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Existence of EF-equivalent non-isomorphic models

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We prove the existence of pairs of models of the same cardinality λ which are very equivalent according to EF games, but not isomorphic. We continue the paper [4], but we do not rely on it.

1 Introduction

There had been much study of equivalence relations between models. When we study such an equivalence relation, the basic question is whether this relation is actually trivial, i. e., if equivalent models are isomorphic. For example, countable models which are elementary equivalent in $L_{\omega_1,\omega}$ are isomorphic. (Scott showed this in [10] for countable vocabulary, and Chang generalized it in [1] for any vocabulary). For $\lambda = \mathrm{cf}(\lambda) > \aleph_0$, Morely gave (without publishing) a counter example – a pair of $L_{\infty,\lambda}$ equivalent models of size λ which are not isomorphic. Shelah [9, Chapter II, § 7] gave such an example for almost every singular λ.

Those questions also relate to classification theory: The existence of "strongly" equivalent models which are not isomorphic is a non-structure property for a class of models. On the other side, if a "not too strong" equivalence relation is actually the isomorphism relation, this is a structure property (see [8] and [2]).

One of the equivalence relations studied in this context is equivalence under EF (Ehrenfeucht-Fraïssé) games. A detailed discussion of EF games and their history can be found in [3] and in [11]. The general structure of an EF game on a pair of models is as follows: There are two players – isomorphism player, whom we call ISO, and anti-isomorphism player, whom we call AIS. During the game, AIS chooses members of the models, and ISO defines "interactively" a partial isomorphism between the models – in every move he has to extend that partial isomorphism so that the elements chosen by AIS will be contained in the domain or in the range. The isomorphism player loses the game if at some point he cannot find a legal move. If he does not lose, he wins. We limit the length of the game and the number of elements that AIS may choose at each move. (Because, if AIS can list all the members of one of the models, then the game is not interesting.) In [4], the games were with fixed length. In this paper, we deal with EF games approximated by trees – the length of the game is limited by adding the demand that in each move, AIS has to choose a node in some fixed tree $\mathcal{T}$ (with certain properties) such that the sequence of nodes formed by his choices is strictly increasing in the order $<^{\mathcal{T}}$. If AIS cannot choose such node, he loses.

We say that *two models are equivalent with respect to some* EF *game* $\eth$ if ISO has a winning strategy in $\eth$ played on those models.

In [4] it was proved that if $\lambda = \mathrm{cf}(\lambda) = \lambda^{\aleph_0}$, then there are non-isomorphic models of size λ which are $\mathrm{EF}_{\alpha,\lambda}$ equivalent for every $\alpha < \lambda$, where $\mathrm{EF}_{\alpha,\lambda}$ equivalence means that they are equivalent under every EF game with α stages such that AIS has to choose $< \lambda$ members of the models at each stage. There was also a result for λ singular, with a necessary change of the equivalence relation.

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Here we generalize the results in two ways: First, we move to EF games approximated by trees instead of fixed-length games (see Hyttinen and Tuuri in [2] who investigated such games in the context of classification theory). Second, we give results also for $\lambda > \beth_\omega$ without the assumption $\lambda = \lambda^{\aleph_0}$, where we use PCF theory to have some "approximation" instead of $\lambda = \lambda^{\aleph_0}$.

In Section 2 we prove that for regular $\lambda = \lambda^{\aleph_0}$ for some class of reasonably large trees (see detailed discussion justifying the choice, in the beginning of Section 2) for every tree from that class there are non-isomorphic models of size λ which are equivalent under EF games approximated by that tree such that in each move AIS is allowed to choose $< \lambda$ members of the models (see Definition 2.1).

In Section 3 we do the parallel for singular λ. But for singular λ, if we allow AIS to choose $< \lambda$ elements in each move, and the tree has a branch of length $\mathrm{cf}(\lambda)$, then the game is not interesting, because AIS can choose all the members of the models during the game. So we have to be more careful – we allow AIS to choose only one element in each move. This is still a generalization of the result for such λ in [4] – see the discussion at the beginning of Section 3.

In Section 4 we prove that for regular $\lambda > \beth_\omega$, for every tree of size λ without a branch of length λ there are non-isomorphic models of size λ which are equivalent under the EF game approximated by that tree such that in each move AIS is allowed to choose $< \lambda$ members of the models.

In Section 5 we prove a similar result for $\lambda > \mathrm{cf}(\lambda) > \beth_\omega$. As we explained above, because of the singularity of λ, we have to restrict the number of elements that AIS is allowed to choose at each move – in stage α, AIS has to choose $< 1 + \alpha$ members of the models. For a further work in preparation of the second author on this subject see F815 in his web site.

2 Games with trees for regular $\boldsymbol{\lambda = \lambda^{\aleph_0}}$

In [2] there is a construction of non-isomorphic models of size λ which are equivalent under EF games approximated by trees of size λ with no λ branch, when $\lambda = \lambda^{<\lambda}$. In [4] there is such a construction under a weaker assumption on λ: $\lambda = \mathrm{cf}(\lambda) = \lambda^{\aleph_0}$, but there the result is for games of any fixed length $< \lambda$, not for games which are approximated by trees. We want to generalize this result to games approximated by trees.

Now, which trees should we consider? If we limit ourselves only to trees of size λ, it seems that the set of trees will be "small". Why? Assume for example that $\lambda = \mathrm{cf}(\lambda) = \lambda^{\aleph_0} < \lambda^{\aleph_1}$. A tree of size λ must drop at least one of the following conditions:

1. Above every node there is an antichain of size λ.
2. Every chain of size $\leq \aleph_1$ has an upper bound.

If $\lambda \gg \aleph_1$, this kind of trees seems to be too degenerate. We could have demanded that the size of the tree will be $\leq 2^{<\lambda}$. But it is possible that $2^{<\lambda} = 2^\lambda$ and it is reasonable to assume that the result is not true in this case.

We take the middle road: We do not limit explicitly the size of the tree, but we demand that the tree will be "definable" enough – the cause of not having a branch of length λ is that the nodes of the tree are actually partial functions from λ to λ which satisfy a certain local condition. By "local" we mean that a function f satisfies the condition iff any restriction of f to a countable set satisfies it. The tree order is inclusion, and there is no function from λ to λ which satisfies the condition. By Remark 2.4 this result is indeed a generalization of "for every tree of size λ and without a λ branch".

Definition 2.1 For a tree $\mathcal{T}$, a cardinal μ, and models with common vocabulary M_1, M_2, we define the game $\Game_{\mathcal{T},\mu}(M_1, M_2)$ between the players ISO and AIS as follows: After stage α in the game we have the sequence $\langle f_\beta : \beta \leq \alpha \rangle$, which is an increasing continuous sequence of partial isomorphisms from M_1 to M_2, and the sequence $\langle z_\beta : \beta \leq \alpha \rangle$ which is an increasing continuous sequence in $\mathcal{T}$.

In stage α of the game, first AIS chooses z_α of level α of $\mathcal{T}$ such that for every $\beta < \alpha$, $z_\alpha >^{\mathcal{T}} z_\beta$. Then:

1. If $\alpha = 0$, then $f_\alpha = \emptyset$.
2. If α is limit, then $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.
3. If $\alpha = \beta + 1$, then AIS chooses $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$ such that $|A_1 \cup A_2| < 1 + \mu$. Then ISO should choose f_α such that f_α is a partial isomorphism from M_1 to M_2, $f_\beta \subseteq f_\alpha$, $A_1 \subseteq \mathrm{Dom}(f_\alpha)$, $A_2 \subseteq \mathrm{Range}(f_\alpha)$.

The first player who cannot find a legal move loses the game. If the isomorphism player ISO has a winning strategy for $\Game_{\mathcal{T},\mu}(M_1, M_2)$, we say that M_1, M_2 *are* $\mathrm{EF}_{\mathcal{T},\mu}$ *equivalent*.

Definition 2.2 We say that $\boxtimes_{\mathcal{F},\lambda}$ *holds* if

1. $\mathcal{F}$ is a set of partial functions from λ to λ;
2. if f is a partial function from λ to λ, then $f \in \mathcal{F}$ iff $f \restriction u \in \mathcal{F}$ for every countable $u \subseteq \mathrm{Dom}(f)$;
3. there is no $f \in \mathcal{F}$ such that $\mathrm{Dom}(f) = \lambda$.

Definition 2.3 If $\boxtimes_{\mathcal{F},\lambda}$ holds, we define a tree $\mathcal{T}_{\mathcal{F}}$ in the following way:

The nodes are functions f such that $f \in \mathcal{F}$ and $\mathrm{Dom}(f)$ is an ordinal;

the order is inclusion.

Note that this tree $\mathcal{T}_{\mathcal{F}}$ does not have a branch of length $\geq \lambda$.

Remark 2.4 If $\mathcal{T}$ is a tree of size λ with no λ branch, we can assume without loss of generality that $\mathcal{T} \subseteq \lambda$. Define $\mathcal{F}$ by $f \in \mathcal{F}$ if f is a partial function from λ to λ such that $x < y$ implies $f(x) <^{\mathcal{T}} f(y)$. We get that $\boxtimes_{\mathcal{F},\lambda}$ holds, and $\mathcal{T}$ can be embedded (as a partial order) in $\mathcal{T}_{\mathcal{F}}$.

Theorem 2.5 *Suppose*

$\mathrm{cf}(\lambda) = \lambda = \lambda^{\aleph_0}$,

$\boxtimes_{\mathcal{F},\lambda}$ *holds,*

$\mathcal{T} = \mathcal{T}_{\mathcal{F}}$.

Then there are non-isomorphic models M_1, M_2 of size λ which are $\mathrm{EF}_{\mathcal{T},\lambda}$ equivalent.

Proof. First, we shall define a tool for constructing models.

Definition 2.6 $\mathfrak{x}$ is a *structure parameter* if it consists of the following objects:

a set I,

a set J_s for each $s \in I$ such that if $s_1 \neq s_2$, then $J_{s_1} \cap J_{s_2} = \emptyset$ (denote $J = \bigcup_{s\in I} J_s$),

sets S, T such that $S \subseteq I \times I$ and $T \subseteq J \times J$.

Definition 2.7 For a given structure parameter $\mathfrak{x}$ we define a model $M = M_{\mathfrak{x}}$ in the following way: First for each $s \in I$ let $\mathbb{G}_s$ be an abelian group generated freely by $\{x_t : t \in J_s\}$ except of the relation $\forall x(2x = 0)$. (We could have also used a free group or a free abelian group, but our choice makes the proof a bit simpler.) We demand also that if $s_1 \neq s_2$, then $\mathbb{G}_{s_1} \cap \mathbb{G}_{s_2} = \emptyset$. For $(s_1, s_2) \in S$, let $\mathbb{G}_{s_1,s_2}$ be the subgroup of $\mathbb{G}_{s_1} \times \mathbb{G}_{s_2}$ generated by $\{(x_{t_1}, x_{t_2}) : (t_1, t_2) \in T \cap (J_{s_1} \times J_{s_2})\}$. The universe of M is $\bigcup_{s\in I} \mathbb{G}_s$. The vocabulary of M consists of

1. for each $a \in M$, a unary function symbol F_a;
2. for each $s \in I$, a unary relation symbol P_s;
3. for each $(s_1, s_2) \in S$, a binary relation symbol Q_{s_1,s_2}.

The interpretation of the symbols in M is as follows:

1. for each $b \in M$, $s \in I$, $a \in \mathbb{G}_s$, if $b \in \mathbb{G}_s$, then $F^M_a(b) = a + b$, else $F^M_a(b) = b$;
2. for each $s \in I$, $P^M_s = \mathbb{G}_s$;
3. for each $(s_1, s_2) \in S$, $Q^M_{s_1,s_2} = \mathbb{G}_{s_1,s_2}$.

Lemma 2.8 *Suppose $I' \subseteq I$ and f is a function, $f : \bigcup_{s\in I'} \mathbb{G}_s \longrightarrow M$. Then f is a partial automorphism of M iff the following hold:*

1. *For each $s \in I'$, $f(0_{\mathbb{G}_s}) \in \mathbb{G}_s$.*
2. *For each $s \in I'$ and $a \in \mathbb{G}_s$ we have $f(a) = f(0_{\mathbb{G}_s}) + a$.*
3. *For each $s_1, s_2 \in I'$, if $(s_1, s_2) \in S$, then $(f(0_{\mathbb{G}_{s_1}}), f(0_{\mathbb{G}_{s_2}})) \in \mathbb{G}_{s_1,s_2}$.*

Proof. Suppose f is a partial automorphism. Then we have:

1. For each $s \in I'$, $0_{\mathbb{G}_s} \in \mathbb{G}_s = P^M_s$, which implies $f(0_{\mathbb{G}_s}) \in P^M_s = \mathbb{G}_s$.
2. For each $s \in I'$ and $a \in \mathbb{G}_s$, $f(a) = f(F^M_a(0_{\mathbb{G}_s})) = F^M_a(f(0_{\mathbb{G}_s})) = f(0_{\mathbb{G}_s}) + a$.
3. For each $s_1, s_2 \in I'$, if $(s_1, s_2) \in S$, then $(0_{\mathbb{G}_{s_1}}, 0_{\mathbb{G}_{s_2}}) \in \mathbb{G}_{s_1,s_2}$ (because it is a subgroup of $\mathbb{G}_{s_1} \times \mathbb{G}_{s_2}$) but $\mathbb{G}_{s_1,s_2} = Q^M_{s_1,s_2}$, therefore we have $(f(0_{\mathbb{G}_{s_1}}), f(0_{\mathbb{G}_{s_2}})) \in \mathbb{G}_{s_1,s_2}$.

Similar arguments show the other direction. $\square$

Now we shall define a structure parameter $\mathfrak{x}$ and put $M = M_{\mathfrak{x}}$. Then we will choose elements $a_*, b_* \in M$, define $M_1 = (M, a_*)$, $M_2 = (M, b_*)$, and show that M_1, M_2 are as required in Theorem 2.5.

Let $\mathfrak{x} = \mathfrak{x}_{\lambda,\mathcal{F}}$ be the following structure parameter:

1. $I = [\lambda]^{\aleph_0}$.
2. For $u \in I$, J_u consists of the quadruples $t = (u, g, h, \zeta)$, where
   (a) g, h are functions from u into λ;
   (b) ζ is a function from $\mathrm{supRange}(g) \cap u$ into λ;
   (c) $\zeta \in \mathcal{F}$;
   (d) g, h are weakly increasing;
   (e) if $g(x) = g(y)$, then $h(x) = h(y)$;
   (f) $h(x) > x$.

For $t = (u, g, h, \zeta)$ we will denote $u = u^t$, $g = g^t$, $h = h^t$, $\zeta = \zeta^t$.

3. $S = \{(u_1, u_2) : u_1, u_2 \in I \text{ and } u_1 \subseteq u_2\}$.
4. $T = \{(t_1, t_2) : t_1, t_2 \in J, u^{t_1} \subseteq u^{t_2}, g^{t_1} \subseteq g^{t_2}, h^{t_1} \subseteq h^{t_2}, \zeta^{t_1} \subseteq \zeta^{t_2}\}$.

Let $M = M_{\lambda,\mathcal{F}} = M_{\mathfrak{x}}$ be the corresponding model. Note that $|I| = \lambda^{\aleph_0} = \lambda$ and for each $u \in I$, $|J_u| = \lambda^{\aleph_0} = \lambda$, therefore $\|M\| = \lambda$. Define $a_* = 0_{\mathbb{G}_\emptyset}$, $b_* = x_{(\emptyset,\emptyset,\emptyset,\emptyset)}$, $M_1 = (M, a_*)$, $M_2 = (M, b_*)$.

Claim 2.9 *M_1, M_2 are $\mathrm{EF}_{\mathcal{T},\lambda}$ equivalent.*

P r o o f. We start with

Definition 2.10 We define a set of functions $\mathcal{G} = \mathcal{G}(\lambda)$ with a partial order $\leq^{\mathcal{G}}$ in the following way:

1. For an ordinal $\alpha < \lambda$, $\mathcal{G}_\alpha$ is the set of functions g which satisfy
   (a) $g : \gamma \longrightarrow \alpha$, $\gamma < \lambda$;
   (b) g is weakly increasing.
2. $\mathcal{G} = \bigcup_{\alpha<\lambda} \mathcal{G}_\alpha$.
3. For each $g \in \mathcal{G}$ such that $\mathrm{Dom}(g) = \gamma$ we define $h_g : \gamma \longrightarrow \gamma + 1$ by

$$h_g(x) = \min(\{y : y < \gamma \wedge g(y) > g(x)\} \cup \{\gamma\}).$$

4. $g_1 \leq^{\mathcal{G}} g_2$ if $g_1 \subseteq g_2$ and $h_{g_1} \subseteq h_{g_2}$.

Claim 2.11

1. *If $g(x) = g(y)$, then $h_g(x) = h_g(y)$.*
2. *$h_g(x) > x$.*
3. *h_g is weakly increasing.*
4. *For every $g_1, g_2 \in \mathcal{G}$, $g_1 \leq^{\mathcal{G}} g_2$ iff*
   (a) *$\mathrm{Dom}(g_1) = \gamma_1 \leq \gamma_2 = \mathrm{Dom}(g_2)$ and $g_1 \subseteq g_2$;*
   (b) *if $\gamma_1 < \gamma_2$, then $g_2(\gamma_1) > g_2(x)$ for every $x < \gamma_1$.*
5. *If $g_1 \in \mathcal{G}_\alpha$ and $\mathrm{Dom}(g_1) < \gamma < \lambda$, then there is $g_2 \in \mathcal{G}_{\alpha+1}$ such that $g_1 \leq^{\mathcal{G}} g_2$ and $\mathrm{Dom}(g_2) = \gamma$.*
6. *If $\delta < \lambda$ and we have $\langle g_\alpha : \alpha < \delta \rangle$ such that $g_\alpha \in \mathcal{G}_\alpha$ and $\beta < \alpha$ implies $g_\beta \leq^{\mathcal{G}} g_\alpha$, then $g = \bigcup_{\alpha<\delta} g_\alpha$ satisfies $g \in \mathcal{G}_\delta$ and $g_\alpha \leq^{\mathcal{G}} g$ for each $\alpha < \delta$.*

P r o o f.

1. – 3. Easy.

4. If there is $x < \gamma_1$ such that $g_2(\gamma_1) = g_2(x)$, then $h_{g_2}(x) = h_{g_2}(\gamma_1) > \gamma_1 \geq h_{g_1}(x)$, so $g_1 \not\leq^{\mathcal{G}} g_2$. On the other direction, if $g_1 \subset g_2$ and $g_2(\gamma_1) > g_2(x)$ for every $x < \gamma_1$, then for every such x: If there is $y < \gamma_1$ such that $g_1(y) > g_1(x)$, let y' be the minimal y which satisfies this. We get $h_{g_1}(x) = h_{g_2}(x) = y'$. If there is no such y, we get $h_{g_1}(x) = h_{g_2}(x) = \gamma_1$. Therefore we have $h_{g_1} \subset h_{g_2}$.

5. Define $g_2 : \gamma \longrightarrow \alpha + 1$ by

$$g_2(x) = \begin{cases} g_1(x) & \text{if } x \in \mathrm{Dom}(g_1), \\ \alpha & \text{if } x \in \gamma \setminus \mathrm{Dom}(g_1). \end{cases}$$

By 4. we get that $g_1 \leq^{\mathcal{G}} g_2$.

6. Remember that λ is regular, therefore $\bigcup_{\alpha<\delta} \mathrm{Dom}(g_\alpha) < \lambda$.

This completes the proof of Claim 2.11. □

Now we will describe a winning strategy for ISO in the game $\Game_{\mathcal{T},\lambda}(M_1, M_2)$.

In stage α of the game, ISO will choose a function g_α such that

1. $g_\alpha \in \mathcal{G}_\alpha$;
2. if $\beta < \alpha$, then $g_\beta \leq^{\mathcal{G}} g_\alpha$;
3. if $\alpha = \beta + 1$ is a successor ordinal and in stage α AIS chose the sets A_1, A_2, then for each $u \in I$ such that $(A_1 \cup A_2) \cap \mathbb{G}_u \neq \emptyset$ we have $u \subseteq \mathrm{Dom}(g_\alpha)$.

The choice of g_α is done in the following way:

1. $g_0 = \emptyset$.
2. If α is limit, then $g_\alpha = \bigcup_{\beta<\alpha} g_\beta$. By Claim 2.11, $g_\alpha \in \mathcal{G}_\alpha$ and if $\beta < \alpha$, then $g_\beta \leq^{\mathcal{G}} g_\alpha$.
3. If $\alpha = \beta + 1$ and in stage α AIS chose the sets A_1, A_2, ISO will choose $\gamma < \lambda$ such that $\mathrm{Dom}(g_\beta) < \gamma$ and $u \subseteq \gamma$ for each $u \in I$ such that $(A_1 \cup A_2) \cap u \neq \emptyset$ (such γ exists since $|A_1 \cup A_2| + \aleph_0 < \lambda$). By Claim 2.11 there is $g \in \mathcal{G}_\alpha$ such that $\mathrm{Dom}(g) = \gamma$ and $g_\beta \leq^{\mathcal{G}} g$. ISO will choose such a function as g_α.

Now remember that if $\alpha = \beta + 1$, then in stage α AIS has to choose a node on level α, which is actually a function $\zeta_\alpha : \alpha \longrightarrow \lambda$, $\zeta_\alpha \in \mathcal{F}$. Then he chooses $A_1 \subset M_1$ and $A_2 \subset M_2$. Then ISO has to choose a partial isomorphism f_α from M_1 to M_2 such that $f_\beta \subseteq f_\alpha$, $A_1 \subseteq \mathrm{Dom}(f_\alpha)$, $A_2 \subseteq \mathrm{Range}(f_\alpha)$ (see Definition 2.1). So, ISO chooses g_α, and then defines f_α according to $f_\beta, A_1, A_2, g_\alpha, \zeta_\alpha$ in the following way:

$$\mathrm{Dom}(f_\alpha) = \mathrm{Dom}(f_\beta) \cup \bigcup\{\mathbb{G}_u \,:\, u \in I, (A_1 \cup A_2) \cap \mathbb{G}_u \neq \emptyset\}.$$

Then, for each $u \in I$ we have $\mathbb{G}_u \subseteq \mathrm{Dom}(f_\alpha)$ or $\mathbb{G}_u \cap \mathrm{Dom}(f_\alpha) = \emptyset$.

If $\mathbb{G}_u \subseteq \mathrm{Dom}(f_\alpha)$, we define $f_\alpha(0_{\mathbb{G}_u}) = x_t$, where $t = (u, g_\alpha \restriction u, h_{g_\alpha} \restriction u, \zeta_\alpha \restriction (u \cap \sup\mathrm{Range}(g_\alpha \restriction u)))$. (Note that because $g_\alpha \in \mathcal{G}_\alpha$, we have $\mathrm{Range}(g_\alpha) \subseteq \alpha = \mathrm{Dom}(\zeta_\alpha)$.)

Next, for every $a \in \mathbb{G}_u$ we define $f_\alpha(a) = f_\alpha(0_{\mathbb{G}_u}) + a$. By the construction we get that if $(u_1, u_2) \in S$, then $(f_\alpha(0_{\mathbb{G}_{u_1}}), f_\alpha(0_{\mathbb{G}_{u_2}})) \in \mathbb{G}_{u_1,u_2}$ (because the corresponding couple of t's lies in T). Therefore by Lemma 2.8, f_α is a partial automorphism of M. We also have:

1. If $\beta < \alpha$, then $g_\beta \subseteq g_\alpha$, $h_{g_\beta} \subseteq h_{g_\alpha}$ and $\zeta_\beta \subseteq \zeta_\alpha$. Therefore $f_\beta \subseteq f_\alpha$.
2. For each $\alpha > 0$,

$$f_\alpha(a_*) = f_\alpha(0_{\mathbb{G}_\emptyset}) = x_{(\emptyset,\emptyset,\emptyset,\emptyset)} = b_*.$$

Therefore f_α is a partial isomorphism from $M_1 = (M, a_*)$ into $M_2 = (M, b_*)$.

This completes the proof of Claim 2.9. □

Claim 2.12 *M_1, M_2 are not isomorphic.*

P r o o f. It is enough to show that M is rigid (i. e. it does not have a non-trivial automorphism).

Assume towards contradiction that $f \neq \mathrm{id}$ is an automorphism of M. For each $u \in I$ we define $c_u = f(0_{\mathbb{G}_u})$. By Lemma 2.8, for each $u \subseteq w \in I$ we have $(c_u, c_w) \in \mathbb{G}_{u,w}$.

For each $u \subset w \in I$ and $t = (w, g, h, \zeta) \in J_w$ we define $\pi_{w,u}(t) \in J_u$ by

$$\pi_{w,u}(t) = (u, g \restriction u, h \restriction u, \zeta \restriction \sup\mathrm{Range}(g \restriction u) \cap u).$$

By the definition of T we have that if $t \in J_w$ and $r \in J_u$, then $(r, t) \in T$ iff $r = \pi_{w,u}(t)$. We define a homomorphism $\widehat{\pi}_{w,u} : \mathbb{G}_w \longrightarrow \mathbb{G}_u$ by $\widehat{\pi}_{w,u}(x_t) = x_r$, where $r = \pi_{w,u}(t)$. We get that $\mathbb{G}_{u,w}$ is the subgroup of $\mathbb{G}_u \times \mathbb{G}_w$ generated by $\{(\widehat{\pi}_{w,u}(x_t), x_t) : t \in J_w\}$. Since $\{x_t : t \in J_w\}$ generates $\mathbb{G}_w$, we get that

$$\mathbb{G}_{u,w} = \{(\widehat{\pi}_{w,u}(c), c) : c \in \mathbb{G}_w\}.$$

Now define $n(u)$ to be the length of the reduced representation of c_u as a sum of the generators $\{x_t : t \in J_u\}$. For $u \subseteq w \in I$ we get $n(u) \leq n(w)$, since $c_u = \widehat{\pi}_{w,u}(c_w)$ and $\widehat{\pi}_{w,u}$ sends one generator to one generator. If for every $u \in I$ there is $w \in I$ such that $n(w) > n(u)$, we can find a sequence $\langle u_n : n < \omega\rangle$ such that $u_n \in I$ and $n(u_n) < n(u_{n+1})$. Define $w = \bigcup_{n<\omega} u_n$, we get that $n(w)$ is infinite – contradiction. Therefore, there is $u_* \in I$ such that $n(u_*)$ is maximal. Since we assumed $f \neq \mathrm{id}$, $n(u_*) > 0$.

Choose $t_* \in J_{u_*}$ such that x_{t_*} appears in the reduced representation of c_{u_*}. For each $u_* \subseteq w \in I$ there is a unique $t(w) \in J_w$ such that $\pi_{w,u_*}(t(w)) = t_*$ and $x_{t(w)}$ appears in the reduced representation of c_w. Such $t(w)$ exists because $c_{u_*} = \widehat{\pi}_{w,u_*}(c_w)$. It is unique because if there were two such t's, t_1, t_2, then

$$\widehat{\pi}_{w,u_*}(x_{t_1}) = \widehat{\pi}_{w,u_*}(x_{t_2}) = x_{t_*}.$$

Since in $\mathbb{G}_{u_*}$, $\forall x(2x = 0)$, it implies $n(w) > n(u_*)$, which contradicts the maximality of $n(u_*)$.

Note that if $u \subseteq w \subseteq z \in I$, then $\pi_{z,u} = \pi_{w,u} \circ \pi_{z,w}$. Therefore, by uniqueness of $t(w)$, if $u_* \subseteq w \subseteq z \in I$, then $t(w) = \pi_{z,w}(t(z))$. For each $u_* \subseteq w \in I$, define $g^w = g^{t(w)}$, $h^w = h^{t(w)}$, $\zeta^w = \zeta^{t(w)}$. If $u_* \subseteq w_1, w_2 \in I$, then the functions $g^{w_1}, h^{w_1}, \zeta^{w_1}$ and $g^{w_2}, h^{w_2}, \zeta^{w_2}$ are respectively compatible, since

$$t(w_1) = \pi_{z,w_1}(t(z)) \quad \text{and} \quad t(w_2) = \pi_{z,w_2}(t(z)),$$

where $z = w_1 \cup w_2$. Define

$$g = \bigcup\{g^w : u_* \subseteq w \in I\}, \quad h = \bigcup\{h^w : u_* \subseteq w \in I\}, \quad \zeta = \bigcup\{\zeta^w : u_* \subseteq w \in I\}.$$

We get:

1. $\mathrm{Dom}(g) = \mathrm{Dom}(h) = \lambda$.
2. g, h are weakly increasing.
3. $h(x) > x$.
4. If $g(x) = g(y)$, then $h(x) = h(y)$.
5. $\zeta \in \mathcal{F}$ (this is by Definition 2.2, 2.).
6. $\mathrm{supRange}(g) \subseteq \mathrm{Dom}(\zeta)$.

By Definition 2.2, 3., $\mathrm{Dom}(\zeta) \neq \lambda$. Therefore by 6., $\mathrm{supRange}(g) < \lambda$. Since g is weakly increasing and λ is regular, there is $\alpha_0 < \lambda$ such that for every $\alpha_0 < \alpha < \lambda$, $g(\alpha) = g(\alpha_0)$. By 4. we get that for every $\alpha_0 < \alpha < \lambda$, $h(\alpha) = h(\alpha_0)$. Choose $\alpha > h(\alpha_0) > \alpha_0$ and get that $h(\alpha) < \alpha$, contradicting 3.

This completes the proof of Claim 2.12. □

The proof of Theorem 2.5 is now finished. □

3 Games with trees for singular $\lambda = \lambda^{\aleph_0}$

It is clear that for λ singular we cannot expect the same result as in the previous section, since the AIS player would be able to list all the members of M_1, M_2. Thus, we prove a weaker result – we allow AIS to choose only one element in each turn. We also remark in Remark 3.2 that this result generalizes the result in [4] for such λ.

Theorem 3.1 *Suppose*

$\mathrm{cf}(\lambda) < \lambda = \lambda^{\aleph_0}$,

$\boxtimes_{\mathcal{F},\lambda}$ *holds,*

$\mathcal{T} = \mathcal{T}_{\mathcal{F}}$.

Then there are non-isomorphic models M_1, M_2 of size λ which are $\mathrm{EF}_{\mathcal{T},1}$ equivalent.

Remark 3.2 We can show that Theorem 3.1 generalizes the result in [4] by choosing appropriate $\mathcal{F}$. The result there shows the existence of two non-isomorphic models of size λ which are equivalent under every EF game of length $< \mathrm{cf}(\lambda)$, which consists of sub-games of length $< \lambda$, such that AIS chooses the length of each sub-game before it starts, and in every sub-game he chooses one element in each move – see the definitions there. Now, an appropriate $\mathcal{F}$ can be chosen by looking at the proof there, but we will take a shortcut – we will use the result instead of the proof. Let us choose a pair of models M_1, M_2 as in the result in [4]. Without loss of generality assume that the universe of M_1 is $\lambda \times \{1\}$ and the universe of M_2 is $\lambda \times \{2\}$. We can take $\mathcal{F}$ to be the set of functions f which satisfy the following conditions:

1. $\mathrm{Dom}(f) \subseteq \lambda$, $\mathrm{Range}(f) \subseteq \lambda$.
2. The partial function f' from M_1 to M_2 defined by

$$\mathrm{Dom}(f') = \mathrm{Dom}(f) \times \{1\}, \text{ where for every } \alpha \in \mathrm{Dom}(f),\ f'((\alpha, 1)) = (f(\alpha), 2),$$

is a partial isomorphism.

Now, it is not hard to see that $\mathrm{EF}_{\mathcal{T}_{\mathcal{F}},1}$ equivalence implies equivalence as in the result of [4].

P r o o f o f T h e o r e m 3.1. Denote $\kappa = \mathrm{cf}(\lambda)$. ($\kappa > \aleph_0$ because $\lambda = \lambda^{\aleph_0}$.) Let $\langle \mu_i : i < \kappa \rangle$ be an increasing and continuous sequence such that $\mu_0 = 0$, ${\mu_i}^+ < \mu_{i+1} = \mathrm{cf}(\mu_{i+1})$, $\mu_i > \aleph_0$ for $i > 0$, and $\bigcup_{i<\kappa} \mu_i = \lambda$. For every $\alpha < \lambda$ there is a unique $i < \kappa$ such that $\alpha \in [\mu_i, \mu_{i+1})$. We denote $i = \boldsymbol{i}(\alpha)$.

We define a structure parameter $\mathfrak{x} = \mathfrak{x}_{\mathcal{F},\lambda}$ in the following way:

1. $I = [\lambda]^{\aleph_0}$.
2. For $u \in I$, J_u is the collection of quadruples $t = (u, g, h, \zeta)$ such that
   (a) g, h are functions from u into λ, ζ is a function from some subset of u into λ;
   (b) $\zeta \in \mathcal{F}$;
   (c) for every $x \in u$, $g(x) \in [\mu_{\boldsymbol{i}(x)}, \mu^+_{\boldsymbol{i}(x)}]$, $h(x) \in [\mu_{\boldsymbol{i}(x)}, \mu_{\boldsymbol{i}(x)+1}]$;
   (d) g, h are weakly increasing;
   (e) if $g(x) = g(y)$, then $h(x) = h(y)$;
   (f) $h(x) > x$;
   (g) $\mathrm{Dom}(\zeta) = u \cap \bigcup\{\mu_{\boldsymbol{i}(x)} : x \in u \text{ and } h(x) = \mu_{\boldsymbol{i}(x)+1}\}$.

For $t = (u, g, h, \zeta)$ we denote $u = u^t$, $g = g^t$, $h = h^t$, $\zeta = \zeta^t$.

3. $S = \{(u_1, u_2) : u_1, u_2 \in I, u_1 \subseteq u_2\}$.
4. $T = \{(t_1, t_2) \in J : u^{t_1} \subseteq u^{t_2}, g^{t_1} \subseteq g^{t_2}, h^{t_1} \subseteq h^{t_2}, \zeta^{t_1} \subseteq \zeta^{t_2}\}$.

Let $M = M_{\mathcal{F},\lambda} = M_{\mathfrak{x}}$ be the corresponding model. Define $a_* = 0_{\mathbb{G}_\emptyset}$, $b_* = x_{(\emptyset,\emptyset,\emptyset,\emptyset)}$. Define $M_1 = (M, a_*)$ and $M_2 = (M, b_*)$.

Claim 3.3 *M_1, M_2 are $\mathrm{EF}_{\mathcal{T},1}$ equivalent.*

P r o o f. We start with

Definition 3.4 A partially ordered set of functions $(\mathcal{W}, \leq^{\mathcal{W}})$, which depends on the sequence $\langle \mu_i : i < \kappa \rangle$, is defined in the following way:

1. We define a set $\mathcal{B}$ such that $\bar{\beta} \in \mathcal{B}$ iff
   (a) $\bar{\beta} = \langle \beta_i : i < \kappa \rangle$, $\mu_i \leq \beta_i \leq \mu_{i+1}$;
   (b) there is $j = \boldsymbol{j}(\bar{\beta}) < \kappa$ such that $i < \boldsymbol{j}(\bar{\beta})$ iff $\beta_i = \mu_{i+1}$.
2. For $\bar{\beta} \in \mathcal{B}$ we define $\mathcal{W}_{\bar{\beta}}$ to be the set of functions g which satisfy
   (a) $\mathrm{Dom}(g) = \bigcup_{i<\kappa} [\mu_i, \beta_i)$;
   (b) g is weakly increasing;
   (c) for every $i < \kappa$, $x \in [\mu_i, \beta_i)$, we have $g(x) \in [\mu_i, \mu_i^+]$, and if $g(x) = \mu_i^+$, then $i < \boldsymbol{j}(\bar{\beta})$.
3. For $j < \kappa$ we define $\mathcal{W}_j = \bigcup\{\mathcal{W}_{\bar{\beta}} : \boldsymbol{j}(\bar{\beta}) \leq j\}$.
4. For $g \in \mathcal{W}_{\bar{\beta}}$ we define a function h_g as follows: $\mathrm{Dom}(h_g) = \mathrm{Dom}(g)$, where for $i < \kappa$ and $x \in [\mu_i, \beta_i)$ we have $h_g(x) = \min(\{y : \mu_i \leq y < \beta_i \wedge g(y) > g(x)\} \cup \{\beta_i\})$.

Claim 3.5

1. *If* $g(x) = g(y)$, *then* $h_g(x) = h_g(y)$.
2. $h_g(x) > x$.
3. h_g *is weakly increasing.*
4. *If* $x \in [\mu_i, \mu_{i+1})$, *then* $h_g(x) \in [\mu_i, \mu_{i+1}]$.
5. *Suppose that* $g_1 \in \mathcal{W}_{\bar\beta^1}$, $g_2 \in \mathcal{W}_{\bar\beta^2}$. *Then* $g_1 \leq^{\mathcal{W}} g_2$ *iff*
   (a) $g_1 \subseteq g_2$ (*therefore for every* $i < \kappa$, $\beta^1_i \leq \beta^2_i$);
   (b) *for every* $i < \kappa$, *if* $\beta^1_i < \beta^2_i$, *then for every* $x \in [\mu_i, \beta^1_i)$, $g_2(x) < g_2(\beta^1_i)$.
6. *If* $g_1 \in \mathcal{W}_j$ *and* $\bar\beta \in \mathcal{B}$, $\boldsymbol{j}(\bar\beta) \leq j$, *then there is* $g_2 \in \mathcal{W}_j$ *such that* $g_1 \leq^{\mathcal{W}} g_2$ *and* $\bigcup_{i<\kappa}[\mu_i, \beta_i) \subseteq \mathrm{Dom}(g_2)$.
7. *If* $\delta < \mu^+_j$ *and* $\langle g_\alpha : \alpha < \delta\rangle$ *is such that* $g_\alpha \in \mathcal{W}_j$ *and* $\alpha < \beta$ *implies* $g_\alpha \leq^{\mathcal{W}} g_\beta$, *then there exists* $g \in \mathcal{W}_j$ *such that if* $\alpha < \delta$, *then* $g_\alpha \leq^{\mathcal{W}} g$.

Proof.
1. – 4. Easy.
5. Like in the proof of Claim 2.11.
6. We may assume that $\mathrm{Dom}(g_1) \subseteq \bigcup_{i<\kappa}[\mu_i, \beta_i)$. Define for $i < \kappa$,

$$\gamma_i = \mu_i + \sup\{g_1(x) : x \in \mathrm{Dom}(g_1) \cap [\mu_i, \mu_{i+1})\}.$$

Since $g_1 \in \mathcal{W}_j$ we have $\gamma_i < \mu^+_i$ for $i \geq j$. Define for $i < \kappa$,

$$\gamma^*_i = \begin{cases} \mu^+_i & \text{if } i < j, \\ \gamma_i + 1 & \text{if } i \geq j. \end{cases}$$

Now define g_2 with $\mathrm{Dom}(g_2) = \bigcup_{i<\kappa}[\mu_i, \beta_i)$, where for every $i < \kappa$ and $x \in [\mu_i, \beta_i)$,

$$g_2(x) = \begin{cases} g_1(x) & \text{if } x \in \mathrm{Dom}(g_1), \\ \gamma^*_i & \text{if } x \notin \mathrm{Dom}(g_1). \end{cases}$$

Since $\boldsymbol{j}(\bar\beta) \leq j$ we have $g_2 \in \mathcal{W}_j$. By 5. we have $g_1 \leq^{\mathcal{W}} g_2$.
7. Define for every $i < \kappa$,

$$\beta_i = \sup(\textstyle\bigcup_{\alpha<\delta} \mathrm{Dom}(g_\alpha) \cap [\mu_i, \mu_{i+1})) + \mu_i, \quad \gamma_i = \sup(\textstyle\bigcup_{\alpha<\delta} \mathrm{Range}(g_\alpha \restriction [\mu_i, \mu_{i+1})) + \mu_i.$$

For every $\alpha < \delta$, $g_\alpha \in \mathcal{W}_j$. Therefore for every $i \geq j$,

$$\sup(\mathrm{Dom}(g_\alpha) \cap [\mu_i, \mu_{i+1})) < \mu_{i+1}, \quad \sup\mathrm{Range}(g_\alpha \restriction [\mu_i, \mu_{i+1})) < \mu^+_i.$$

Therefore, since $\delta < \mu^+_j \leq \mu^+_i < \mu_{i+1} = \mathrm{cf}(\mu_{i+1})$, we get that for $i \geq j$ we have $\beta_i < \mu_{i+1}$ and $\gamma_i < \mu^+_i$.
Define for $i < \kappa$,

$$\beta^*_i = \begin{cases} \mu_{i+1} & \text{if } i < j, \\ \beta_i & \text{if } i \geq j, \end{cases} \qquad \gamma^*_i = \begin{cases} \mu^+_i & \text{if } i < j, \\ \gamma_i + 1 & \text{if } i \geq j. \end{cases}$$

Denote $g' = \bigcup_{\alpha<\delta} g_\alpha$. Define $g \in \mathcal{W}_j$ with $\mathrm{Dom}(g) = \bigcup_{i<\kappa}[\mu_i, \beta^*_i)$ for $i < \kappa$ and $x \in [\mu_i, \beta^*_i)$ by

$$g(x) = \begin{cases} g'(x) & \text{if } x \in \mathrm{Dom}(g'), \\ \gamma^*_i & \text{if } x \notin \mathrm{Dom}(g'). \end{cases}$$

By 5. we get that $\alpha < \delta$ implies $g \geq^{\mathcal{W}} g_\alpha$.
This completes the proof of Claim 3.5. □

Now we will describe a winning strategy for ISO:

In every stage α in the game ISO will choose a function g_α such that

1. $g_\alpha \in \mathcal{W}_{\boldsymbol{i}(\alpha)+1}$;
2. if $\varepsilon < \alpha$, then $g_\varepsilon \leq^{\mathcal{W}} g_\alpha$;
3. if in stage α AIS chose an element from $\mathbb{G}_u$, then $u \subseteq \mathrm{Dom}(g_\alpha)$.

ISO can choose such g_α in the following way:

1. For $\alpha = 0$, $g_0 = \emptyset$.
2. For α limit, since $\alpha < \mu_{\boldsymbol{i}(\alpha)+1}$ and for every $\varepsilon < \alpha$, $g_\varepsilon \in \mathcal{W}_{\boldsymbol{i}(\alpha)+1}$, we can use Claim 3.5, 7.
3. If $\alpha = \varepsilon + 1$ and in stage α AIS chose an element from $\mathbb{G}_u$, then we choose $\bar{\beta} = \langle \beta_i : i < \kappa \rangle$ in the following way: If $i < \boldsymbol{i}(\alpha) + 1$, then $\beta_i = \mu_{i+1}$. Else $\mu_{i+1} > \alpha$. We choose $\beta_i < \mu_{i+1}$ such that

$$u \cap [\mu_i, \mu_{i+1}) \subseteq [\mu_i, \beta_i).$$

Now $\boldsymbol{j}(\bar{\beta}) = \boldsymbol{i}(\alpha) + 1$, so by Claim 3.5, 6. we can find $g \in \mathcal{W}_{\boldsymbol{i}(\alpha)+1}$ such that

$$g_\varepsilon \leq^{\mathcal{W}} g \quad \text{and} \quad \textstyle\bigcup_{i<\kappa} [\mu_i, \beta_i) \subseteq \mathrm{Dom}(g).$$

Define $g_\alpha = g$.

Now if $\alpha = \varepsilon + 1$ and in stage α AIS chose an element from $\mathbb{G}_u$ and the node $\zeta_\alpha \in \mathcal{T}$, ISO will define the automorphism f_α according to g_α, ζ_α with $\mathrm{Dom}(f_\alpha) = \mathrm{Dom}(f_\varepsilon) \cup \mathbb{G}_u$. For every w such that $\mathbb{G}_w \subseteq \mathrm{Dom}(f_\alpha)$, $f_\alpha(0_{\mathbb{G}_w}) = x_t$, where

$$t = (w, g_\alpha \restriction w, h_{g_\alpha} \restriction w, \zeta_\alpha \restriction (w \cap \{\mu_{\boldsymbol{i}(x)} : x \in w \wedge h_{g_\alpha}(x) = \mu_{\boldsymbol{i}(x)+1}\})).$$

(Note that $v \subseteq \alpha = \mathrm{Dom}(\zeta_\alpha)$, because $g_\alpha \in \mathcal{W}_{\boldsymbol{i}(\alpha)+1}$.) As in Section 2 we get that f_α is a partial isomorphism and $\varepsilon < \alpha$ implies $f_\varepsilon \subseteq f_\alpha$. This completes the proof of Claim 3.3. □

Claim 3.6 *M_1, M_2 are not isomorphic.*

Proof. We imitate the proof of Claim 2.12. It is enough to show that M is rigid. Assume towards contradiction that $f \neq \mathrm{id}$ is an automorphism of M. For each $u \subset w \in I$ and $t = (w, g, h, \zeta) \in J_w$ we define $\pi_{w,u}(t) \in J_u$ by $\pi_{w,u}(t) = (u, g^t \restriction u, h^t \restriction u, \zeta^t \restriction v)$, where $v = \bigcup\{\mu_{\boldsymbol{i}(x)} : x \in u \wedge h^t(x) = \mu_{\boldsymbol{i}(x)+1}\} \cap u$.

We proceed as in the proof of Claim 2.12, and we get that we can find functions g, h, ζ such that the following hold:

1. $\mathrm{Dom}(g) = \mathrm{Dom}(h) = \lambda$, $\mathrm{Dom}(\zeta) \subseteq \lambda$.
2. If $\boldsymbol{i}(x) = i$, then $g(x) \in [\mu_i, \mu_i^+]$, $h(x) \in [\mu_i, \mu_{i+1}]$.
3. g, h are weakly increasing.
4. If $g(x) = g(y)$, then $h(x) = h(y)$.
5. $h(x) > x$.
6. If $h(x) = \mu_{\boldsymbol{i}(x)+1}$, then $\mu_{\boldsymbol{i}(x)} \subseteq \mathrm{Dom}(\zeta)$.
7. $\zeta \in \mathcal{F}$.

By 7. we get that $\mathrm{Dom}(\zeta) \neq \lambda$, therefore by 6. there exists $i < \kappa$ such that if $\boldsymbol{i}(x) = i$, then $\boldsymbol{i}(h(x)) = i$. By 2., $\boldsymbol{i}(x) = i$ implies $g(x) \leq \mu_i^+$. By 3., g is weakly increasing. Since $\mu_{i+1} = \mathrm{cf}(\mu_{i+1}) > \mu_i^+$, we can find α_0 such that if $\alpha_0 \leq x < \mu_{i+1}$, then $g(x) = g(\alpha_0)$. By 5., $h(\alpha_0) > \alpha_0$. By the choice of i we get that $h(\alpha_0) < \mu_{i+1}$. Choose $h(\alpha_0) < x < \mu_{i+1}$. We get $h(x) > x > h(\alpha_0)$ but $g(x) = g(\alpha_0)$. This contradicts 4. Therefore we proved that M is rigid.

This completes the proof of Claim 3.6. □

The proof of Theorem 3.1 is now also completed. □

4 λ regular and $> \beth_\omega$

In this section we show a result which holds for every λ being regular and $> \beth_\omega$. In the previous sections we used the assumption $\lambda = \lambda^{\aleph_0}$. Here we use instead of it the existence of a set $\mathcal{P} \subset [\lambda]^{\aleph_0}$ of size λ which is "dense". By "dense" we mean that for every $A \in [\lambda]^{\beth_\omega}$ there is $B \subset A$, $B \in \mathcal{P}$.

Remark 4.1

1. Looking at the proof, one can see that instead of $\lambda > \beth_\omega$ it is enough to assume the following:
   (a) $\lambda > 2^{\aleph_0}$.
   (b) There is $\mathcal{P} \subset [\lambda]^{\aleph_0}$ such that
      i. $|\mathcal{P}| = \lambda$;
      ii. for every $A \in [\lambda]^{\lambda}$, there is $B \in \mathcal{P}$ such that $B \subset A$.
2. It is possible that it can be proved in ZFC that every $\lambda > 2^{\aleph_0}$ satisfies 1.(b) (it is a problem in cardinal arithmetic).

Theorem 4.2 *Suppose*
$\lambda = \mathrm{cf}(\lambda) > \beth_\omega$,
$\mathcal{T}$ *is a tree of size* λ *with no branch of length* λ.
Then there are models M_1, M_2 *of size* λ *which are* $\mathrm{EF}_{\mathcal{T},\lambda}$ *equivalent but not isomorphic.*

Proof. Let χ be a large enough cardinal (for example $\chi = \beth_7(\lambda)$).

Claim 4.3 *We can find* $\mathfrak{M}$ *such that the following hold:*

1. $\mathfrak{M}$ *is an elementary sub-model of* $\mathcal{H}(\chi)$.
2. $\lambda + 1 \subseteq \mathfrak{M}$.
3. $\|\mathfrak{M}\| = \lambda$.
4. *For every* $\langle (x_i, z_i) : i < \lambda \rangle$ *such that* $x_i \in \mathfrak{M}$ *and* $z_i \in \mathcal{T}$ *for every* $i < \lambda$ *there exists an increasing sequence* $\langle i_n : n < \omega \rangle$ *such that*
   (a) $\langle (x_{i_n}, z_{i_n}) : n < \omega \rangle \in \mathfrak{M}$;
   (b) *if in addition for* $i < j < \lambda$ *the level of* z_i *(in* $\mathcal{T}$*) is strictly less than the level of* z_j*, then* $\langle z_{i_n} : n < \omega \rangle$ *is an antichain in the order* $\leq^{\mathcal{T}}$.

In the proof of Claim 4.3 we use a partial version of the RGCH Theorem (see Shelah [5]).

Theorem 4.4 (RGCH Theorem, partial version) *If* $\lambda \geq \beth_\omega$*, then there is regular* $\kappa < \beth_\omega$ *and* $\mathcal{P} \subseteq [\lambda]^{<\beth_\omega}$ *such that*

1. $|\mathcal{P}| = \lambda$,
2. *for every* $A \in [\lambda]^{\beth_\omega}$*, we can find* $\langle A_i : i < \varepsilon \rangle$ *such that* $\varepsilon < \kappa$*,* $A_i \in \mathcal{P}$ *for every* $i < \varepsilon$*, and* $A = \bigcup_{i<\varepsilon} A_i$.

Corollary 4.5 *If* $\lambda \geq \beth_\omega$*, then we can find a set* $\mathcal{P}^* \subseteq [\lambda]^{\aleph_0}$ *such that* $|\mathcal{P}^*| = \lambda$ *and for every* $A \in [\lambda]^{\beth_\omega}$ *there is* $B \in \mathcal{P}^*$ *such that* $B \subseteq A$.

Proof. Choose κ and $\mathcal{P}$ as in Theorem 4.4 and define $\mathcal{P}^* = \bigcup\{[A]^{\aleph_0} : A \in \mathcal{P}\}$. □

Proof of Claim 4.3. We construct $\mathfrak{M}_n$ for every $n < \omega$ such that

1. $\mathfrak{M}_0$ is an elementary sub-model of $\mathcal{H}(\chi)$ such that $\|\mathfrak{M}_0\| = \lambda$, $\lambda + 1 \subseteq \mathfrak{M}_0$, and for every $A \in [\lambda]^{\beth_\omega}$ there is $B \in \mathfrak{M}_0 \cap [\lambda]^{\aleph_0}$ such that $B \subset A$ (this is possible by Corollary 4.5);
2. $\|\mathfrak{M}_n\| = \lambda$;
3. $\mathfrak{M}_n$ is an elementary sub-model of $\mathcal{H}(\chi)$;
4. if $A \in \mathfrak{M}_n$ and $|A| \leq \lambda$, then $A \subseteq \mathfrak{M}_{n+1}$;
5. $\mathfrak{M}_n \in \mathfrak{M}_{n+1}$ and $\mathfrak{M}_n \subset \mathfrak{M}_{n+1}$.

Now, let $\mathfrak{M} = \bigcup_{n<\omega} \mathfrak{M}_n$. We will prove that $\mathfrak{M}$ satisfies the conclusion of Claim 4.3.

Suppose that $\langle (x_i, z_i) : i < \lambda \rangle \subseteq \mathfrak{M} \times \mathcal{T}$ for every $i < \lambda$. We may assume without loss of generality that there is $n_0 < \omega$ such that $\{(i, x_i, z_i) : i < \lambda\} \subseteq \mathfrak{M}_{n_0}$. If the condition in Claim 4.3, 4.(b) is not satisfied, then we are done, because we can find $A \in [\lambda]^{\aleph_0}$ such that $\{(i, x_i, z_i) : i \in A\} \in \mathfrak{M}_{n_0+1}$ (because in $\mathfrak{M}_{n_0+1}$ there is a one to one correspondence between $\lambda \times \mathfrak{M}_{n_0} \times \mathcal{T}$ and λ, and every subset of λ of size $\beth_\omega$ has an infinite countable subset that is a member of $\mathfrak{M}_0$).

If the condition in Claim 4.3, 4.(b) is satisfied, then we have two cases:

Case (1): We can find $A \in [\lambda]^{\beth_\omega}$ such that $\langle z_i : i \in A \rangle$ is an antichain in $\leq^{\mathcal{T}}$.

Case (2): We cannot find such A.

If we are in Case (1), then we are done in the same way as before.

Suppose we are in Case (2).

Claim 4.6 *For every $j < \lambda$, we can find $j < i_0 < i_1 < i_2 < \lambda$ such that $z_{i_0} <^{\mathcal{T}} z_{i_1}, z_{i_2}$ and z_{i_1}, z_{i_2} are not comparable in $\leq^{\mathcal{T}}$.*

Proof. Assume towards contradiction that there is $j^* < \lambda$ such that we cannot find $j^* < i_0 < i_1 < i_2 < \lambda$ which are as in the claim. Define $C = \{z_i : j^* < i < \lambda\}$. Then comparability in $\leq^{\mathcal{T}}$ is an equivalence relation on C. Since λ is regular, either there are λ equivalence classes or there is an equivalence class of size λ. In other words, C contains an antichain or a chain of size λ. Both options are not possible, the first since we are in Case (2) and the second since $\mathcal{T}$ does not have a λ branch. Contradiction. □

By Claim 4.6 we can choose for every $j < \lambda$ a triple $i_0(j), i_1(j), i_2(j)$ such that

1. $i_0(j) < i_1(j) < i_2(j) < \lambda$;
2. $j < j'$ implies $i_2(j) < i_0(j')$;
3. $z_{i_0(j)} <^{\mathcal{T}} z_{i_1(j)}, z_{i_2(j)}$;
4. $z_{i_1(j)}$ and $z_{i_1(j)}$ are not comparable in $\leq^{\mathcal{T}}$.

We choose $A \in [\lambda]^{\aleph_0}$ such that $\{(j, i_0(j), i_1(j), i_2(j), x_j, z_j) : j \in A\} \in \mathfrak{M}_{n_0+1}$. Using the Ramsey Theorem in $\mathfrak{M}_{n_0+1}$, we can find an increasing sequence $\langle j_n : n < \omega \rangle$ such that

1. $j_n \in A$ for every $n < \omega$;
2. $\langle j_n : n < \omega \rangle \in \mathfrak{M}_{n_0+1}$;
3. $\{z_{i_1(j_n)} : n < \omega\}$ is a chain or an antichain in $\mathcal{T}$;
4. $\{z_{i_2(j_n)} : n < \omega\}$ is a chain or an antichain in $\mathcal{T}$.

Now we are done, since either $\{z_{i_1(j_n)} : n < \omega\}$ or $\{z_{i_2(j_n)} : n < \omega\}$ must be an antichain. Because if both are chains, we get that $z_{i_1(j_0)} <^{\mathcal{T}} z_{i_1(j_1)}, z_{i_2(j_0)} <^{\mathcal{T}} z_{i_2(j_1)}$. Since $z_{i_0(j_1)}$ is on higher level than $z_{i_1(j_0)}, z_{i_2(j_0)}$ and it is $<^{\mathcal{T}} z_{i_1(j_1)}, z_{i_2(j_1)}$ we get that $z_{i_1(j_0)}, z_{i_2(j_0)} <^{\mathcal{T}} z_{i_0(j_1)}$ – contradiction, since by the construction they are not comparable.

This completes the proof of Claim 4.3. □

We choose $\mathfrak{M}$ as in Claim 4.3 and we define a structure parameter $\mathfrak{x} = \mathfrak{x}(\mathfrak{M})$ in the following way:

Definition 4.7

1. I consists of the objects of the form (u, Λ), where
   (a) $u \in \lambda^{<\aleph_0}$;
   (b) $\Lambda \in \mathfrak{M}$, $|\Lambda| \leq \aleph_0$, Λ is a set of partial functions from λ to λ with finite domain.

For $s = (u, \Lambda)$ we denote $u = u^s$ and $\Lambda = \Lambda^s$. We define $\Gamma(s) = u^s \cup \bigcup\{\mathrm{Dom}(f) : f \in \Lambda^s\}$. Note that this a countable set.

2. For $s = (u, \Lambda) \in I$, J_s consists of all the objects of the form $t = (u, \Lambda, g, h, F, z)$, where
   (a) g, h are functions from u to λ;
   (b) F is a function from Λ^2 to $\{0, 1\}$;
   (c) $z \in \mathcal{T}$;

(d) the level α of z in the tree $\mathcal{T}$ is minimal under the condition $\alpha > y$ for every y such that $y \in \mathrm{Range}(g)$ or there are $f_1, f_2 \in \Lambda$ such that $F(f_1, f_2) = 1$ and $y \in \mathrm{Range}(f_1)$;

(e) there is a witness $(\boldsymbol{g}, \boldsymbol{h})$ for t, which means that

i. $\mathrm{Dom}(\boldsymbol{g}) = \mathrm{Dom}(\boldsymbol{h}) \subseteq \lambda$, $\mathrm{Range}(\boldsymbol{g}) \cup \mathrm{Range}(\boldsymbol{h}) \subseteq \lambda$,

ii. $\Gamma(s) \subseteq \mathrm{Dom}(\boldsymbol{g})$,

iii. $\boldsymbol{g}, \boldsymbol{h}$ are weakly increasing,

iv. $\boldsymbol{h}(x) > x$,

v. if $\boldsymbol{g}(x) = \boldsymbol{g}(y)$, then $\boldsymbol{h}(x) = \boldsymbol{h}(y)$,

vi. $g \subseteq \boldsymbol{g}$, $h \subseteq \boldsymbol{h}$,

vii. for every $(f_1, f_2) \in \Lambda^2$, $F(f_1, f_2) = 1$ iff $f_1 \subseteq \boldsymbol{g}$ and $f_2 \subseteq \boldsymbol{h}$.

3. $S = I^2$.
4. T consists of the pairs $(t_1, t_2) \in J^2$, where

(a) t_1, t_2 have a common witness;

(b) z^{t_1}, z^{t_2} are comparable in the order $\leq^{\mathcal{T}}$.

Fact 4.8 *Suppose*

$s \in I$, $z \in \mathcal{T}$,

$\boldsymbol{g}, \boldsymbol{h}$ *satisfy conditions* i. – v. *from Definition* 4.7*,* 2.(e)*,*

$\mathrm{Dom}(\boldsymbol{g}) \subset \alpha$*, where* α *is the level of* z*.*

Then the following hold:

1. *There is a unique* $t \in J_s$ *such that* $(\boldsymbol{g}, \boldsymbol{h})$ *is a witness for* t *and* $z^t \leq^{\mathcal{T}} z$*. We denote* $t = t(s, \boldsymbol{g}, \boldsymbol{h}, z)$*.*
2. *If*

(a) $\boldsymbol{g}', \boldsymbol{h}', z'$ *also satisfy the conditions in* 1.*,*

(b) z, z' *are comparable in* $\leq^{\mathcal{T}}$*,*

(c) $\boldsymbol{g}', \boldsymbol{h}'$ *are compatible with* $\boldsymbol{g}, \boldsymbol{h}$*, respectively,*

then $t(s, \boldsymbol{g}, \boldsymbol{h}, z) = t(s, \boldsymbol{g}', \boldsymbol{h}', z')$*.*

Let $M = M_{\mathfrak{x}}$ be the corresponding model. We can check that $\|M\| = \lambda$. Let $a_* = 0_{\mathbb{G}_{(\emptyset,\emptyset)}}$, $b_* = x_{(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,z_*)}$, where z_* is the root of $\mathcal{T}$ (without loss of generality there is a root). Define $M_1 = (M, a_*)$, $M_2 = (M, b_*)$.

Claim 4.9 M_1, M_2 *are* $\mathrm{EF}_{\mathcal{T},\lambda}$ *equivalent.*

We describe a winning strategy for ISO – this is very similar to the proof of Claim 2.9, so we will omit the details. We are using the definitions in Definition 2.10.

In every stage α of the game ISO will choose a function $\boldsymbol{g}_\alpha$ such that the following hold:

1. $\boldsymbol{g}_0 = \emptyset$.
2. $\boldsymbol{g}_\alpha \in \mathcal{G}_\alpha$ (see definition of $\mathcal{G}_\alpha$ and $\leq^{\mathcal{G}}$ in Definition 2.10).
3. $\beta < \alpha$ implies $\boldsymbol{g}_\beta \leq^{\mathcal{G}} \boldsymbol{g}_\alpha$.
4. If in stage α AIS chose the sets A_1, A_2, then for each $s \in I$, if $\mathbb{G}_s \cap (A_1 \cup A_2) \neq \emptyset$, then $\Gamma(s) \subseteq \mathrm{Dom}(\boldsymbol{g}_\alpha)$.

Now if $\alpha = \beta + 1$ and in stage α AIS chose the sets A_1, A_2 and the node z_α, ISO will define $\boldsymbol{h}_\alpha = h_{\boldsymbol{g}_\alpha}$ and then define f_α by

1. $\mathrm{Dom}(f_\alpha) = \bigcup\{\mathbb{G}_s : \Gamma(s) \subseteq \mathrm{Dom}(\boldsymbol{g}_\alpha)\}$,
2. for each s such that $\mathbb{G}_s \subseteq \mathrm{Dom}(f_\alpha)$, $f_\alpha(0_{\mathbb{G}_s}) = x_t$, where $t = t(s, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha, z_\alpha)$.

Claim 4.10 M_1, M_2 *are not isomorphic.*

Proof. It is enough to show that M is rigid. Assume towards contradiction that $f \neq \mathrm{id}$ is an automorphism of M. Denote $c_s = f(0_{\mathbb{G}_s})$ for $s \in I$. Denote $W_s = \{t \in J_s : x_t \text{ is in the reduced representation of } c_s\}$. Since $f \neq \mathrm{id}$ there is $s^* = (u^*, \Lambda^*)$ such that $W_{s^*} \neq \emptyset$. Note also that if $u^{s^*} \subseteq u^s$ and $\Lambda^* \subseteq \Lambda^s$, then there is a natural projection π_{s,s^*} from J_s into J_{s^*} such that $W_{s^*} \subseteq \mathrm{Range}(\pi_{s,s^*} \restriction W_s)$ (see the proof of Claim 2.12), therefore $W_s \neq \emptyset$.

Choose s_i, t_i, α_i for $i < \lambda$ such that the following hold:

1. $s_i \in I$, $s_i = (u^* \cup \{\alpha_i\}, \Lambda^*)$.
2. $t_i \in W_{s_i}$.
3. $\alpha_i < \lambda$.
4. If $i < j$, then $h^{t_i}(\alpha_i) < \alpha_j$.

Case (*1): $\sup\{g^{t_i}(\alpha_i) : i < \lambda\} = \lambda$. Then, since the level of z^{t_i} in $\mathcal{T}$ must be greater than $g^{t_i}(\alpha_i)$, we may assume that if $i < j$, then the level of z^{t_i} is strictly less than the level of z^{t_j}.

Case (*2): $\sup\{g^{t_i}(\alpha_i) : i < \lambda\} < \lambda$. Then by regularity of λ, we may assume that for every $i, j < \lambda$, $g^{t_i}(\alpha_i) = g^{t_j}(\alpha_j)$.

Now, no matter in which case we are, we proceed in the following way: By the properties of $\mathfrak{M}$ (see Claim 4.3) we can find a set $A \subset \lambda$ such that

1. $|A| = \aleph_0$;
2. $\{W_{s_i} : i \in A\} \in \mathfrak{M}$;
3. if we are in Case (*1), $\{z^{t_i} : i \in A\}$ is an antichain (we can have that because in Case (*1) the level of z^{t_i} is strictly increasing with i – see Claim 4.3).

We define $s^+ = (u^*, \{g^t, h^t : t \in W_{s_i}, i \in A\} \cup \Lambda^*)$. (Note that $\bigcup_{i \in A} W_{s_i} \in \mathfrak{M}$, therefore $s^+ \in I$.)

Claim 4.11 *For every $i \in A$, if $r \in J_{s^+}$, $t \in W_{s_i}$, $(r, t) \in T$, then we have:*

1. *If $(\boldsymbol{g}, \boldsymbol{h})$ is a witness for r, then $g^t \subseteq \boldsymbol{g}$, $h^t \subseteq \boldsymbol{h}$.*
2. *If $t \neq t' \in J_{s_i}$, then $(r, t') \notin T$.*

Proof.

1. Let $(\boldsymbol{g}_0, \boldsymbol{h}_0)$ be a common witness for r, t. Then we have $g^t \subseteq \boldsymbol{g}_0$, $h^t \subseteq \boldsymbol{h}_0$. Now $g^t, h^t \in \Lambda^{s^+}$, therefore $(g^t, h^t) \in \mathrm{Dom}(F^r)$. Since $(\boldsymbol{g}_0, \boldsymbol{h}_0)$ is a witness for r and $g^t \subseteq \boldsymbol{g}_0$, $h^t \subseteq \boldsymbol{h}_0$, then $F^r(g^t, h^t) = 1$. Therefore for any witness $(\boldsymbol{g}, \boldsymbol{h})$ of r, we have $g^t \subseteq \boldsymbol{g}$, $h^t \subseteq \boldsymbol{h}$.

2. There are three cases:

(a) $g^t \neq g^{t'}$ or $h^t \neq h^{t'}$. Then, since all those functions have the same domain, we get that r, t' cannot have a common witness $(\boldsymbol{g}, \boldsymbol{h})$ because by 1. we must have $g^t \subseteq \boldsymbol{g}$, $h^t \subseteq \boldsymbol{h}$.

(b) $F^t \neq F^{t'}$. Then, since $\mathrm{Dom}(F^t) = \Lambda^* \subseteq \Lambda^{s^+} = \mathrm{Dom}(F^r)$ and $(r, t) \in T$ we know that $F^t \subseteq F^r$. Since $F^t \neq F^{t'}$ and $\mathrm{Dom}(F^t) = \mathrm{Dom}(F^{t'})$, we get that F^r and $F^{t'}$ are not compatible (and therefore there is no common witness).

(c) $z^t \neq z^{t'}$. By the previous cases we may assume that $F^t = F^{t'}$, $g^t = g^{t'}$, and $h^t = h^{t'}$, therefore $z^t, z^{t'}$ are on the same level (see Definition 4.7, 2.(d)). We can also see that z^r must be on a greater level (remember that $F^t \subseteq F^r$ and $F^r(g^t, h^t) = 1$). Since $(r, t) \in T$, z^t, z^r are comparable in $\leq^{\mathcal{T}}$. It follows that $z^{t'}, z^r$ are not comparable, thus $(r, t') \notin T$. □

Claim 4.12 *For every $i \in A$ there is $r \in W_{s^+}$ such that $(r, t_i) \in T$.*

Proof. Since $(c_s, c_{s^+}) \in \mathbb{G}_{s,s^+}$ and this group is generated by $\{(x_t, x_{t'}) : (t, t') \in T \cap (J_s \times J_{s^+})\}$, there are representations (not necessarily reduced) $c_{s_i} = x_{w_1} + \cdots + x_{w_n}$, $c_{s^+} = x_{r_1} + \cdots + x_{r_n}$ with $(r_n, w_n) \in T$.

We may assume that if $1 \leq \ell_1 < \ell_2 \leq n$, then either $r_{\ell_1} \neq r_{\ell_2}$ or $w_{\ell_1} \neq w_{\ell_2}$. (Otherwise, we can reduce both representations – remember that in those groups $2x = 0$.) Since x_{t_i} appears in the reduced representation of c_{s_i}, t_i must appear among the w's. Let ℓ be such that $w_\ell = t_i$. Now we show that if $\ell_1 \neq \ell$, then $r_{\ell_1} \neq r_\ell$. Assume towards contradiction that $r_{\ell_1} = r_\ell$. By our assumption, $w_{\ell_1} \neq w_\ell$. Now, we have:

1. $(r_{\ell_1}, w_{\ell_1}), (r_\ell, w_\ell) \in T$;
2. $w_\ell \in W_{s_i}$;
3. $w_\ell \neq w_{\ell_1}$.

This contradicts Claim 4.11.

We got that for every $\ell_1 \neq \ell$, $r_{\ell_1} \neq r_\ell$, which implies that x_{r_ℓ} does not cancel. Hence $r_\ell \in W_{s^+}$ and we are done. □

Now choose $r_i \in W_{s+}$ for each $i \in A$ such that $(r_i, t_i) \in T$.

Claim 4.13 *If $i < j$, then $r_i \neq r_j$.*

Proof.
If we are in Case (*1), then $\{z^{t_i} : i \in A\}$ is an antichain. So, z^{t_i}, z^{t_j} are not comparable. Since $z^{r_i} \geq^{\mathcal{T}} z^{t_i}$ and $z^{r_j} \geq^{\mathcal{T}} z^{t_j}$ (see the proof of Claim 4.11 – z^{r_i}, z^{t_i} are comparable and z^{r_i} is on greater level), we must have $r_i \neq r_j$.

If we are in Case (*2), assume towards contradiction that $r = r_i = r_j$. Let $(\boldsymbol{g}, \boldsymbol{h})$ be a witness for r. Then by Claim 4.11, $g^{t_i}, g^{t_j} \subseteq \boldsymbol{g}$, $h^{t_i}, h^{t_j} \subseteq \boldsymbol{h}$. Since we are in Case (*2) we get that $\boldsymbol{g}(\alpha_i) = \boldsymbol{g}(\alpha_j)$ but by the construction $\boldsymbol{h}(\alpha_i) < \alpha_j < \boldsymbol{h}(\alpha_j)$, which contradicts the definition of a witness (see Definition 4.7, 2.(e)). □

We got that W_{s+} is infinite – contradiction. Therefore M must be rigid, and hence the proof of Claim 4.10 is finished. □

With the proof of Claim 4.10 the proof of Theorem 4.2 is also completed. □

5 $\lambda > \mathrm{cf}(\lambda) > \beth_\omega$

Clearly, for λ being singular and $> \beth_\omega$ we cannot prove the same result as for regular $\lambda > \beth_\omega$ (since in such game AIS will be able to list all the elements of the two models). Therefore, we define another type of game.

Definition 5.1 Let M_1, M_2 be models with common vocabulary. Let $\mathcal{T}$ be a tree. The game $\Game^*_{\mathcal{T}}(M_1, M_2)$ is defined in the same way as the definition of $\Game_{\mathcal{T},\mu}$ (see Definition 2.1) except that in stage α we demand that the sets A_1, A_2 chosen by AIS will satisfy $|A_1 \cup A_2| < 1 + \alpha$ instead of $|A_1 \cup A_2| < 1 + \mu$. We say that M_1, M_2 *are* $\mathrm{EF}^*_{\mathcal{T}}$ *equivalent* if ISO has a winning strategy for $\mathrm{EF}^*_{\mathcal{T}}(M_1, M_2)$.

Remark 5.2 Note that in Theorem 3.1, if we replace $\mathrm{EF}_{\mathcal{T},1}$ with $\mathrm{EF}^*_{\mathcal{T}}$, we do not get a stronger result, because for every tree $\mathcal{T}$ which satisfies the conditions there, we can construct another tree $\mathcal{T}'$ which satisfies the conditions, so that $\mathrm{EF}_{\mathcal{T}',1}$ equivalence would imply $\mathrm{EF}^*_{\mathcal{T}}$ equivalence.

Theorem 5.3 *Suppose that*

1. $\lambda > \mathrm{cf}(\lambda) = \kappa > \beth_\omega$*;*
2. $\mathcal{T}$ *is a tree of size λ without a λ branch.*

Then there are non-isomorphic models M_1, M_2 of size λ which are $\mathrm{EF}^*_{\mathcal{T}}$ equivalent.

Proof. Let χ be a large enough cardinal (for example $\chi = \beth_7(\lambda)$).

Claim 5.4 *We can find $\mathfrak{M}$ such that the following hold:*

1. $\mathfrak{M}$ *is an elementary sub-model of $\mathcal{H}(\chi)$.*
2. $\lambda + 1 \subseteq \mathfrak{M}$.
3. *For every $\langle (x_i, z_i) : i < \kappa \rangle$ such that $x_i \in \mathfrak{M}$ and $z_i \in \mathcal{T}$ for every $i < \lambda$ there exists an increasing sequence $\langle i_n : n < \omega \rangle$ such that:*
   (a) $\langle (x_{i_n}, z_{i_n}) : n < \omega \rangle \in \mathfrak{M}$*;*
   (b) *if in addition for every $\alpha < \lambda$ there is $i < \kappa$ such that the level of z_i is greater than α, then we can also have that $\langle z_{i_n} : n < \omega \rangle$ is an antichain in $\leq^{\mathcal{T}}$.*

Proof. The same proof as the proof of Claim 4.3 (we are using the fact that κ is regular and $\kappa > \beth_\omega$). □

Let $\mathfrak{M}$ be as in Claim 5.4. Let $\langle \mu_i : i < \kappa \rangle$ be an increasing and continuous sequence such that $\mu_0 = 0$, $\mu_i^+ + \aleph_0 < \mu_{i+1} = \mathrm{cf}(\mu_{i+1})$, and $\bigcup_{i<\kappa} \mu_i = \lambda$.

For every $\alpha < \lambda$ there is a unique $i < \kappa$ such that $\alpha \in [\mu_i, \mu_{i+1})$. We denote this i by $\boldsymbol{i}(\alpha)$.

We define a structure parameter $\mathfrak{x}$ in the following way:

Definition 5.5

1. I consists of the objects of the form (u, Λ), where
   (a) $u \in \lambda^{<\aleph_0}$;
   (b) $\Lambda \in \mathfrak{M}$, $|\Lambda| \leq \aleph_0$, Λ is a set of partial functions from λ to λ with finite domain.

For $s = (u, \Lambda)$ we denote $u = u^s$, $\Lambda = \Lambda^s$. We define $\Gamma(s) = u^s \cup \bigcup\{\mathrm{Dom}(f) : f \in \Lambda^s\}$. Note that this a countable set.

2. For $s = (u, \Lambda) \in I$, J_s consists of the objects of the form $t = (u, \Lambda, g, h, F, z)$, where
   (a) g, h are functions from u to λ;
   (b) F is a function from Λ^2 to $\{0, 1\}$;
   (c) $z \in \mathcal{T}$;
   (d) the level α of z in the tree $\mathcal{T}$ is minimal with regard to the condition that $\alpha \geq \mu_{\boldsymbol{i}(x)}$ for every x such that $h(x) = \mu_{\boldsymbol{i}(x)+1}$ or there are $f_1, f_2 \in \Lambda$ such that $F(f_1, f_2) = 1$ and $f_2(x) = \mu_{\boldsymbol{i}(x)+1}$;
   (e) there is a witness $(\boldsymbol{g}, \boldsymbol{h})$ for t, which means that
      i. $\mathrm{Dom}(\boldsymbol{g}) = \mathrm{Dom}(\boldsymbol{h}) \subseteq \lambda$, $\mathrm{Range}(\boldsymbol{g}) \cup \mathrm{Range}(\boldsymbol{h}) \subseteq \lambda$,
      ii. $\Gamma(s) \subseteq \mathrm{Dom}(\boldsymbol{g})$,
      iii. $g \subseteq \boldsymbol{g}$, $h \subseteq \boldsymbol{h}$,
      iv. for every $(f_1, f_2) \in \Lambda^2$, $F(f_1, f_2) = 1$ iff $f_1 \subseteq \boldsymbol{g}$ and $f_2 \subseteq \boldsymbol{h}$,
      v. $\boldsymbol{g}, \boldsymbol{h}$ are weakly increasing,
      vi. $\boldsymbol{h}(x) > x$,
      vii. if $\boldsymbol{g}(x) = \boldsymbol{g}(y)$, then $\boldsymbol{h}(x) = \boldsymbol{h}(y)$,
      viii. $\boldsymbol{g}(x) \in [\mu_{\boldsymbol{i}(x)}, \mu^+_{\boldsymbol{i}(x)}]$,
      ix. $\boldsymbol{h}(x) \in [\mu_{\boldsymbol{i}(x)}, \mu_{\boldsymbol{i}(x)+1}]$.
3. $S = I^2$.
4. T consists of the pairs $(t_1, t_2) \in J^2$, where
   (a) t_1, t_2 have a common witness;
   (b) z^{t_1}, z^{t_2} are comparable in the order $\leq^{\mathcal{T}}$.

Fact 5.6 *Suppose*

$s \in I$, $z \in \mathcal{T}$,

$\boldsymbol{g}$, $\boldsymbol{h}$, *and* s *satisfy* i. – ii. *and* v. – ix. *from Definition* 5.5, 2.(e),

$\bigcup\{\mu_{\boldsymbol{i}(x)} : \boldsymbol{h}(x) = \mu_{\boldsymbol{i}(x)+1}\} \subset \alpha$, *where* α *is the level of* z.

Then the following hold:

1. *There is a unique* $t \in J_s$ *such that* $(\boldsymbol{g}, \boldsymbol{h})$ *is a witness for* t *and* $z^t \leq^{\mathcal{T}} z$. *We denote* $t = t(s, \boldsymbol{g}, \boldsymbol{h}, z)$.
2. *If*
   (a) $\boldsymbol{g}', \boldsymbol{h}', z'$ *satisfy the conditions in* 1.,
   (b) z, z' *are comparable in* $\leq^{\mathcal{T}}$,
   (c) $\boldsymbol{g}', \boldsymbol{h}'$ *are compatible with* $\boldsymbol{g}, \boldsymbol{h}$, *respectively,*

then $t(s, \boldsymbol{g}, \boldsymbol{h}, z) = t(s, \boldsymbol{g}', \boldsymbol{h}', z')$.

Let $M = M_{\mathfrak{x}}$ be the corresponding model. We can check that $\|M\| = \lambda$. Let $a_* = 0_{\mathbb{G}_{(\emptyset,\emptyset)}}$, $b_* = x_{(\emptyset,\emptyset,\emptyset,\emptyset,\emptyset,z_*)}$, where z_* is the root of $\mathcal{T}$ (without loss of generality there is a root). Define $M_1 = (M, a_*)$, $M_2 = (M, b_*)$.

Claim 5.7 *M_1, M_2 are $\mathrm{EF}^*_{\mathcal{T}}$ equivalent.*

We describe a winning strategy for ISO – this is very similar to the proof of Claim 3.3, so we omit the details. We use the definitions in Definition 3.4. In every stage α of the game ISO will choose a function $\boldsymbol{g}_\alpha$ such that:

1. $\boldsymbol{g}_0 = \emptyset$.
2. $\boldsymbol{g}_\alpha \in \mathcal{W}_{\boldsymbol{i}(\alpha)+1}$.
3. $\beta < \alpha$ implies $\boldsymbol{g}_\beta \leq^{\mathcal{W}} \boldsymbol{g}_\alpha$.
4. If in stage α AIS chose the sets A_1, A_2, then for each $s \in I$, if $\mathbb{G}_s \cap (A_1 \cup A_2) \neq \emptyset$, $\Gamma(s) \subseteq \mathrm{Dom}(\boldsymbol{g}_\alpha)$.

Now if $\alpha = \beta + 1$ and in stage α AIS chose the sets A_1, A_2 and the node z_α, ISO will define $\boldsymbol{h}_\alpha = h_{\boldsymbol{g}_\alpha}$ and then define f_α by

1. $\mathrm{Dom}(f_\alpha) = \bigcup\{\mathbb{G}_s : \Gamma(s) \subseteq \mathrm{Dom}(\boldsymbol{g}_\alpha)\}$,
2. for each s such that $\mathbb{G}_s \subseteq \mathrm{Dom}(f_\alpha)$, $f_\alpha(0_{\mathbb{G}_s}) = x_t$, where $t = t(s, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha, z_\alpha)$.

Claim 5.8 *M_1, M_2 are not isomorphic.*

Proof. It is enough to show that M is rigid. The proof is very similar to the proof of Claim 4.10. Assume towards contradiction that $f \neq \mathrm{id}$ is an automorphism of M. Denote

$$W_s = \{t \in J_s : x_t \text{ is in the reduced representation of } c_s\}.$$

Since $f \neq \mathrm{id}$ there is $s^* = (u^*, \Lambda^*)$ such that $W_{s^*} \neq \emptyset$.

Case (*1): We can find $\langle s_\theta, t_\theta, \alpha_\theta : \theta < \kappa\rangle$ such that

1. $s_\theta \in J$, $s_\theta = (u^* \cup \{\alpha_\theta\}, \Lambda^*)$;
2. $t_\theta \in W_{s_\theta}$;
3. $h^{t_\theta}(\alpha_\theta) = \mu_{\boldsymbol{i}(\alpha_\theta)+1}$;
4. if $\theta < \varepsilon < \kappa$, then $\boldsymbol{i}(\alpha_\theta) < \boldsymbol{i}(\alpha_\varepsilon)$.

In this case, note that the level of z^{t_θ} must be $\geq \mu_{\boldsymbol{i}(\alpha_\theta)}$.

Case (*2): We cannot find such a sequence. Therefore, for every large enough $i < \kappa$, for every α such that $\boldsymbol{i}(\alpha) = i$, for $s(\alpha) = (u^* \cup \{\alpha\}, \Lambda^*)$, for every $t \in W_{s(\alpha)}$ we have $h^t(\alpha) < \mu_{i+1}$. Choose i^* which satisfies this and $\mu_{i^*} > \kappa$. We can find $\langle t_\theta, s_\theta, \alpha_\theta : \theta < \mu_{i^*+1}\rangle$ such that

1. $s_\theta \in I$, $t_\theta \in W_{s_\theta}$;
2. $\boldsymbol{i}(\alpha_\theta) = i^*$;
3. if $\theta < \varepsilon$, then $h^{t_\theta}(\alpha_\theta) < \alpha_\varepsilon$ $(< h^{t_\varepsilon}(\alpha_\varepsilon))$.

Since $\mu_{i^*+1} = \mathrm{cf}(\mu_{i^*+1}) > \mu^+_{i^*}$ and for every θ we have $g^t_\theta(x) \leq \mu^+_{i^*}$ (this is by Definition 5.5, 2.(e)viii.), we may assume that $g^{t_\theta}(\alpha_\theta)$ is constant.

Now, in both cases, we proceed in a similar way to the proof of Claim 4.10. Using Claim 5.4, we choose $A \subset \kappa$ such that

1. $|A| = \aleph_0$;
2. $\langle W_{s_\theta} : \theta \in A\rangle \in \mathfrak{M}$;
3. if we are in Case (*1), then $\langle z^{t_\theta} : \theta \in A\rangle$ is an antichain in $\leq^{\mathcal{T}}$ (we can demand this because in Case (*1) the levels of the z^{t_θ}'s are not bounded in λ – see Claim 5.4).

Define $s^+ \in I$ by $s^+ = (\emptyset, \Lambda^* \cup \{g^t, h^t : t \in W_{s_\theta}, \theta \in A\})$.

Claim 5.9 *For every $\theta \in A$ the following holds: If $r \in J_{s^+}$, $t \in W_{s_\theta}$, $(r, t) \in T$, then*

1. *if $(\boldsymbol{g}, \boldsymbol{h})$ is a witness for r, then $g^t \subseteq \boldsymbol{g}$ and $h^t \subseteq \boldsymbol{h}$;*
2. *if $t \neq t' \in J_{s_\theta}$, then $(r, t') \notin T$.*

Proof. See the proof of Claim 4.11. □

Claim 5.10 *For every $\theta \in A$ there is $r \in W_{s^+}$ such that $(r, t_\theta) \in T$.*

Proof. See the proof of Claim 4.12. □

Now, using Claim 5.10, we choose for each $\theta \in A$ an $r_\theta \in W_{s+}$ such that $(t_\theta, r_\theta) \in T$.

Claim 5.11 *If $\theta < \varepsilon$, then $r_\theta \neq r_\varepsilon$.*

Proof.

If we are in Case (*1): $z^{t_\theta}, z^{t_\varepsilon}$ are not comparable. But $z^{r_\theta} \geq^{\mathcal{T}} z^{t_\theta}$ because they are comparable and z^{r_θ} is on greater level, since that level is determined by Definition 5.5, 2.(d). By the same argument, $z^{r_\varepsilon} \geq^{\mathcal{T}} z^{t_\varepsilon}$. Therefore, $z^{r_\varepsilon}, z^{r_\theta}$ are not comparable, so $r_\theta \neq r_\varepsilon$.

If we are in Case (*2): Assume towards contradiction that $r = r_\theta = r_\varepsilon$. Let $(\boldsymbol{g}, \boldsymbol{h})$ be a witness for r. Then by Claim 5.9, $g^{t_\theta}, g^{t_\varepsilon} \subseteq \boldsymbol{g}$ and $h^{t_\theta}, h^{t_\varepsilon} \subseteq \boldsymbol{h}$. Since we are in Case (*2) we obtain that

$$\boldsymbol{g}(\alpha_\theta) = \boldsymbol{g}(\alpha_\varepsilon) \quad \text{and} \quad \boldsymbol{h}(\alpha_\theta) < \alpha_\varepsilon < \boldsymbol{h}(\alpha_\varepsilon),$$

which contradicts the definition of a witness (see Definition 5.5, 2.(e)). □

We got that W_{s+} is infinite – contradiction. Therefore, M must be rigid, which proves Claim 5.8. □

The proof of Theorem 5.3 is now complete. □

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