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Canonical models for $\aleph_1$-combinatorics

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Abstract

We define the property of Π_2-compactness of a statement ϕ of set theory, meaning roughly that the hard core of the impact of ϕ on combinatorics of $\aleph_1$ can be isolated in a canonical model for the statement ϕ. We show that the following statements are Π_2-compact: "dominating number $= \aleph_1$," "cofinality of the meager ideal $= \aleph_1$", "cofinality of the null ideal $= \aleph_1$","bounding number $= \aleph_1$", existence of various types of Souslin trees and variations on uniformity of measure and category $= \aleph_1$. Several important new metamathematical patterns among classical statements of set theory are pointed out.

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0. Introduction

One of the oldest enterprises in higher set theory is the study of combinatorics of the first uncountable cardinal. It appears that many phenomena under investigation in this area are Σ_2 statements in the structure $\langle H_{\aleph_2}, \in, \mathfrak{I}\rangle$, where $H_{\aleph_2}$ is the collection of sets of hereditary cardinality $\aleph_1$ and $\mathfrak{I}$ is a predicate for nonstationary subsets of ω_1. For example:

(1) the Continuum Hypothesis – or, "there exists an ω_1 sequence of reals such that every real appears on it";

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(2) the negation of Souslin Hypothesis – or, "there exists an ω_1-tree without an uncountable antichain" [14];
(3) $\mathfrak{d} = \aleph_1$ – or, "there is a collection of $\aleph_1$ many functions in ${}^{\omega}\omega$ such that any other such function is pointwise dominated by one of them";
(4) indeed, every equality $= \aleph_1$ for a classical invariant of the continuum is a Σ_2 statement – $\mathfrak{b} = \aleph_1$, $\mathfrak{s} = \aleph_1$, additivity of measure $= \aleph_1 \ldots$ [3];
(5) there is a partition $h : [\omega_1]^2 \to 2$ without an uncountable homogeneous set [17];
(6) the nonstationary ideal is $\aleph_1$-dense [20].

It appears that Σ_2 statements generally assert that the combinatorics of $\aleph_1$ is complex. Therefore, given a sentence ϕ about sets, it is interesting to look for models where ϕ and as few as possible Σ_2 statements hold, in order to isolate the real impact of ϕ to the combinatorics of $\aleph_1$. The whole machinery of iterated forcing [10] and more recently the $P_{\max}$ method [20] were developed explicitly for this purpose. This paper is devoted to constructing such canonical Σ_2-poor (or Π_2-rich) models for a number of classical statements ϕ.

We consider cases of ϕ being $\mathfrak{d} = \aleph_1$, cofinality of the meager ideal $= \aleph_1$, cofinality of the null ideal $= \aleph_1$, $\mathfrak{b} = \aleph_1$, existence of some variations of Souslin trees, variations on uniformity of measure and category $= \aleph_1$ and for all of these we find canonical models. It is also proved that $\phi =$ "reals can be covered by $\aleph_1$ many meager sets" does not have such a model. But let us first spell out exactly what makes our models canonical.

Fix a sentence ϕ. Following the $P_{\max}$ method developed in [20], we shall aim for a σ-closed forcing P_ϕ definable in $L(\mathbb{R})$ so that the following holds:

Theorem Scheme 0.1. *Assume the Axiom of Determinacy in $L(\mathbb{R})$. Then in $L(\mathbb{R})^{P_\phi}$, the following holds*:
(1) *ZFC, $\mathfrak{c} = \aleph_2$, the nonstationary ideal is saturated, $\delta^1_2 = \aleph_2$.*
(2) *ϕ.*

Theorem Scheme 0.2. *Assume that ψ is a Π_2 statement for $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle$ and*
(1) *the Axiom of Determinacy holds in $L(\mathbb{R})$;*
(2) *there is a Woodin cardinal with a measurable above it;*
(3) *ϕ holds;*
(4) *$\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \psi$.*
Then in $L(\mathbb{R})^{P_\phi}$, $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \psi$.

If these two theorems can be proved for ϕ, we say that ϕ is Π_2-compact.

What exactly is going on? Recall that granted large cardinals, the theory of $L(\mathbb{R})$ is invariant under forcing [19] and so must be the theory of $L(\mathbb{R})^{P_\phi}$. Now varying the ZFC universe enveloping $L(\mathbb{R})$ so as to satisfy various Π_2 statements ψ, from Theorem Scheme 0.2 it follows that necessarily $L(\mathbb{R})^{P_\phi}$ must realize all such Π_2 sentences ever achievable in conjunction with ϕ by forcing in presence of large cardinals. In particular, roughly if ψ_i: $i \in I$ are Π_2-sentences one by one consistent with ϕ then even their

Sh:610

conjunction is consistent with ϕ. And $L(\mathbb{R})^{P_\phi}$ is *the* model isolating the impact of ϕ on combinatorics of $\aleph_1$.

It is proved in [20] that ϕ = "true", "the nonstationary ideal is $\aleph_1$-dense" and others are Π_2-compact assertions. This paper provides many classical Σ_2 statements which are Π_2-compact as well as examples of natural noncompact statements. In general, our results appear to run parallel with certain intuitions related to iterated forcing. The Π_2-compact assertions often describe phenomena for which good preservation theorems [3, Chapter 6] are known. This is not surprising given that in many cases the $P_{\max}$ machinery can serve as a surrogate to the preservation theorems – see Theorem 1.15(5) – and that many local arguments in $P_{\max}$ use classical forcing techniques – see Lemma 4.4 or Theorem 5.6. There are many open questions left:

Question 0.3. *Is it possible to define a similar notion of* Π_2*-compactness without reference to large cardinals?*

Question 0.4. *Is the Continuum Hypothesis* Π_2*-compact? Of course,* (1) *of Theorem Scheme* 0.1 *would have to be weakened to accomodate the Continuum Hypothesis.*

The first section outlines the proof scheme using which all the Π_2 compactness results in this paper are demonstrated. The scheme works subject to verification of three combinatorial properties – Lemma schemes 1.10, 1.13, 1.16, of independent interest – of the statement ϕ in question, which is done in Sections 2–5. These sections can be read and understood without any knowledge of [20]. The only indispensable – and truly crucial – reference to [20] appears in the first line of the proof of Theorem 1.15. At the time this paper went into print, a draft version of [20] could be obtained from its author. There were cosmetical differences in the presentation of $P_{\max}$ in this paper and in [20].

Our notation follows the set-theoretical standard as set forth in [6]. The letter $\mathfrak{I}$ stands for the nonstationary ideal on ω_1. A system a of countable sets is stationary if for every function $f:(\bigcup a)^{<\omega} \to \bigcup a$ there is some $x \in a$ closed under f. H_κ denotes the collection of all sets of hereditary size $<\kappa$. By a "model" we always mean a model of ZFC if not explicitly said otherwise. The symbol $\diamondsuit$ stands for the statement: there is a sequence $\langle A_\alpha\colon \alpha \in \omega_1\rangle$ such that $A_\alpha \subset \alpha$ for each $\alpha \in \omega_1$ and for every $B \subset \omega_1$ the set $\{\alpha \in \omega_1\colon B \cap \alpha = A_\alpha\} \subset \omega_1$ is stationary. ω_1-trees grow downward, are always infinitely branching, are considered to consist of functions from countable ordinals to ω ordered by reverse inclusion, and if $T, \leqslant$ is such a tree then $T_\alpha = \{t \in T\colon$ ordertype of the set $\{s \in T\colon t \leqslant s\}$ under $\geqslant$ is just $\alpha\}$ and $T_{<\alpha} = \bigcup_{\beta \in \alpha} T_\beta$. For $t \in T$, $lev(t)$ is the unique ordinal α such that $t \in T_\alpha$. For trees S of finite sequences, we write $[S]$ to mean the set $\{x\colon \forall n \in \omega\ x \restriction n \in S\}$. When we compare open sets of reals sitting in different models then we always mean to compare the open sets given by the respective definitions. δ^1_2 is the supremum of lengths of boldface Δ^1_2 prewellorderings of reals, Θ is the supremum of lengths of all prewellorderings of reals in $L(\mathbb{R})$. In forcing, the western convention of writing $q \leqslant p$ if q is more informative than p is utilized. $\lessdot$ denotes the relation of complete embedding between complete Boolean algebras or partial orders. $RO(P)$ is the complete Boolean algebra determined by a partially ordered set P, and $\mathbb{C}_\kappa$ is the Cohen algebra on κ coordinates. The algebra $\mathbb{C} = \mathbb{C}_\omega$ is construed as having a dense set ${}^{<\omega}\omega$ ordered by reverse inclusion. Lemma and Theorem "schemes" indicate that we shall attempt to prove some of their instances later.

1. General comments

This section sets up a framework in which all Π_2-compactness results in this paper will be proved. Subsection 1.0 introduces a crucial notion of an iteration of a countable transitive model of ZFC. In Subsection 1.1, a uniform in a sentence ϕ way of defining the forcing P_ϕ and proving instances of Theorem schemes 0.1, 0.2 is provided. In this *proof scheme* there are three combinatorial lemmas, 1.10, 1.13, 1.16, which must be demonstrated for each ϕ separately, and that is done in the section of the paper dealing with that particular assertion ϕ. In Subsection 1.2 it is shown how subtle combinatorics of ϕ can yield regularity properties of the forcing P_ϕ. And finally Subsection 1.3 gives some examples of failure of Π_2-compactness.

1.0. Iterability

The cornerstone of the $P_{\max}$ method is the possibility of finding generic elementary embeddings of the universe with critical point equal to ω_1. This can be done in several ways from sufficiently large cardinals. Here is our choice:

Definition 1.1 (*Woodin* [19]). Let δ be a Woodin cardinal. The nonstationary tower forcing $\mathbb{Q}_{<\delta}$ is defined as the set $\{a \in V_\delta$: a is a stationary system of countable sets$\}$ ordered by $b \leqslant a$ if for every $x \in b$, $x \cap \bigcup a \in a$.

The important feature of this notion is the following. Whenever δ is a Woodin cardinal and $G \subset \mathbb{Q}_{<\delta}$ is a generic filter then in $V[G]$ there is an ultrapower embedding $j: V \to M$ such that the critical point of j is ω_1^V, $j(\omega_1^V) = \delta$ and M is closed under ω sequences; in particular M is wellfounded. All of this has been described and proved in [19]. We shall be interested in iterations of this process.

Definition 1.2 (*Woodin* [20]). Let M be a countable transitive model of ZFC, $M \models \delta$ is a Woodin cardinal. An iteration of M of length γ based on δ is a sequence $\langle M_\alpha$: $\alpha \in \gamma\rangle$ together with commuting maps $j_{\alpha\beta}: M_\alpha \to M_\beta : \alpha \in \beta \in \gamma$ so that

(1) $M = M_0$;
(2) each M_α is a model of ZFC, possibly not transitive. Moreover, $j_{\alpha\beta}$ are elementary embeddings;
(3) for each α with $\alpha + 1 \in \gamma$ there is a M_α generic filter $G_\alpha \subset (\mathbb{Q}_{<j_{0\alpha}(\delta)})^{M_\alpha}$. The model $M_{\alpha+1}$ is the generic ultrapower of M_α by G_α and $j_{\alpha\alpha+1}$ is the ultrapower embedding;
(4) at limit ordinals $\alpha \in \gamma$ a direct limit is taken.

Convention 1.3. If the models M_α are well-founded we replace them with their transitive isomorphs. Everywhere in this paper, in the context of one specific iteration we keep the indexation system as in the above definition. We write $\theta_\alpha = \omega_1^{M_\alpha}$ and $\mathbb{Q}_\alpha = (\mathbb{Q}_{<j_{0\alpha}(\delta)})^{M_\alpha}$.

Definition 1.4 (*Woodin* [20]). An iteration j of a model M is called full if it is of length $\omega_1 + 1$ and for every pair $\langle x, \beta \rangle$ with $x \in \mathbb{Q}_\beta$ and $\beta \in \omega_1$ the set $\{\alpha \in \omega_1 : j_{\beta\alpha}(x) \in G_\alpha\} \subset \omega_1$ is stationary.

If all models in an iteration $j : M \to N$ of length $\omega_1 + 1$ are wellfounded then j can be thought of as stretching $\mathscr{P}(\omega_1)^M$ into a collection of subsets of the real ω_1. The fullness of j is then a simple bookkeeping requirement on it, making sure in particular that the model N is correct about the nonstationary ideal, that is $\mathfrak{I} \cap N = \mathfrak{I}^N$.

Definition 1.5 (*Woodin* [20]). A countable transitive model M is said to be *iterable* with respect to its Woodin cardinal δ if all of its iterations based on δ produce only wellfounded models. M is called *stable iterable* with respect to δ if all of its generic extensions by forcings of size $<\delta$ are iterable with respect to δ.

It is not a priori clear whether iterability and stable iterability are two different notions. We shall often neglect the dependence of the above definition on the ordinal δ.

Of course, a problem of great interest is to produce many rich iterable models. The following Lemma and its two corollaries record the two methods of construction of such models used in this paper.

Lemma 1.6 (Woodin [20]). *Let N be a transitive model of* ZFC *such that $\omega_1 = On \cap N$ and $N \models$ "$\delta < \kappa$ are a Woodin and an inaccessible cardinal respectively". Then $M = N \cap V_\kappa$ is stable iterable with respect to δ.*

Corollary 1.7 (Woodin [20]). *Suppose that the Axiom of Determinacy holds in $L(\mathbb{R})$. Then for every real x there is a stable iterable model containing x.*

Proof. The determinacy assumption provides a model N as in the Lemma containing every real x given beforehand [14]. Then $x \in N \cap V_\kappa = M$ is the desired countable stable iterable model. □

Corollary 1.8 (Woodin [20]). *Suppose $\delta < \kappa$ are a Woodin and a measurable cardinal respectively. Then for every real x there is a stable iterable model elementarily embeddable into V_κ which contains x.*

Proof. Fix a real x and choose a countable elementary substructure $Z \prec V_{\kappa+2}$ containing x, δ, κ and a measure U on κ. Let $\pi : Z \to \bar{Z}$ be the transitive collapse. Then the model $\bar{Z}$ is iterable in Kunen's sense [Ku] with respect to its measure $\pi(U)$, since its iterations lift those of the universe using the measure U. Let N^* be the ω_1-th iterand of $\bar{Z}$ using the measure $\pi(U)$, let $N = N^* \cap V_{\omega_1}$ and let $M = N \cap V_{\pi(\kappa)}$. The Lemma applied to $N, \pi(\delta)$ and $\pi(\kappa)$ shows that the model M is stable iterable; moreover, $M = \bar{Z} \cap V_{\pi(\kappa)}$ and so the map $\pi \restriction M$ elementarily embeds M into V_κ. Since $x \in M$, the proof is complete. $\square$

Proof of lemma. We shall show that M is iterable; the iterability of its small generic extensions $M[G]$ follows from an application of our proof to the model $N[G]$.

For contradiction, assume that there is an iteration $j : M \to M^*$ which yields an illfounded model. Since $j''(On \cap M)$ is cofinal in the ordinals of M^*, there must be some $\beta \in M$ such that $j(\beta)$ is illfounded. Choose the iteration j of the minimal possible length γ_0 and so that the least ordinal β_0 with $j(\beta_0)$ illfounded is smallest possible among all iterations of length γ_0. Note that γ_0 must be a successor of a countable limit ordinal.

Now γ_0, β_0 are definable in the model N as the unique solutions to the formula $\psi(x, y, M) =$ "for every large enough cardinal $\lambda, Coll(\omega, \lambda) \Vdash \chi(\check{x}, \check{y}, \check{M})$", where $\chi(x, y, z)$ says "x is the minimal length of a bad iteration of z and y is the minimal bad ordinal among such iterations of length x". The point is that whenever $\kappa, \gamma, \beta < \lambda < \omega_1$ and $G \subset Coll(\omega, \lambda)$ is an N-generic filter, then in the model $N[G]$ $\chi(\gamma_0, \beta_0, M)$ is a Σ^1_2 property of hereditarily countable objects and therefore evaluated correctly.

There must be ordinals $\gamma_1 < \gamma_0$ and $\beta_1 < j_{0\gamma_1}(\beta_0)$ such that $j_{\gamma_1\gamma_0}(\beta_1)$ is illfounded. Since κ is an inaccessible cardinal of N, the iteration $j_{0\gamma_1}$ can be copied to an iteration of the whole model N using the same nonstationary tower generic filters. Write $j_{0,\gamma_1} : N \to N'$ for this extended version of j_{0,γ_1} again and note that $M_{\gamma_1} = j_{0,\gamma_1}(M) = N' \cap V_{j_{0,\gamma_1}(\kappa)}$. By elementarity, the ordinals $j_{0,\gamma_1}(\gamma_0), j_{0,\gamma_1}(\beta_0)$ are the unique solution to the formula $\psi(x, y, j_{0,\gamma_1}(M))$ in the model N'. However, an application of the previous paragraph to N' shows that this cannot be, since $\gamma_0 - \gamma_1, \beta_1$ are better candidates for such a solution. Contradiction. $\square$

1.1. The $P_{\max}$ method

In this subsection we present a proof scheme used in this paper to show that various Σ_2 sentences ϕ for the structure $\langle H_{\aleph_2}, \in, \omega_1, \Im \rangle$ are Π_2 compact. For the record, all statements ϕ considered here are consequences of $\diamondsuit$ and therefore easily found consistent with large cardinals.

Definition 1.9. The set P_ϕ is defined by induction on rank of its elements. $p \in P_\phi$ if $p = \langle M_p, w_p, \delta_p, H_p \rangle$ where if no confusion is possible we drop the subscript p and
(1) M is a countable transitive model of ZFC iterable with respect to its Woodin cardinal δ;
(2) $M \models w$ is a witness for ϕ;
(3) $H \in M$ is the *history* of the condition p; it is a set (possibly empty) of pairs $\langle q, j \rangle$ where $q \in P_\phi$ and j is in M a full iteration of the model M_q based on δ_q such that $j(w_q) = w$ and $j(H_q) \subset H$;
(4) if $\langle q, j \rangle, \langle q, k \rangle$ are both in H then $j = k$.
The ordering on P_ϕ is defined by $q \leqslant p$ just in case $\langle p, j \rangle \in H_q$ for some j.

The notion of a witness for ϕ used above is the natural one; if $\phi = \exists x\, \forall y\, \chi(x, y)$ with χ a Σ_0 formula, then $x \in H_{\aleph_2}$ is a witness for ϕ whenever $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \forall y\, \chi(x, y)$. However, for obviously equivalent versions of the sentence ϕ this notion can vary a little. A special care will always be taken as to what variation of ϕ we are working with.

The idea behind the definition of the forcing P_ϕ is to construct $H_{\aleph_2}$ of the resulting model as a sort of direct limit of its approximations in countable models taken under iterations – which are recorded in the histories – and extensions.

The possibility of Π_2-compactness of ϕ depends on the validity of three combinatorial Lemmas which show how witnesses for ϕ in countable transitive models can be stretched by iterations of these models into real witnesses for ϕ. These Lemmas are used in Theorem 1.15 for σ-closure and various density arguments about P_ϕ.

The first combinatorial fact to be proved is:

Lemma scheme 1.10 (Simple iteration lemma). *Suppose $\diamondsuit$ holds. If M is a countable transitive model of* ZFC *iterable with respect to its Woodin cardinal δ and $M \models$ "w is a witness for ϕ" then there is a full iteration j based on δ of the model M such that $j(w)$ is a witness for ϕ.*

Certainly there is a need for some assumption of the order of $\diamondsuit$, since a priori ϕ does not have to hold at all and then $j(w)$ could not be a witness for it! Later we shall try to optimalise this assumption to $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \phi$, the weakest possible.

For a detailed analysis of the forcing P_ϕ a more involved variant of this Lemma will be necessary. Essentially, the iteration j is to be built cooperatively by two players, one of whom attempts to make $j(w)$ into a witness for ϕ. The other one stages various local obstacles to that goal. The relevant definitions:

Definition 1.11. A sequence $\vec{N}$ of models with a witness is a system $\langle w, N_i, \delta_i \colon i \in \omega\rangle$ where

(1) N_i are countable transitive models of ZFC$+\delta_i$ is a Woodin cardinal$+w$ is a witness for ϕ; we set $\mathbb{Q}_i = \mathbb{Q}_{<\delta_i}$ as computed in N_i
(2) $N_i \in N_{i+1}$ and $\omega_1^{N_i}$ is the same for all $i \in \omega$
(3) if $N_i \models$ "$a \in V_{\delta_i}$ is a stationary system of coutable sets" then $N_{i-1} \models$ "a is a stationary system of countable sets"; so $\mathbb{Q}_0 \subset \mathbb{Q}_1 \subset \ldots$
(4) if $N_i \models$ "$A \subset \mathbb{Q}_i$ is a maximal antichain" then $N_{i+1} \models$ "$A \subset \mathbb{Q}_{i+1}$ is a maximal antichain".

We say that the sequence begins with the triple $\langle N_0, w, \delta_0\rangle$, set $\mathbb{Q}_{\vec{N}} = \bigcup_{i\in\omega} \mathbb{Q}_i$ and $\omega_1^{\vec{N}} = \omega_1^{N_0}$. A filter $G \subset \mathbb{Q}_{\vec{N}}$ is said to be $\vec{N}$ generic if it meets all maximal antichains of $\mathbb{Q}_{\vec{N}}$ which happen to belong to $\bigcup_{i\in\omega} N_i$.

This definition may seem a little artificial, an artifact of the machinery of [20]. The really interesting information a sequence of models carries is the model $\bigcup_{i\in\omega} N_i$ with its first-order theory. This model can be viewed as a w-correct extension of N_0. It is

important that
(1) $\langle \bigcup_i N_i, \in \rangle \models w$ is a witness for ϕ
(2) $\langle \bigcup_i N_i, \in \rangle$ satisfies all Π_2-consequences of ZFC in the language $\in, \mathfrak{S}$, where $\mathfrak{S}$ is the predicate for stationary systems of countable sets
(3) N_0 is in $\bigcup_i N_i$ correct about stationary systems of countable sets in V_{δ_0} and their maximal antichains.

It should be noted that though $\mathbb{Q}_{\vec{N}}$ is not an element of $\bigcup N_i$, it is a class in that model – the class of all stationary systems of countable sets. If a filter $G \subset \mathbb{Q}_{\vec{N}}$ is $\vec{N}$-generic then the filters $G \cap \mathbb{Q}_i$ are N_i-generic by (3,4) of the above definition. However, not every N_i-generic filter on $\mathbb{Q}_i$ can be extended into an $\vec{N}$-generic filter on $\mathbb{Q}_{\vec{N}}$.

Definition 1.12. $\mathcal{G}_\phi$ is a two-person game of length ω_1 between players Good and Bad. The rules are:

Round 0: The player Bad plays M, w, δ such that M is a countable transitive model of ZFC iterable with respect to its Woodin cardinal δ and $M \models$ "w is a witness for ϕ"

Round $\alpha > 0$: an ordinal γ_α and an iteration $j_{\gamma_\alpha} : M \to M_{\gamma_\alpha}$ of length $\gamma_\alpha + 1$ based on δ are given.

- Bad plays a sequence $\vec{N}$ of models beginning with $M_{\gamma_\alpha}, j_{\gamma_\alpha}(w), j_{\gamma_\alpha}(\delta)$ and a condition $p \in \mathbb{Q}_{\vec{N}}$.
- Good plays an $\vec{N}$-generic filter $G \subset \mathbb{Q}_{\vec{N}}$ with $p \in G$.
- Bad plays an ordinal $\gamma_{\alpha+1} > \gamma_\alpha$ and an iteration $j_{\gamma_{\alpha+1}}$ of M of length $\gamma_{\alpha+1} + 1$ which prolongs the iteration j_{γ_α} and such that the γ_α-th ultrapower on it is taken using the filter $G \cap \mathbb{Q}_{\gamma_\alpha}$.

Here, $\gamma_1 = -1, j_1 = id$ and at limit α's, j_{γ_α} is the direct limit of the iterations played before α.

In the end, let j be the direct limit of the iterations played. The player Good wins if either the player Bad cannot play at some stage or the iteration j is not full or $j(w)$ is a witness for ϕ.

Thus the player Bad is responsible for the bookkeeping to make the iteration full and has a great freedom in prolonging the iteration on a nonstationary set of steps. The player Good has a limited access on a closed unbounded set of steps to steering $j(w)$ into a witness for ϕ. In the real life, the player Bad can easily play all the way through ω_1 and make the resulting iteration full.

We shall want to prove

Lemma scheme 1.13 (Strategic iteration lemma). *Suppose $\diamond$ holds. Then the player Good has a winning strategy in the game $\mathcal{G}_\phi$.*

Now suppose that the relevant instances of Lemma schemes 1.10, 1.13 are true for ϕ. Then, granted the Axiom of Determinacy in $L(\mathbb{R})$, the model $L(\mathbb{R})^{P_\phi}$ can be completely analysed using the methods of [20] to verify Theorem scheme 0.1. Let $G \subset P_\phi$ be a generic filter.

Definition 1.14. In $L(\mathbb{R})[G]$, for any $p \in G$ define
(1) k_p is the iteration of M_p which is the direct limit of the system $\{j\colon \exists q \in G \langle p, j\rangle \in H_q\}$.
(2) $W = k_p(w_p)$.

It is obvious from the definition of the poset P_ϕ that the system $\{j : \exists q \in G\langle p, j\rangle \in H_q\}$ is directed, and that the definition of W does not depend on the particular choice of $p \in G$.

Theorem 1.15. *Assume the Axiom of Determinacy in $L(\mathbb{R})$ and the relevant instances of Lemma schemes* 1.10 *and* 1.13 *hold. Then P_ϕ is a σ-closed notion of forcing and in $L(\mathbb{R})[G]$, the following hold*:
(1) ZFC;
(2) *for every $X \in H_{\aleph_2}$ there are $p \in G$ and $x \in M_p$ such that $X = k_p(x)$*;
(3) $\mathfrak{c} = \aleph_2$, *the nonstationary ideal is saturated,* $\delta^1_2 = \omega_2$;
(4) $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \phi$ *and W is a witness for ϕ*;
(5) *Suppose $\psi = \forall x\ \exists y\ \chi(x, y)$ for some Σ_0-formula χ is a Π_2-statement for $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle$. Suppose that* ZFC+$\diamond$ *proves that for each $x \in H_{\aleph_2}$ there is a forcing P preserving witnesses to ϕ and stationary subsets of ω_1 such that $P \Vdash \exists y \chi(\check{x}, \dot{y})$. Then $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \psi$.*

Proof. Parts (1,2,3) are straigthforward generalizations of Section 4.3 in [20]. Work in $L(\mathbb{R})[G]$ and prove (4).

First note that whenever $p \in G$ then $k_p''(\mathfrak{I})^{M_p} = \mathfrak{I} \cap k_p''M_p$. To see it, suppose $p \in G$, $M_p \models$ "$s \subset \omega_1$ is a stationary set" and fix a club $C \subset \omega_1$. By (2), there is a condition $q \in G$ and $c \in M_q$ so that $C = k_q(c)$. Let $r \in G$ be a common lower bound of p, q with $\langle p, i\rangle, \langle q, j\rangle \in H_r$. Then
(1) $M_r \models j(c) \subset \omega_1$ is a club
(2) $M_r \models i(s)$ is stationary, since the iteration i is full in M_r.

Therefore $j(c) \cap i(s) \neq 0$. By absoluteness, $k_r j(c) \cap k_r i(s) \neq 0$ and since $k_r j = k_q$ and $k_r i = k_p$, we have $C \cap k_p(s) = 0$ and $k_p(s)$ is stationary.

Now suppose (4) fails; so $\phi = \exists x\ \forall y\ \upsilon(x, y)$ for some Σ_0 formula υ, and W is not a witness for ϕ, and $\neg\upsilon(W, Y)$ for some $Y \in H_{\aleph_2}$. By (2), there is a condition $p \in G$ and $y \in M_p$ such that $Y = k_p(y)$.

But now, $M_p \models \langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \upsilon(w_p, y)$, since w_p is a witness for ϕ in the model M_p. By elementarity of k_p, absoluteness and the previous paragraph $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \upsilon(j_p(w_p) = W, j_p(y) = Y)$, contradiction.

To prove (5), fix the formula χ and note that by (2) it is enough to show that for $p \in P_\phi$ and $x \in M_p$ there is $q \leqslant p$ which forces an existence of Y such that $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \chi(k_p(x), Y)$. And indeed, using Corollary 1.7 choose a countable transitive stable iterable model M with $M \models$ "$\diamond$ + δ is a Woodin cardinal" with $p \in M$. Apply the Iteration Lemma 1.10 in M to get a full iteration j of M_p such that $j(w_p)$ is a witness for ϕ in M. By the assumptions on χ applied in $M \cap V_\kappa$, where κ is the least inaccessible cardinal of M, there is a generic extension $M[K]$ of M by a forcing of size $<\kappa$ preserving $j(w_p)$ and stationary subsets of ω_1 such that there is $y \in M[K]$ with $\chi(j(x), y)$. Obviously, setting $q = \langle M[K], j(w_p), \delta, H\rangle$, where $H = j(H_p) \cup \{\langle p, j\rangle\}$, we have $q \leqslant p$ and $q \Vdash \chi(\dot{k}_p(x), k_q(y))$ as desired. $\square$

The rudimentary comparison of the cardinal structure of $L(\mathbb{R})$ and $L(\mathbb{R})[G]$ carries over literally from [20]: namely $\aleph_1, \aleph_2$ are the same in these models, $\Theta = \aleph_3^{L(\mathbb{R})[G]}$ and all cardinals above Θ are preserved. This will not be used anywhere in this paper.

It should be remarked that under the assumptions of the Theorem, the model $K = L(\mathscr{P}(\omega_1))$ as evaluated in $L(\mathbb{R})[G]$ satisfies (1)–(5) and it can be argued that K is the "canonical model" in view of its minimal form and Theorem 1.23. In fact, $L(\mathbb{R})[G]$ is a generic extension of K by the poset $(\omega_2^{<\omega_2})^K$.

The final point in the analysis of the model $L(\mathbb{R})[G]$ is the proof of Theorem Scheme 0.2 for P_ϕ. We know of only one approach for doing this, namely to prove

Lemma scheme 1.16 (Optimal iteration lemma). *Suppose $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \phi$. Whenever M is a countable transitive model iterable with respect to its Woodin cardinal δ and $M \models$ "w is a witness for ϕ" then there is a full iteration j of M based on δ such that $j(w)$ is a witness for ϕ.*

It is crucial that the assumption of this optimal iteration lemma is truly the weakest possible. Provided Lemma schemes 1.10, 1.13 and 1.16 are true for ϕ, we can conclude

Corollary 1.17. *Suppose instances of Iteration Lemmas* 1.10, 1.13 *and* 1.16 *for ϕ are true. Then ϕ is Π_2-compact.*

Proof. We shall prove the relevant instance of Theorem Scheme 0.2. Assume that ψ is a Π_2 sentence, $\psi = \forall x \exists y\ \chi(x, y)$ for some Σ_0 formula χ. Assume that there is a Woodin cardinal δ with a measurable cardinal $\dot{\kappa}$ above it, and $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \psi \wedge \phi$. It must be proved that $L(\mathbb{R})^{P_\phi} \models \langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \psi$.

For contradiction suppose that $p \in P_\phi$ forces $\neg\psi = \exists x\ \forall y\ \neg\chi(x, y)$ holds in $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle$ of the generic extension. By Theorem 1.15(2), by eventually strengthening the condition p we may assume that there is $x \in \langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle^{M_p}$ so that $p \Vdash \forall y \neg\chi(k_p(x), y)$ holds in $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle$ of the generic extension.

Following Corollary 1.8, there is a countable transitive iterable model M elementarily embeddable into V_κ containing p, which is a hereditarily countable object in M. By the relevant instance of the Optimal Iteration Lemma applied within M there is a full iteration j of the model M_p such that $j(w_p)$ is a witness for ϕ in $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle^M$. It follows that the quadruple $q = \langle M, j(w_p), \bar{\delta}, H\rangle$, where $H = j(H_p) \cup \{\langle p, j\rangle\}$ and $\bar{\delta}$ is a Woodin cardinal of M, is a condition in P_ϕ and $q \leqslant p$. Since in M, $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \psi$, necessarily there is $y \in \langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle^M$ such that $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle^M \models \chi(j(x), y)$. It follows that $q \Vdash \chi(k_q j(x) = k_p(x), k_q(y))$ in $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle$ of the generic extension, a contradiction to our assumptions on p, x. $\square$

Predicates other than $\Im$ can be added to the language of $H_{\aleph_2}$ keeping the amended version of Theorem scheme 0.2 valid. For example, if Iteration Lemmas 1.10, 1.13 and 1.16 hold for ϕ then the relevant instance of Theorem scheme 0.2 can be shown to hold with the richer structure $\langle H_{\aleph_2}, \in, \Im, X\colon X \subset \mathbb{R}, X \in L(\mathbb{R})\rangle$; however, the proof is a little involved and we omit it. See [20]. In certain cases a predicate for witnesses for ϕ can be added keeping Theorem scheme 0.2 true for ϕ. This increases the expressive power of the language a little. Such a possibility will be discussed on a case-by-case basis.

Let us recapitulate what we proved in this subsection. Let ϕ be a Σ_2 sentence for $\langle H_{\aleph_2}, \in, \omega_1, \Im\rangle$, a consequence of $\diamondsuit$. If instances of Lemma schemes 1.10 and 1.13 are shown to hold, then the model $L(\mathbb{R})^{P_\phi}$ has the properties listed in Theorem 1.15 or Theorem scheme 0.1. And if the optimal iteration Lemma 1.16 for ϕ is proved then the relevant instance of Theorem scheme 0.2 is true and ϕ is Π_2-compact. It should be noted that Iteration Lemma 1.10 follows from both Lemma 1.13 and Lemma 1.16. We include it because it is frequently much easier to prove and because it is often the first indication that a Π_2-compactness type of result can be proved.

1.2. Order of witnesses

After an inspection of the proofs of iteration lemmas in the subsequent sections the following notion comes to light:

Definition 1.18. Let ϕ be a Σ_2 sentence. For $v, w \in H_{\aleph_2}$ we set $v \leqslant_\phi w$ if in every forcing extension of the universe whenever v is a witness to ϕ then w is such a witness.

Of course, the formally impermissible consideration of all forcing extensions can be expressed as quantification over partially ordered sets. While restricting ourselves to just *forcing* extensions may seem to be somewhat artificial, it is logically the easiest way and the resulting notion fits all the needs of the present paper. It should be noted that $\leqslant_\phi$ is sensitive to the exact definition of a witness as it was the case for P_ϕ. Obviously $\leqslant_\phi$ is a quasiorder and the nonwitnesses form the $\leqslant_\phi$ smallest $\leqslant_\phi$-equivalence class.

Example 1.19. For $\phi =$"there is a Souslin tree" and S, T such trees the relation $T \leqslant_\phi S$ is equivalent to the assertion "for every $s \in S$ there are $s' \leqslant s, t \in T$ such that $RO(S \restriction s')$ can be completely embedded into $RO(T \restriction t)$". For then, preservation of the Souslinity of T implies the preservation of the Souslinity of S. On the other hand, if the assertion fails, there must be $s \in S$ such that $T \Vdash$ "$S \restriction s$ is an Aronszajn tree" because every cofinal branch through S is generic. By the c.c.c. productivity theorem then, the finite condition forcing specializing the tree $S \restriction s$ preserves the Souslinity of T and collapses the Souslinity of S; ergo, $T \nleqslant_\phi S$.

Note that in the above example the relation $\leqslant_\phi$ was Σ_1 on the set of all witnesses. It is not clear whether this behavior is typical; the proofs of the iteration lemmas in this paper always use a Σ_1 phenomenon to guarantee the relation $\leqslant_\phi$ or the $\leqslant_\phi$-equivalence of two witnesses.

Definition 1.20. Suppose $\phi = \exists x\ \forall y\ \chi(x,y)$ for some Σ_0 formula χ and let $\psi(x_0,x_1)$ be Σ_1. We say that ψ is a *copying procedure* for ϕ if ZFC $\vdash \langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle \models \forall x_0, x_1\ (\psi(x_0, x_1) \rightarrow (\forall y\ \chi(x_0,y) \leftrightarrow \forall y\ \chi(x_1,y)))$. In other words, $\psi(x_0,x_1)$ guarantees that x_1 is a witness for ϕ iff x_0 is a witness for ϕ.

Example 1.21. Let $\phi =$ "there is a nonmeager set of reals of size $\aleph_1$". One possible copying procedure for ϕ is $\psi(x_0,x_1) =$ "there is a continuous category-preserving function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f''(x_0) = x_1$". Note that this is really a statement about a code for f, which is essentially a real, and it can be cast in a Σ_1 form.

The following theorems, quoted without proof, are applications of the above concepts. The first implies that the forcings P_ϕ for sentences ϕ considered in this paper are all homogeneous and therefore the Σ_n theory of $L(\mathbb{R})^{P_\phi}$ is a (definable) element of $L(\mathbb{R})$ for every $n \in \omega$. The second shows that models in Sections 2, 3 and 5 not only optimalize the Σ_2-theory of $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle$ but are in fact characterized by this property. The choice of copying procedures necessary for its proof will always be clear from the arguments in the section dealing with that particular ϕ.

Theorem 1.22. *Suppose the axiom of determinacy holds in $L(\mathbb{R})$, suppose ϕ is a Σ_2 sentence for which iteration Lemmas* 1.10, 1.13 *hold and ψ is a copying procedure for ϕ such that* ZFC *proves one of the following*:
(1) *for every two witnesses x_0, x_1 for ϕ there is a forcing P preserving stationary subsets of ω_1 and witnesses for ϕ such that $P \Vdash \psi(x_0,x_1)$;*
(2) *for every witness x_0 and every countable transitive iterable model M with $M \models$ "w is a witness for ϕ" there is a full iteration j of M such that $\psi(x_0, j(w))$ holds.*
Then P_ϕ is a homogeneous notion of forcing.

Theorem 1.23. *Suppose the axiom of determinacy holds in $L(\mathbb{R})$, suppose ϕ is a Σ_2 sentence for which iteration Lemmas* 1.10, 1.13 *hold and ψ is a copying procedure for ϕ such that* ZFC *proves "for every two witnesses x_0, x_1 to ϕ there is a forcing P preserving stationary subsets of ω_1 and witnesses to ϕ such that $P \Vdash \psi(x_0,x_1)$*. **If** *the Σ_2-theory of the structure $\langle H_{\aleph_2}, \in, \mathfrak{I}, X\colon X \subset \mathbb{R}, X \in L(\mathbb{R})\rangle$ is the same in V as in $L(\mathbb{R})^{P_\phi}$* **then** *$\mathscr{P}(\omega_1) = \mathscr{P}(\omega_1) \cap L(\mathbb{R})[G]$ for some possibly external $L(\mathbb{R})$-generic filter $G \subset P_\phi$.*

1.3. Limitations

Of course by far not every Σ_2 sentence ϕ can be handled using the proof scheme outlined in Subsection 1.1. Each of the three iteration lemmas can prove to be a problem; in some cases, it is possible to show that the statement ϕ is not Π_2-compact by exhibiting Π_2 assertions ψ_i: $i \in I$ each of whom is consistent with ϕ yet $\bigwedge_{i \in I} \psi_i \vdash \neg\phi$.

Example 1.24. The simple iteration lemma for $\phi =$ "the Continuum Hypothesis" fails. The reason is that whenever M is a countable transitive iterable model and j is an iteration of M then $j(\mathbb{R} \cap M) \neq \mathbb{R}$-namely, the real coding the model M is missing from $j(\mathbb{R} \cap M)$.

Example 1.25. The simple iteration lemma for $\phi =$ "there is a maximal almost disjoint family (MAD) of sets of integers of size $\aleph_1$" fails. Note that if $A \subset \mathscr{P}({}^{<\omega}\omega)$ is a MAD extending the set of all branches in ${}^{<\omega}\omega$ then A is collapsed as a MAD whenever a new real is added to the universe. Thus if M is a countable transitive iterable model with $M \models$ "the Continuum Hypothesis holds and A is a MAD as above" then no iteration j of M makes $j(A)$ into a MAD for the same reason in the previous example.

Example 1.26. The strategic iteration lemma for $\phi =$ "there is a nonmeager set of reals of size $\aleph_1$" with the natural notion of witness fails. The reason is somewhat arcane and we omit it.

Example 1.27. The optimal iteration lemma for $\phi =$ "the reals can be covered with $\aleph_1$ many meager sets" cannot be proved. For suppose that M is a countable transitive iterable model and $M \models$ "the Continuum Hypothesis holds and $\mathscr{C} = \{X_f \colon f \in {}^{\omega}\omega\}$, where $X_f \subset {}^{\omega}\omega$ is the set of all reals pointwise dominated by the function f. So $\mathscr{C}$ constitutes a covering of the real line by $\aleph_1$ meager sets." Also suppose that $\phi \wedge \mathfrak{b} > \aleph_1$ holds in the universe – this is consistent and happens after adding $\aleph_2$ Laver reals to a model of GCH [9]. Then no iteration j of the model M can make $j(\mathscr{C})$ into a covering of the real line, because there always will be a function in ${}^{\omega}\omega$ eventually dominating all of $j(\mathbb{R} \cap M)$. Note that in this case, $\mathscr{C}$ should be thought of as a collection of Borel codes as opposed to a set of sets of reals.

The previous example suggests that ϕ is not Π_2-compact, and indeed, it is not. For consider Π_2 sentences

$\psi_0 =$ "$\mathfrak{b} > \aleph_1$",

$\psi_1 =$ "for every bounded family $A \subset {}^{\omega}\omega$ of size $\aleph_1$

there is a function infinitely many times equal to every function in A".

Now $\phi \wedge \psi_0$ holds after iterating Laver reals, $\phi \wedge \psi_1$ holds after iterating proper ${}^{\omega}\omega$-bounding forcings [12, Proposition 2.10], and $\psi_0 \wedge \psi_1 \vdash \neg\phi$ can be derived easily from the combinatorial characterization of ϕ in [2, 3].

Example 1.28. The optimal iteration lemma for $\phi =$ "$\mathfrak{t} = \aleph_1$" cannot be proved. Recall that $\mathfrak{t}$ is the minimal length of a tower and a tower is a decreasing sequence of infinite subsets of ω without lower bound in the modulo finite inclusion ordering. To see the reason for the failure, suppose M is a countable transitive iterable model and $M \models$ "the Continuum Hypothesis holds and t is a tower of height ω_1 consisting of sets of asymptotic density one." That such towers exist under CH has been pointed out to us by W. Hugh Woodin. Suppose that in the universe $\mathfrak{t}=\aleph_1$ holds and no towers consist of sets from a fixed Borel filter – such a situation can be attained by iterating Souslin c.c.c. forcings over a model of CH. Then no iteration j of M can make $j(t)$ into a tower.

It seems that the two Π_2 assertions

$$\psi_0 = \text{"no towers consist of sets from a fixed Borel filter"},$$

$$\psi_1 = \text{"every tower consists of sets from some Borel filter"}$$

provide a witness for non-Π_2-compactness of ϕ, however, the consistency of $\phi \wedge \psi_1$ seems to be a difficult open problem.

Example 1.29. The optimal iteration lemma for $\phi =$ "there is a Souslin tree" cannot be proved. For suppose that M is a countable transitive iterable model with $M \models$ "$\diamondsuit$ and T is a homogeneous Souslin tree". Suppose that in the universe there are Souslin trees but none of them are homogeneous–this was proved consistent in [1]. Then no iteration j of M can make $j(T)$ into a Souslin tree, since $j(T)$ is necessarily homogeneous.

Again, the above example provides natural candidates to witness the non-Π_2-compactness of ϕ. Let

$$\psi_0 = \text{"for every finite set } T_i : i \in I \text{ of Souslin trees there are } t_i \in T_i \text{ such that } \prod_i T_i \restriction t_i \text{ is c.c.c."},$$

$$\psi_1 = \text{"for every Souslin tree } T \text{ there are finitely many } t_i : i \in I \text{ in } T \text{ such that } \prod_i T \restriction t_i \text{ is nowhere c.c.c."}.$$

The sentence $\phi \wedge \psi_0$ was found consistent by [1], but the consistency of $\phi \wedge \psi_1$ is an open problem. In view of the results of Subsection 4.0 the sentences ψ_0, ψ_1 are the *only* candidates for noncompactness of ϕ.

2. Dominating number

The proof of Π_2-compactness of the sentence "there is a family of $\aleph_1$ many functions in ${}^{\omega}\omega$ such that any function in ${}^{\omega}\omega$ is modulo finite dominated by one in the family" or $\mathfrak{d}=\aleph_1$, is in some sense prototypical, and the argument will be adapted to other invariants in Section 3. The important concept we isolate to prove the iteration lemmas is that of *subgenericity*. It essentially states that the classical Hechler forcing is the optimal way for adding a dominating real. To our knowledge, this concept has not been explicitly defined before.

Sh:610

2.0. The combinatorics of $\mathfrak{d}$

There is a natural Souslin [7] forcing associated to the order of eventual dominance on ${}^{\omega}\omega$ designed to add a "large" function:

Definition 2.1. The Hechler forcing $\mathbb{D}$ is the set $\{\langle a, A\rangle : \mathrm{dom}(a) = n$ for some $n \in \omega$, $\mathrm{rng}(a) \subset \omega$ and A is a finite subset of ${}^{\omega}\omega\}$. The order is defined by $\langle a, A\rangle \leqslant \langle b, B\rangle$ if
(1) $b \subset a, B \subset A$;
(2) $\forall n \in dom(a \backslash b)\ \forall f \in B\ a(n) \geqslant f(n)$.

For a condition $p = \langle a, A\rangle$ the function $\mathrm{body}(p) \in {}^{\omega}\omega$ is defined as $\mathrm{body}(p)(n) = a(n)$ if $n \in \mathrm{dom}(a)$ and $\mathrm{body}(p)(n) = \max\{f(n)\colon f \in A\}$ if $n \notin dom(a)$.

If $G \subset \mathbb{D}$ is a generic filter, the Hechler real d is defined as $\bigcup\{a\colon \langle a, 0\rangle \in G\}$.

Below, we shall make use of restricted versions of $\mathbb{D}$. Say $f \in {}^{\omega}\omega$. Then define $\mathbb{D}(f)$ to be the set of all $p \in \mathbb{D}$ with $\mathrm{body}(p)$ pointwise dominated by the function f, with the order inherited from $\mathbb{D}$. Note that $\mathbb{D}$ as defined above is not a separative poset.

Obviously, all forcings defined above are c.c.c. The important combinatorial fact about $\mathbb{D}$ is that a Hechler real is in fact an "optimal" function eventually dominating every ground model function. This will be immediately made precise:

Definition 2.2. Let M be a transitive model of ZFC and $f \in {}^{\omega}\omega$. We say that the function f $\mathbb{D}$-dominates M if every $g \in M \cap {}^{\omega}\omega$ is eventually dominated by f.

Lemma 2.3. *Let M be a transitive model of ZFC and let $f \in {}^{\omega}\omega$ $\mathbb{D}$-dominate M. If $D \subset \mathbb{D} \cap M$ is a dense set in M, then $D \cap \mathbb{D}(f)$ is dense in $\mathbb{D}(f) \cap M$.*

Note that $\mathbb{D} \cap M$ is $\mathbb{D}$ as computed in M; also $\mathbb{D}(f) \cap M \notin M$.

Proof. Fix a dense set $D \subset \mathbb{D} \cap M$ in M and a condition $p = \langle a, A\rangle$ in $\mathbb{D}(f) \cap M$. We shall produce a condition $q \leqslant p$ in $D \cap \mathbb{D}(f)$. Working in M, it is easy to construct a sequence $\langle a_i, A_i\rangle\colon i \in \omega$ of conditions in D with $\mathrm{dom}(a_i) = n_i$ and
(1) $\langle a_i, A_i\rangle \leqslant p$
(2) $\langle a_{i+1}, A_{i+1}\rangle$ is any element of D stronger than $\langle \mathrm{body}(p) \restriction n_i, A_i\rangle$.

Define a function $g \in {}^{\omega}\omega$ as follows: for $n \leqslant n_0$, let $g(n) = a(n)$. For $n \geqslant n_0$, find an integer $i \in \omega$ with $n_i \leqslant n < n_{i+1}$ and set $g(n) = \max\{a_{i+1}(n), h(n)\colon h \in A_i\}$. Since $g \in {}^{\omega}\omega \cap M$, the function f dominates g pointwise starting from some n_i. Then $q = \langle a_{i+1}, A_{i+1}\rangle \in D \cap \mathbb{D}(f)$ is the desired condition. $\square$

Corollary 2.4 (Subgenericity). *Let P be a forcing and $\dot{g}$ a P-name such that*
(1) *$P \Vdash \dot{g} \in {}^{\omega}\omega$ $\mathbb{D}$-dominates the ground model*
(2) *for every $f \in {}^{\omega}\omega$ the boolean value $\|\check{f} \leqslant \dot{g}$ pointwise$\|_P$ is non-zero.*
Then there is a complete embedding $RO(\mathbb{D}) \lessdot P * (\mathbb{D}(\dot{g}) \cap$ the ground model$) = R$ so that $R \Vdash$ "$\dot{d} \leqslant \dot{g}$ pointwise", where $\dot{d}$ is the $\mathbb{D}$-generic real.

Thus under every $\mathbb{D}$-dominating real, a Hechler real is lurking behind the scenes.

Proof. The Hechler real $\dot{d}$ will be read off the second iterand in the natural way, and Lemma 2.3 will guarantee its genericity. By some Boolean algebra theory, that yields a complete embedding of $RO(\mathbb{D} \restriction p)$ to $RO(R)$, for some $0 \neq p \in \mathbb{D}$. We must prove that $p = 1$. But fix an arbitrary $q = \langle a, A \rangle \in \mathbb{D}$ and let $g = \text{body}(q)$. Then $\langle \|\check{g} \leqslant \dot{f}$ pointwise$\|_P, \check{q} \rangle \in R$ is a nonzero element of $RO(R)$ forcing that the $\mathbb{D}$-generic real meets the condition q. Thus with the given embedding, any condition in $\mathbb{D}$ can be met, consequently $p = 1$ and the proof is complete. □

Corollary 2.5. *Let M be a countable transitive model of* ZFC *such that $M \models$ "P is a forcing adding a dominating function $\dot{f}$ as in Corollary* 2.4", *let $p \in P$ and let $g \in {}^{\omega}\omega$ be any function, not necessarily in the model M. Then there is an M-generic filter $G \subset B$ containing p such that g is eventually dominated by $\dot{f}/G$.*

Proof. Apply the previous Corollary in the model M and find the forcing R and the relevant embeddings $P \lessdot R$, $\mathbb{D} \lessdot R$. Assume $P \subset R$ and set x to be the projection of r into $\mathbb{D}$ via the above embedding. Step out of the model M and find a filter $H \subset \mathbb{D}$ such that

(1) $x \in H$;

(2) H meets every maximal antichain of $\mathbb{D}$ which is an element of M;

(3) the Hechler real e derived from H eventually dominates the function g.

This is easily done. Now the key point is that the model M computes maximal antichains of $\mathbb{D}$ correctly: if $M \models$ "$A \subset \mathbb{D}$ is a maximal antichain" then this is a Π^1_1 fact about A under suitable coding and therefore A really is a maximal antichain of $\mathbb{D}$. Consequently, the filter $H \cap M \subset \mathbb{D}^M$ is M-generic.

Choose an M-generic filter $K \subset R$ with $H \cap M \subset K$ under the embedding of $\mathbb{D}$ mentioned above. Let $G = K \cap P$. The filter $G \subset P$ is M-generic and has the desired properties: the function g is eventually dominated by e which is pointwise smaller than $\dot{f}/G$. □

2.1. A model for $\mathfrak{d} = \aleph_1$

A natural notion of a witness for $\mathfrak{d} = \aleph_1$ to be used in the definition of $P_{\mathfrak{d} = \aleph_1}$ is that of an eventual domination cofinal subset of ${}^{\omega}\omega$ of size $\aleph_1$. We like to consider an innocent strengthening of this notion in order to later ensure that assumptions of Corollary 2.5 are satisfied.

Lemma 2.6. *The following are equivalent*:

(1) $\mathfrak{d} = \aleph_1$;

(2) *there is a sequence $d : \omega_1 \to {}^{\omega}\omega$ increasing in the eventual domination order such that for every $f \in {}^{\omega}\omega$ the set $S_f = \{\alpha \in \omega_1 \colon f \leqslant d(\alpha)$ pointwise$\}$ is stationary.*

A sequence d as in (2) will be called a *good* dominating sequence and will be used as a witness for $\mathfrak{d} = \aleph_1$.

Proof. Only (1) → (2) needs an argument. Choose an arbitrary eventual domination cofinal set $\{f_\alpha : \alpha \in \omega_1\} \subset {}^\omega\omega$ and a sequence $\langle S_{a,\alpha} : a \in {}^{<\omega}\omega, \alpha \in \omega_1 \rangle$ of pairwise disjoint stationary subsets of ω_1. By a straightforward induction on $\beta \in \omega_1$ it is easy to build the sequence $d : \omega_1 \to {}^\omega\omega$ so that

(1) $d(\beta)$ eventually dominates all f_α: $\alpha \in \beta$ and $d(\alpha)$: $\alpha \in \beta$;

(2) if $\beta \in S_{a,\alpha}$ then $d(\beta)$ pointwise dominates both a and f_α.

The sequence d is as required. For choose $f \in {}^\omega\omega$. There are $a \in {}^{<\omega}\omega, \alpha \in \omega_1$ so that f is pointwise dominated by the function g taking maxima of functional values of a and f_α. The set S_f is then a superset of $S_{a,\alpha}$ and therefore stationary. □

Suppose now that M is a countable transitive model of ZFC, $M \models$ "$d : \omega_1 \to {}^\omega\omega$ is a good dominating sequence and δ is a Woodin cardinal". Working in M a simple observation is that $\mathbb{Q}_{<\delta} \Vdash$ "$(j_{\mathbb{Q}}d)(\omega_1^M)$ $\mathbb{D}$-dominates M", where $j_{\mathbb{Q}}$ is the term for the generic nonstationary tower embedding. Moreover, by (2) above the pair $\mathbb{Q}_{<\delta}, (j_{\mathbb{Q}}d)(\omega_1^M)$ satisfies requirements of Corollary 2.5. Thus there is a generic ultrapower of M lifting $(j_{\mathbb{Q}}d)(\omega_1^M)$ arbitrarily high in the eventual domination order in V. Also, whenever j is a full iteration of the model M such that $j(d)$ is a dominating sequence, it is really a good dominating sequence.

Optimal Iteration Lemma 2.7. *Suppose $\mathfrak{d} = \aleph_1$. Whenever M is a countable transitive model iterable with respect to its Woodin cardinal δ and $M \models$ "d is a good dominating sequence" there is a full iteration j of M so that $j(d)$ is a good dominating sequence.*

Proof. Let $\{f_\alpha : \alpha \in \omega_1\}$ be an eventual domination cofinal family of functions. We shall produce a full iteration j of the model M based on δ with $\theta_\alpha = \omega_1^{M_\alpha}$ such that the function $jd(\theta_\alpha)$ eventually dominates the function f_α, for every $\alpha \in \omega_1$. This will prove the lemma.

The iteration j will be constructed by induction on $\alpha \in \omega_1$. First, fix a partition $\{S_\xi : \xi \in \omega_1\}$ of ω_1 into pairwise disjoint stationary sets. By induction on $\alpha \in \omega_1$ build models M_α together with the elementary embeddings plus an enumeration $\{\langle x_\xi, \beta_\xi \rangle : \xi \in \omega_1\}$ of all pairs $\langle x, \beta \rangle$ with $x \in \mathbb{Q}_\beta$. The induction hypotheses at α are:

(1) the function $jd(\theta_\alpha)$ eventually dominates f_α.

(2) the initial segment $\{\langle x_\xi, \beta_\xi \rangle : \xi \in \theta_\alpha\}$ of the enumeration under construction has been built and it enumerates all pairs $\langle x, \beta \rangle$ with $x \in \mathbb{Q}_\beta$, $\beta \in \alpha$.

(3) for $\gamma \in \alpha$ if $\theta_\gamma \in S_\xi$ for some (unique) $\xi \in \theta_\gamma$ then $j_{\beta_\xi,\gamma}(x_\xi) \in G_\gamma$.

The hypothesis (1) ensures that the resulting sequence jd will be dominating. The enumeration together with (3) will imply the fullness of the iteration.

At limit stages, the direct limit of the previous models and the union of the enumerations constructed so far is taken. The successor step is handled easily using a version of Corollary 2.5 below the condition $j_{\beta_\xi,\gamma}(x_\xi) \in \mathbb{Q}_\alpha$ if $\alpha \in S_\xi$ for some $\xi \in \theta_\alpha$ and the observation just before the formulation of Lemma 2.7. Let j be the direct limit of the iteration system constructed in the induction process. □

This lemma could have been proved even without subgenericity, since after all, the forcing $\mathbb{Q}$ adds a Hechler real by design. With the sequences of models entering the stage though it is important to have a sort of a uniform term for this real.

Strategic Iteration Lemma 2.8. *Suppose $\mathfrak{d}=\aleph_1$. The player Good has a winning strategy in the game $\mathcal{G}_{\mathfrak{d}=\aleph_1}$.*

Proof. Let $\vec{N}=\langle d, N_i, \delta_i\colon i\in\omega\rangle$ be a sequence of models with a good dominating sequence d, let $y_0\in\mathbb{Q}_{\vec{N}}$ and let $f\in{}^{\omega}\omega$ be an arbitrary function. We shall show that there is an $\vec{N}$-generic filter $G\subset\mathbb{Q}_{\vec{N}}$ with $y_0\in G$ such that $(j_{\mathbb{Q}}d)(\omega_1^{\vec{N}})$ eventually dominates the function f, where $j_{\mathbb{Q}}$ is the generic ultrapower embedding of the model N_0 using the filter $G\cap\mathbb{Q}^{N_0}_{<\delta_0}$. With this fact a winning strategy in the game $\mathcal{G}_{\mathfrak{d}=\aleph_1}$ for the good player consists of an appropriate bookkeeping using a fixed dominating sequence as in the previous lemma.

First, let us fix some notation. Choose an integer $i\in\omega$, work in N_i and set $\mathbb{Q}_i=\mathbb{Q}^{N_i}_{<\delta_i}$. Consider the $\mathbb{Q}_i$-term j_i for a $\mathbb{Q}_i$-generic ultrapower embedding of the model N_i. The function $j_i d(\omega_1^{\vec{N}})$ is forced to be represented by the function $\alpha\mapsto d(\alpha)$ and to $\mathbb{D}$-dominate the model N_i. Applying Corollary 2.4 in N_i to $\mathbb{Q}_i$ and $j_i d(\omega_1^{\vec{N}})$ it is possible to choose a particular dense subset R_i of the iteration found in that Corollary, namely $R_i=\{\langle y,a,A\rangle\colon\ y\in\mathbb{Q}_i, \langle a,A\rangle\in\mathbb{D},\ A\subset\operatorname{rng}(d)$ and for every $x\in y$ the function $d(x\cap\omega_1)$ pointwise dominates body$(\langle a,A\rangle)\}$ ordered by $\langle z,b,B\rangle\leqslant\langle y,a,A\rangle$ just in case $z\leqslant y$ in $\mathbb{Q}_i$ and $\langle b,B\rangle\leqslant\langle a,A\rangle$ in $\mathbb{D}$. It is possible to restrict ourselves to sets $A\subset\operatorname{rng}(d)$ since $rng(d)$ is an eventual domination cofinal family in N_i. As in that corollary, the $\mathbb{Q}_i$-generic will be read off the first coordinate and the $\mathbb{D}$-generic real $e\in{}^{\omega}\omega$ will be read off the other two, with $j_i d(\omega_1)$ pointwise dominating the function e. With this embedding of $\mathbb{D}$ into the poset R_i, we can compute the projection $pr_{i\mathbb{D}}(\langle y,a,A\rangle)=\Sigma_{\mathbb{D}}\{\langle b,B\rangle\in\mathbb{D}\colon\ \langle b,B\rangle\leqslant\langle a,A\rangle, B\subset\operatorname{rng}(d)$ and the system $z=\{x\in y\colon d(x\cap\omega_1)$ pointwise dominates the function body$(\langle b,B\rangle)\}$ is stationary$\}$.

Now step out of the model N_i. There are two key points, capturing the uniformity of the above definitions in $i\in\omega$:

(1) $R_0\subset R_1\subset\cdots$;

(2) $pr_{i\mathbb{D}}(\langle y,a,A\rangle)$ computes the same value in $\mathbb{D}$ in all models N_i with $y\in\mathbb{Q}_i$.

Therefore we can write $pr_{\mathbb{D}}(\langle y,a,A\rangle)$ to mean the constant value in $\mathbb{D}$ of this expression without any danger of confusion. Another formulation of (2) is that N_0 computes a function from $\mathbb{D}$ into R_0 which constitutes a complete embedding of $\mathbb{D}$ into all R_i in the respective models N_i. Note that R_0 is *not* a complete suborder of the R_i's.

Now everything is ready to construct the filter $G\subset\mathbb{Q}_{\vec{N}}$. First, let us choose a sufficiently generic filter $H\subset\mathbb{D}$. There are the following requirements on H:

(3) $pr_{\mathbb{D}}(\langle y_0,0,0\rangle)\in H$;

(4) H meets all maximal antichains of $\mathbb{D}$ which happen to be elements of $\bigcup_i N_i$;

(5) the Hechler real $e\in{}^{\omega}\omega$ given by the filter H eventually dominates the function $f\in{}^{\omega}\omega$.

This is easily arranged. It follows from (4) that $H \cap N_i$ is an N_i-generic subset of $\mathbb{D}^{N_i}$ since the model N_i is Σ^1_1 correct and therefore computes maximal antichains of $\mathbb{D}$ correctly.

Let X_k: $k \in \omega$ be an enumeration of all maximal antichains of $\mathbb{Q}_{\vec{N}}$ which are elements of $\bigcup_i N_i$. By induction on $k \in \omega$ build a descending sequence $y_0 \geqslant y_1 \geqslant \cdots \geqslant y_k \geqslant \cdots$ of conditions in $\mathbb{Q}_{\vec{N}}$ so that

(6) y_{k-1} has an element of X_k above it;

(7) $pr_{\mathbb{D}}(\langle y_{k+1}, 0, 0\rangle) \in H$.

This is possible by the genericity of the filter H. Suppose y_k is given. There is an integer $i \in \omega$ such that $y_k \in \mathbb{Q}_i$ and $X_k \in N_i$ is a maximal antichain in $\mathbb{Q}_i$. Now $H \cap N_i$ is a Hechler N_i-generic filter and $\bar{X}_k = \{\langle z, 0, 0\rangle : z \in X_k\} \subset R_i$ is a maximal antichain in R_i. Therefore, there must be a condition $y_{k+1} \leqslant y_k$ as required in (6,7).

In the end, let $G \subset \mathbb{Q}_{\vec{N}}$ be the filter generated by the conditions $y_k : k \in \omega$. It is an $\vec{N}$-generic filter by (6) and the function $j_0 d(\omega_1^{\vec{N}})$ pointwise dominates the Hechler function e by (7) and therefore – from (5) – eventually dominates the function $f \in {}^\omega\omega$ as desired. □

Conclusion 2.11. *The sentence $\phi = \mathfrak{d} = \aleph_1$ is Π_2-compact, moreover, in Theorem Scheme 0.2 we can add a predicate for dominating sequences of length ω_1 to the language of $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle$.*

Proof. All the necessary iteration lemmas have been proved. To see that the dominating predicate $\mathfrak{D}$ can be added, go through the proof of Corollary 1.15 again and note that if j is a full iteration of a countable transitive iterable model M such that $j(M \cap \mathbb{R})$ is cofinal in the eventual domination ordering then $\mathfrak{D} \cap M_{\omega_1} = \mathfrak{D}^{M_{\omega_1}}$. □

3. Other $\mathfrak{d}$-like cardinal invariants

The behavior of the dominating number seems to be typical for a number of other cardinal invariants. We present here two cases which can be analysed completely. Recall that for an arbitrary ideal, the cofinality of that ideal is defined as the minimal size of a collection of small sets such that any small set is covered by one in that collection. This is an important cardinal characteristic of that ideal [3].

3.0. Cofinality of the meager ideal

In this subsection we prove that the statement "the cofinality of the meager ideal is $\aleph_1$" is Π_2-compact. It is not difficult to see and will be proved below that it is enough to pay attention to the nowhere dense ideal. As in the previous section, there is a canonical forcing related to this ideal.

Definition 3.1. (1) NWD is the set of all perfect nowhere dense trees on $\{0, 1\}$;

(2) for a tree $T \in \mathrm{NWD}$ and a finite set $x \subset T$ the tree $T \restriction x$ is defined as the set of all elements of T $\subset$-comparable with some element of x;
(3) for a tree $S \in \mathrm{NWD}$ and a sequence $\eta \in S$ the tree $S(\eta)$ is the set $\{\tau: \eta^\frown\tau \in S\}$;
(4) $\mathbb{UM}$, the universal meager forcing [S, Definition 4.2], is the set $\{\langle n,S\rangle: n\in\omega, S \in \mathrm{NWD}\}$ ordered by $\langle n,S\rangle \leqslant \langle m,T\rangle$ if $n \geqslant m, T \subset S$ and $S \cap {}^m 2 = T \cap {}^m 2$;
(5) for a tree $U \in \mathrm{NWD}$ we write $\mathbb{UM}(U) = \{\langle n,S\rangle \in \mathbb{UM}: S \subset U\}$; this set has an order on it inherited from $\mathbb{UM}$.

Obviously, the collection $\{[T] : T \in \mathrm{NWD}\}$ is a base for the nowhere dense ideal. $\mathbb{UM}$ is a σ-centered Souslin forcing designed so as to produce a very large nowhere dense tree: if $G \subset \mathbb{UM}$ is a generic filter, then this tree is $U_G = \bigcup\{T: \langle 0,T\rangle \in G\}$; it is nowhere dense and it codes the generic filter. The following is the instrumental weakening of genericity:

Definition 3.2. Let M be a transitive model of ZFC and $U \in \mathrm{NWD}$. We say that the tree U $\mathbb{UM}$-dominates the model M if there is some element $T \in M \cap \mathrm{NWD}$ included in U and for every $T \in M \cap \mathrm{NWD}$ there is an integer $n \in \omega$ such that setting $x = T \cap U \cap {}^n 2$, the inclusion $T \restriction x \subset U \restriction x$ holds.

This notion has certain obvious monotonicity properties. Suppose $S \subset T$ and U are perfect nowhere dense trees and n is an integer such that setting $x = T \cap U \cap {}^n 2$, $T \restriction x \subset U \restriction x$ holds. Then with $y = S \cap U \cap {}^n 2$ we have $S \restriction y \subset U \restriction y$ and for any integer $m > n$ and $z = T \cap U \cap {}^m 2$ we have $T \restriction z \subset U \restriction z$.

It is immediate that if $U \in \mathrm{NWD}$ is $\mathbb{UM}$-generic then it $\mathbb{UM}$-dominates the ground model. On the other hand, any $\mathbb{UM}$-dominating tree covers an $\mathbb{UM}$-generic tree:

Lemma 3.3. *Let M be a transitive model of* ZFC *and let $U \in \mathrm{NWD}$ $\mathbb{UM}$-dominate the model M. If $D \subset \mathbb{UM} \cap M$ is a dense set in M then $D \cap \mathbb{UM}(U)$ is dense in $\mathbb{UM}(U) \cap M$.*

Proof. First, the set $\mathbb{UM}(U) \cap M$ is nonempty. Now let $\langle n,S\rangle \in \mathbb{UM}(U) \cap M$ and let $D \in M$ be a dense subset of $\mathbb{UM} \cap M$ which is an element of the model M. We shall produce a condition $p \in D \cap \mathbb{UM}(U), p \leqslant \langle n,S\rangle$, proving the lemma.

Work in M. By induction on $i \in \omega$, build conditions $\langle n_i, T_i\rangle, p_{x,i} \in \mathbb{UM}$ so that
(1) $\langle n_0, T_0\rangle = \langle n,S\rangle$ and $\langle n_{i+1}, T_{i+1}\rangle \leqslant \langle n_i, T_i\rangle$;
(2) for every integer $i > 0$, for every sequence $\eta \in {}^i 2$ there is a sequence $\tau \in {}^{n_i} 2$ with $\eta \subset \tau$ and $\tau \notin T_i$;
(3) to produce $\langle n_{i+1}, T_{i+1}\rangle$ from $\langle n_i, T_i\rangle$, for every nonempty set $x \subset {}^{n_i} 2 \cap T_i$ find a condition $p_{x,i} = \langle n_{x,i}, S_{x,i}\rangle \leqslant \langle n_i, T_i \restriction x_i\rangle$ in the dense set D. Set $T_{i+1} = \bigcup S_{x,i}$ and let n_{i+1} be arbitrary so that (2) is satisfied.

After this is done, let $T_\omega = \bigcup_i T_i$. The induction hypothesis (2) implies that $T_\omega \in \mathrm{NWD} \cap M$ and therefore, there is an integer $i \in \omega$ such that setting $x = {}^{n_i} 2 \cap T \cap U$ we get $T_\omega \restriction x \subset U \restriction x$. Note that the set x is nonempty, because it includes $S \cap {}^{n_i}2$. Now $p_{x,i}$ is the desired condition. □

Corollary 3.4 (Subgenericity). *Let P be a forcing and $\dot{S}$ a P-name such that*
(1) *$P \Vdash$ the tree $\dot{S}$ $\mathbb{UM}$-dominates the ground model;*
(2) *for every $T \in$ NWD the boolean value $\|\check{T} \subset \dot{S}\|_P$ is non-zero.*
Then there is a complete embedding $RO(\mathbb{UM}) \lessdot P * (\mathbb{UM}(\dot{S}) \cap$ the ground model$) = R$ such that $R \Vdash$ "$\dot{U} \subset \dot{S}$", where $\dot{U}$ is the $\mathbb{UM}$-generic tree.

Corollary 3.5. *Let M be a countable transitive model of* ZFC *such that $M \models$ "P is a forcing adding a $\mathbb{UM}$-dominating tree $\dot{S}$ as in Corollary 3.4", let $p \in P$ and let $T \in$ NWD be any tree, not necessarily in the model M. Then there is an M-generic filter $G \subset P$ containing p such that for some sequence $\eta \in \dot{S}/G$ we have $T \subset \dot{S}/G(\eta)$.*

Set $\phi =$ "cofinality of the meager ideal is $\aleph_1$". The analysis of the forcing P_ϕ is now completely parallel to the treatment in Section 2.

Definition 3.6. A witness for ϕ is an ω_1-sequence s of perfect nowhere dense trees such that
(1) for every NWD tree T the set $\{\alpha \in \omega_1 : T \subset s(\alpha)\} \subset \omega_1$ is stationary;
(2) for every NWD tree T the set $C_T = \{\alpha \in \omega_1$: there is $n \in \omega$ such that if $x = {}^n2 \cap T \cap s(\alpha)$ then $T \restriction x \subset s(\alpha) \restriction x\} \subset \omega_1$ contains a club;
(3) there is a NWD tree T which is contained in all $s(\alpha)$: $\alpha \in \omega_1$.

Of course, it is important to verify that this notion deserves its name.

Lemma 3.7. *The following are equivalent:*
(1) *the cofinality of the meager ideal is $\aleph_1$;*
(2) *the cofinality of the nowhere dense ideal is $\aleph_1$;*
(3) *there is a witness for ϕ.*

Proof. (1) $\rightarrow$ (2): Let $\{Y_\alpha : \alpha \in \omega_1\}$ be a base for the meager ideal, $Y_\alpha \subset \bigcup \{[T^i_\alpha] : i \in \omega\}$ for some sequence $T^i_\alpha : i \in \omega$ of NWD trees with $T^i_\alpha \subset T^{i+1}_\alpha$. We shall show that the collection $\{[T^i_\alpha(\eta)] : \alpha \in \omega_1,\ i \in \omega,\ \eta \in T^i_\alpha\}$ is a base for the nowhere dense ideal, proving the lemma. Indeed, let S be a nowhere dense tree on ω. We shall produce α, i, η so that $S \subset T^i_\alpha(\eta)$ and therefore $[S] \subset [T^i_\alpha(\eta)]$.

It is a matter of an easy surgery on the tree S to obtain a nowhere dense tree $\bar{S}$ so that for every $\eta \in \bar{S}$ there is $\tau \in \bar{S}$ with $\eta \subset \tau$ and $\bar{S}(\tau) = S$. Choose a countable ordinal α so that $[\bar{S}] \subset Y_\alpha$ and attempt to build a descending sequence $\eta_i : i \in \omega$ of elements of $\bar{S}$ so that $\eta_i \notin T^i_\alpha$. There must be an integer $i \in \omega$ such that the construction cannot proceed past η_i – otherwise the branch $\bigcup_{i \in \omega} \eta_i \in [\bar{S}]$ would lie outside of the set Y_α. But then, $\bar{S}(\eta_i) \subset T^i_\alpha(\eta_i)$ and if $\tau \in \bar{S}$ is such that $\eta_i \subset \tau$ and $\bar{S}(\tau) = S$ then necessarily $S = \bar{S}(\tau) \subset T^i_\alpha(\tau)$.

(2)$\to$(3): Let T_ξ: $\xi \in \omega_1$ be a $\subset$-cofinal family of NWD trees. Fix a partition S_ξ: $\xi \in \omega_1$ of ω_1 into disjoint stationary sets and a NWD tree T. For each $\alpha \in \omega_1$ choose a filter $G_\alpha \subset \mathbb{UM}$ such that

(1) $\langle 0, T \cup T_\xi \rangle \in G_\alpha$ whenever $\alpha \in S_\xi$

(2) G_α meets the dense sets $D_\beta = \{\langle n, U \rangle \in \mathbb{UM}$: setting $x = U \cap {}^n 2$ we have $T_\beta \restriction x \subset U \restriction x$ for all $\beta \in \alpha$.

Using the remarks after Definition 3.2 it is easy to see that the sequence s: $\omega_1 \to$ NWD defined by $s(\alpha) = \bigcup \{U$: $\langle 0, U \rangle \in G_\alpha\}$ is the desired witness for ϕ.

(3)$\to$(1): Let s: $\omega_1 \to$ NWD be a witness for ϕ. Obviously, the family $Y_\alpha = \bigcup_{\beta \in \alpha} [s(\beta)]$: $\alpha \in \omega_1$ is $\subset$-cofinal in the nowhere dense ideal. $\square$

Now if M is a transitive model of ZFC with $M \models$ "s is a witness for ϕ and δ is a Woodin cardinal" then in M, $\mathbb{Q}_{<\delta} \Vdash$ "$js(\omega_1)$ is a NWD tree $\mathbb{UM}$-covering the model M", where j is the term for the generic nonstationary tower embedding; also $M, \mathbb{Q}_{<\delta}, js(\omega_1)$ satisfy the assumptions of Corollary 3.5. The proof of Π_2-compactness of ϕ translates now literally from the previous section. We prove the strategic iteration lemma from optimal assumptions.

Strategic Iteration Lemma 3.8. *Suppose that the cofinality of the meager ideal is equal to $\aleph_1$. The good player has a winning strategy in the game $\mathcal{G}_\phi$.*

Proof. Let $\vec{N} = \langle s, N_i, \delta_i$: $i \in \omega \rangle$ be a sequence of models with a witness for ϕ, let $y_0 \in \mathbb{Q}_{\vec{N}}$ and let T be an arbitrary NWD tree. We shall show that there is an $\vec{N}$-generic filter $G \subset \mathbb{Q}_{\vec{N}}$ such that letting $S = j_{\mathbb{Q}} s(\omega_1^{\vec{N}})$, where $j_{\mathbb{Q}}$ is the $\mathbb{Q}^{N_0}_{<\delta_0}$-generic ultrapower embedding using the filter $G \cap \mathbb{Q}^{N_0}_{<\delta_0}$, we have that for some $\eta \in S, T \subset S(\eta)$. With this fact in hand, the winning strategy for the good player consists just from an appropriate bookkeping:

Since cofinality of the meager ideal is $\aleph_1$, it is possible to choose a $\subset$-cofinal family T_α: $\alpha \in \omega_1$ of NWD trees. So the good player can easily play the game so that with the resulting embedding j of the initial iterable model M, for every $\alpha \in \omega_1$ there is $\gamma \in \omega_1$ and a sequence η in the tree $js(\gamma)$ such that $T_\alpha \subset js(\gamma)(\eta)$. It is immediate that if this is the case and the iteration j is full, the sequence $j(s)$ is a witness for ϕ and the good player won the run of the game. For let $S \in$ NWD be an arbitrary tree. Then there are α, γ and η such that $S \subset T_\alpha \subset js(\gamma)(\eta)$ and so

(1) the set $\{\beta \in \omega_1$: $S \subset js(\beta)\}$ contains the set $\{\beta \in \omega_1$: $js(\gamma)(\eta) \subset js(\beta)\}$ which is in the target model of the iteration j, is stationary there from Definition 3.6(1) and so is stationary in V by the fullness of the iteration

(2) the set $\{\beta \in \omega_1$: for some $n \in \omega$, $S \restriction x \subset js(\beta) \restriction x$ holds with $x = {}^n 2 \cap S \cap js(\beta)\}$ contains the set $\{\beta \in \omega_1$: for some $n \in \omega, js(\gamma)(\eta) \restriction x \subset js(\beta) \restriction x$ holds with $x = {}^n 2 \cap js(\gamma)(\eta) \cap js(\beta)\}$, which is in the target model of the iteration j and contains a club by Definition 3.6(2).

Therefore Definition 3.6(1,2) are verified for $j(s)$ and (3) of that definition follows from elementarity of the embedding j. Thus js is a witness for ϕ as desired.

The proof of the local fact about the sequence of models carries over from Lemma 2.8 with the following changes:

(1) d is replaced with s, $\mathbb{D}$ is replaced with $\mathbb{UM}$, the Hechler real e is replaced with a NWD tree U

(2) the step (5) of that proof is replaced with: there is a sequence $\eta \in U$ such that $T \subset U(\eta)$. Note that the set $\{\langle n, S\rangle \in \mathbb{UM}: \exists \eta \in S\ T \subset S(\eta)\}$ is dense in $\mathbb{UM}$.

(3) the ordering R_i is defined as follows: $R_i = \{\langle y, n, S\rangle: y \in \mathbb{Q}_i, \langle n, S\rangle \in \mathbb{UM},\ S = s(\alpha) \restriction z$ for some $\alpha \in \omega_1^{N}$ and finite set z such that $\forall x \in y\ S \subset s(x \cap \omega_1)\}$. It is possible to restrict ourselves to the trees S of the above form, since the sequence $s(\alpha)$: $\alpha \in \omega_1$ is $\subset$-cofinal in $\mathrm{NWD} \cap N_i$. $\square$

Conclusion 3.9. *The sentence ϕ =cofinality of the meager ideal $= \aleph_1$ is Π_2-compact. Theorem Scheme* 0.2 *holds even with an extra predicate for ω_1-sequences of meager sets cofinal in the ideal.*

3.1. Cofinality of the null ideal

In this subsection we shall show that "cofinality of the null ideal $= \aleph_1$" is a Π_2-compact statement. The following textbook equality will be used:

Lemma 3.10. *Cofinality of the null ideal is equal to the cofinality of the poset of the open subsets of reals of finite measure ordered by inclusion.*

Therefore we will really care about large open sets of finite measure.

Definition 3.11. The amoeba forcing $\mathbb{A}$ is the set $\{\langle \mathcal{O}, \varepsilon\rangle: \mathcal{O}$ is an open set of finite measure and ε is a positive rational greater than $\mu(\mathcal{O})\}$ ordered by $\langle \mathcal{O}, \varepsilon\rangle \leqslant \langle \mathcal{P}, \delta\rangle$ if $\mathcal{P} \subset \mathcal{O}$ and $\varepsilon \leqslant \delta$. The restricted poset $\mathbb{A}(\mathcal{O})$ for an open set $O \subset \mathbb{R}$ is $\{\langle \mathcal{P}, \varepsilon\rangle \in A: \mathcal{P} \subset \mathcal{O}\}$ with the inherited ordering.

It is not a priori clear why the different versions of the amoeba forcing should be isomorphic, see [18]. The amoeba poset is a σ-linked Souslin forcing notion designed to add a "large" open set of finite measure. If $G \subset \mathbb{A}$ is generic then the set $\mathcal{O}_G = \bigcup\{\mathcal{P}: \langle \mathcal{P}, \varepsilon\rangle \in G$ for some $\varepsilon\}$ is this open set and it determines the generic filter. Again, there is a natural weakening of the notion of genericity. Fix once and for all a sequence f_i: $i \in \omega$ of measure-preserving functions from $\mathbb{R}$ to $\mathbb{R}$ so that the sets $f_i''\mathbb{R}$ are pairwise disjoint and the sequence is arithmetical.

Definition 3.12. Let M be a transitive model of ZFC and $\mathcal{O}$ be an open set of reals. We say that $\mathcal{O}$ $\mathbb{A}$-dominates the model M if for every open set $\mathcal{P}$ of finite measure in the model M for all but finitely many integers $m \in \omega$, $f_m^{-1}\mathcal{P} \subset \mathcal{O}$.

Obviously, the amoeba generic open set does $\mathbb{A}$-dominate the ground model. We aim for the subgenericity theorems.

Lemma 3.13. *Let M be a transitive model of* ZFC *and let $\mathcal{O}$ dominate M. For every dense set $D \subset \mathbb{A} \cap M$ which is in the model M the set $D \cap \mathbb{A}(\mathcal{O})$ is dense in $M \cap \mathbb{A}(\mathcal{O})$.*

Proof. Let $M, \mathcal{O}, D$ be as in the lemma and let $\langle \mathscr{P}, \varepsilon \rangle \in M \cap \mathbb{A}(\mathcal{O})$. We shall produce a condition $p \in D \cap \mathbb{A}(\mathcal{O})$ below $\langle \mathscr{P}, \varepsilon \rangle$, proving the lemma.

Work in the model M. By induction on $i \in \omega$ build conditions $\langle \mathscr{R}_i, \delta_i \rangle \leqslant \langle \mathscr{P}, \varepsilon_i \rangle$ so that

(1) $\varepsilon = \varepsilon_0, \langle \mathscr{R}_i, \delta_i \rangle \in D$;

(2) for every integer $i > 0$ the inequality $\varepsilon_i - \mu(\mathscr{P}) < 2^{-i}$ holds.

Let $\mathscr{S} \in M$ be any open set of finite measure which covers the set $\bigcup_{i \in \omega} f_i(\mathscr{R}_i \setminus \mathscr{P})$. Since $\mathcal{O}$ $\mathbb{A}$-dominates the model M, there must be an integer $i \in \omega$ such that $f_i^{-1} \mathscr{S} \subset \mathcal{O}$, and so $\mathscr{R}_i \subset \mathcal{O}$. Then $\langle \mathscr{R}_i, \delta_i \rangle \leqslant \langle \mathscr{P}, \varepsilon_i \rangle$ is the desired condition. $\square$

Corollary 3.14 (Subgenericity). *Let P be a forcing and $\dot{\mathcal{O}}$ a P-name such that*

(1) *$P \Vdash$ "the open set $\dot{\mathcal{O}} \subset \mathbb{R}$ $\mathbb{A}$-dominates the ground model";*

(2) *for every open set $\mathscr{P}$ of finite measure the boolean value $\|\check{\mathscr{P}} \subset \dot{\mathcal{O}}\|_P$ is nonzero.*

Then there is a complete embedding $RO(\mathbb{A}) < P * (\mathbb{A}(\dot{O}) \cap$ the ground model$) = R$ such that $R \Vdash$ "$\dot{\mathscr{R}} \subset \dot{\mathcal{O}}$", where $\dot{\mathscr{R}}$ is the name for the A-generic open set.

A witness for $\phi =$ "cofinality of the null ideal $= \aleph_1$" is an ω_1-sequence o of open sets of finite measure such that

(1) for every open set $\mathscr{P}$ of finite measure the set $\{\alpha \in \omega_1 \colon \mathscr{P} \subset o(\alpha)\} \subset \omega_1$ is stationary;

(2) for every open set $\mathscr{P}$ of finite measure the set $\{\alpha \in \omega_1 \colon$ for all but finitely many integers $m \in \omega$ $f_m^{-1} \mathscr{P} \subset o(\alpha)\}$ contains a club.

Again, it is very simple to prove using Lemma 3.10 that ϕ is equivalent with the existence of a witness. The analysis of the forcing P_ϕ almost literally translates from Section 2. We leave all of this to the reader.

Conclusion 3.15. *The statement $\phi =$ "cofinality of the null ideal is $\aleph_1$" is Π_2-compact. Theorem Scheme* 0.2 *holds even with a predicate for cofinal families of null sets added to the language of $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I} \rangle$.*

4. Souslin trees

The assertion "there is a Souslin tree" does not seem to be Π_2-compact as outlined in Subsection 1.3; however, some of its variations are. A $P_{\max}$-style model in which many Souslin trees exist was constructed in [20] and in the course of the argument the following theorem, which implies the strategic iteration lemmas for all sentences considered in this section, was proved.

Sh:610

Let $\mathcal{G}_S$ be a two-person game played along the lines of $\mathcal{G}_\phi$ – in Definition 1.12 – with the following modifications:
(1) in the 0th move the player Bad specifies a collection $\mathcal{S}$ of Souslin trees in the model M instead of just one witness;
(2) in the αth step the player Bad must choose a sequence $\langle N_i: i \in \omega\rangle$ of models such that $j_{\gamma_\alpha}(\mathcal{S})$ consists of Souslin trees as seen from each N_i: $i \in \omega$;
(3) the player Good wins if $j_{\omega_1}(\mathcal{S})$ is a collection of Souslin trees.

Strategic Iteration Lemma 4.1 (Woodin [20]). *Assume $\diamond$. Then the player Good has a winning strategy in the game $\mathcal{G}_S$.*

Proof. Recall that ω_1-trees are by our convention sets of functions from countable ordinals to ω with some special properties. Fix a diamond sequence $\langle A_\beta: \beta \in \omega_1\rangle$ guessing uncountable subsets of such trees. The player Good wins the game as follows. Suppose we are at αth stage of the play and let $\beta = \omega_1^{M_{\gamma_\alpha}}$ and $\mathcal{S}_\alpha = j_{0,\gamma_\alpha}(\mathcal{S})$. Suppose Bad played a sequence $\vec{N} = \langle N_i, \delta_i: i \in \omega\rangle$ of models according to the rules – so $N_0 = M_{\gamma_\alpha}$ – and some $p \in \mathbb{Q}_{\vec{N}}$. Let us call an $\vec{N}$-generic filter $G \subset \mathbb{Q}_{\vec{N}}$ good if setting $j_{\mathbb{Q}}$ to be the ultrapower embedding of N_0 derived from $G \cap \mathbb{Q}_0$ we have: for every tree $S \in \mathcal{S}_\alpha$, if $A_\beta \subset S$ is a maximal antichain then every node at β-th level of $j_{\mathbb{Q}}(S)$ has an element of A_β above it in the tree ordering.

If the player Good succeeds in playing a good filter containing p at each stage $\alpha \in \omega_1$ of the game then he wins: every tree in the collection $j_{0\omega_1}(\mathcal{S})$ can then be shown Souslin by the usual diamond argument. Thus the following claim completes the proof.

Claim 4.2. *At stage α there is a good filter $G \subset \mathbb{Q}_{\vec{N}}$ containing p.*

Proof. Actually, any sufficiently generic filter is good. Note that every $\mathbb{Q}_0$ name $\dot{y} \in N_0$ for a cofinal branch of any tree $S \in \mathcal{S}_\alpha$ is in fact a $\mathbb{Q}_{\vec{N}}$-name for a generic subset of S – this follows from the fact that S is a Souslin tree in every model N_i: $i \in \omega$. Thus if a filter $G \subset \mathbb{Q}_{\vec{N}}$ meets every dense set recursive in some fixed real coding $\vec{N}$ and A_β, necessarily the branch $\dot{y}/G$ meets the set A_β if $A_\beta \subset S$ is a maximal antichain. Consequently, such a filter is good, since every $\mathbb{Q}_0$ name $\dot{y} \in N_0$ for an element of βth level of $j_{\mathbb{Q}}(S)$ can be identified with a name for a cofinal branch of the tree S. □

4.0. Free Souslin trees

The first Π_2-compact sentence considered in this section is $\phi =$ "there is a free tree" as clarified in the following definition:

Definition 4.3. A Souslin tree S is free if for every finite collection s_i: $i \in I$ of distinct elements of the same level of S the forcing $\prod_{i \in I} S \restriction s_i$ is c.c.c.

Thus every finitely many pairwise distinct cofinal branches of a free tree are mutually generic. It is not difficult to prove that both of the classical methods for forcing a Souslin tree [5, 15] in fact provide free trees. It is an open problem whether existence of Souslin trees implies existence of free trees.

The following observation, pointed out to us by W. Hugh Woodin, greatly simplifies the proof of the optimal iteration lemma for ϕ: any sufficiently rich (external) collection of cofinal branches of a free tree determines a *symmetric extension* of the universe in the appropriate sense.

Lemma 4.4. *Suppose that M is a countable transitive model of a rich fragment of* ZFC, *$M \models$ "S is a free Souslin tree" and $B = \{b_i : i \in I\}$ is a countable collection of cofinal branches of S such that $\bigcup B = S$. Then there is an enumeration $b_j : j \in \omega$ of B such that the equations $b_j = \dot{c}_j$ determine an M-generic filter on P_S.*

Here, $P_S \in M$ is the finite support product of countably many copies of the tree S, with $\dot{c}_j : j \in \omega$ being the canonical P_S-names for the added ω branches of S.

Corollary 4.5. *Suppose M, S and B are as in the Lemma and suppose that $M \models$ "P is a forcing, $p \in P$ and $P \Vdash \dot{C}$ is a collection of cofinal branches of the tree S such that $\bigcup \dot{C} = \check{S}$". Then there is an M-generic filter $G \subset P$ with $p \in G$ and $\dot{C}/G = B$.*

Proof. Work in M. Without loss of generality we may assume that $p = 1$ and that the forcing P collapses both $\kappa = (2^{\aleph_1})^+$ and $|\dot{C}|$ to $\aleph_0$. (Otherwise switch to $P \times Coll(\omega, \lambda)$ for some large enough ordinal λ.) There is a complete embedding of $RO(P_S)$ into $RO(P)$ such that $P \Vdash$ "$\dot{C}$ is the canonical set of branches of S added by P_S under this embedding". This follows from Lemma 4.4 applied in M^P to $M \cap H_\kappa$, S and $\dot{C}$. Another application of the Lemma to M, S and B gives an M-generic filter $H \subset P_S$ such that B is the canonical set of branches of the tree S added by H. Obviously, any M-generic filter $G \subset P$ extending H – via the abovementioned embedding – is as desired. $\square$

Proof of lemma. Say that the conditions in P_S have the form of functions from some $n \in \omega$ to S with the natural ordering. We shall show that for each injective $f : n \to B$ and every open dense set $O \subset P_S$ in the model M there is an injection $g : m \to B$ extending f and a condition $p \in O$ with $\text{dom}(p) = m$ and $\bigwedge_{k \in m} p(k) \in g(k)$. Granted that, a construction of the desired enumeration is straigthforward by the obvious bookkeeping argument using the countability of both M and B.

So fix f and O as above. There is an ordinal $\alpha \in \omega_1^M$ such that the branches $f(k) : k \in n$ pick pairwise distinct elements $s_k : k \in n$ from αth level of the tree S. Let $D = \{z \in \prod_{k \in n} S \restriction s_k : \exists p \in O \ \ p \restriction n = z\} \in M$. Since $O \subset P_S$ is dense below the condition $\langle s_k : k \in n \rangle \in P_S$, the set D must be dense in $\prod_{k \in n} S \restriction s_k$. Since this product is c.c.c. in the model M, the branches $f(k) : k \in n$ determine an M-generic filter on it and there must be $z \in D$ such that $\bigwedge_{k \in n} z(k) \in f(k)$. Choose a condition $p \in O$ with

$\mathrm{dom}(p)=m$ and $p\restriction n=z$. Since $\bigcup B=S$, it is possible to find branches $g(k)$: $n\leqslant k<m$ in the set B which are pairwise distinct and do not occur on the list $f(k)$: $k\in n$ such that $\bigwedge_{n\leqslant k<m} p(k)\in g(k)$. The branches $f(k)$: $k\in n$ and $g(k)$: $n\leqslant k<m$ together give the desired injection. $\square$

Optimal Iteration Lemma 4.6. *Assume there is a free tree. Whenever M is a countable transitive model of* ZFC *iterable with respect to its Woodin cardinal δ and $M\models$"U is a free tree" there is a full iteration j of M such that $j(U)$ is a free tree.*

Proof. Let T be a free Souslin tree and let M, U, δ be as above; so $M\models$"U is a free Souslin tree". We shall produce a full iteration j of M such that there is a club $C\subset\omega_1$ and an isomorphism $\pi: T\restriction C\to j(U)\restriction C$. Then, since the trees $T, j(U)$ are isomorphic on a club, necessarily $j(U)$ is a free Souslin tree. This will finish the proof of the lemma.

The iteration will be constructed by induction on $\alpha\in\omega_1$ and we will have $\theta_\alpha=\omega_1^{M_\alpha}$ and $C=\{\theta_\alpha:\ \alpha\in\omega_1\}$. Also, we shall write U_α for the image of the tree U under the embedding $j_{0,\alpha}$. This is not to be confused with the α-th level of the tree U. In this proof, levels of trees are never indexed by the letter α.

First, fix a partition $\{S_\xi:\ \xi\in\omega_1\}$ of the set of countable limit ordinals into disjoint stationary sets. By induction on $\alpha\in\omega_1$, build the models together with the elementary embeddings, plus an isomorphism $\pi: T\restriction C\to jU\restriction C$, plus an enumeration $\{\langle x_\xi,\beta_\xi\rangle:\ \xi\in\omega_1\}$ of all pairs $\langle x,\beta\rangle$ with $x\in\mathbb{Q}_\beta$. The induction hypotheses at $\alpha\in\omega_1$ are

(1) the function $\pi\restriction T\restriction\{\theta_\gamma:\ \gamma\in\alpha\}$ has been defined and it is an isomorphism of $T\restriction\{\theta_\gamma:\ \gamma\in\alpha\}$ and $U_\alpha\restriction\{\theta_\gamma:\ \gamma\in\alpha\}$;

(2) the initial segment $\{\langle x_\xi,\beta_\xi\rangle:\ \xi\in\theta_\alpha\}$ has been constructed and every pair $\langle x,\beta\rangle$ with $\beta\in\alpha$ and $x\in\mathbb{Q}_\beta$ appears on it;

(3) if $\gamma\in\alpha$ belongs to some – unique – set S_ξ then $j_{\beta_\xi,\gamma}(x_\xi)\in G_\gamma$.

At limit steps, we just take direct limits and unions. At successor steps, given $M_\alpha, U_\alpha, \delta_\alpha$ and $\pi\restriction\{\theta_\gamma:\ \gamma\in\alpha\}$, we must provide an M_α-generic filter $G_\alpha\subset\mathbb{Q}_\alpha$ such that setting $U_{\alpha+1}=j_{\mathbb{Q}}U_\alpha$, where $j_{\mathbb{Q}}$ is the generic ultrapower of M_α by G_α, it is possible to extend the isomorphism π to θ_αth levels of T and $U_{\alpha+1}$.

First suppose α is a successor ordinal, $\alpha=\beta+1$. Let G_α be an arbitrary M_α-generic filter; we claim that G_α works. Simply let for every $t\in T_{\theta_\beta}$ $\pi\restriction(T\restriction t)_{\theta_\alpha}$ to be a bijection of $(T\restriction t)_{\theta_\alpha}$ and $(U_{\alpha+1}\restriction\pi(t))_{\theta_\alpha}$. This is clearly possible since both of these sets are infinite and countable. Induction hypothesis (1) continues to hold, induction hypothesis (2) is easily arranged by extending the enumeration properly and (3) does not say anything about successor ordinals.

Finally, suppose α is a limit ordinal. In this case, the θ_αth level of the tree $U_{\alpha+1}$ is determined by $\pi\restriction T\restriction\{\theta_\gamma:\ \gamma\in\alpha\}$ and the necessity of extending the isomorphism π to the θ_αth level of the tree T. Namely we must have θ_αth level of $U_{\alpha+1}$ equal to the set $D=\{d_t:\ t\in T_{\theta_\alpha}\}$ where $d_t=\bigcup\{\pi(r):\ r\in T\restriction\{\theta_\gamma:\ \gamma\in\alpha\}, t\leqslant_T r\}$. Corollary 4.5 applied to $M_\alpha, S_\alpha, \mathbb{Q}_\alpha, (S_{\alpha+1})_{\theta_\alpha}$ and D shows that an appropriate generic filter on $\mathbb{Q}_\alpha$ can be found containing the condition $j_{\beta_\xi,\alpha}(x_\xi)$ if $\alpha\in S_\xi$. The isomorphism π then extends in the obvious unique fashion mapping t to d_t. $\square$

Conclusion 4.7. *The sentence $\phi=$"there is a free tree" is Π_2-compact.*

Another corollary to the proof of Lemma 4.6 is the fact that Σ^1_1 theory of free trees is *complete* and *minimal* in the following sense. Suppose ψ is a Σ^1_1 property of trees T which depends only on the Boolean algebra $RO(T)$, that is, ZFC$\vdash$"$RO(S)=RO(T)$ implies $T\models\psi$ iff $S\models\psi$". Then, granted large cardinals, the sentence ψ is either true on all free trees in all set-generic extensions of the universe or it fails on all such trees. Moreover, if ψ fails on *any* ω_1-tree in any set-generic extension then it fails on all free trees. Here, by Σ^1_1 property we mean a formula of the form $\exists A\subset T\chi$, where all quantifiers of χ range over the elements of T only.

It should be noted that it is impossible to add a predicate $\mathfrak{S}$ for free trees to the language of $\langle H_{\aleph_2},\in,\omega_1,\mathfrak{I}\rangle$ and preserve the compactness result. For consider the following two Π_2 sentences for $\langle H_{\aleph_2},\in,\mathfrak{I},\mathfrak{S}\rangle$:

$$\psi_0=\forall S,\quad T\in\mathfrak{S}\quad \exists s\in S,\quad t\in T\ S\restriction s\times T\restriction t\quad \text{is c.c.c.},$$

$$\psi_1=\text{for every Aronszajn tree } T \text{ there is a tree } S\in\mathfrak{S} \text{ such that } T\Vdash S \text{ is special.}$$

It is immediate that ψ_0 and ψ_1 together imply that $\mathfrak{S}$ is empty, i.e. $\neg\phi$. Meanwhile, $\psi_0\wedge\phi$ was found consistent in [AS]–and in fact holds in our model – and $\psi_1\wedge\phi$ holds after adding $\aleph_2$ Cohen reals to any model of GCH, owing to the following Lemma:

Lemma 4.8. *For every aronszajn tree T, $\mathbb{C}_{\aleph_1}\Vdash$"there is a free tree which is specialized after forcing with T".*

Note that Cohen algebras preserve Souslin trees.

Proof. Let T be an Aronszajn tree. Define a forcing P as a set of pairs $p=\langle s_p,f_p\rangle$ where

(1) s_p is a finite tree on $\omega_1\times\omega$ such that $\langle\alpha,n\rangle<_{s_p}\langle\beta,m\rangle$ implies $\beta<\alpha\ldots$ this is a finite piece of the tree S under construction;

(2) f_p is a finite function with domain contained in T and each $f_p(t)$ a function from $\mathrm{dom}(s)\cap(\{\alpha\}\times\omega)$ to ω where $\alpha=lev(t)\ldots$ this is a finite piece of the S-specializing T-name;

(3) for every $i<_{s_p}j$ and $t<_T u$ the inequality $f_p(t)(i)\neq f_p(u)(j)$ holds, if the relevant terms are defined... this is the specializing condition.

The ordering is defined by $q\leqslant p$ if $\mathrm{dom}(s_p)\subset\mathrm{dom}(s_q)$ and $s_q\cap\mathrm{dom}(s_p)\times\mathrm{dom}(s_p)=s_p$ and $f_p(t)\subset f_q(t)$ whenever $t\in\mathrm{dom}(f_p)$.

Let $G\subset P$ be a generic filter and in $V[G]$ define a tree S on $\omega_1\times\omega$ as the unique tree extending all s_p: $p\in G$, and a function τ on the tree T to be $\tau(t)=\bigcup_{p\in G}f_p(t)$. Obviously, τ represents a T-name for a specializing function on S: if $b\subset T$ is a cofinal branch then the function g: $S\to\omega$, $g=\bigcup_{t\in b}\tau(t)$ specializes the tree S due to the condition (3) in the definition of the forcing P. To complete the proof, we have to verify that $RO(P) = \mathbb{C}_{\aleph_1}$ and that $P \Vdash S$ is a free Souslin tree. This is done in the following two claims.

Claim 4.9. *$RO(P)$ is isomorphic to $\mathbb{C}_{\aleph_1}$.*

Proof. Obviously, P has uniform density $\aleph_1$, therefore it is enough to prove that P has a closed unbounded collection of regular subposets [8]. Let $\alpha \in \omega_1$ be a limit ordinal and let $P_\alpha = \{p \in P\colon\ dom(s_p) \subset \alpha \times \omega\}$. It is easy to verify that all P_α's are regular subposets of P and that they constitute an increasing continuous chain exhausting all of P, proving the lemma. □

Claim 4.10. *$P \Vdash S$ is a free Souslin tree.*

Proof. Assume that $p \Vdash$ "$\dot{A} = \{a_\alpha\colon\ \alpha \in \omega_1\}$ is a family of pairwise distinct elements of $\dot{S} \restriction i_0 \times \dot{S} \restriction i_1 \times \cdots \times \dot{S} \restriction i_n$", for some integer n and pairwise s_p-incompatible elements $i_0 \ldots i_n$ of $\mathrm{dom}(s_p)$. To prove the Lemma, it is enough to produce a condition $q \leqslant p$ and ordinals $\alpha < \beta$ such that $q \Vdash \dot{a}_\alpha$ and $\dot{a}_\beta$ are compatible.

Pick p_α, a_α: $\alpha \in \omega_1$ such that each p_α is a condition stronger than p and it decides the value of the name $\dot{a}_\alpha$ to be a_α, regarded as an $n+1$-element subset of $\mathrm{dom}(s_{p_\alpha}) \restriction i_0 \cup \mathrm{dom}(s_{p_\alpha}) \restriction i_1 \cup \cdots \cup \mathrm{dom}(s_{p_\alpha}) \restriction i_n$.

By a repeated use of counting arguments and a Δ-system lemma, thinning out the collection of p_α, a_α's we may assume that

(1) $\mathrm{dom}(s_{p_\alpha})$ form a Δ-system with root r and $s_{p_\alpha} \restriction r \times r$ is the same for all α;

(2) even the sets $lev(s_{p_\alpha}) = \{\beta \in \omega_1\colon\ \mathrm{dom}(s_{p_\alpha}) \cap \{\beta\} \times \omega \neq 0\}$ form a Δ system with root $lev(r) = \{\beta \in \omega_1\colon\ r \cap \{\beta\} \times \omega \neq 0\}$;

(3) $f_{p_\alpha} \restriction T_\beta$ are the same for all α, this for all $\beta \in lev(r)$;

(4) a_α form a Δ-system with root $b \subset r$.

Now let $x_\alpha = \mathrm{dom}(f_{p_\alpha}) \setminus \bigcup_{\beta \in lev(r)} T_\beta$. Thus x_α are pairwise disjoint finite subsets of the Aronszajn tree T, and it is possible to find countable ordinals $\alpha < \beta$ such that every $t \in x_\alpha$ is T-incompatible with every $u \in x_\beta$. It follows that any tree s_q with $\mathrm{dom}(s_q) = \mathrm{dom}(s_{p_\alpha}) \cup \mathrm{dom}(s_{p_\beta})$, $s_q \restriction \mathrm{dom}(s_{p_\alpha}) \times \mathrm{dom}(s_{p_\alpha}) = s_{p_\alpha}$ and $s_q \restriction \mathrm{dom}(s_{p_\beta}) \times \mathrm{dom}(s_{p_\beta}) = s_{p_\beta}$, together with the function $f_q = f_{p_\alpha} \cup f_{p_\beta}$ give a condition $q = \langle s_q, f_q \rangle$ in the forcing P which is stronger than both p_α and p_β. It is a matter of an easy surgery on s_{p_α} and s_{p_β} to provide such a condition q so that a_α, a_β are compatible in $s_q \restriction i_0 \times s_q \restriction i_1 \times \cdots \times s_q \restriction i_n$. Then $p \geqslant q \Vdash$ "$\dot{a}_\alpha$ and $\dot{a}_\beta$ are compatible elements of $\dot{A}$" as desired. □

4.1. Strongly homogeneous Souslin trees

In this subsection it is proved that the assertion $\phi =$ "there is a strongly homogeneous Souslin tree" is Π_2-compact, where

Definition 4.11. Let T be an ω_1-tree. A family $\{h(s_0, s_1)\colon\ s_0, s_1 \in T$ are elements of the same level of $T\}$ is called *coherent* if

(1) $h(s_0, s_1)$ is a level-preserving isomorphism of $T \restriction s_0$ and $T \restriction s_1$; $h(s, s) = id$;

(2) (*commutativity*) let s_0, s_1, s_2 be elements of the same level of T and $t_0 \leqslant s_0$. Then $h(s_1, s_2)h(s_0, s_1)(t_0) = h(s_0, s_2)(t_0)$;
(3) (*coherence*) let s_0, s_1 be elements of the same level of T and $t_0 \leqslant s_0$. Let $t_1 = h(s_0, s_1)(t_0) \leqslant s_1$. Then $h(t_0, t_1) = h(s_0, s_1) \restriction T \restriction t_0$;
(4) (*transitivity*) if α is a limit ordinal and t_0, t_1 are two different elements at αth level of T then there are $s_0, s_1 \in T_{<\alpha}$ such that $h(s_0, s_1)(t_0) = t_1$.

A tree is called *strongly homogeneous* if it has a coherent family of isomorphisms.

The existence of strongly homogeneous Souslin trees can be proved from $\diamondsuit$ by a standard argument. Also, Todorcevic's term for a Souslin tree in one Cohen real extension provides in fact for a strongly homogeneous tree:

Theorem 4.12. $\mathbb{C} \Vdash$ *there is a strongly homogeneous Souslin tree.*

Proof. An elaboration on Todorcevic's argument [16]. Let T be a family of functions such that
(1) every $f \in T$ is of the form $f : \alpha \to \omega$, finite-to-one for some countable ordinal α;
(2) for each $\alpha \in \omega_1$ there is $f \in T$ with $\alpha = \mathrm{dom}(f)$;
(3) every two functions $f, g \in T$ are modulo finite equal on the intersection of their domains;
(4) T is closed under finite changes of its elements.

Such a family is built as in [16] and it can be understood as a special Aronszajn tree under the reverse inclusion order. If $c \in {}^{\omega}\omega$ is a Cohen real then [16] the tree $T_c = \{c \circ f : f \in T\}$ ordered by reverse inclusion is a Souslin tree in $V[c]$. To conclude the proof of the theorem, we shall find a coherent family of isomorphisms of the tree T which is easily seen to lift to the tree T_c. Namely, let $f, g \in T, \mathrm{dom}(f) = \mathrm{dom}(g)$. Define $h(f, g)(e) = g \cup (e \setminus f)$ for $e \in T$ with $f \subset e$. By (3) and (4) above this is a well-defined function and an isomorphism of the trees $T \restriction f$ and $T \restriction g$. The easy proof that these isomorphisms form a coherent family on a tree T which lifts to the tree T_c is left to the reader. $\square$

Paul Larson proved that every strongly homogeneous Souslin tree contains a regularly embedded free tree. In fact, every strongly homogeneous Souslin tree can be written as a product of two free trees.

Optimal Iteration Lemma 4.13. *Assume there is a strongly homogeneous Souslin tree. Whenever M is a countable transitive model of ZFC iterable with respect to its Woodin cardinal δ with $M \models$ "U is a strongly homogeneous Souslin tree" there is a full iteration j of M such that $j(U)$ is a strongly homogeneous Souslin tree.*

Proof. Let T be a strongly homogeneous Souslin tree with a coherent family $\{g(t_0)(t_1) : t_0, t_1 \in T_\alpha \text{ for some } \alpha \in \omega_1\}$ of isomorphisms and let M, U, δ be as above and $M \models$ "U is a strongly homogeneous Souslin tree as witnessed by a family $h = \{h(s_0)(s_1) : s_0, s_1 \in U_\xi$

Sh:610

for some $\xi \in \omega_1^M\}$". We shall produce a full iteration j of M such that there is a club $C \subset \omega_1$ and an isomorphism $\pi : T \restriction C \to j(U) \restriction C$ which commutes with the internal isomorphisms of the trees: $\pi g(t_0, t_1)(u) = jh(\pi t_0, \pi t_1)(\pi u)$ whenever the relevant terms are defined. Then, since the trees $T, j(U)$ are isomorphic on a club, necessarily $j(U)$ is a Souslin tree, and it is strongly homogeneous as witnessed by the family $j(h)$. This will finish the proof of the lemma. Again, below U_α denotes the tree $j_{0\alpha}U$ and not the αth level of U. Levels of trees are never indexed by α.

The iteration will be constructed by induction on $\alpha \in \omega_1$ and we will have $\theta_\alpha = \omega_1^{M_\alpha}$ and $C = \{\theta_\alpha : \alpha \in \omega_1\}$. First, fix a partition $\{S_\xi : \xi \in \omega_1\}$ of the set of countable limit ordinals into disjoint stationary sets. By induction on $\alpha \in \omega_1$, we shall build the models together with the elementary embeddings, plus an isomorphism $\pi : T \restriction C \to jU \restriction C$, plus an enumeration $\{\langle x_\xi, \beta_\xi \rangle : \xi \in \omega_1\}$ of all pairs $\langle x, \beta \rangle$ with $x \in \mathbb{Q}_\beta$. The induction hypotheses at $\alpha \in \omega_1$ are:

(1) the function $i \restriction T \restriction \{\theta_\gamma : \gamma \in \alpha\}$ has been defined, it is an isomorphism of $T \restriction \{\theta_\gamma : \gamma \in \alpha\}$ and $U_\alpha \restriction \{\theta_\gamma : \gamma \in \alpha\}$ and it commutes with the internal isomorphisms of the trees, i.e. $\pi g(t_0, t_1)(u) = j_{0,\alpha} h(\pi t_0, \pi t_1)(\pi u)$ whenever the relevant terms are defined;

(2) the initial segment $\{\langle x_\xi, \beta_\xi \rangle : \xi \in \theta_\alpha\}$ has been constructed and every pair $\langle x, \beta \rangle$ with $\beta \in \alpha$ and $x \in \mathbb{Q}_\beta$;

(3) if $\gamma \in \alpha$ belongs to some – unique – set S_ξ then $j_{\beta_\xi, \gamma}(x_\xi) \in G_\gamma$.

As before, (1) is the crucial condition ensuring that the tree T is copied to $j(U)$ properly. (2,3) are just bookkeping requirements for making the resulting iteration full.

At limit steps, we just take direct limits and unions of the isomorphisms and enumerations constructed so far. At successor steps, given M_α, we must produce a M_α-generic filter $G_\alpha \subset \mathbb{Q}_\alpha$ such that setting $\langle M_{\alpha+1}, U_{\alpha+1}, \delta_{\alpha+1} \rangle$ to be the generic ultrapower of $\langle M_\alpha, U_\alpha, \delta_\alpha \rangle$ by G_α, the isomorphism π can be extended to θ_αth level of the trees T and $U_{\alpha+1}$ preserving the induction hypothesis (1).

• *Case* 1. α is a successor ordinal, $\alpha = \beta + 1$. Choose an arbitrary M_α-generic filter $G_\alpha \subset \mathbb{Q}_\alpha$. We shall show how the isomorphism π can be extended to the θ_αth level of the trees T and $U_{\alpha+1}$ preserving the induction hypothesis (1).

Let $t \in T_{\theta_\alpha}$ be arbitrary. The *θ_β-orbit* of t is the set $\{u \in T_{\theta_\alpha} : \exists t_0, t_1 \in T_{\theta_\beta}\ u = g(t_0, t_1)(t)\}$. By the commutativity property of the isomorphisms g, the θ_αth level of the tree T partitions into countably many disjoint γ-orbits O_k: $k \in \omega$. Also, for every $u \in T_{\theta_\beta}$ and integer $k \in \omega$ there is a unique $t \in O_k$ with $t \leqslant_T u$. The same analysis applies to the tree $U_{\alpha+1}$ and isomorphisms h. The θ_αth level of the tree $U_{\alpha+1}$ partitions into countably many disjoint γ-orbits N_k: $k \in \omega$.

Now it is easy to see that there is a unique way to extend the function π to T_{θ_α} so that $\pi'' O_k = N_k$ and π is order-preserving. Such an extended function will satisfy the induction hypothesis (1). The induction hypothesis (2) is easily managed and the induction hypothesis (3) does not say anything about successor ordinals α.

• *Case* 2. α is a limit ordinal. In this case, θ_αth level of the tree $U_{\alpha+1}$ is already predetermined by $\pi \restriction T \restriction \{\theta_\gamma : \gamma \in \alpha\}$ and the necessity of extending π. Namely, we must have $(U_{\alpha+1})_{\theta_\alpha} = \{u :$ there is $t \in T_{\theta_\alpha}$ such that $u = \bigcup\{\pi r : t \leqslant_T r\}\}$. The challenge is to find an M_α-generic filter $G_\alpha \subset \mathbb{Q}_\alpha$ such that $(\dot{U}_{\alpha+1})_{\theta_\alpha}/G_\alpha$ is of the above described form.

Then necessarily the only possible orderpreserving extension of π to T_{θ_α} will satisfy the induction hypothesis (1). We shall use the fact that it is enough to know one element of $(U_{\alpha+1})_{\theta_\alpha}$ in order to determine the whole level – by transitivity, Definition 4.2(4).

Work in M_α. Fix $\dot{u}$, an arbitrary $\mathbb{Q}_\alpha$-name for an element of $(U_{\alpha+1})_{\theta_\alpha}$, which will be identified with the cofinal – and therefore M_α-generic – branch of the tree U_α it determines. Let $b_0 \in \mathbb{Q}_\alpha$ be defined as $j_{\beta_\xi,\alpha}(x_\xi)$ if α belongs to some – unique – set S_ξ with $\xi \in \theta_\alpha$, otherwise let $b_0 = 1$ in $\mathbb{Q}_\alpha$. Let B be the complete subalgebra of $RO(\mathbb{Q}_\alpha)$ generated by the name $\dot{u}$. By some Boolean algebra theory, there must be $b_1 \leqslant b_0$ and $s \in U_\alpha$ so that $pr_B b_1 = [[\check{s} \in \dot{u}]]_B = b_2$ and $B \restriction b_2$ is isomorphic to $RO(U_\alpha \restriction s)$ by an isomorphism generated by the name $\dot{u}$. Without loss of generality $lev(s) = \theta_\gamma$ for some $\gamma \in \alpha$, since the set $\{\theta_\gamma \colon \gamma \in \alpha\}$ is cofinal in θ_α.

Now pick an element $t \in T_{\theta_\alpha}$ such that $t \leqslant_T \pi^{-1} s$. Then $c = \bigcup \{\pi(r) \colon t \leqslant_T r\}$ is a cofinal M_α-generic branch through U_α containing s. Let $H \subset B$ be the M_α-generic filter determined by the equation $c = \dot{u}$ and let $G_\alpha \subset RO(\mathbb{Q}_\alpha)$ be any M_α generic filter with $H \subset G_\alpha, b_1 \in G_\alpha$. We claim that G_α works.

Let $\langle M_{\alpha+1}, U_{\alpha+1}, \delta_{\alpha+1}\rangle$ be the generic ultrapower of $\langle M_\alpha, U_\alpha, \delta_\alpha\rangle$ using the filter G_α. Define $\pi \restriction T_{\theta_\alpha}$ by $\pi g(r_0, r_1)(t) = j_{0,\alpha+1} h(s_0, s_1)(c)$ where $r_0, r_1 \in T_{\theta_\gamma}$ for some $\gamma \in \alpha$ and $t \leqslant_T r_0$, $\pi(r_0) = s_0$, $\pi(r_1) = s_1$ and t is the element of T_{θ_α} used to generate c in the previous paragraph. By the induction hypothesis (1) and coherence – Definition 4.2(4)-π is well-defined, and by transitivity applied to both T and U side $\pi \restriction T_{\theta_\alpha} : T_{\theta_\alpha} \to (U_{\alpha+1})_{\theta_\alpha}$ is a bijection. It is now readily checked that π commutes with the internal isomorphisms g, h and the induction hypothesis continues to hold at $\alpha + 1$. The hypothesis (2) is easily managed by suitably prolonging the enumeration, and the induction hypothesis (3) is maintained by the choice of $b_0 \in G_\alpha$. $\square$

Conclusion 4.14. *The sentence "there is a strongly homogeneous Souslin tree" is* Π_2-*compact.*

Again, the proof of Lemma 4.13 shows that the Σ^1_1 theory of strongly homogeneous Souslin trees is complete in the same sense as explained in the previous Subsection. The first order theory of the model obtained in this Subsection has been independently studied by Paul Larson.

4.2. Other types of Souslin trees

One can think of a great number of Σ_1 constraints on Souslin trees. With each of them, the first two iteration Lemmas can be proved for $\phi =$ "there is a Souslin tree with the given constraint" owing to Lemma 4.1. However, the absoluteness properties of the resulting models as well as the status of Π_2-compactness of such sentences ϕ are unknown. Example:

Definition 4.15. A Souslin tree T is self-specializing if $T \Vdash$ "$\check{T} \backslash \dot{b}$ is special, where $\dot{b}$ is the generic branch".

A selfspecializing tree can be found under $\diamondsuit$ or after adding $\aleph_1$ Cohen reals. Such a tree is obviously neither free nor strongly homogeneous and no such a tree exists in the models from the previous two subsections.

5. The bounding number

In this section it will be proved that the sentence $\mathfrak{b}=\aleph_1$ is Π_2-compact even in the stronger sense with a predicate for unbounded sequences added to the language of $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I}\rangle$. Everywhere below, by an *unbounded sequence* we mean a modulo finite increasing ω_1-sequence of increasing functions in ${}^\omega\omega$ without an upper bound in the eventual domination ordering of ${}^\omega\omega$.

5.0. *Combinatorics of* $\mathfrak{b}$

A subgenericity theorem similar to the one obtained in the dominating number section can be proved here too, in this case essentially saying that adding a Cohen real is an optimal way of adding an unbounded real. It is just a restatement of a familiar fact from recursion theory and is of limited use in what follows.

We abuse the notation a little writing $\mathbb{C} = {}^{<\omega}\omega$ ordered by reverse extension and for a function $f\in {}^\omega\omega$, $\mathbb{C}(f)=\{\eta\in\mathbb{C}\colon \eta$ is on its domain pointwise $\leqslant f\}$ ordered by reverse extension as well.

Lemma 5.1. *Let M be a transitive model of* ZFC *and $f\in {}^\omega\omega$ be an increasing function which is not bounded by any function in M. Whenever $D\in M$, $D\subset\mathbb{C}$ is a dense set, then $D\cap\mathbb{C}(f)\subset\mathbb{C}(f)$ is dense.*

Proof. Fix a dense subset $D\in M$ of $\mathbb{C}$ and a condition $p\in\mathbb{C}(f)$. We shall produce $q\leqslant p$, $q\in D\cap\mathbb{C}(f)$, proving the Lemma.

Work in the model M. By induction on $n\in\omega$ build a sequence $p_0, p_1 \ldots, p_n, \ldots$ of conditions in $\mathbb{C}$ so that

(1) $p_0 = p$, $\mathrm{dom}(p_{n+1}) > \mathrm{dom}(p_n)$;

(2) p_{n+1} is any element of the set D below the condition $r = p^\frown\langle 0, 0\ldots 0\rangle$ with as many zeros as necessary to get $\mathrm{dom}(r)=m_n$.

Let $X\subset\omega$ be the set $\{\mathrm{dom}(p_n)\colon n\in\omega\}$ and let $g: X\to\omega$ be given by $g(m_n) = \max(\mathrm{rng}(p_{n+1}))$. Then $X, g\in M$ and since the function f is increasing and unbounded over the model M, there must be an integer n such that $f(\mathrm{dom}(p_n)) > g(\mathrm{dom}(p_n))$. Then obviously $q = p_{n+1}\leqslant p$ is the desired condition in $D\cap\mathbb{C}(f)$. $\square$

Corollary 5.2 (Subgenericity). *Let P be a forcing and $\dot f$ a P-name such that*

(1) $P\Vdash$ "*$\dot f\in {}^\omega\omega$ is an increasing unbounded function*";

(2) *for every finite sequence η of integers the boolean value $\|\check\eta$ is bounded on its domain by $\dot f\|_P$ is nonzero.*

Then there is a P-name $\dot{Q}$ and a complete embedding $\mathbb{C} < RO(P * \dot{Q})$ such that $P * \dot{Q} \Vdash$ "$\dot{f}$ pointwise dominates $\dot{c}$, the $\mathbb{C}$-generic function".

Proof. Set $\dot{Q} = \mathbb{C}(\dot{f})$ and use the previous lemma. □

It follows that a collection $A \subset {}^{\omega}\omega$ of increasing functions is unbounded just in case the set $X = \{f \in {}^{\omega}\omega$: some $g \in A$ eventually dominates the function $f\}$ is nonmeager. For if A is bounded by some $h \in {}^{\omega}\omega$ then $X \subset \{f \in {}^{\omega}\omega$: h eventually dominates $f\}$ and the latter set is meager; on the other hand, if A is unbounded then Lemma 5.1 provides sufficiently strong Cohen reals in the set X to prove its nonmeagerness. A posteriori, a forcing preserving nonmeager sets preserves unbounded sequences as well.

A more important feature of the bounding number is that every two unbounded sequences can be made in some sense isomorphic. Recall the quasiordering $\leqslant_{\mathfrak{b}}$ defined in Subsection 1.2.

Definition 5.3. For $b, c \in H_{\aleph_2}$ set $b \leqslant_{\mathfrak{b}} c$ if in every forcing extension of the universe b is an unbounded sequence implies c is an unbounded sequence.

While under suitable assumptions (for example the Continuum Hypothesis) the behavior of this quasiorder is very complicated, in the model for $\mathfrak{b} = \aleph_1$ we will eventually build there will be exactly two classes of $\leqslant_{\mathfrak{b}}$-equivalence. The key point is the introduction of the following $\Sigma_1 \langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I} \rangle$ concept to ensure $\leqslant_{\mathfrak{b}}$-equivalence of two unbounded sequences.

Definition 5.4. Unbounded sequences b, c are locked if there is an infinite set $x \subset \omega$ such that for every $\alpha \in \omega_1$ there is $\beta \in \omega_1$ with $b(\beta) \restriction x$ eventually dominating $c(\alpha) \restriction x$; and vice versa, for every $\alpha \in \omega_1$ there is $\beta \in \omega_1$ with $c(\beta) \restriction x$ eventually dominating $b(\alpha) \restriction x$.

It is immediate that locked sequences are $\leqslant_{\mathfrak{b}}$-equivalent. Note that any bound on an infinite set $x \subset \omega$ of a collection of increasing functions in ${}^{\omega}\omega$ easily yields a bound of that collection on the whole ω.

Now it is possible to lock unbounded sequences using one of the standard tree forcings of [3]:

Definition 5.5. The Miller forcing $\mathbb{M}$ is the set of all nonempty trees $T \subset {}^{<\omega}\omega$ consisting of increasing sequences for which

(1) for every $t \in T$ there is a splitnode s of T which extends t;

(2) if a sequence s is a splitnode of T then s has in fact infinitely many immediate successors in T.

$\mathbb{M}$ is ordered by inclusion.

The Miller forcing is proper, $\mathscr{M}$-friendly – see Definition 6.6 – and as such preserves nonmeager sets of reals and unbounded sequences of functions by the argument following Corollary 5.2. If $G \subset \mathbb{M}$ is a generic filter then $f = U \bigcup \bigcap G \in {}^{\omega}\omega$ is an increasing function called a Miller real.

Theorem 5.6. $\mathbb{M} \Vdash$ "*every two unbounded sequences from the ground model are locked*".

Corollary 5.7. *It is consistent with* ZFC *that there are exactly two classes of* $\leqslant_{\mathfrak{b}}$*-equivalence.*

Proof. Start with a model of ZFC + GCH and iterate Miller forcing ω_2 times with countable support. The resulting poset has $\aleph_2$-c.c., it is proper and $\mathcal{M}$-friendly [3], therefore it does not collapse unbounded sequences and forces $\mathfrak{b} = \aleph_1$. In the resulting model, there are exactly two classes of $\leqslant_{\mathfrak{b}}$-equivalence: the objects which are not unbounded sequences and the unbounded sequences, which are pairwise locked by the above theorem and a chain condition argument. □

It also follows from the Theorem that whenever there are two unbounded sequences, one of length ω_1 and the other of length ω_2, Miller forcing necessarily collapses $\aleph_2$ to $\aleph_1$.

Proof of Theorem 5.6. Let $\dot{x}$ be an $\mathbb{M}$-name for the range of the Miller real. We shall show that every two unbounded sequences $b, c : \omega_1 \to {}^{\omega}\omega$ in the ground model are forced to be locked by $\dot{x}$. To this end, given $T \in \mathbb{M}$ and $\alpha \in \omega_1$ a tree $S \in \mathbb{M}$, $S \subset T$ and an ordinal $\beta \in \omega_1$ will be produced such that $S \Vdash$ "$c(\alpha) \restriction \dot{x}$ is eventually dominated by $b(\beta) \restriction \dot{x}$". The theorem then follows by the obvious density and symmetricity arguments.

So fix b, c, T and α as above. Let $\beta \in \omega$ be an ordinal such that $b(\beta)$ is not eventually dominated by any function recursive in $c(\alpha)$ and T. A tree $S \in \mathbb{M}$, $S \subset T$ with the same trunk t as T will be found such that

$$s \in S,\ n \in \mathrm{dom}(s) \setminus \mathrm{dom}(t) \text{ implies } c(\alpha)(n) \leqslant b(\beta)(n). \tag{$*$}$$

This will complete the proof. Let S be defined by $s \in S$ iff $s \in T$ and if s' is the least splitnode of T above or equal to s then for every $n \in \mathrm{dom}(s') \setminus \mathrm{dom}(t)$ it is the case that $c(\alpha)(n) \leqslant b(\beta)(n)$.

Obviously $t \in S \subset T$ and S has property (*), moreover S is closed under initial segment and if $s \in S$ then the least splitnode of T above or equal to s belongs to S as well. We must show that $S \in \mathbb{M}$, and this will follow from the fact that if $s \in S$ is a splitnode of the tree T then s has infinitely many immediate successors in S. And indeed, let $y \subset \omega$ be the infinite set of all integers $n \in \omega$ with $s^\frown\langle n \rangle \in T$ and let $g : y \to \omega$ be a function defined by $g(n) = c(\alpha)(s'(m-1))$, where s' is the least splitnode of T above or equal to $s^\frown\langle n \rangle$ and $m = \mathrm{lth}(s')$. Then by the choice of the ordinal $\beta \in \omega_1$ the set $z = \{n \in y\colon\ g(n) \leqslant b(\beta)(n)\} \subset \omega$ is infinite and every sequence $s^\frown\langle n \rangle$: $n \in z$ belongs to the tree S. □

5.1. A model for $\mathfrak{b}=\aleph_1$

The $P_{\max}$ variant for $\mathfrak{b}=\aleph_1$ will be built using the following notion of a witness: $b:\omega_1\to{}^\omega\omega$ is a good unbounded sequence if it is unbounded and for every sequence $\eta\in{}^{<\omega}\omega$ the set $\{\alpha\in\omega_1\colon b(\alpha)$ pointwise dominates η on its domain$\}\subset\omega_1$ is stationary. Note that if j is a full iteration of a model M, $M\models$ "b is a good unbounded sequence" and $j(b)$ is unbounded then in fact $j(b)$ is a good unbounded sequence. Also, whenever δ is a Woodin cardinal of M and $j_{\mathbb{Q}}$ is the $\mathbb{Q}_{<\delta}$-term for the canonical ultrapower embedding of M then $j_{\mathbb{Q}}b(\omega_1^M)$ is a name for an unbounded function.

Optimal Iteration Lemma 5.8. *Assume $\mathfrak{b}=\aleph_1$. Whenever M is a countable transitive model of* ZFC *iterable with respect to its Woodin cardinal δ and $M\models$ "b is a good unbounded sequence" there is a full iteration j of M based on δ such that $j(b)$ is an unbounded sequence.*

Proof. Drawing on the asumption, choose an unbounded sequence c of length ω_1 and fix an arbitrary iterable model M with $M\models$ "b is an unbounded sequence and δ is a Woodin cardinal". Two full iterations j_0,j_1 of the model M will be constructed simultaneously so that the function $n\mapsto\max\{j_0b(\theta_{0,\alpha})(n),j_1b(\theta_{1,\alpha})(n)\}$ eventually dominates the function $c(\alpha)$, this for every $\alpha\in\omega_1$. Here $\theta_{0,\alpha}$ is ω_1 in the sense of the αth model on the iteration j_0; similarly for $\theta_{1,\alpha}$.

It follows immediately that one of the sequences $j_0(b),j_1(b)$ must be unbounded, since if both were bounded-say by functions f_0,f_1 respectively – then the sequence c would be bounded as well by the function $n\mapsto\max\{f_0(n),f_1(n)\}$, contrary to the choice of c.

Now the iterations j_0,j_1 can be constructed easily using standard bookkeeping arguments and the following claim.

Claim 5.9. *Let M_0,M_1 be countable transitive models of* ZFC *and let*
(1) *$M_0\models P$ is a poset, $p\in P$, and $p\Vdash_P\dot x\in{}^\omega\omega$ is an increasing function unbounded over M_0.*
(2) *$M_1\models Q$ is a poset, $q\in Q$, and $q\Vdash_Q\dot y\in{}^\omega\omega$ is and increasing function unbounded over M_1.*

Suppose $f\in{}^\omega\omega$ is an arbitrary function. Then there are M_0 (M_1, respectively) generic filters $p\in G\subset P$, $q\in H\subset Q$ such that the function $n\mapsto\max\{(\dot x/G)(n),(\dot y/H)(n)\}$ eventually dominates f.

Proof. Without loss of generality assume that f is an increasing function. Let $C_k\colon k\in\omega$ and $D_k\colon k\in\omega$ enumerate all open dense subsets of P in the model M_0 and of Q in M_1, respectively. By induction on $k\in\omega$ simultaneously build sequences $p=p_0\geqslant p_1\geqslant\cdots\geqslant p_k\geqslant\cdots$ of conditions in P, $q=q_0\geqslant q_1\geqslant\cdots\geqslant q_k\geqslant\cdots$ of conditions in Q and integers $0=n_0=m_0$, $m_k\leqslant n_k<m_{k+1}$ so that

(1) $p_{k+1}\in C_k$, $q_{k+1}\in D_k$ for all $k\in\omega$;

(2) p_{k+1} decides $\dot{x} \restriction m_k$, q_{k+1} decides $\dot{y} \restriction n_k$;
(3) for $k>0$, for infinitely many integers $i \in \omega$ there is a condition $r_i \leqslant p_k$ such that $r_i \Vdash_P$ "$\dot{x}(m_k) = i$"; similarly, for infinitely many integers $i \in \omega$ there is a condition $s_i \leqslant q_k$ such that $s_i \Vdash_Q \dot{y}(n_k) = i$;
(4) for $k>0$, $p_{k-1} \Vdash \dot{x}(m_k) > f(n_k)$ and $q_{k+1} \Vdash \dot{y}(n_k) > f(m_{k-1})$.

This is easily arranged – (3) is made possible by the fact that $\dot{x}, \dot{y}$ are terms for unbounded functions. Let $G \subset P$, $H \subset Q$ be the filters generated by the p_i's or q_i's respectively. These filters have the desired genericity properties by (1) above and the function $n \mapsto \max\{(\dot{x}/G)(n), (\dot{y}/H)(n)\}$ pointwise dominates f from n_1 on. This can be argued from the induction hypothesis (4) and the fact that $\dot{x}/G$, $\dot{y}/H$ and f are all increasing functions. □

Strategic Iteration Lemma 5.10. *Assume the Continuum Hypothesis. The good player has a winning strategy in the game $\mathcal{G}_{\mathfrak{b}=\aleph_1}$.*

Proof. Let $\vec{N} = \langle b, N_i, \delta_i \colon i \in \omega \rangle$ be a sequence of models with a good unbounded sequence, let $y_0 \in \mathbb{Q}_{\vec{N}}$ and let $f \in {}^{\omega}\omega$ be an arbitrary function. We shall show that there is an $\vec{N}$-generic filter $G \subset \mathbb{Q}_{\vec{N}}$ with $y_0 \in G$ such that the function $j_{\mathbb{Q}} b(\omega_1^{\vec{N}})$ is not eventually dominated by the function f, where $j_{\mathbb{Q}}$ is the $\mathbb{Q}^{N_0}_{<\delta_0}$-generic ultrapower of the model N_0 using the filter G. Then the winning strategy of the good player consists essentially only of a suitable bookkeeping using the Continuum Hypothesis.

Now the proof of the existence of such a filter G is in fact very easy. We indicate a slightly inefficient though conceptual proof. Just use the subgenericity Corollary 5.2 and the proof of Lemma 2.8 with the following changes:
(1) d is replaced with b, the Hechler forcing is replaced with Cohen forcing and the real $e \in {}^{\omega}\omega$ will be taken sufficiently $\mathbb{C}$-generic;
(2) step (5) in the proof of Lemma 2.8 is replaced by: e is not eventually dominated by the function f;
(3) the poset R_i is now the set of all pairs $\langle y, \eta \rangle$ with $y \in \mathbb{Q}_i$, $\eta \in {}^{<\omega}\omega$ and for every $x \in y$ the sequence η is everywhere on its domain dominated by the function $b(x \cap \omega_1)$. □

Conclusion 5.11. *The sentence $\mathfrak{b} = \aleph_1$ is Π_2-compact, even with the predicate $\mathfrak{B}$ for unbounded sequences added to the language of $\langle H_{\aleph_2}, \in, \omega_1, \mathfrak{I} \rangle$.*

Proof. Π_2-compactness follows from Lemmas 5.8 and 5.10. To show that Π_2 statements of $\langle H_{\aleph_2}, \in, \mathfrak{I}, \mathfrak{B} \rangle$ reflect to the model $L(\mathbb{R})^{P_{\mathfrak{b}=\aleph_1}}$, proceed as in Corollary 1.17.

Suppose in V, a Π_2-sentence ψ-equal to $\forall x\, \exists y\, \chi(x, y)$ for some Σ_0 formula χ-holds in $\langle H_{\aleph_2}, \in, \mathfrak{I}, \mathfrak{B} \rangle$ together with $\mathfrak{b} = \aleph_1$ and let $\delta < \kappa$ be a Woodin and a measurable cardinal respectively. For contradiction, suppose that $p \in P_{\mathfrak{b}=\aleph_1}$ forces $\neg\psi$ to hold; strengthening the condition p if necessary we may assume that for some $x \in M_p$, $p \Vdash \forall y \neg\chi(\dot{k}_p(x), y)$ where $\dot{k}_p$ is the term for the canonical iteration of M_p as defined in 1.14. By Corollary 1.8, there is a countable transitive model M elementarily embeddable into V_κ such that

$p \in M$ and M and all of its generic extensions by posets of size $\leqslant \mathfrak{c}^M$ are iterable. So $M \models$ "$\mathfrak{b} = \aleph_1$ and $\langle H_{\aleph_2}, \in, \mathfrak{I}, \mathfrak{B}\rangle \models \psi$ and some cardinal $\bar{\delta}$ is Woodin". The optimal iteration Lemma applied in M yields there a full iteration j of the model M_p such that $j(b_p)$ is a good unbounded sequence. Let N be an $\mathbb{M}$-generic extension of the model M_p and let $q = \langle N, j(b_p), \bar{\delta}, j(H_p) \cup \{\langle j, p\rangle\}\rangle$.

First, $q \in P_{\mathfrak{b} = \aleph_1}$ is a condition strengthening p. To see that note that the model N is iterable, the iteration j is full in N, since the forcing $\mathbb{M}$ preserves stationary sets, and $j(b_p)$ is a good unbounded sequence in N since $\mathbb{M}$ preserves such sequences.

Second, $q \Vdash$ "$\mathfrak{B}$ in the sense of the structure $\dot{k}_q(H_{\aleph_2})^M$ is just $\mathfrak{B} \cap \dot{k}_q(H_{\aleph_2})^M$". Observe that $N \models$ "all sequences in $(H_{\aleph_2})^M$ which there are unbounded are locked with $j(b_p)$", by Theorem 5.6 applied in the model M. Therefore q forces even the following stronger statement, by the elementarity of the embedding $\dot{k}_q$: "whenever $\dot{k}_q(H_{\aleph_2})^M \models$ 'c is an unbounded sequence' then c and $\dot{k}_q j(b_p)$ are locked; since $\dot{k}_q j(b_p)$ is unbounded, the sequence c must be unbounded as well".

Third, $q \Vdash$ "$\exists y \chi(\dot{k}_q j(x) = \dot{k}_p(x), y)$", giving the final contradiction with our choice of p and x. Since ψ holds in the model M, there must be some $y \in (H_{\aleph_2})^M$ such that $\langle H_{\aleph_2}, \in, \mathfrak{I}, \mathfrak{B}\rangle^M \models \chi(j(x), y)$. Then $q \Vdash \langle H_{\aleph_2}, \in, \mathfrak{I}, \mathfrak{B}\rangle^M \models \chi(\dot{k}_q j(x), \dot{k}_q(y))$ by the previous paragraph and absoluteness of Σ_0 formulas. □

6. Uniformity of the meager ideal

The sentence "there is a nonmeager set of reals of size $\aleph_1$" does not seem to be compact, however, a similar a bit stronger assertion is.

Definition 6.1. A sequence $\langle r_\alpha \colon \alpha \in \omega_1\rangle$ of real numbers is called weakly Lusin if for every meager $X \subset \mathbb{R}$ the set $\{\alpha \in \omega_1 \colon r_\alpha \in X\} \subset \omega_1$ is nonstationary.

Thus the existence of a weakly Lusin sequence is a statement intermediate between a nonmeager set of size $\aleph_1$ and a Lusin set. It is equivalent to neither of them, as will be shown below.

6.0. A model for a weakly Lusin sequence

We will prove the iteration Lemmas necessary to conclude that the sentence $\phi =$ "there is a weak Lusin sequence" is Π_2-compact. Note that if $\langle M, k, \delta\rangle$ is a triple such that $M \models$ "k is a weak Lusin sequence and δ is a Woodin cardinal" then in M, $\mathbb{Q}_{<\delta} \Vdash$ "$j_{\mathbb{Q}} k(\omega_1^M)$ is a Cohen real over M", and so it generates a natural Cohen subalgebra of $\mathbb{Q}_{<\delta}$. The following abstract copying lemma will be relevant:

Lemma 6.2. *Let N be a countable transitive model of* ZFC, *$N \models$ "P is a partially ordered set and $P \Vdash \dot{r}$ is a Cohen real". Suppose that $p \in P$ and $s \in {}^\omega\omega$ is a Cohen real over N. Then there is an N-generic filter $G \subset P$ so that $p \in G$ and $\dot{r}/G = s$ modulo finite.*

Proof. In the model N, let $P = \mathbb{C} * \dot{Q}$ where $\mathbb{C}$ is the Cohen subalgebra of $\mathbb{B}$ generated by the term $\dot{r}$. It follows from the assumptions and some boolean algebra theory in N that there are $q \leqslant p$ in $RO(P)$ and a finite sequence η such that $pr_{\mathbb{C}}(q) = [[\eta \subset \dot{r}]]_{\mathbb{C}}$. Let $t \in {}^{\omega}\omega$ be the function defined by $\eta \subset t$ and $s(n) = t(n)$ for $n \neq dom(\eta)$. Since the real s is Cohen over N and $s = t$ modulo finite, even t is Cohen and the filter $H \subset \mathbb{C}$ generated by the equation $\dot{r} = t$ is N-generic. Finally, choose an N-generic filter $G \subset P$ with $H \subset G$ and $q \in G$. This is possible since $pr_{\mathbb{C}}(q) \in H$. The filter $G \subset P$ is as desired. $\square$

Optimal Iteration Lemma 6.3. *Assume that there is a weakly Lusin sequence. Whenever M is a countable transitive model iterable with respect to its Woodin cardinal δ such that $M \models$ "k is a weakly Lusin sequence", then there is a full iteration j of M such that $j(k)$ is a weakly Lusin sequence.*

Proof. Fix a Lusin sequence J and M, k, δ as in the Lemma. We shall produce a full iteration j of M such that there is a club $C \subset \omega_1$ with $\forall \alpha \in C\ jk(\alpha) = J(\alpha)$ modulo finite. Then jk really is a Lusin sequence and the Lemma is proved.

The desired iteration j will be constructed by induction on $\alpha \in \omega_1$. Let S_ξ: $\xi \in \omega_1$ be a partition of ω_1 into pairwise disjoint stationary sets. By induction on $\alpha \in \omega_1$, models M_α together with the elementary embeddings will be built plus an enumeration $\{\langle x_\xi, \beta_\xi \rangle : \xi \in \omega_1\}$ of all pairs $\langle x, \beta \rangle$ with $x \in \mathbb{Q}_\beta$. The induction hypotheses at α are
(1) if $\gamma < \alpha$ and $J(\theta_\gamma)$ is a Cohen real over M_γ then $k_{\gamma+1}(\theta_\gamma) = J(\theta_\gamma)$ modulo finite;
(2) if $\gamma \leqslant \alpha$ then $\{\langle x_\xi, \beta_\xi \rangle : \xi \in \theta_\gamma\}$ enumerates all pairs $\langle x, \beta \rangle$ with $\beta < \gamma$, $x \in \mathbb{Q}_\beta$;
(3) if $\gamma < \alpha$ and $\theta_\gamma \in S_\xi$ for some $\xi \in \theta_\gamma$ then $j_{\beta_\xi, \gamma}(x_\xi) \in G_\gamma$.

As before, the hypothesis (1) makes sure that the sequence J gets copied onto jk properly and (2,3) are just bookkeeping tools for making the resulting iteration full.

At limit ordinals just direct limits are taken and the new enumeration is the union of all old ones. The successor step is handled easily using the previous Lemma applied for $N = M_\alpha$, $P = \mathbb{Q}_\alpha$, $\dot{r} = j_{\mathbb{Q}} k_\alpha(\theta_\alpha)$, $s = J(\theta_\alpha)$ and $b = j_{\beta_\xi \alpha}(x_\xi)$, the last two in the case that s is Cohen over M_α and $\theta_\alpha \in S_\xi$ for some (unique) $\xi \in \theta_\alpha$.

To prove that the resulting iteration is as desired, note that it is full and that the set $D = \{\alpha \in \omega_1 : J(\theta_\alpha)$ is a Cohen real over $M_\alpha\}$ contains a closed unbounded set. For assume otherwise. Then the complement S of D is stationary and for every limit ordinal $\alpha \in S$ there is a nowhere dense tree in some M_β, $\beta \in \alpha$ such that the real $J(\omega_1^{M_\alpha})$ is a branch of this tree – this is because a direct limit is taken at step α. By a simple Fodor-style argument there is a nowhere dense tree and a stationary set $T \subset S$ such that every $J(\theta_\alpha)$: $\alpha \in T$ is a branch of this tree. This contradicts the assumption of J being a weakly Lusin sequence.

Strategic Iteration Lemma 6.4. *Assume the Continuum Hypothesis. The good player has a winning strategy in the game $\mathscr{G}_\phi$ connected with weakly Lusin sequences.*

Proof. Given a sequence $\vec{N} = \langle k, N_i, \delta_i : i \in \omega \rangle$ of models with a weakly Lusin sequence k, a condition $y_0 \in \mathbb{Q}_{\vec{N}}$ and nowhere dense trees T_n: $n \in \omega$, we shall show that there is an $\vec{N}$-generic filter $G \subset \mathbb{Q}_{\vec{N}}$ with $y_0 \in G$ such that the real $j_{\mathbb{Q}}k(\omega_1^{\vec{N}})$ is not a branch through any of the trees T_n, where $j_{\mathbb{Q}}$ is the generic ultrapower embedding of the model N_0 using the filter $G \cap \mathbb{Q}^{N_0}_{<\delta_0}$. With this fact in hand, a winning strategy for the good player consists of just a suitable bookkeping using the Continuum Hypothesis.

Again, we provide maybe a little too conceptual proof of the existence of the filter G, using the ideas from Lemma 2.8. No subgenericity theorems are needed this time. Let $i \in \omega$ and work in N_i. Let $\mathbb{Q}_i = \mathbb{Q}_{\delta_i}$ and j_i to be the $\mathbb{Q}_i$ term for the generic ultrapower embedding of the model N_i. So the term $j_ik(\omega_1^{\vec{N}})$ is a term for a Cohen real, and it gives a complete embedding of $\mathbb{C}$ into $\mathbb{Q}_i$. The key point is that with this embedding, the computation of the projection $pr_{\mathbb{C}}(y)$ gives the same value in $\mathbb{C}$ in every model N_i with $y \in \mathbb{Q}_i$, namely $\Sigma_{\mathbb{C}}\{\eta \in {}^{<\omega}\omega$: the system $\{x \in y: \eta \subset k(x \cap \omega_1)\}$ is stationary$\}$.

First, fix a suitably generic filter $H \subset \mathbb{C}$. The requirements are

(1) $pr_{\mathbb{C}}(y_0) \in H$;

(2) H meets all the maximal antichains that happen to belong to $\bigcup_i N_i$;

(3) the Cohen real $c \in {}^{\omega}\omega$ given by the filter H does not constitute a branch through any of the nowhere dense trees T_n.

This is easily done. Now let X_n: $n \in \omega$ be an enumeration of all maximal antichains of $\mathbb{Q}_{\vec{N}}$ in $\bigcup N_i$ and by induction on $n \in \omega$ build a decreasing sequence y_n: $n \in \omega$ of conditions in $\mathbb{Q}_{\vec{N}}$ so that

(1) $pr_{\mathbb{C}}(y_n) \in H$;

(2) y_{n+1} has an element of X_n above it.

This can be done since the filter H is $\mathbb{C}$-generic over every model N_i. In the end, let G be the filter on $\mathbb{Q}_{\vec{N}}$ generated by the conditions y_n: $n \in \omega$. This filter is as desired. Note that c is the uniform value of $j_ik(\omega_1^{\vec{N}})$ as evaluated according to this filter. $\square$

Conclusion 6.5. *The sentence "there is a weakly Lusin sequence of reals" is Π_2-compact.*

It is unclear whether it is possible to add a predicate for witnesses in this case.

6.1. Combinatorics of weakly Lusin sequences

In this subsection it is proved that the existence of a Lusin set, weakly Lusin sequence and a nonmeager set of size $\aleph_1$ are nonequivalent assertions.

The following regularity property of forcings will be handy:

Definition 6.6 (*Bartoszyński and Judah* [3, 6.3.15]). A forcing P is called $\mathcal{M}$-friendly if for every large enough regular cardinal λ, every condition $p \in P$, every countable elementary submodel M of H_λ with $p, P \in M$ and every function $h \in {}^{\omega}\omega$ Cohen-generic over M there is a condition $q \leqslant p$ such that q is master for M and $q \Vdash$ "$\check{h}$ is Cohen-generic over $M[G]$".

It is not difficult to prove that $\mathcal{M}$-friendly forcings preserve nonmeager sets. Moreover $\mathcal{M}$-friendliness is preserved under countable support iterations [3, Section 6.3.C].

The following fact, pointed out to us by Tomek Bartoszyński, replaces our original more complicated argument.

Lemma 6.7. (1) *The Miller forcing* $\mathbb{M}$ – *see Definition* 5.5 – *is* $\mathcal{M}$*-friendly.*
(2) $\mathbb{M}$ *destroys all weakly Lusin sequences from the ground model.*

Corollary 6.8. *It is consistent with* ZFC *that there is a nonmeager set of size* $\aleph_1$ *but no weakly Lusin sequences.*

Proof. Iterate Miller forcing over a model of GCH ω_2 times. □

Proof of Lemma. Suppose M is a countable transitive model of a rich fragment of ZFC, $T \in \mathbb{M} \cap M$ and $f \in {}^{\omega}\omega$ is a Cohen real over M. We shall produce M-master conditions S_0, $S_1 \subset T$ in $\mathbb{M}$ such that
(1) $S_0 \Vdash \check{f}$ is a Cohen real over M;
(2) $S_1 \Vdash \check{f}$ is eventually dominated by the Miller real.

This will finish the proof: (1) shows that $\mathbb{M}$ is $\mathcal{M}$-friendly and (2) by a standard argument implies that for any weakly Lusin sequence $\langle r_\alpha \colon \alpha \in \omega_1 \rangle$ of elements of ${}^{\omega}\omega$ the set $\{\alpha \in \omega_1 \colon r_\alpha$ belongs to the meager set of all functions in ${}^{\omega}\omega$ eventually dominated by the Miller real$\} \subset \omega_1$ is $\mathbb{M}$-forced to be stationary.

By a mutual genericity argument there is an M-generic filter $G \subset Coll(\omega, (2^{\mathfrak{c}})^M)$ such that the function f is still Cohen generic over $M[G]$. Work in $M[G]$. Let $\eta_k \colon k \in \omega$ enumerate the Cohen forcing ${}^{<\omega}\omega$, let $\dot{O}_k \colon k \in \omega$ enumerate all the $\mathbb{M} \cap M$-names for dense subsets of the Cohen forcing in M and let $D_k \colon k \in \omega$ enumerate the open dense subsets of $\mathbb{M} \cap M$ in M. Build a fusion sequence $T = T_0 \geqslant T_1 \geqslant T_2 \geqslant \cdots$ of trees in $\mathbb{M} \cap M$ so that if s is a kth level splitnode of T_k and $\{i_n \colon n \in \omega\}$ is an enumeration of the set of all integers i with $s^\frown\langle i \rangle \in T_k$ then for every $n \in \omega$ the sequence $s^\frown\langle i_n \rangle$ belongs to T_{k+1}, $T_{k+1} \restriction s^\frown\langle i_n \rangle \in D_k$ and there is some extension η of η_n in the Cohen forcing ${}^{<\omega}\omega$ such that $T_{k+1} \Vdash \check{\eta} \in \dot{O}_k$. This is readily done. Let $S = \bigcup_i T_i \in \mathbb{M} \cap M[G]$. The tree S is an M-master condition in $\mathbb{M}$ and since $f \in {}^{\omega}\omega$ is Cohen generic over the model $M[G]$, the following two subtrees S_0, S_1 of S are still in $\mathbb{M}$:
(1) $S_0 = \{s \in S \colon$ whenever s is a proper extension of a kth level splitnode $t \in T_k$ then $T_{k+1} \restriction s \Vdash \check{\eta} \in \dot{O}_k$ for some $\eta \subset f\}$;
(2) $S_1 = \{s \in S \colon \forall n \in dom(s \setminus \text{the trunk of } T)\ f(n) < s(n)\}$.

It is easy to see that the trees S_0, S_1 are as desired. □

Question 6.9 *Does the saturation of the nonstationary ideal plus the existence of a nonmeager set of reals of size* $\aleph_1$ *imply the existence of a weakly Lusin sequence?*

Next it will be proved that the existence of a weakly Lusin sequence does not imply that of a Lusin set. A classical forcing argument can be tailored to fit this need; instead, we shall show that there are no Lusin sets in the model built in the previous

Subsection. This follows from Theorem 1.15(2) and a simple density argument using the following fact:

Lemma 6.10. ($AD^{L(\mathbb{R})}$) *Let M be a countable transitive iterable model, $M \models$ "K is a weakly Lusin sequence and X is a Lusin set". Then there is a countable transitive iterable model N and an iteration $j \in N$ such that $N \models$ "j is a full iteration of M, $j(K)$ is a weakly Lusin sequence and $j(X)$ is not a Lusin set".*

Proof. Fix M, K and X and for notational reasons assume that X is really an injective function from ω_1^M to ${}^\omega\omega$ enumerating that Lusin set. Working in the model M, if δ is a Woodin cardinal, $\mathbb{Q}_{<\delta}$ is the nonstationary tower and $j_{\mathbb{Q}}$ is the $\mathbb{Q}_{<\delta}$-name for the canonical embedding of M then both $j_{\mathbb{Q}} K(\omega_1^M)$ and $j_{\mathbb{Q}} X(\omega_1^M)$ are $\mathbb{Q}_{<\delta}$ terms for Cohen reals.

Using the determinacy assumption, choose a countable transitive model N_0 such that $M \in N_0$ is countable there and N_0 and all of its generic extensions by forcings of size $\aleph_1^{N_0}$ are iterable. Work in N_0. Force two ω_1-sequences $\langle c_\beta\colon \beta \in \omega_1\rangle, \langle d_\beta\colon \beta \in \omega_1\rangle$ of Cohen reals – elements of ${}^\omega\omega$ – and a function $e \in {}^\omega\omega$ eventually dominating every $d_\beta\colon \beta \in \omega_1$ with finite conditions.

Set $N = N_0[\langle c_\beta\colon \beta \in \omega_1\rangle, \langle d_\beta\colon \beta \in \omega_1\rangle, e]$ In the model N, the reals $\langle c_\beta\colon \beta \in \omega_1\rangle$ constitute a weakly Lusin sequence, indeed a Lusin set, because their sequence is Cohen generic over the model $N_0[\langle d_\beta\colon \beta \in \omega_1\rangle, e]$. In the model $N_0[\langle c_\beta\colon \beta \in \omega_1\rangle, \langle d_\beta\colon \beta \in \omega_1\rangle]$ build a full iteration j of M so that

(1) if $\alpha \in \omega_1$ is limit then whenever made possible by the model M_α we have $jK(\theta_\alpha) = c_{\theta_\alpha}$ modulo finite;

(2) if $\alpha \in \omega_1$ is successor then $jX(\theta_\alpha)$ is equal to one of the reals $d_\beta\colon \beta \in \omega_1$ modulo finite.

By the arguments from the previous subsection and the fact that $\langle c_\beta\colon \beta \in \omega_1\rangle, \langle d_\beta\colon \beta \in \omega_1\rangle$ are mutually generic sequences of Cohen reals over N_0, (2) is always possible to fulfill and the set $\{\beta \in \omega_1\colon jK(\beta) = c_\beta \text{ modulo finite}\} \subset \omega_1$ will contain a club.

Now N, j are as desired. In the model N, the iteration j is full since N is a c.c.c. extension of $N_0[\langle c_\beta\colon \beta \in \omega_1\rangle, \langle d_\beta\colon \beta \in \omega_1\rangle]$ in which j was constructed to be full; the sequence jK is on a club modulo finite equal to a weakly Lusin sequence $\langle c_\beta\colon \beta \in \omega_1\rangle$ and so is weakly Lusin itself; and the set $\{jX(\theta_\alpha) : \alpha \in \omega_1 \text{ successor}\}$ is an uncountable subset of $\operatorname{rng}(jX)$ contained in the meager set of all reals eventually dominated by $e \in {}^\omega\omega$, consequently $\operatorname{rng}(jX)$ is not a Lusin set. □

6.2. *The null ideal*

The methods of this paper can be adapted to give a parallel result about the null ideal.

Definition 6.11. A sequence $\langle r_\alpha\colon \alpha \in \omega_1\rangle$ of real numbers is called a weakly Sierpiński sequence if for every null set S, the set $\{\alpha \in \omega\colon r_\alpha \in S\}$ is nonstationary.

Theorem 6.12. *The sentence "there is a weakly Sierpiński sequence" is* Π_2*-compact.*

By an argument parallel to 6.7(2) it can be proved that existence of a nonnul set of size $\aleph_1$ and of a weakly Sierpiński sequence are nonequivalent statements.

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