\mathbb{B}}_S/\dot I)=\lambda$". Then for each $i<\mathrm{cf}(\lambda)$ we find a $\mathbb{Q}_S$-name $\langle\dot b_{i,\xi}\colon\xi<\chi^+_i\rangle$ for a sequence of elements of $\dot{\mathbb{B}}_S$ such that

$$p\Vdash_{\mathbb{Q}_S}\text{"}(\forall\xi<\zeta<\chi^+_i)(0/\dot I<\dot b_{i,\xi}/\dot I<\dot b_{i,\zeta}/\dot I)\text{"}.$$

Repeat the procedure applied in the previous clause, now with $\dot b_{i,\xi}$ instead of $\dot b_{\chi_i+\xi}$ there, and get $i_0,i_1,\xi^0_0,\xi^0_1,\xi^1_0,\xi^1_1$ as there (and we use the same notation $p^k_\ell,\tau^k_\ell,H^{k_0,\ell_0}_{k_1,\ell_1}$ as before). Now we define a condition q stronger than all the p^k_ℓ. Naturally we let $\gamma^q=\gamma^{p^0_0}$, $w^q=w^{p^0_0}\cup w^{p^0_1}\cup w^{p^1_0}\cup w^{p^1_1}$, $u^q=u^{p^0_0}\cup u^{p^0_1}\cup u^{p^1_0}\cup u^{p^1_1}$. Suppose $(j,\zeta)\in u^q$ and $\alpha<\gamma^q$. We define $f^q_{j,\zeta,\alpha}\colon u^q\to 2$ declaring that

$$f^q_{j,\zeta,\alpha}\restriction(\{j\}\times[0,\zeta))\cap u^q\equiv 0\quad\text{and}\quad f^q_{j,\zeta,\alpha}\restriction(\{j\}\times[\zeta,\chi^+_j])\cap u^q\equiv 1,$$

and

– if $(j,0)\in u^*$ then $f^q_{j,\zeta,\alpha}\supseteq\bigcup_{\ell,k<2} f^{p^k_\ell}_{j,0,\alpha}\restriction(u^{p^k_\ell}\setminus\{j\}\times\chi^+_j)$,

– if $(j,0)\in u^{p^k_\ell}$ but $(j,0)\notin u^{p^{k'}_{\ell'}}$ for $(k',\ell')\neq(k,\ell)$ then

$$f^q_{j,\zeta,\alpha}=\bigcup_{k',\ell'<2} f^{p^{k'}_{\ell'}}_{H^{k,\ell}_{k',\ell'}(j,\zeta),\alpha},$$

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– if $(j,0)\in u_{i_1}\setminus u^*$ then

$$f^q_{j,\zeta,\alpha}\supseteq\bigcup_{\ell<2} f^{p^1_\ell}_{j,0,\alpha}\restriction(u^{p^1_\ell}\setminus\{j\}\times\chi^+_j)\cup\bigcup_{\ell<2} f^{p^0_\ell}_{H^{1,0}_{0,0}(j,0),\alpha},$$

– if $(j,0)\in u_{i_0}\setminus u^*$ then first take $\xi^\ell=\min\{\xi\leqslant\chi^+_j\colon (j,\xi)\in u^{p^0_\ell}\ \&\ \zeta\leqslant\xi\}$ (for $\ell<2$) and next put

$$f^q_{j,\zeta,\alpha}=f^{p^0_0}_{j,\xi^0,\alpha}\cup f^{p^0_1}_{j,\xi^1,\alpha}\cup f^{p^1_0}_{H^{0,1}_{1,0}(j,\xi^1),\alpha}\cup f^{p^1_1}_{H^{0,0}_{1,1}(j,\xi^0),\alpha}$$

(remember that $H^{0,1}_{1,0}[u_{i_0}]=H^{0,0}_{1,1}[u_{i_0}]=u_{i_1}$ and both isomorphisms are the identity on u^*).

It should be a routine to verify that

$$q=\langle\gamma^q,w^q,u^q,\langle f^q_{j,\zeta,\alpha}\colon (j,\zeta)\in u^q,\ \alpha<\gamma^q\rangle\rangle\in\mathbb{Q}_S$$

is a stronger than all p^k_ℓ. Note that the only case when possibly $f^q_{j,\zeta,\alpha}(\tau^0_0)\neq f^q_{j,\zeta,\alpha}(\tau^0_1)$ is $(j,0)\in u_{i_0}\setminus u^*$. But then $f^q_{j,\zeta,\alpha}(\tau^1_0)=f^q_{j,\zeta,\alpha}(\tau^0_1)$ and $f^q_{j,\zeta,\alpha}(\tau^1_1)=f^q_{j,\zeta,\alpha}(\tau^0_0)$. Hence (by Proposition 2.2) $\mathbb{B}_q\models\tau^0_1\wedge(-\tau^0_0)\leqslant\tau^1_0\wedge(-\tau^1_1)$ and therefore

$$q\Vdash_{\mathbb{Q}_S}\text{“}\dot b_{i_0,\xi^0_1}\wedge(-\dot b_{i_0,\xi^0_0})\leqslant\dot b_{i_1,\xi^1_0}\wedge(-\dot b_{i_1,\xi^1_1})\text{”}.$$

Now, $q\Vdash$ “$\dot b_{i_1,\xi^1_0}/\dot I\leqslant\dot b_{i_1,\xi^1_1}/\dot I$” so we conclude $q\Vdash$ “$\dot b_{i_0,\xi^0_1}\wedge(-\dot b_{i_0,\xi^0_0})\in\dot I$”. But the last statement contradicts $q\Vdash$ “$\dot b_{i_0,\xi^0_0}/\dot I<\dot b_{i_0,\xi^0_1}/\dot I$”, finishing the proof. □

Conclusion 7.6. *It is consistent that there is a Boolean algebra $\mathbb{B}$ of size λ such that there is an ultrafilter $x\in\mathrm{Ult}(\mathbb{B})$ of tightness λ, there is no free λ-sequence in $\mathbb{B}$ and* $\mathrm{t}(\mathbb{B})=\lambda\notin\mathrm{Depth}_{\mathrm{Hs}}(\mathbb{B})$ (*i.e. no homomorphic image of $\mathbb{B}$ has depth λ*).

Let us note that in the universe $\mathbf{V}^{\mathbb{Q}_S}$ we have $2^{\mathrm{cf}(\lambda)}\geqslant\lambda$. This is a real limitation – we can prove that $2^{\mathrm{cf}(\lambda)}$ cannot be small in this context. In the proof we will use the following theorem cited here from [17].

Theorem 7.7 (see Shelah [17, Lemma 5.1(3)]). *Assume that* $\lambda=\sup_{i<\mathrm{cf}(\lambda)}\chi_i$, $\mathrm{cf}(\lambda)<\chi_i<\lambda$, $\mu=(2^{\mathrm{cf}(\lambda)})^+$. *Let X be a $T_{3\frac{1}{2}}$ topological space with a basis $\mathcal{B}$. Suppose that φ is a function assigning cardinal numbers to subsets of X such that*:

(i) $\varphi(A)\leqslant\varphi(A\cup B)\leqslant\varphi(A)+\varphi(B)+\omega_0$ *for* $A,B\subseteq X$,

(ii) *for each $i<\mathrm{cf}(\lambda)$ there is a sequence $\langle u_\alpha\colon \alpha<\mu\rangle\subseteq\mathcal{B}$ such that*

$$(\forall g:\mu\to 2^{\mathrm{cf}(\lambda)})(\exists\alpha\neq\beta)(g(\alpha)=g(\beta)\ \&\ \varphi(u_\alpha\setminus\mathrm{cl}_X(u_\beta))\geqslant\chi_i),$$

(iii) *for sufficiently large $\chi<\lambda$, if $\langle A_\alpha\colon \alpha<\mu\rangle$ is a sequence of subsets of X such that $\varphi(A_\alpha)\leqslant\chi$ then $\varphi(\bigcup_{\alpha<\mu}A_\alpha)\leqslant\chi$.*

Then there is a sequence $\langle u_i:i<\mathrm{cf}(\lambda)\rangle\subseteq\mathcal{B}$ such that

$$(\forall i<\mathrm{cf}(\lambda))\left(\varphi(u_i\setminus\bigcup_{j\neq i}u_j)\geqslant\chi_i\right).$$

Theorem 7.8. *Suppose that $\mathbb{B}$ is a Boolean algebra satisfying $2^{\mathrm{cf}(\mathrm{t}(\mathbb{B}))} < \mathrm{t}(\mathbb{B})$. Then for some ideal I on $\mathbb{B}$ we have* $\mathrm{Depth}(\mathbb{B}/I) = \mathrm{t}(\mathbb{B})$.

Proof. Let $\lambda = \mathrm{t}(\mathbb{B})$ and let $\langle \chi_i : i < \mathrm{cf}(\lambda)\rangle$ be an increasing cofinal in λ sequence of successor cardinals, $\chi_0 > (2^{\mathrm{cf}(\lambda)})^+ = \mu$. Further, let X be the Stone space $\mathrm{Ult}(\mathbb{B})$ and thus we may think that $\mathbb{B} = \mathscr{B}$ is a basis of the topology of X. Now define a function φ on subsets of X by

$$\varphi(Y) = \sup\{\kappa\colon \text{there are sequences } \langle y_\zeta\colon \zeta < \kappa\rangle \subseteq Y \text{ and } \langle u_\zeta\colon \zeta<\kappa\rangle \subseteq \mathscr{B} \\ \text{such that } (\forall \zeta, \xi < \kappa)(y_\zeta \in u_\xi \Leftrightarrow \xi < \zeta)\}.$$

We are going to apply Theorem 7.7 to these objects and for this we should check the assumptions there. The only not immediate demands might be (ii) and (iii). So suppose $i < \mathrm{cf}(\lambda)$. Since $\chi_i < \lambda = \mathrm{t}(\mathbb{B})$ we can find a free sequence $\langle u^*_\xi\colon \xi < \chi_i^+\rangle \subseteq \mathbb{B}$. Next, for each $\xi < \chi_i^+$ we may choose an ultrafilter $y_\xi \in X$ such that

$$(\forall \zeta < \chi_i^+)(y_\xi \in u^*_\zeta \quad\Leftrightarrow\quad \zeta < \xi).$$

Now, for $\alpha < \mu$, let $u_\alpha = u^*_{\chi_i\cdot\alpha}$. Suppose $g : \mu \to 2^{\mathrm{cf}(\lambda)}$ and take any $\alpha < \beta < \mu$ such that $g(\alpha) = g(\beta)$. Note that

$$u_\alpha \setminus \mathrm{cl}_X(u_\beta) = u^*_{\chi_i\cdot\alpha} \setminus u^*_{\chi_i\cdot\beta} \supseteq \{y_\xi : \chi_i\cdot\alpha < \xi < \chi_i\cdot(\alpha+1)\}$$

and easily $\varphi(\{y_\xi\colon \chi_i\cdot\alpha < \xi < \chi_i\cdot(\alpha+1)\}) = \chi_i$. Thus $\varphi(u_\alpha\setminus \mathrm{cl}_X(u_\beta)) \geqslant \chi_i$ and the demand 7.7(ii) is verified. Assume now that $\mu < \chi < \lambda$ and $A_\alpha \subseteq X$ (for $\alpha < \mu$) are such that $\varphi(\bigcup_{\alpha<\mu} A_\alpha) > \chi$. Let sequences $\langle y_\xi\colon \xi < \chi^+\rangle \subseteq \bigcup_{\alpha<\mu} A_\alpha$ and $\langle u_\xi\colon \xi<\chi^+\rangle \subseteq \mathbb{B}$ witness this. Then for some $C \in [\chi^+]^{\chi^+}$ and $\alpha < \mu$ we have $\langle y_\xi\colon \xi \in C\rangle \subseteq A_\alpha$ and therefore $\langle y_\xi, u_\xi : \xi \in C\rangle$ witness $\varphi(A_\alpha) \geqslant \chi^+$. This finishes checking the demand 7.7(iii).

So we may use Theorem 7.7 and we get a sequence $\langle u_i\colon i < \mathrm{cf}(\lambda)\rangle \subseteq \mathbb{B}$ such that

$$(\forall i < \mathrm{cf}(\lambda))\left(\varphi(u_i \setminus \bigcup_{j\neq i} u_j) \geqslant \chi_i\right).$$

Then for each $i < \mathrm{cf}(\lambda)$ we may choose sequences $\langle y^i_\xi\colon \xi < \chi_i\rangle \subseteq u_i \setminus \bigcup_{j\neq i} u_j$ and $\langle w^i_\xi\colon \xi < \chi_i\rangle \subseteq \mathbb{B}$ such that

$$y^i_\xi \in w^i_\zeta \;\Leftrightarrow\; \zeta < \xi,$$

and we may additionally demand that $w^i_\xi \subseteq u_i$ (for each $\xi < \chi_i$). Now let

$$I \stackrel{\mathrm{def}}{=} \{b \in \mathbb{B}\colon (\forall i < \mathrm{cf}(\lambda))(\forall \xi < \chi_i)(y^i_\xi \notin b)\}.$$

It should be clear that I is an ideal in the Boolean algebra $\mathbb{B}$ (identified with the algebra of clopen subsets of X). Fix $i < \mathrm{cf}(\lambda)$ and suppose that $\zeta < \xi < \chi_i$. By the choices of the w^i_ξ's we have $y^i_\xi \in w^i_\zeta \setminus w^i_\xi$ and no y^i_ρ belongs to $w^i_\xi \setminus w^i_\zeta$. As $w^i_\xi \subseteq u_i$ we conclude $\mathbb{B}/I \models w^i_\xi/I < w^i_\zeta/I$. Thus the sequence $\langle w^i_\xi/I\colon \xi < \chi_i\rangle$ (for $i < \mathrm{cf}(\lambda)$) is strictly decreasing in $\mathbb{B}/I$ and consequently Depth$(\mathbb{B}/I) \geqslant \lambda$. Since there is λ many y^i_ξ's only, we may easily check that there are no decreasing λ^+-sequences in $\mathbb{B}/I$ (remember the definition of I), finishing the proof. □

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