Classification of Non Elementary Classes II

Abstract Elementary Classes

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Introduction

No knowledge of the first part is assumed; we rather start to redo it, eliminating or weakening set theoretic assumptions, and generalizing the context. In [Sh1], proving a conjecture of Baldwin, we show that

$(*)_1$ no $\psi \in L_{\omega_1,\omega}(Q))$ has a unique uncountable model up to isomorphism,

by showing that

$(*)_2$ categoricity (of $\psi \in L_{\omega_1,\omega}(Q))$ in $\aleph_1$ implies the existence of a model of ψ of power $\aleph_2$.

Unfortunately this was not proved (i.e. in ZFC), diamond of $\aleph_1$ was assumed. In [Sh2] this was weakened to $2^{\aleph_0} < 2^{\aleph_1}$; here we shall prove it in ZFC (see §3). (However, for getting the conclusion from the weaker assumption $I(\aleph_1, \psi) < 2^{\aleph_1}$ as there we still need $2^{\aleph_0} < 2^{\aleph_1}$).

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The main result of [Sh2] was:

$(*)_3$ if $n > 0$, $2^{\aleph_0} < 2^{\aleph_1} < \cdots < 2^{\aleph_n}$, $\psi \in L_{\omega_1,\omega}$, $1 \le I(\aleph_\ell, \omega) < \mu(\aleph_\ell)$ for $\ell \le n$, $\ell \ge 1$ (where $\mu(\aleph_\ell)$ is usually $2^{\aleph_\ell}$ and always $> 2^{\aleph_{\ell-1}}$)
then ψ has a model of cardinality $\aleph_{n+1}$.

$(*)_4$ if $n > 0$, $2^{\aleph_0} < 2^{\aleph_1} < \ldots$, $\psi \in L_{\omega_1,\omega}$, $1 \le I(\aleph_\ell, \psi) < \mu(\aleph_\ell)$ for $\ell < \omega$
then ψ has a model in every infinite cardinal (and satisfies Los Conjecture), (note that $(*)_3$ for $n = 1$, assuming $\Diamond_{\aleph_1}$ was proved in [Sh1].)

Why in [Sh2] ψ was assumed to be just in $L_{\omega_1,\omega}$ and not more generally in $L_{\omega_1,\omega}(Q)$? Mainly because we feel that in [Sh1], the logic $L_{\omega_1,\omega}(Q)$ was incidental. We delay the search for the right context to this sequel. So here we are working in "abstract elementarily class" (so no logic is present in the context) whose main feature is the absence of amalgamation. So if $\mathcal{L}$ is a fragment of $L_{\infty,\omega}$, (for a fixed vocabulary), $T \subseteq \mathcal{L}$ a theory, $K = \{M : M \models T\}$, $M <_K N$ if and only if $M <_{\mathcal{L}} N$, we get such a class. So $\psi \in L_{\omega_1,\omega}(Q)$ is not represented directly, but can be with minor adaptation; see 3.9(2) and for other applications Makowsky and Shelah [MSh]. Surprising (and easily), every such K can be represented as a pseudo elementary class if we allow omitting types, (see 1.8). We introduce a replacement for saturated models (for first order T) and full models (for excellent classes, see [Sh2]): limit models; really some variants of it. See Definition 3.1. The strongest is M superlimit: $(\exists N)(M < N \wedge M \neq N)$ and if $M_i \cong M$ for $i < \delta \le \| M \|$ then $\bigcup_{i<\delta} M_i \cong M$. Such M exists for a first order T for some pairs λ, δ; and it exists for every $\lambda \ge 2^{|T|}$ if and only if T is superstable. But we can prove something under those circumstances: if K is categorical in λ (or just have a superlimit model M^* in λ), but the λ-amalgamation property fail for M^* and $2^\lambda < 2^{\lambda^+}$ *then* $I(\lambda^+, K) = 2^{\lambda^+}$ (see 3.5). With some restriction on λ and K, we can prove e.g. $I(\lambda, K) = I(\lambda^+, K) = 1 \Rightarrow I(\lambda^{++}, K) \ge 1$, (see 3.7, 3.8).

However our main aim was to do the parallel of [Sh2] in our context, and it is natural to assume K is $PC_{\aleph_0}$.

Sections 4,5 present work toward this goal (§5 assuming $2^{\aleph_0} < 2^{\aleph_1}$; §4 without it). We should note that dealing with superlimit models rather than full ones make problems, as well as the fact that the class is not necessarily elementary in some reasonable logics. Because of the second we were driven to use the forcing, and "the type $\bar{a}$ materialize": $gtp(\bar{a}, N, M)$. We also (necessarily) encounter the case $|D(\aleph_0)| = \aleph_1$. Because of the first, the scenario for getting a full model in $\aleph_1$ (which can be adapted to $(\aleph_1, \{\aleph_1\})$-superlimit - see 5.9) does not seem to be enough for getting superlimit models in $\aleph_1$ (see 5.24).

We had felt that arriving at enough conclusions on the models of cardinality $\aleph_1$ to start dealing with models of cardinality $\aleph_2$, will be a strong indication that we can complete the generalization of [Sh2], so getting superlimits in $\aleph_2$ is the culmination of this paper and a natural stopping point. The rest (of the parallel to [Sh2]) was delayed, and Grossberg had taken it on himself.

Grossberg and Shelah [GSh2] will do parallel work replacing $\aleph_0$ by any cardinal.

Much remains to be done.

1. Proving $(*)_3$, $(*)_4$ in our context.

2. Parallel results in ZFC; e.g. prove $(*)_3$ for $n = 1$, $2^{\aleph_0} = 2^{\aleph_1}$.

Note that if $2^{\aleph_0} = 2^{\aleph_1}$, assuming $1 \le I(\aleph_1, K) < 2^{\aleph_1}$ give really less-new phenomena arise (see §6). See §4 (and its concluding remarks).

3. Construct examples; e.g. K (or $\psi \in L_{\omega_1,\omega}$), categorical in $\aleph_0, \aleph_1, \ldots, \aleph_n$ but not in $\aleph_{n+1}$.[1]

4. If K is PC_λ, categorical in λ, λ^+, does it necessarily have a model in λ^{++}?

The work was done in 1977, and a preprint was circulated. Meanwhile an expository article of Makowsky [Ma] represent, give background and explain the easy parts of the paper. The author have corrected and replaced some proofs and added mainly §6.

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[1] In late 85 much was done on this [Sh 10]

§1. Axioms and Simple Properties for Classes of Models

1.1 Conventions.

Here K will be a class of L-models and $<_K$ a two-place relation on the models in K (we usually omit K). We do not distinguish between K and $(K,<_K)$. We shall assume that K, $<_K$ are fixed, and $M <_K N \Rightarrow N, N \in K$; and we assume that the following axioms hold. When we use $<$ in the usual sense we write $<_{L_{\omega,\omega}}$.

1.2 Definition: Ax 0: The holding of $M \in K$, $N < M$ depend on N, M only up isomorphism i.e. $[M \in K, M \cong N \Rightarrow K]$, and [if $N < M$ and f is an isomorphism from M onto the L-model M' mapping N' onto N' *then* $N' < M'$].

Ax I: If $M < N$ then $M \subseteq N$ (i.e. M is a submodel of N).

Ax II: $M_0 < M_1 < M_2$ implies $M_0 < M_2$, and $M < M$ for $M \in K$.

Ax III: If λ is a regular cardinal, M_i $(i < \lambda)$ is a $<$-increasing (i.e. $i < j < \lambda$ implies $M_i < M_j$) and continuous (i.e. for $\delta < \lambda$, $M_\delta = \bigcup_{i<\delta} M_i$) *then* $M_0 < \bigcup_{i<\lambda} M_i$.

Ax IV: If λ is a regular cardinal M_i $(i < \lambda)$ is $<$-increasing continuous, $M_i < N$ *then* $\bigcup_{i<\lambda} M_i < N$.

Ax V: If $N_0 \subseteq N_1 < M, N_0 < M$ *then* $N_0 < N_1$.

Ax VI: If $A \subseteq N$; $|A| \leq \lambda(K)$ *then* for some $M < N$, $A \subseteq |M| \leq \lambda(K)$ (we assume for simplicity $|L| \leq \lambda(K)$).

Notation: $K_\lambda = \{M \in K : \|M\| = \lambda\}$, $K_{<\lambda} = \bigcup_{\mu<\lambda} K_\mu$.

1.3 Definition: The embedding $f : N \to M$ is a K-embedding if its range is the universe of a model $N' < M$,

(so $f : N \to N'$ is an isomorphism onto).

1.4 Definition: Let T_1 be a theory in L_1, $L \subseteq L_1$, Γ a set of types in L_1.

(1) $EC(T_1,\Gamma) = \{M : M$ an L_1-model of T_1 which omits every $p \in \Gamma\}$. (so L_1 is reconstructible from T_1,Γ)

$PC(T_1,\Gamma,L) = \{M : M$ is an L-reduct of some $M_1 \in EC(T_1,\Gamma)\}$.

(3) We say K is PC_λ^μ if for some $T_1,T_2,\Gamma_1,\Gamma_2, L$: $K = PC(T_1,\Gamma_1,L)$, and $\{(M, N) : M \prec N,\ M, N \in K\} = PC(T_2, \Gamma_2, L')$ where $L' = L \cup \{P\}$ (P a new one place predicate), $|T_\ell| \le \lambda$, $|\Gamma_\ell| \le \mu$ for $\ell = 1,2$. If $\mu = \lambda$ we omit it.

1.5 Example: If $L_1 = L$, T_1,Γ as above, then $K \overset{def}{=} EC(T_1,\Gamma)$, $\prec_K \overset{def}{=} \prec_{L_{\omega,\omega}}$ satisfy the Axioms from 1.2 (for $\lambda(K) \overset{def}{=} |T_1| + \aleph_0$).

1.6 Lemma: Let I be a directed set (i.e. partially ordered by $\le$, such that any two elements have a common upper bounded).

(1) If M_t is defined for $t \in I$, and $t \le s \in I$ implies $M_t \prec M_s$ *then* for every $t \in I$, $M_t \prec \bigcup_{s\in I} M_s$.

(2) If in addition $t \in I$ implies $M_t \prec N$ *then* $\bigcup_{s\in I} M_s \prec N$.

Proof: By induction on $|I|$ (simultaneously for (1) and (2)).

If I is finite, then I has a maximal element $t(0)$, hence $\bigcup_{t\in I} M_s = M_{t(0)}$, so there is nothing to prove.

So suppose $|I| = \mu$ and we have proved the assertion when $|I| < \mu$. Let $\lambda = cf\,\mu$ so λ is a regular cardinal. We can find I_α ($\alpha < \lambda$) such that $|I_\alpha| < |I|$, $\alpha < \beta < \lambda$ implies $I_\alpha \subseteq I_\beta \subseteq I$, $\bigcup_{\alpha<\lambda} I_\alpha = I$, for limit $\delta < \lambda$, $I_\delta = \bigcup_{\alpha<\delta} I_\alpha$ and I_α is directed and non-empty. Let $M^\alpha = \bigcup_{t\in I_\alpha} M_t$; so by the induction hypothesis on (1), $t \in I_\alpha$ implies $M_t \prec M^\alpha$; [and if we are proving (2) by the induction hypothesis on (2), $M^\alpha \prec N$]. If $\alpha < \beta$ then $t \in I_\alpha$ implies $M_t \prec M^\beta$; hence by the induction hypothesis on (2) $M^\alpha = \bigcup_{t\in I_\alpha} M_t \prec M^\beta$. So by Ax III

$M^{\alpha} < \bigcup_{\beta<\lambda} M^{\beta} = \bigcup_{t\in I} M_t$, and as $t \in I_{\alpha}$ implies $M_t < M^{\alpha}$, by Ax II $t \in I$ implies $M_t < \bigcup_{s\in I} M_s$. [If we are proving (2) by Ax IV, $\bigcup_{s\in I} M_s = \bigcup_{\alpha<\lambda} M^{\alpha} < N$].

1.7 Lemma: Let $L_1 = L \cup \{F_i^n : i < \lambda(K), n < \omega\}$, F_i^n an n-place function symbol (assuming, of course, $F_i^n \notin L$).

Every model M (in K) can be expanded to an L_1-model M_1 such that:

(1) $M_{\bar{a}} < M$ when $\bar{a} \in {}^n|M|$ and where $M_{\bar{a}}$ is the submodel of M with universe $\{F_i^n(\bar{a}) : i < \lambda(K)\}$,

(2) if $\bar{a} \in {}^n|M_{\bar{a}}|$ then $\|M_{\bar{a}}\| \le \lambda(K)$,

(3) if $\bar{b}$ is a subsequence of $\bar{a}$, then $M_{\bar{b}} < M_{\bar{a}}$,

(4) for every $N_1 \subseteq M_1, N_1 \restriction L < M$.

Proof: We define by induction on n, the values of $F_i^n(\bar{a})$ for every $i < \lambda(K)$, $\bar{a} \in {}^n|M|$. By Ax VI there is an $M_{\bar{a}} < M$, $\|M_{\bar{a}}\| \le \lambda(K)$, $|M_{\bar{a}}|$ include $\bigcup\{M_{\bar{b}} : \bar{b}$ a subsequence of $\bar{a}$ of length $< n\} \cup \bar{a}$ and $M_{\bar{a}}$ does not depend on the order of $\bar{a}$. Let $|M_{\bar{a}}| = \{c_i : i < i_0 \le \lambda(K)\}$, and define $F_i^n(\bar{a}) = c_i$ for $i < i_0$ and c_0 for $i_0 \le i < \lambda(K)$.

Clearly our conditions are satisfied if $\bar{b}$ is a subsequence of $\bar{a}$, $M_{\bar{b}} < M_{\bar{a}}$ by Ax V.

Remark: This is the only place we use Ax V, VI (except in 2.7 which is not used later); and it is clear that we can omit Ax V if we strengthen somewhat Ax VI.

1.8 Lemma: 1) There is a set Γ of types in L_1 (from Lemma 1.7) such that $K = PC(\emptyset, \Gamma, L)$. Moreover if $M_1 <_{L_{\omega,\omega}} N_1 \in EC(\emptyset, \Gamma)$, M, N the L-reducts of M_1, N_1 resp. then $M <_K N$.

2) Similar results hold for $\{(M, N) : N < M \,;\, N, M \in K\}$.

Proof: 1) Let Γ_n be the set of complete quantifier free n-types in L_1, $p(x_0, \ldots, x_{n-1})$, such that: if M_1 is an L_1-model, $\bar{a}$ realizes p in M_1 and M is the L-reduct of M_1, then $M_{\bar{b}} \prec_K M_{\bar{a}}$ for any subsequence $\bar{b}$ of $\bar{a}$; where $M_{\bar{c}}(\bar{c} \in {}^m|M_1|)$ is the submodel of M whose universe is $\{F_i^m(\bar{c}) : i < \lambda(K)\}$ (and there are such submodels).

Let Γ be the set of p which are complete quantifier free n-types (in L_1) which do not belong to Γ_n for some $n < \omega$.

By 1.6 $PC_t(\emptyset,\Gamma,L) \subseteq K$ and by 1.7 $K \subseteq PC(\emptyset,P,L)$

2) Applying 1.7 to M, when $N \prec M$ w.l.o.g. $\bar{a} \in {}^nN \Rightarrow M_a \subseteq N$.

We let Γ'_n be the set of complete quantifier free n-types in $L'_1 \stackrel{def}{=} L_1 \cup \{P\}$ (P a monadic predicate), $p(x_0, \ldots, x_{n-1})$ such that:

(*) (α) if M_1 is an L_1-model, $\bar{a}$ realizes p in M_1, M the L-reduct of M_1, *then* $M_{\bar{b}} \prec_K M_{\bar{a}}$ for any subsequence $\bar{b}$ of $\bar{a}$ where $M_{\bar{c}}$ ($\bar{c} \in |M_1|$) is the submodel of M whose universe is $\{F_i^m(\bar{c}) : i < \lambda(K)\}$, (and there are such models),

(β) $\bar{b} \subseteq P^{M_1} \Rightarrow M_{\bar{b}} \subseteq P$ for $\bar{b} \subseteq \bar{a}$.

We leave the rest to the reader.

1.9 Conclusion: There is $L_1, L \subseteq L_1$, $|L_1| \leq \lambda(K)$ such that: for any $M \in K$ and any L_1-expansion M_1 of M which is in $PC_t(\emptyset,\Gamma)$,

$$N_1 \prec_{L_{\omega,\omega}} M_1 \Rightarrow N_1 \restriction L \prec_K M$$

$$N_1 \prec_{L_{\omega,\omega}} N_2 \prec_{L_{\omega,\omega}} M_1 \Rightarrow N_1 \restriction L \prec_K N_2 \restriction L.$$

1.10 Conclusion: If for every $\alpha < (2^{\lambda(K)})^+$ K has a model of cardinality $\geq \beth_\alpha$ *then* K has a model in every cardinality $\geq \lambda(K)$.

Proof. Use 1.8 and the value of the Hanf number for: first order theory and omitting any set of types, for languages of cardinality $\lambda(K)$.

§2 Amalgamation Properties and Homogeneity

2.1 Definition: $\mathcal{D}(N) = \{N/\cong : N \prec M,\ |||N||| \le \lambda(K)\}$.

$\mathcal{D}(K) = \{N/\cong : N \in K,\ |||N||| \le \lambda(K)\}$.

2.2 Definition: Let $\lambda > \lambda(K)$

(1) A model M is λ-model homogeneous *if* whenever $N_0 \prec N_1 \prec M$, $|||N_1||| < \lambda$, f an $\prec$-embedding of N_0 into M, *then* some $\prec$-embedding $f' : N_1 \to M$ extend f.

(2) M is λ-strongly model homogeneous if: for every $N \in K_{<\lambda}$ such that $N \prec M$ and $f : N \to M$ is a K-embedding there exist a K-automorphism $g : M \to M$ extending f.

(3) M is λ-*saturated homogeneous* if for every $N_\ell \in K_{<\lambda} (\ell = 0,1)$ such that $N_0 \prec N_1$ if there exists an K-embedding $f : N_0 \to M$ *then* there exists an K-embedding $g : N_1 \to M$ extending f.

4) For each of the above three properties, if M has power λ and has the λ-property then we say for short that M has the property (i.e. omitting λ). (saturated homogeneous is usually called universal homogeneous).

5) M is (D,λ)-sequence-homogeneous if: $D = \{tp(\bar{a},\emptyset, M): \bar{a} \in |M|\}$ and if $a_i \in M$ for $i \le \alpha < \lambda$, $b_j \in M$ for $j < \alpha$ and $tp(\langle a_i : i < \alpha\rangle,\emptyset,M) = tp(\langle b_i : i < \alpha\rangle,\emptyset, M)$, *then* for some $b_\alpha \in M$, $tp(\langle a_i : i \le \alpha\rangle,\emptyset, M) = tp(\langle b_i : i \le \alpha\rangle,\emptyset,M)$. We omit the "sequence" sometimes.

2.3 Theorem: Assume N is λ-model-homogeneous, $\mathcal{D}(M) \subseteq \mathcal{D}(N)$, ($\lambda(K) < \lambda$ of course). Then

(1) If $M_0 \prec M_1 \prec M$, $|||M_0||| < \lambda$, $|||M_1||| \le \lambda$, f a K-embedding of M_0 into N, *then* we can extend f to a K-embedding of M_1 into N.

(2) If $M_1 \prec M$, $|||M_1||| \le \lambda$ then there is a K-embedding of M_1 into N.

Proof: We prove by induction $\mu \le \lambda$ that

(i)$_\lambda$ for every $M_1 \prec M$, $|||M_1||| \le \mu$ there is a K- embedding into N.

(ii)$_\lambda$ if $M_0 \prec M_1 \prec M$, $|||M_1||| \le \mu$, $|||M_0||| < \mu$ then any K-embedding of M_0 into N can be extended to a K-embedding of M into N.

Then clearly (i)$_\lambda$ is (2) and (ii)$_\lambda$ is (1).

Proof of (i)$_\lambda$:

If $\mu \le \lambda(K)$, this follows by $\mathcal{D}(M) \subseteq \mathcal{D}(N)$.

If $\mu > \lambda(K)$ then by 1.9 $M_1 = \bigcup_{\alpha<\mu} M_1^\alpha$, $M_1^\alpha < M_1$, M_1^α is $<$-increasing and continuous, and $|||M_1^\alpha||| < \mu$. We define by induction on α, a K-embedding $f_\alpha : M_1^\alpha \to N$, such that for $\beta < \alpha$, f_α extend f_β. We can define f_0 by $(i)_{\chi(\beta)}$ where $\chi(\beta) = |||M_1^\beta|||$. We then define f_α for $\alpha = \gamma+1$: by $(i)_{\chi(\alpha)}$ there is an K-embedding g_α of M_1^α) is a K-embedding of $M_{1,b}^\alpha$) is a K-embedding of $M_{1,b}^\alpha$ into N; Now let $M_{1,a}^\alpha = g_\alpha(M_1^\alpha)$, $M_{1,b}^\alpha = g_\alpha(M_1^\gamma)$, so $(f_\gamma \circ g_\alpha^{-1}$ into N, so there is a K- embedding h_α of $M_{1,a}^\alpha$ into N extending $(f_\gamma \circ g_\alpha^{-1})$. Now $(h_\alpha \circ g_\alpha)$ is a K-embedding of M_1^α into N extending f_γ, as required, and for limit α, $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$, f_α is a K-embedding into N by 1.6. So we finish the induction and $\bigcup_{\alpha<\mu} f_\alpha$ is as required.

Proof of (ii)$_\mu$:

Let g be a K-embedding of M_1 into N, it exists by $(i)_\mu$ which we have just proved. Let g be onto $N'_1 < N$, and let $g \restriction M_0$ be onto $N'_0 < N'_1$, and let f be onto $N_0 < N$. So clearly $h : N'_0 \to N_0$ define by $hg(a) = f(a)$ for $a \in |M_0|$ is an isomorphism from N'_0 onto N_0. As $N_0, N'_0, N'_1 < N$, *if* $|||N'_1||| < \lambda$, we can extend h to an isomorphism h' from N'_1 onto $N_1 < N$, so $hg : M_1 \to N$, where $h_0 = h$; and let $h' = h_\lambda$.

2.4 Conclusion: (1) If M, N are model-homogeneous, of the same cardinality and $\mathcal{D}(M) = \mathcal{D}(N)$ *then* M, N are isomorphic. Moreover if $M_0 < M$, $|||M_0||| < |||M|||$, *then* any K-embedding of M_0 into N can be extended to an isomorphism from M onto N.

(2) The number of model homogeneous models of cardinality λ is $\leq 2^{2^{\lambda(K)}}$.

(3) If M is λ-model-homogeneous, $\mathcal{D}(M) = \mathcal{D}(K)$ *then* M is λ-universal, i.e., every model M (in K) of cardinality $\leq \lambda$, has a K-embedding into M.

(4) If M is λ-model-homogeneous then it is λ-saturated homogeneous for $\{M \in K_{\leq\lambda} : \mathcal{D}(N) \subseteq \mathcal{D}(M)\}$.

(5) If M is λ-model homogeneous, $\mathcal{D}(M) = \mathcal{D}(K)$ *then* M is λ-saturated homogeneous for K.

Proof: (1) Immediate by 2.3(2), using the standard hence-and-forth argument.

(2) The number of models (in K) of power $\leq \lambda(K)$ is, up to isomorphism, $\leq 2^{\lambda(K)}$. Hence the number of possible $\mathcal{D}(M)$ is $\leq 2^{2^{\lambda(K)}}$. So by 2.4(1) we finish.

(3),(4),(5) Immediate.

2.5 Definition: (1) A model M has the (λ,μ)- amalgamation property (= am. p.) if: *for every* M_1, M_2 such that $\|M_1\| = \lambda$, $\|M_2\| = \mu$, $M < M_1$, $M < M_2$, *there* is a model N, and K-embedding $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$, such that $f_1 \restriction |M| = f_2 \restriction |M|$. Now the meaning of e.g. the $(\leq\lambda, <\mu)$- amalgamation property is clear. Always $\lambda,\mu \geq \lambda(K)$.

(2) K has the (κ,λ,μ)-amalgamation property *if* every model M (in K) of cardinality κ has the (λ,μ)- amalgamation property. The (κ,λ)-amalgamation property is just the (κ,κ,λ)-amalgamation property. The κ- amalgamation property is just the (κ,κ,κ)-amalgamation property.

(3) K the (λ,μ)-JEP (joint embedding property) *if* for any $M_1 \in K$, $M_2 \in K$ of cardinality λ,μ respectively there is $N \in K$ into which M_1 and M_2 are K-embeddable.

4) λ-JEP is the (λ,λ)-JEP.

5) The amalgamation property *means* the (κ,λ,μ)-amalgamation property for every $\lambda,\mu \geq \kappa$ $(\geq \lambda(K))$.

6) The JEP means the (λ,μ)-JEP for every λ,μ.

Remark: Clearly in all cases, the roles of λ,μ are symmetric.

2.6 Theorem: 1) If $\lambda(K) < \kappa \leq \lambda$, $\lambda = \lambda^{<\kappa}$, $K_\lambda \neq \emptyset$, and K has the $(<\kappa,\lambda)$-amalgamation property *then* for every model M of cardinality λ, there is a κ-model homogeneous model N of cardinality λ, $M < N$.

2) So in (1) if $\kappa = \lambda$, there is a universal, homogeneous model of cardinality λ, provided that for some $M \in K_{\leq\lambda}$, $\mathcal{D}(M) = \mathcal{D}(K)$.

2.6A Remark: 1) The last assumption of 2.6(2) holds e.g. if $(\leq\lambda(K), < 2^{\lambda(K)})$-JEP holds and $|D(K)| \leq \lambda$

2) If for some $M \in K$, $\mathcal{D}(M) = \mathcal{D}(K)$ *then* we can have such M of power $\leq 2^{\lambda(K)}$.

3) We can 2.6 replace the assumption "$(<\kappa,\lambda)$-amalgamation property" by "$(<\kappa,<\lambda)$-amalgamation property" *if* e.g. no $M \in K_{<\lambda}$ is maximal.

Proof: Immediate.

2.6B Remark: Also the corresponding converses holds.

2.7 Lemma: (1) If K has the κ-amalgamation property then K has the (κ,κ^+)-amalgamation property and even the $(\kappa,\kappa^+,\kappa^+)$-amalgamation property.

(2) If λ_i $(i \leq \alpha)$ is increasing and continuous, $\lambda(K) \leq \lambda_0$, and for every $i < \alpha$, K has the $(\lambda_i,\mu + \lambda_i,\lambda_{i+1})$-amalgamation property then K has the $(\lambda_0,\mu+\lambda_\alpha,\lambda_\alpha)$-amalgamation property.

(3) If $\kappa \leq \mu \leq \lambda$, K has the (κ,μ,μ)- amalgamation property and the (μ,λ)-amalgamation property then K has the (κ,λ,μ)-amalgamation property.

(4) If $\kappa \leq \mu_1 \leq \mu$, and for every M, $\|M\| = \mu_1$, there is N, $M \prec N$, $\|N\| = \mu$, *then* the (κ,μ,λ)-amalgamation property (for K) implies the (κ,μ_1,λ)-amalgamation property of M.

Proof: Straightforward.

2.8 Conclusion: If K has the κ-amalgamation property for every $\chi_1 \leq \kappa \leq \chi_2$ then K has the (κ,λ,μ)-amalgamation property whenever $\chi_1 \leq \kappa \leq \lambda \leq \chi_2$, $\kappa \leq \mu \leq \chi_2$.

* * *

It may be interesting to note that even waiving AX IV we can say something.

2.9 Context: For the remainder of this section, Ax IV is not assumed.

2.10 Definition: Let $M \in K$ have power λ, a regular uncountable cardinal. We say M is *smooth* if there are $\langle M_i : i < \lambda\rangle$, M_i is increasing continuous $M_i \prec M$, $\|M_i\| < \lambda$, $M = \bigcup_{i<\lambda} M_i$.

2.10A Remark: We can define S/D-smooth, for S a subset $\mathcal{P}(\lambda)$, D a filter on $\mathcal{P}(\lambda)$

(naturally such that for every one- to-one function from λ to λ, $\{a \in \mathcal{P}(\lambda)$: a closed under $f\} \in D$, and usually a normal $\lambda(K)^+$- complete filter)

2.11 Lemma : If $M, N \in K_\lambda$ $(\lambda > \lambda(K))$ are smooth, model homogeneous, and $\mathcal{D}(M) = \mathcal{D}(N)$ *then* $M \cong N$.

2.11A Remark: It is reasonable to consider

(*) If $M \in K_\lambda$, $(\lambda > \lambda(K))$ is smooth and model homogeneous, and $N \in K_\lambda$ is smooth, $\mathcal{D}(N) \subseteq \mathcal{D}(M)$ *then* N can be K-embedded into M.

This can be proved in the context of "universal classes (e.g AxFr_1).

Proof: Left to the reader.

2.12 Fact: If $(K_i, \prec_i)$ satisfies the axioms with $\lambda_i = \lambda_i(K_i, \prec_i)(\geq \aleph_0)$ for $i < \alpha$, $K = \bigcap_{i<\alpha} K_i$ and $\prec$ is defined by $M \prec N$ *if and only if* for $i < \alpha$, $M \prec_i N$, *then* $(K, \prec)$ satisfies the axioms with $\lambda(K, \prec) \leq \sum_{i<\alpha} \lambda_i$. We can add Ax IV (to assumption and conclusion).

Proof: Easy.

§3 Limit Models and Other Results

In this section we introduce various variants of limit models. We prove that if K has a superlimit model M^* of power λ for which the λ-amalgamation property fails, $2^\lambda < 2^{\lambda^+}$ *then* $I(\lambda,K) = 2^\lambda$ (see 3.5). We then prove a generalization of: if $\psi \in L_{\omega_1,\omega}(Q)$ is categorical in $\aleph_1$ then it has model in $\aleph_2$. (see 3.7, 3.8). Now that.

The reader can read 3.1(1), ignore the other definitions, and continue with 3.4(2),(5), and everything from 3.5, (interpretating all variants as superlimits).

Example: Let λ have cofinality $\geq \aleph_1$, then

$K = \{(A,<) : (A,<)$ a well order of order type $\leq \lambda^+\}$
$<_K = \{(M, N) : M, N \in K, N$ an end extension of $M\}$

is an abstract elementary class, categorical in λ^+.

Note that if we are dealing with classes which are categorical or simple in some sense, we have a good chance to find limit models, and they are useful in constructions.

3.1 Definition: Let λ be a cardinal $\geq \lambda(K)$.

1) $M \in K_\lambda$ is *superlimit* if

(a) *for every* $N \in K_\lambda$ such that $M < N$ *there* is $M' \in K_\lambda, N < M'$ and $N \neq M'$.

(b) if $\delta < \lambda^+$ is limit $\langle M_i : i < \delta\rangle$ is $<$-increasing, and (for $i < \delta$) $M_i \cong M$ *then* $\bigcup_{i<\delta} M_i \cong M$.

2) For $S \subseteq \{\mu : \aleph_0 \leq \mu \leq \lambda, \mu$ regular$\}$, $M \in K_\lambda$ in (λ,S)-superlimit if:

(a) from above holds and

(b) $M_i \cong M$ is $(<-)$ increasing for $i < \mu \in S$ *then* $\bigcup M_i \cong M$.

3) Let $S \subseteq \lambda^+$ be stationary. We call $M(\in K_\lambda)$ *S-strongly limit* if for some function $F : K_\lambda \to K_\lambda$:

(a) for $N \in K_\lambda, N \prec F(N), N \neq F(N)$

(b) if $\delta \in S$, $\langle M_i : i < \delta \rangle$ is an increasing continuous sequence of members of K_λ, $M_0 \cong M, F(M_{i+1}) \prec M_{i+2}$ then $M \cong \bigcup_{i<\delta} M_i$

4) Let $S \subseteq \lambda^+$ be stationary. We call $M\ (\in K_\lambda)$ *S-limit* if for some function $F : K_\lambda \to K_\lambda$:

(a) for $N \in K_\lambda, N \prec F(N), N \neq F(N)$

(b) if $\langle M_i : i < \lambda^+ \rangle$ is an increasing continuous sequence of members of K_λ, $M_0 \cong M, F(M_{i+1}) \prec M_{i+2}$ *then* for a closed unbounded subset C of λ^+,

$$[\delta \in S \cap C \Rightarrow M_\delta \cong M]$$

5) We define "S-weakly limit", "S-medium limit" like "S-limit", "S-strongly limit" reps. replacing "$F(M_{i+1}) \prec M_{i+2}$" by "$M_{i+1} \neq F(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$".

6) If $S = \lambda^+$ then we omit S (in parts (3),(4),(5)). We call M weakly limit if it is S-weakly limit for a dense family of stationary $S \subseteq \|M\|^+$.

7) For $S \subseteq \{\mu : \aleph_0 \leq \mu \leq \lambda,\ \mu \text{ regular}\}$, M is (λ,S)-strongly-limit *if* M is $\{\delta < \lambda^+ : cf\, \delta \in S\}$-strongly-limit. Similarly for the other motions.

Obvious Implication Diagram

(where $S \subseteq \{\mu : \mu \text{ regular } \leq\lambda\}$ $S_1 \subseteq \{\delta < \lambda^+ : cf\,\delta \in S\}$ is a stationary subset of λ^+)

superlimit = $(\lambda, \{\mu : \mu\leq\lambda \text{ regular}\})$-superlimit

(λ, S)-superlimit

S_1-strongly limit

S_1-medium limit $\qquad$ S_1-limit

S_1-weakly limit

$\{\mu{:}\mu\leq\lambda \text{ regular}\}$-strongly limit

weakly limit

3.2 Lemma: 1) If $S_i \subseteq \lambda^+$, for $i < \lambda^+$, $S = \{\alpha < \lambda^+ : (\exists i < \alpha)\alpha \in S_i\}$, $S_i \cap i = \emptyset$ for $i < \lambda$ *then:* M is S_i-strongly limit for each $i< \lambda$ *if and only if* M is S-strongly limit.

2) Suppose $\kappa \leq \lambda$ is regular and $S \subseteq \{\delta < \lambda^+ \colon cf\,\delta = \kappa\}$ is a stationary set and $M \in K_\lambda$ *then* the following are equivalent:

a) M is S-strongly limit

b) M is $(\lambda, \{\kappa\})$-strongly limit

c) there is a function $F : K_\lambda \to K_\lambda$, $(\forall N \in K_\lambda)$ $[N <_K F(N) \wedge N \neq F(N)]$ such that if $M_i \in K_\lambda$ for $i < \kappa$, $[i < j \Rightarrow M_i < M_j]$, $F(M_{i+1}) < M_{i+2}$, $M_0 \cong M$ *then* $\bigcup_{i<\kappa} M_i \cong M$.

3) In part (1) we can replace "strongly limit" by "limit", "medium limit" and "weakly limit".

4) Suppose $\kappa \leq \lambda$ is regular, $S \subseteq \{\delta < \lambda^+ : cf\,\delta = \kappa\}$ is a stationary set which is good (see below in the appendix and [Sh6]), and $M \in K_\lambda$.

The following are equivalent:

a) M is S-medium limit

b) there is a function from $\bigcup_{\alpha<\kappa}{}^{\alpha}(K_\lambda)$ to K such that

(α) for any $\langle M_i : i \leq \alpha \rangle$, $M_0 = M$, *if* $\alpha < \kappa$, M_i is $\prec$-increasing, $M_i \in K_\lambda$, then $M_\alpha \prec F(\langle M_i : i \leq \alpha \rangle)$

(β) if $\langle M_i : i < \kappa \rangle$ is $\prec$- increasing, $M_0 = M$, $M_i \in K_\lambda$, and for $i < \kappa$, $M_{i+1} \prec F(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$ *then* $\bigcup_{i<\kappa} M_i \cong M$.

3.3 Lemma: Let T be a first order complete theory, K its class of models, $\prec_K = \prec_{L_{\omega,\omega}}$.

1) If λ is regular, M a saturated model of T of power λ, *then* M is $(\lambda, \{\lambda\})$-super limit.

2) If λ is singular, M a special model of T of power λ (i.e. $M = \bigcup_{i<cf\,\lambda} M_i$, M_i is λ_i-saturated, $\langle M_i : i < cf\,\lambda \rangle$ increasing, $\lambda = \sum_{i<cf\,\lambda} \lambda_i$) *then* M is $(\lambda, \{cf\,\lambda\})$-strongly limit.

3) If T is stable, and M a saturated model of T of cardinality λ *then* M is $(\lambda, \{\mu : \kappa(T) \leq \mu \leq \lambda, \mu \text{ regular}\})$-superlimit (on $\kappa(T)$-see [Sh3, III §3]). (note that by [Sh3] if λ is singular and T has a saturated model of cardinality λ then T is stable, $cf\,\lambda \geq \kappa(T)$).

4) If T stable, λ singular $> \kappa(T)$, M a special model of T of power λ, $S \subseteq \{\delta < \lambda^+ : cf\,\delta \geq \kappa(T)\}$ is good (see [Sh6] or appendix) *then* M is (λ, S)-strongly limit.

Proof: 1) If M_i is a λ-saturated model of T for $i < \delta$, $cf\,\delta \geq \lambda$ *then* $\bigcup_{i<\delta} M_i$ is λ-saturated. Remembering the uniqueness of a λ-saturated model of T of power λ we finish.

2) We use the (well known) uniqueness of the special model. Note that an increasing union of special models of length $cf\,\lambda$ seem not to be necessary special, however if: for $i < cf\,\lambda$, M_i is a model of T of power λ, $M_i = \bigcup_{\xi<cf\,\lambda} M_{i,\xi}$, $M_{i,\xi}$ increasing in ξ, $M_{i,\xi}$ λ_i-saturated and $\lambda = \sum_{i<cf\,\lambda} \lambda_i$, and $(\forall i<j<cf\,\lambda)\ (\exists \xi < cf\,\lambda)(\forall \zeta)[\xi \leq \zeta < cf\,\lambda \Rightarrow M_{i,\zeta} \subseteq M_{j,\xi}]$ *then* $\bigcup_{i<cf\,\lambda} M_i$ is special.

3) Use [Sh3 , III 3.11]: if M_i is a λ-saturated model of T,, $\langle M_i : i < \delta \rangle$ increasing

$cf\ \delta \geq \kappa(T)$ *then* $\bigcup_{i<\delta} M_i$ is λ-saturated.

4) Left to the reader.

3.4 Claim: 1) If $M_\ell \in K_\lambda$ are S_ℓ- weakly limit, $S_0 \cap S_1 \neq \emptyset\ mod\ D_{\lambda^+}$ then $M_0 \cong M_1$, or M_0, M_1 cannot be embedded into one model.

2) K has at most one weakly limit model of cardinality λ provided K has (λ,λ)-JEP.

3) If $M \in K_\lambda$, then $\{S \subseteq \lambda^+ : M$ is S-weakly limit or S not stationary$\}$ is a normal ideal over λ^+.

Instead "S-weakly" limit "S-medium limit", "S-limit" "S-strongly limit" can be used.

4) In Definition 3.1 w.l.o.g. $F(M) \cong M$ or $F(\overline{M}) \cong M$).

5) If K is categorical in λ, *then* the $M \in K_\lambda$ is superlimit limit provided that $K_{\lambda^+} \neq \emptyset$ (or, what is equivalent, M has a proper K-extension).

Theorem 3.5: If $M \in K_\lambda$ is S-weakly limit, S is not small (see [DSh]) and M does not have the λ- amalgamation property *then* $I(\lambda^+, K) = 2^{\lambda^+}$, and there is no universal member in K_{λ^+}. Also there are 2^{λ^+} models $M \in K_{\lambda^+}$ no one K-embeddable into another.

Remark: 1) By [DSh], and see more [Sh 7 ,Ch XIV §1] if $2^\lambda < 2^{\lambda^+}$ *then* $S = \lambda^+$ is not small.

2) We can define a limit family of models (i.e. the result should be in the family). But then the family should satisfy that any member does not have the amalgamation property. But this complicated the situation, and the gain is not clear, so we abandon this.

Remark: A subsequent work is Grossberg and Shelah [GSh1]. Now we work in a certain framework. There the framework is changed and a full proof appears.

Proof: Similar to [Sh2] 2.7, 6.3.

We can define by induction on $\alpha < \lambda^+$, models M_η for $\eta \in {}^\alpha 2$ such that:

(i) $M_\eta \in K_\lambda, M_{<>} = M$,

(ii) for $\beta < \alpha, \eta \in {}^\alpha 2, M_{\eta \restriction \beta} \prec M_\eta$.

(iii) for $i+2 \leq \alpha$, $(F \mid \langle M_{\eta \restriction j} : j \leq i+1 \rangle) \prec M_{i+2}$ (F from Definition 3.1(5)).

(iv) if $\alpha = \beta+1$, β non limit, $\eta \in {}^\alpha 2$, then $M_{\eta \restriction \beta} \neq M_\eta$.

(v) if α is limit $\eta \in {}^\alpha 2$ *then:* if M_η fail the λ-amalgamation property then $M_{\eta \hat{} \langle 0 \rangle}, M_{<\eta \hat{} \langle 1 \rangle>}$ cannot be amalgamated over M_η, i.e. for no N, $M_\eta \prec N \in K$, and $M_{\eta \hat{} \langle 0 \rangle}$, $M_{\eta \hat{} \langle 1 \rangle}$ can be K-embedded into N over M_η.

For $\alpha = 0$, α limit, we have no problem, for $\alpha+1$, α-limit: if M_η fail the λ-amalgamation property - use its definition, otherwise let $M_{\eta \hat{} \langle 1 \rangle} = M_\eta$; for $\alpha+1$, α non limit- use F.

Let for $\eta \in {}^{\lambda^+}2$, $M_\eta = \bigcup_{\alpha < \lambda^+} M_{\eta \restriction \alpha}$. By changing names we can assume that for $\eta \in {}^\alpha 2 (\alpha < \lambda)$ the universe of M_η is an ordinal $< \lambda^+$ (or even $\subseteq \lambda \times (1+\ell(\eta))$ and we could even demand equality). So (by (iv), for $\eta \in {}^{\lambda^+} 2$, M_η has universe λ^+.

First why is there no universal member in K_λ? if $N \in K_\lambda$ is universal, w.l.o.g. its universe is λ^+. For $\eta \in {}^{\lambda^+}2$ as $M_\eta \in K_{\lambda^+}$, there is a K-embedding f_η of M_η into N. So f_η is a function from λ^+ to λ^+. Let $\eta \in {}^{\lambda^+}2$, by the choice of F and of $\langle M_{\eta \restriction \alpha} : \alpha < \lambda^+ \rangle$ there is a closed unbounded $C_\eta \subseteq \lambda^+$ such that for $\alpha \in C_\eta$ $M_{\eta \restriction \alpha} \cong M$, hence $M_{\eta \restriction \alpha}$ fail the λ-amalgamation property. W.l.o.g. for $\delta \in C_\eta$, $M_{\eta \restriction \delta}$ has universe δ. Now by [DSh], if for $\eta \in {}^{\lambda^+}2$, $f_\eta : \lambda^+ \to \lambda^+$, $C_\eta \subseteq \lambda^+$ closed unbounded *then* for some $\eta \neq \nu \in {}^{\lambda^+}2$, and $\delta \in C_\eta \cap S$, $\eta \restriction \delta = \nu \restriction \delta$, $\eta(\delta) \neq \nu(\delta)$ and $f_\eta \restriction \delta = f_\nu \restriction \delta$.

Now as $\delta \in S \cap C_\eta$, $M_{\eta \restriction \delta} \cong M$ hence fail λ-amalgamation property. Also $M_{\eta \restriction \delta}$ has universe δ as $\delta \in C_\eta$.

So $f_\eta \restriction M_{\eta \restriction \delta} = f_\eta \restriction \delta = f_\nu \restriction \delta = f_\nu \restriction M_{\nu \restriction \delta}$. So $f_\eta \restriction M_{\eta \restriction (\delta+1)}$, $f_\nu \restriction M_{\nu \restriction (\delta+1)}$ show that $M_{\eta \restriction (\delta+1)}, M_{\nu \restriction (\delta+1)}$, can be amalgamated over $M_{\eta \restriction \delta}$ contradicting (v) of the construction.

It takes some more effort to get 2^λ pairwise non isomorphic model.

Case A: There is $M^* \in K_\lambda$, $M < M^*$ such that for every N, $M^* < N \in K_\lambda$ there are $N^1, N^2 \in K_\lambda, N < N^1, N < N^2$ and N^2, N^2 cannot be amalgamated over M^* (not just N).

In this case we do not need "M is S-weakly limit".

We redefine M_η, $\eta \in {}^\alpha 2$, $\alpha < \lambda^+$: if $\alpha = 0$ $M_{<>} = M^*$; *if* α limit $\eta \in {}^\alpha 2 : M_\eta = \bigcup_{\beta<\alpha} M_{\eta\restriction\beta}$; *if* $\eta \in {}^\beta 2$, $\alpha = \beta + 1$, use the assumption for $N = M_\eta$, now obviously $N^1 \neq N$, and $N^2 \neq N$, so we can define $M_\eta < M_{\eta\char`^<1>} \in K_\lambda$, $M_{\eta\char`^<1>} \neq M_\eta$, $M_{\eta\char`^<0>} \neq M_\eta$, such that $M_{\eta\char`^<0>}$ $M_{\eta\char`^<1>}$ cannot be amalgamated over M^*.

Now $2^\lambda < 2^{\lambda^+}$ (this is equivalent to "there is a non small $S \subseteq \lambda^+$"). Obviously, the $M_\eta = \bigcup_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$, for $\eta \in {}^{\lambda^+}2$ are pairwise non isomorphic over M^*, and by [Sh3, VIII 1.3] we finish.

Note also that for each $\eta \in {}^{\lambda^+}2$ the set $\{\nu \in {}^{\lambda^+}2$: M_ν can be K- embedded into $M_\eta\}$ has power $\leq |\{f : f$ a K- embedding of M^* into $M_\eta\}| \leq 2^\lambda$. So if $(2^\lambda)^+ < 2^{\lambda^+}$, then by Hajnal free subset theorem, there are 2^{λ^+} models $M_\eta \in K_{\lambda^+} (\eta \in {}^{\lambda^+}2)$ no one K-embeddable into another. If $(2^\lambda)^+ = 2^{\lambda^+}$ - then repeat the proof in case B below with the M_η's we have constructed here.

Case B: Not case A.

Now we return to the first construction, but we can add

(vi) if $\eta \in {}^{(\alpha+1)}2$, *then* if $M_\eta < N^1, N^2$ both in K_λ, *then* N^1, N^2 can be amalgamated over $M_{\eta\restriction\alpha}$.

As $\{W \subseteq \lambda^+ : W$ is small$\}$ is a normal ideal (see [DSh]) (and it is on a successor cardinal) it is well known that we can find λ^+ pairwise disjoint non small $S_\zeta \subseteq S$ for $\zeta < \lambda^+$. We define a function F:

$F(\eta,\nu,\beta)$ *is* one if for some limit $\delta < \lambda^+$, $\eta \in {}^\delta 2$, $\nu \in {}^\delta 2$, M_η, M_ν has universe δ, f is a K-embedding of M_η into M_ν, and for some ρ, $\nu\char`^<0> \not\leq \rho \in {}^{\lambda^+ \geq}2$, $f$ can be extended to a $K$-embedding of $M_{\eta\char`^<0>}$ into M_ρ

$F(\eta,\nu,f)$ is zero otherwise.

For each ζ, as S_ζ is not small, by simple coding, for every $\zeta < \lambda^+$ there is $h_\zeta : S_\zeta \to \{0,1\}$ such that:

(*) for every $\eta \in {}^{\lambda^+}2$, $\nu \in {}^{\lambda^+}2$, $f : \lambda^+ \to \lambda^+$, for a stationary set of $\delta \in S_\zeta$,

$$F(\eta \restriction \delta, \nu \restriction \delta, f \restriction \delta) = h_\zeta(\delta).$$

Now for every $W \subseteq \lambda^+$ we define $\eta_W \in {}^{\lambda^+}2$:

$\eta_W(\alpha)$ is $h_\zeta(\alpha)$, *if* $\zeta \in W$ (note that there is at most one ζ) *if* $\alpha \in S_\zeta$

$\eta_W(\alpha)$ is zero *if* there is no such ζ.

Now we can show (chasing the definitions) that for $W(1), W(2) \subseteq \lambda^+$, $W(1) \not\subseteq W(2)$, $M_{\eta_{W(1)}}$ cannot be K-embedded into $M_{\eta_{W(2)}}$. This clearly suffices.

Remark: We can many times (and in paraticular in 3.5) strengthen "there is not universal $M \in K_\lambda$" to "there is no $M \in K_\mu$ into which every $M \in K_{\lambda^+}$ can be K-embedded". We need $\neg Unif(\lambda^+, S, 2, \mu)$ (see [Sh 7, Ch XIV §1]).

Theorem 3.6: (1) Suppose K is PC_ω or has models of arbitrarily large cardinals and $I(\aleph_1, K) < 2^{\aleph_1}$. Then there is K_1 such that

A) $M \in K_1 \Rightarrow M \in K$, and $M <_{K_1} N \Rightarrow M <_K N$ and $\lambda(K_1) = \lambda(K)$.

B) If K has models of arbitrarily large cardinality then so does K_1.

C) K_1 is $PC_{\aleph_0}$.

D) If K is PC_ω, then $K_{\aleph_1} \neq \emptyset \Rightarrow (K_1)_{\aleph_1} \neq \emptyset$, Also $K_{\aleph_0} \neq \emptyset \Rightarrow (K_1)_{\aleph_0} \neq \emptyset$.

E) All models of K_1 are $L_{\infty,\omega}$-equivalent, and $M <_{K_1} N \Rightarrow M <_{\infty,\omega} N$ and if $K_{\aleph_0} \neq \emptyset$, K_1 is categorical in $\aleph_0$.

2) If in (1) we added $\lambda(K)$ names to formulas in $L_{\infty,\omega}$, we can assume each member of K is $\aleph_0$-homogeneous.

Proof: Like [Sh1] 1.1 (using 1.12 for $\alpha = 2$)

3.7 Theorem: Suppose

A) K has a super limit member M^* of cardinality λ, $(\lambda \geq \lambda(K))$ (if K is categorical in λ then by assumption B) there is such M; really λ^+-strong limit suffine).

B) K is categorical in λ^+.

C) K is PC_λ, $\lambda = \aleph_0$ or $\beth_\delta$, cf $\delta = \omega$ *or* $\lambda = \aleph_1$, K is $PC_{\aleph_0}$.

Then K has a model cardinality λ^{++}.

Remark: 1) If $\lambda = \aleph_0$ we can wave hypothesis (A) by the previous theorem.

2) Hypothesis C) can be weakened to:

(*) K is PC_μ, and any $\psi \in L_{\mu^+,\omega}$ which has a model M of order-type λ^+, $|P^M| = \lambda$, has a non-well-ordered model of N cardinality λ, $|P^M| = \lambda$, and *{*$M \in K_\lambda : M$ superlimit*}* is PC_μ (among models in K_λ) and similarly *{*$(M, F(M)) : M \in K_\lambda$*}*.

It is well known, see e.g. [Sh3] VII §5 why hypothesis C) implies this.

Proof: It is well known that the instances of (*) needed for (C) are true (see e.g. [Sh3, VII §5].

Stage a: If suffices to find $N_0 \prec N_1$, $|||N_0||| = \lambda^+$, $N_0 \neq N_1$.

We define by induction on $\alpha < \lambda^{++}$ a model $N_\alpha \in K_{\lambda^+}$ such that $\beta < \alpha$ implies $N_\beta \prec N_\alpha$ and $N_\beta \neq N_\alpha$, N_0, N_1 are defined [w.l.o.g. $|||N_1||| = \lambda^+$ as $\lambda \geq \lambda(K)$], for limit $\delta < \lambda^{++}$, $\bigcup_{\alpha<\delta} N_\alpha$ is as required. For $\alpha = \beta + 1$, by the λ^+-categoricity, N_0 is isomorphic to N_β, say by f, and we define $N_{\beta+1}$ such that f can be extended to an isomorphism from N_1 onto $N_{\beta+1}$, so clearly $N_{\beta+1}$ is as required. Now $\bigcup_{\alpha<\lambda^{++}} N_\alpha \in K_{\lambda^{++}}$.

Hence the following theorem completes the proof of 3.7 (use $F =$ the identiy).

3.8 Theorem: Suppose

(A) K has a λ^+-strongly limit member M^* cardinality λ, as exemplified by $F : K_\lambda \to K_\lambda$.

(B) $(I(\lambda^+, K_{\lambda^+}) < 2^{\lambda^+}$, or even just $I(\lambda^+, K^F_{\lambda^+}) < 2^{\lambda^+}$ (see below).

(C) K and $\{F(M), M) : M \in K_\lambda\}$ (and $\{(M, N) : N < N\}$ of course) are PC_μ, $\mu = \lambda = \aleph_0$, or $\mu = \lambda = \beth_\delta$, $cf\,\delta = \aleph_0$ or $\mu = \aleph_0, \lambda = \aleph_1$.

(D) If $\mu = \aleph_0$, $\lambda = \aleph_1$ then F and the superlimits in $K_{\aleph_1}$ are $PC_{\aleph_0}$.

Then we can find $N_0 < N_1$, $N_0 \neq N_1$ such that $N_0, N_1 \in K^F_{\lambda^+}$, where $K^F_{\lambda^+} = \{\bigcup M_i : M_i \in K_\lambda,\ \langle M_i : i < \lambda^+\rangle$ increasing continuous $M^* \cong M_{i+1} \neq M_i$, $F(M_{i+1}) < M_{i+2}\}$.

Remark: Theorem 3.8 is good for classes which are not exactly as required, see e.g. 3.9.

Proof of 3.8: (hence of 3.7). The reader may do well to read it with $F =$ the identity in mind.

Stage b: We now try to find N_0, N_1 as mentioned above by approximations of cardinality λ. A triple will denote here (M, N, a), $M, N \cong M^*$ (see hypothesis A), $N < M$, and $a \in M - N$. Let $<$ be the following order among triples: $(M, N, a) < (M', N', a')$ if $a = a'$, $N < N'$, $M < M'$, $N \neq N'$ and even $(\exists N^*)[N < N^* \wedge F(N^*) < N']$ $(\exists M^*)[M < M^* \wedge F(M^*) < N']$. (It is tempting to omit a and require $N = M \cap N'$, but this apparently does not work).

We first note there is at least one triple (as M^* has a proper elementary extension which is isomorphic to it, because it is a limit model).

Stage c: We show that if there is no maximal triple, our conclusion follows. First we omit F. We define by induction on α a triple (M_α, N_α, a) increasing by $<$. For $\alpha = 0$ see the end of lst stage, for $\alpha = \beta + 1$, we can define (M_α, N_α, a) by the hypothesis of this stage. For limit $\delta < \lambda^+$, (M_δ, N_δ, a) will be $(\bigcup_{\alpha<\delta} M_\alpha, \bigcup_{\alpha<\delta} N_\alpha, a)$ (notice $N_\delta \prec M_\delta$ by AX IV). Now similarly $N = \bigcup_{\alpha<\lambda^+} N_\alpha \prec M = \bigcup_{\alpha<\lambda^+} M_\alpha$, and the element a exemplifiers $M \neq N$, so by stage a we finish.

This suffice, and there is no problem to do this.

Stage d: There are $M_i \cong M^*$ for $i \leq \omega$ such that $[i < j \leq \omega \Rightarrow M_j^* \prec M_i^*]$, $F(M_i) \prec M_{i+1}$ and $|M_\omega| = \bigcap_{n<\omega} |M_n|$ and each M_i is superlmit.

Remark: See [Sh1, 2.3A].

Proof: As M^* as superlimit, there is an $\prec_K$- increasing continuous sequence $\langle M_i : i < \lambda^+ \rangle$, $M_i \cong M^*$, $M_i \cong M^*$ and $F(M_{i+1}) \prec M_{i+2}$. So w.l.o.g. $\bigcup_{i<\lambda^+} M_i$ has universe λ^+, M_0 has universe λ.

Define a model $\mathcal{A}$: its universe is λ^+.

Relations and Functions: a) those of $\bigcup_{i<\lambda^+} M_i$.

b) R. two place: a R i *if and only if* $a \in M_i$.

c) P (monadic relation) $P = \lambda$

c) g two place function such that for each i, $g(i,-)$ is an isomorphism from M_0 onto M_i.

d) < (two place relation - the usual ordering.

e) relations with parameter i witnessing $M_i \prec \bigcup_{j<\lambda^+} M_j$.

f) relations with parameters i witnessing each M_δ is superlimit, $F(M_{i+1}) \prec M_{i+2}$,

Let $\psi \in L_{\mu^+,\omega}$ describe this. So ψ has a non-well ordered model $\mathcal{A}^*$, $|P^{\mathcal{A}^*}| = \lambda$. So let

$$\mathcal{A}^* \models "a_{n+1} < a_n" \quad \text{for} \quad n < \omega.$$

Let for $a \in \mathcal{A}^*$, $A_a = \{x \in \mathcal{A}^* : \mathcal{A}^* \models xRa_n\}$

$$M_a = (\mathcal{A}^* \restriction L(K)) \restriction A_a.$$

Easily $M_a \prec \mathcal{A}^* \restriction L(K)$ (use (e)), $\|M_a\| = \lambda$. In fact M_a is superlimit.

So $M_{a_n} \prec \mathcal{A}^* \restriction L(K), M_{a_{n+1}} \subseteq M_{a_n}$ hence $M_{a_{n+1}} \prec_K M_{a_n}$. Let $M_n \stackrel{def}{=} M_{a_n}$.

Let $I = \{b \in \mathcal{A}^* : \bigwedge_{n<\omega} [\mathcal{A}^* \models b < a_n]\}$.

Also as for $b \in I$, $M_b < \mathcal{A}^* \restriction L(K)$, and $M_{b_1} \prec M_{b_2}$ for $b_1 <^{\mathcal{A}^*} b_2$, clearly $M_\omega \stackrel{def}{=} (\mathcal{A}^* \restriction (L(K)) \restriction \bigcup_{b \in I} A_b$ satisfies $M_\omega \prec \mathcal{A}^* \restriction L(K)$ hence $M_\omega \prec M_n$ for $n < \omega$. Obviously $M_\omega \subseteq \bigcap_{n<\omega} M_n$, and equality holds as ψ guarantee

(*) for every $y \in \mathcal{A}^*$ there is a minimal $x \in \mathcal{A}^*$ such that $y \in M_x$.

As each M_b is superlimit of cardinality λ, also M_ω is.

Stage e: Suppose there is a maximal triple, then we shall show $I(\lambda^+, K) = 2^{\lambda^+}$, and even $I(\lambda^+, K^F_{\lambda^+}) = 2^{\lambda^+}$, and so we shall get a contradiction.

So there is a maximal triple (M^0, N^0, a). So for each super limit $M \in K_\lambda$, there are $M', a, M \prec M'$, $a \in M' - M$, such that if $M'' \in K_\lambda, M' \prec M''$, and $N \in K_\lambda$, $(\exists N^+)(M \prec N^+ \wedge F(N^+) \prec M \neq N \wedge N \prec M'')$ then $a \in N$. (That is, in some sense a is algebraic over M). On the other hand by stage d for each super limit $M \in K_\lambda$, there are M'_n $(n < \omega)$ such that $M \prec M'_{n+1} \prec M'_n \in K_\lambda$, and $\bigcap_{n<\omega} M'_n = M$.

Now for each $S \subseteq \lambda^+$ we define by induction on $\alpha \leq \lambda^+$, M^S_α, increasing (by $\prec_K$) and continuous with universe an ordinal $< \lambda^+$. Let $M^S_0 = M^*$ and for limit $\delta < \lambda^+$, $M^S_\delta = \bigcup_{\alpha<\delta} M^S_\alpha$; by the induction assumption M^S_δ is limit, hence isomorphic to M^*. For $\alpha = \beta + 1$, β successor $M^S_\alpha = F(M^S_\beta)$. So we are left with the case $\alpha = \delta + 1$, δ limit (or zero).

Now if $\delta \in S$, choose $M_{\delta+1}$, a^S_δ such that $(M^S_{\delta+1}, M^S_\delta, a^S_\delta)$ is a maximal pair (possible as by the hypothesis of this case there is a maximal triple, and there is a unique limit model). If $\delta \neq S$ we choose $M^{S,n}_\delta \in K_\lambda$, $M^S_\delta \prec M^{S,n+1}_\delta \prec M^{S,n}_\delta$ for $n < \omega$ and $M^S_\delta = \bigcap_{n<\omega} M^{S,n}$; and let $M^S_{\delta+1} = M^{S,0}_\delta$, (again possible as $M_\delta \cong M^*$ and an assertion above).

Now clearly it suffices to prove that if $S^0, S^1 \subseteq \lambda^+$, $S^1 - S^0$, is stationary then $M^{S^1} \neq M^{S^0}$. Now suppose f is a K-embedding from M^{S^1} into M^{S^0}. Then $S^2 = \{\delta < \lambda^+ : M^{S^1_\delta}, M^{S^2_\delta}$ has universe δ, and for $i < \lambda^+ [i < \delta \Leftrightarrow f(i) < \delta]\}$ is a closed unbounded subset of λ^+, hence there is a limit $\delta \in (S^1 - S^0) \cap S^2$. Let us look at $f(a^{S^1}_\delta)$, as $\delta \in S^1$, $a^{S^1}_\delta \in M^{S^1}_{\delta+1} - M^{S^1}_\delta$, $f(a^{S^1}_\delta)$ belongs to $M^{S^0} - M^{S^0}_\delta$, but $M^{S^0}_\delta = \bigcap_{n<\omega} M^{S^0,n}_\delta$ (as $\delta \notin S^0$). Hence for some n $f(a^{S^1}_\delta) \notin M^{S^0,n}_\delta$. But then $M^{S^1}_\delta < f^{-1}(M^{S^0,n}_\delta) < M^{S^1}_\beta$ (for some large enough β), and $a^{S^1}_\delta \notin f^{-1}(M^{S^0,n}_\delta)$; contradicting the choice of $(M^{S^1}_{\delta+1}, M^{S^1}_\delta, a^{S^1}_\delta)$. If F is trivial, we finish. Otherwise

3.8A Observation: We have been innaccurate: we should consider $\{(M, N) : F(M) = N\}$ as a relation, closed under isomorphism and assume f is onto M^{S^0}.

3.9 Conclusion: 1) If $\lambda(K) = \aleph_0$, K is $PC_{\aleph_0}$ and $I(\aleph_1, K) = 1$ *then* K has a model of cardinality $\aleph_2$.

2) If $\psi \in L_{\omega_1,\omega}(Q)$ (Q is the quantifier "there are uncountably many") has one and only one model of power $\aleph_1$ up to isomorphism *then* ψ has a model in $\aleph_2$.

Proof: 1) By 3.6 we get suitable K_1 (as in the conclusion) and by 3.7 K_1 has a model in $\aleph_2$, hence K has a model in $\aleph_1$.

2) We can replace ψ by a countable theory $T \subseteq L_{\omega_1,\omega}(Q)$.

Let $\mathcal{L}$ be a fragment of $L_{\omega_1,\omega}(Q)$ in which T is included. W.l.o.g. T "says" that every formula of $\mathcal{L}$ is equivalent to a relation, and T is complete

$K = \{M : M$ an atomic $L(T)$-model of $T \cap L_{\omega,\omega}\}$.

$M <_K N$ *if* $M \in K$, $N \in K$ and if $\bar{a} \in M$, $b \in N - M$ $N \models R[b, \bar{a}]$, then for some P, $N \models P[\bar{a}]$ and $(\forall \bar{x})[P(\bar{x}) \equiv QyR(y, \bar{x})] \in T$. By [Sh1] w.l.o.g. K is categorical in $\aleph_0$.

Let F be such that for $M \in K_{\aleph_0}$, $N = F(M)$ means: *if* $\bar{a} \in M$, $M \models P[\bar{a}]$, $\forall \bar{x}[P(\bar{x}) \equiv QyR(y, \bar{x})] \in T$ *then* for some $b \in N - M$, $N \models R[b, \bar{a}]$.

Note that every $M \in K^F_{\aleph_1}$ is a model of ψ.

So 3.8 give that some $M \in K^F_{\aleph_1}$ has a proper extension in $K^F_{\aleph_1}$.

The rest should be easy.

Remark: Proving 3.9(2), we can get $M \in K_{\aleph_2}$, such that $M \models P[\bar{a}]$, $\forall x[P(\bar{x}) = QyR(y,\bar{x})] \in T$ then $\{b \in M : M \models R[b,\bar{a}]\}$ has cardinality $\aleph_2$. This is because in the proof of 3.8 we show that no triple is maximal.

Problem: If K is PC_λ, K categorical in λ, and λ^+, does it necessarily have a model in λ^{++}?

Remark: The problem is proving (*).

§4 Forcing and Categoricity

The main aim in this section is, for K as in §1, what we can deduce from $I(\aleph_1, K) < 2^{\aleph_1}$, first without assuming $2^{\aleph_0} < 2^{\aleph_1}$.

We can build a model of power $\aleph_1$ by an ω_1 sequence of countable approximations. There are models which are the union of quite generic sequence $\langle N_i : i < \omega_1 \rangle$ ($\prec$- increasing) of countable models, so it is natural to look at them (e.g. if K is cateogorical in $\aleph_1$, every model in $K_{\aleph_1}$ is like that). More exactly, we look at countable models and figure out properties of the quite generic models in $K_{\aleph_1}$. The main results are 4.8(a), (f).

4.1 Definition: For λ and $N_0 \in K_{<\lambda}$ let

1) $L^0_{\mu,\kappa}$ be first order logic enriched by conjunctions (and disjunctions) of length $< \mu$, homogeneous strings of existential quantifiers or of universal quantifiers of length $< \kappa$, and the cardinality quantifiers $\exists^{\geq\lambda}$ (denoted also by Q). But we apply those operations such that any formula has $< \kappa$ free variables, and the non logical symbols are from $L(K)$.

2) $\underset{\sim}{L}(N_0, A_i; A)_{i<\alpha}$ is the language, with the logic $\underset{\sim}{L}$, and with the non-logical symbols of $L(K)$, the predicates $x \in N_0$, $x \in A_i$, and the individual constants, a, $a \in A$. (If we omit N_0, or A, or A_i it is omitted here, so $L_{\omega,\omega}(\,)$ has the language $L(K)$.

3) $\underset{\sim}{L}^1_{\mu,\kappa}$ is as in 1), but we have also variables (and quantification) over relations of cardinality $< \lambda$.

4) $(N, N_0, A_i; A)_{i<\alpha}$ is the model N expanded by monadic predicates for N_0, $A_i (i < \alpha)$ and individual constants for every $c \in A$. For N_0 we use the predicate P, so we may write $\underset{\sim}{L}(P)$ instead $\underset{\sim}{L}(N_0)$, but writing $\underset{\sim}{L}(N_0)$ we fix the interpretation of P.

4.2 Definition:

1) For $N \in K_{<\lambda}$, $\varphi(x_0,\ldots) \in L^1_{\mu,\kappa}(N)$ we define by induction on φ when $N_0 \Vdash^{\lambda}_{K} \varphi[a_0,\ldots]$ (where $N \prec N_0 \in K_{<\lambda}$, $a_0,\ldots$ are elements of N_0, or appropriate relations over it, depending on the kind of x_i) (thus clearly the forcing is define for weaker languages such as

$L^0_{\mu,\kappa}(N_0,A_i;A_0)$, when $|A_i| < \lambda$.

For φ atomic this means $N_0 \models \varphi[a_0,...]$.

For $\varphi = \bigwedge_i \varphi_i$ this means

$$N_0 \Vdash^{\lambda}_{K} \varphi_i[a_0,...] \text{ for each } i.$$

For $\varphi = \exists \bar{x}\psi(\bar{x},a_0,...)$ this means for every N_1, $N_0 < N_1 \in K_{<\lambda}$ there is N_2, $N_1 < N_2 \in K_{<\lambda}$ and $\bar{b}$ from N_2 of the appropriate kind such that $N_2 \Vdash^{\lambda}_{K}\psi[\bar{b},\bar{a}]$.

For $\varphi = \neg\psi$ this means for no $N_1, N_0 < N_1 \in K_{<\lambda}$ and $N_1 \Vdash^{\lambda}_{K} \psi[a_0,...]$.

For $\varphi(x_0,...) = (Qy)\psi(y,x_0,...)$ this means for every $N_1,N_0 < N_1 \in K_{<\lambda}$ there is N_2, $N_0 < N_2 \in K_{<\lambda}$ and $a \in N_2-N_1$ such that $N_2 \Vdash^{\lambda}_{K}\psi[a,a_0,...]$.

2) The $\underset{\sim}{L}$-generic-type of $\bar{a}$ in N is $\{\varphi(\bar{x}) \in \underset{\sim}{L} : N \Vdash^{\lambda}_{K}\varphi[\bar{a}]\}$, where $\underset{\sim}{L}$ is a language for N (or some expansion of it). We say "$\bar{a}$ materialize p (or φ) if p (or $\{\varphi\}$) is a subset of the $\underset{\sim}{L}$-generic type of $\bar{a}$ in N.

4.3 Definition: Let $N_i (i < \lambda)$ be an increasing (by $<$) continuous sequence, $N = \bigcup_{i<\lambda} N_i$, $|||N_i||| < \lambda$, $\underset{\sim}{L}^*$ a fragment of the logic $L^1_{\infty,\kappa}$.

1) N is $\underset{\sim}{L}^*$-generic, if for any $\varphi(x_0,...) \in \underset{\sim}{L}^*$ $a_0,... \in N$:

$N \models \varphi[a_0,...] \Leftrightarrow$ for some $\alpha < \lambda$, $N_\alpha \Vdash^{\lambda}_{K} \psi[a_0,...]$.

2) The presentation $\langle N_i : i < \lambda\rangle$ of N is $\underset{\sim}{L}^*$-generic if for any $\alpha < \lambda$ with cofinality $\geq \kappa$, $\psi(x_0,...) \in \underset{\sim}{L}^*(N_\alpha,N_i)_{i\in I}$, $I \subseteq \alpha$, $|I| < \kappa$ and $a_0,... \in N$

$$N \models \varphi[a_0,...] \Leftrightarrow \text{for some } \alpha < \lambda, N_\alpha \Vdash^{\lambda}_{K} \psi[a_0,...]$$

and for each $\beta \geq \alpha$, with cofinality $\geq \kappa$, N_β is almost $\underset{\sim}{L}^*(N_\alpha, N_i; |N_\alpha|)_{i \in I}$-generic (see part 4).

3) N is strongly $\underset{\sim}{L}^*$-generic if it has an $\underset{\sim}{L}^*$- generic presentation (in this case, if λ is regular, then for any presentation $\langle N_i : i < \lambda \rangle$ of N there is a closed unbounded $S \subseteq \lambda$ such that $\langle N_i : i \in S \rangle$ is an L^*-generic presentation).

4) We add "almost" to all the above defined notions if for $\Vdash$, the inductive definitions of truth works except possibly for Q (e.g. $N \Vdash_K^\lambda \exists x \varphi(x, \ldots)$ iff for some $a \in N$, $N \Vdash_K \varphi(a, \ldots)$).

4.3A Remark: 1) Notice we can choose $N_i = N_0 = N$, so $\|N\| < \lambda$. In particular almost L^*-generic models of cardinality $< \lambda$ may well exist.

2) So we concentrate on $\lambda = \aleph_1$, and fragments of $L^0_{\infty,\omega}$ (mainly $L^0_{\omega_1,\omega}$ and its countable fragments).

3) There are obvious implications, and forcing is preserved by isomorphism.

There are obvious theorems no the existence of generic models, e.g.

4.4 Theorem:

1) If $N_0 \in K_{<\lambda}$, $\lambda = \mu^+$, $\mu^{<\kappa} = \mu$, $\underset{\sim}{L} \subseteq \underset{\sim}{L}_{\infty,\kappa}$, $|\underset{\sim}{L}| < \lambda$. *Then* there are $N_i (i < \lambda)$ such that $\langle N_i : i < \lambda \rangle$ is an $\underset{\sim}{L}$-generic representation of $N = \bigcup_{i<\lambda} N_i$, (hence N is $\underset{\sim}{L}$-generic).

4.4A Remark: If $\underset{\sim}{L} = \bigcup_{i<\lambda} \underset{\sim}{L}_i$, $|\underset{\sim}{L}_i| < \lambda$, that we can get "$\langle N_i : j < i < \lambda \rangle$ is an $\underset{\sim}{L}_j$ -generic representation of N for each j.

From time to time we add some hypothesis and prove a series of claims; such that the hypothesis holds, at least w.l.o.g., in the case we are interested in. We are interested in the case $I(\aleph_1, \kappa) < 2^{\aleph_1}$, so by 3.6 it is reasonable to make:

4.5 Hypothesis: K is PC_ω, $<$ refine $<_{\infty,\omega}$, and K is categorical in $\aleph_0$ and $1 \leq I(\aleph_1, K) < 2^{\aleph_1}$.

Claim 4.6: For each $\bar{a} \in N \in K_{\aleph_0}$, and $\varphi(\bar{x}) \in L^0_{\infty,\omega}(P)$ ($\bar{a}$ finite), $(N_0, N_0) \Vdash^{\aleph_1}_K \varphi[\bar{a}]$ or $(N_0, N_0) \Vdash^{\aleph_1}_K \neg\varphi[\bar{a}]$ (i.e P is interpreted as N_0).

Proof: Suppose not, and for each $S \subseteq \omega_1$, we define by induction on α, $N^S_\alpha \in K_{\aleph_0}(\alpha < \omega_1)$, increasing (by $<$) and continuous. $N^S_0 = N$, and for limit α, $N^S_\alpha = \bigcup_{\beta<\alpha} N^S_\beta$. For $\alpha = \beta + 1$, β limit remember $(N_\beta, \bar{a}) \cong (N, \bar{a})$ as $N = N_0 < N_\beta$ hence $N_0 <_{\infty,\omega} N_\beta$. So (N_β, N_β) does not force ($\Vdash^{\aleph_1}_K$) $\varphi[\bar{a}]$ nor $\neg\varphi[\bar{a}]$. So there are $M_\ell (\ell = 0,1)$, $N^S_\beta < M_\ell \in K_{N_0}$, $(M_0, N_\beta) \Vdash^{\aleph}_K \varphi[\bar{a}]$, $M_1 \Vdash^{\aleph_1}_K \neg\varphi[\bar{a}]$. Now if $\beta \in S$ let $N^S_\alpha = M_0$, and if $\beta \in S$, $N^S_\alpha = M_1$. For $\alpha = \beta+1$, β non limit we take care to guarantee that $\langle N^S_\alpha : \alpha < \omega_1\rangle$ will be an $\underset{\sim}{L}$-generic presentation. Let $N^S = \bigcup_{\alpha<\omega_1} N^S_\alpha$. Now if $S(0) - S(1)$ is stationary, $(M^{S(0)}, \bar{a}) \not\cong (M^{S(1)}, \bar{a})$,, for if $f : M^{S(0)} \to M^{S(1)}$ is an isomorphism, for a closed unbounded set of α's f maps $M^{S(0)}_\alpha$ onto $M^{S(1)}_\alpha$, so this holds for some $\alpha \in S(0) - S(1)$, and we get a contradiction. By [Sh3], VIII 1.3, we get $I(\aleph_1, K) = 2^{\aleph_1}$, contradiction.

4.7 Claim: For each countable $\underset{\sim}{L} = \underset{\sim}{L}(P) \subseteq \underset{\sim}{L}^0_{\omega_1,\omega}(P)$, and $N \in K_{\aleph_0}$ the number of complete $\underset{\sim}{L}(N)$-types p (with no parameters) such that $N \Vdash^{\aleph_1}_K (\exists\bar{x})\wedge p$, is countable.

Proof: At first glance it seemed that Keisler [K1] will implies this. However, here we need the parameter N, so we need to work a little. Suppose the conclusion fails. First we define by induction N_α, $(\alpha < \omega_1)$ increasing by $<$, $|N_\alpha| = \omega\alpha$, such that

(i) $\langle N_\alpha : \alpha < \omega_1\rangle$ is $\underset{\sim}{L}$- generic,

(ii) for each $\beta < \alpha$, there is $a^\beta_\alpha \in N_{\alpha+1} - N_\alpha$ materializing an $\underset{\sim}{L}(N_\beta)$-type not realized in N_α, (i.e. in (N_α, N_β)) (possible by 4.6).

Now we expand N by all relevant information: the order $<$, $c (c \in N_0)$, enough "set theory", "witness" for $N_\beta <_K N_\alpha$ for $\beta < \alpha$, and F, $F(\beta,\alpha) = a^\beta_\alpha$, and names for all formulas in $\underset{\sim}{L}(N_\alpha)$ (with α as a parameter). We get a model $\mathcal{B}$. By Keisler [K1] we get models $\mathcal{B}_i (i < 2^{\aleph_1})$, of power $\aleph_1$, so that the set of $\underset{\sim}{L}(N_0)$-types realizes in N^i (the $L(K)$-reduct of $\mathcal{B}_i$) are distinct for distinct i's. So $(N^i, c)_{c \in N_0}$ are pairwise non- isomorphic. If $2^{\aleph_0} < 2^{\aleph_1}$ we finish by [Sh1] VIII 1.3.

So we can assume $2^{\aleph_0} = 2^{\aleph_1}$. We can define $N_i \in K_{\aleph_0}$ $<$-increasing continuous, $\langle N_i : j < i < \aleph_1 \rangle$ is $\underset{\sim}{L}(N_j)$-generic for every j and for $j < i$, $a_{j,i} \in N_{i+1}$ materializing in N_{i+1} a complete $\underset{\sim}{L}(N_j)$-type $p_{j,i}$ not materializing in N_i. W.l.o.g. N_j has universe $\omega \times (j+1)$. So $\bigcup_{i<\omega_1} N_i$ realize the types $\{p_{j,i}: j \le i < \omega_1\}$ which are distinct complete $\underset{\sim}{L}(N_j)$-type. Let $\mathcal{A}$ be an expansion of $\bigcup_{i<\omega_1} N_i$, coding enough set theory. We define by induction on $\alpha < 2^{\aleph_0}$, $N_\alpha = \bigcup_{i<\omega_1} N_{\alpha,i}$, $N_{\alpha,i}$ countable and increasing continuous in i. For α, we define by induction on $i < \omega_1$, a countable model $\mathcal{A}_{\alpha,i}$. For $i = 0$, it is some countable elementary submodel of $\mathcal{A}$. For $i =$ limit $\mathcal{A}_{\alpha,i} = \bigcup_{j<i} \mathcal{A}_{\alpha,j}$. for $i = \delta+1, \delta$ limit - it is an end extension of $\mathcal{A}_{\alpha,\delta}$, with a first new element (used extensively in [Sh1], see [Sh4]). For $i = \delta + n$, $n < \omega$, δ limit or zero: there is $a_{i,\delta} \in \mathcal{A}_{\alpha,i}$ which $\mathcal{A}_{\alpha,i}$ say it realizes an $\underset{\sim}{L}(\mathcal{A}_{\alpha,\delta} \restriction L(\delta))$- type p, and this type is not realized in N_β interpreting P as $N_{\beta,i}$ for any $\beta < \alpha$.

Remark: An alternative way is to note that choosing pairwise disjoint stationary $S_i \subsetneq \omega_1$ for $i < \omega_1$ we can build $\bar{N} = \langle N_i : i < \omega_1 \rangle$ ($<$-increasing continuous sequence of members of $K_{\aleph_0}$, $N_i \ne N_{i+1}$) such that: if $i < \delta \in S_i$ then $(N_{\delta+1}, N_\delta) \cong (N_\delta, N_i)$. So for a complete $\underset{\sim}{L}(P)$-type p, for every i, (N, N_i) realize p *if and only if* $\{\delta < \omega_1 : (N, N_\delta)\text{-realize } p\}$ is stationary.

The rest should be clear.

4.8 Lemma: There are countable $\underset{\sim\alpha}{L}(P) \subseteq \underset{\sim\omega_1,\omega}{L^0}(P)$ increasing continuous in α, closed under finitary operations, such that:

a) For each $N \in K_0$ and every complete $\underset{\sim\alpha}{L}(N)$-type p, $N \Vdash_K^{\aleph_1} (\exists \bar{x}) \wedge p \Rightarrow \wedge p \in \underset{\sim\alpha+1}{L}(P)$. Hence for every $\underset{\sim\omega_1,\omega}{L^0}(P)$-formula $\psi(\bar{x})$ there are formulas $\varphi_n(\bar{x}) \in \bigcup_{\alpha<\omega_1} \underset{\sim\alpha}{L^0}(P)$ such that $(N, N) \Vdash_K^{\aleph_1} (\forall \bar{x})[\psi(\bar{x}) \equiv \bigvee_n \varphi_n(\bar{x})]$.

b) For every $N_0 \prec N_1 \in K_{\aleph_0}$ there is $N_2, N_1 \prec N_2 \in K_{\aleph_0}$, such that for every $\bar{a} \in N_2$, $\varphi(\bar{x}) \in \underset{\sim}{L}^0_{\omega_1,\omega}(N_0), N_1 \Vdash^{\aleph_1}_K \varphi[\bar{a}]$ or $N_1 \Vdash^{\aleph_1}_K \neg\varphi[\bar{a}]$.

c) If $N \prec N_\ell \in K_{\aleph_0} (\ell = 1,2)$, $\bar{a}_\ell \in N_\ell$, and the $\bigcup_\alpha \underset{\sim}{L}_\alpha(N)$-generic types of $\bar{a}_\ell$ in N_ℓ are equal, then so are the $\underset{\sim}{L}^0_{\infty,\omega}(N)$-generic types. In fact there is M, $N \prec M$, and K-embeddings $f_\ell : N_\ell \to M$ such that f_ℓ maps N onto itself, and $f_1(\bar{a}_1) = f_2(\bar{a}_2)$.

d) For each $N \in K_{\aleph_0}$, complete $L^0_{\omega_1,\omega}(N)$-type p, the class $\{\langle M, N, \bar{a}\rangle : M \in K_{\aleph_0}$, $N \prec M$, $\bar{a}$ materialize p in $(M, N)\}$ is a $PC_{\aleph_0}$ class.

e) Let $\underset{\sim}{L}^2(N)$ be the set of formulas in $L^0_{\omega_1,\omega}(N)$ in which the quantifier (Qx) does not appear.

If $N \prec M \in K_{\aleph_0}$, $\bar{a} \in M$, and for some complete $\underset{\sim}{L}^2(N)$-type p, $\bar{a}$ materialize p in (M, N) *then* for some complete $L^0_{\omega_1,\omega}(N)$-type q_p, $\bar{a}$ materialize q_p in (M, N).

f) The number of complete $L^0_{\omega_1,\omega}(N)$-types p which for some $\bar{a} \in M \in K_{\aleph_0}$, $N \prec M$, $\bar{a}$ materialize in (M, N) is $\le \aleph_1$.

g) If in f) we get there are $\aleph_1$ such types then $I(\aleph_1, K) \ge \aleph_1$.

4.8A Remark: We cannot get rid of the case of $\aleph_1$ types (but see 5.16, 5.17). For let $K = \{(A,E,<): E$ an equivalence relation on A, each E-equivalence class is countable, $x < y \Rightarrow xEy$, and on each E-equivalence class $<$ is a 1-transitive linear order, and $xEy \Rightarrow (x/E,<,x) \cong (y/E,<,y)\}$ and $M \prec N$ if $M \subseteq N$, and $[x \in M \wedge y \in N \wedge xEy \Rightarrow y \in M]$.

Proof: a) We define $\underset{\sim}{L}_\alpha(P)$ by induction on α using 4.7. The second phrase is proved by induction on the depth of the formula.

b) By iterating ω times, it suffices to prove this for each $\bar{a} \in N_1$, so again by iterating ω times it suffices to prove this for a fix $\bar{a} \in N_1$.

If the conclusion fail we can define by induction on $n < \omega$ for every $\eta \in {}^n2$, model M_η and $\varphi_n(\bar{x}) \in L^0_{\omega_1,\omega}(N)$ such that:

(i) $M_{<>} = N_1$

(ii) $M_\eta \prec M_{\eta\hat{}<\ell>} \in K_{\aleph_0}$ for $\ell = 0,1$

(iii) $(M_\eta, N) \Vdash^{\aleph_1}_K \varphi_n(\bar{a})$

(iv) $\varphi_{\eta\hat{}<1>}(\bar{x}) = \neg\, \varphi_{\eta\hat{}<0>}(\bar{x})$

Now for $\eta \in {}^\omega2$, let $M_\eta = \bigcup_{n<\omega} M_{\eta\restriction n}$. Clearly for $\eta \in {}^\omega2$, $M_\eta \Vdash^{\aleph}_K (\exists\bar{x})[\bigwedge_{n<\omega} \varphi_{\eta\restriction n}(\bar{x})]$, and, after slight work, we get contradiction to 4.7.

c) By a) we can find $M_\ell \in K_{\aleph_1}$, $N_\ell \prec M_\ell$, M_ℓ are $L_\alpha(N_0)$-equivalent for each α, hence by (a), $L^0_{\omega_1,\omega}(N_0)$-equivalent. As in [Sh1], extend $M_1 \cup M_2$ by enough set theory, (and names to $c \in N_1 \cup N_2$) and find a non-well ordered countable model satisfying enough of the properties of the previous model. We find countable M_ℓ, $N_\ell \prec M_\ell$, $(M_1, N_0) \cong (M_2, N_0)$ as required there.

d) Let $N_0 \prec M_0 \in K_{\aleph_0}$ and $\bar{a}_0 \in M_0$ be such that $(M_0, N_0) \Vdash^{\aleph_1}_K \bigwedge_{\varphi(\bar{x})\in p} \varphi[\bar{a}_0]$. clearly $\{(M, N, a) : M \in K_{\aleph_0}, N \in K_{\aleph_0}, N \prec M$, and there are $M'' \in K_{\aleph_0}$, $M \prec M''$ and K-embedding $f : M_0 \to M''$, such that $f(N_0) = N$, $g(a_0) = a\}$ is a $PC_{\aleph_0}$ class. But by 4.8(b) (and trivialities) it is the required class.

e) If this fails, then (by (b)) for some $N_\ell \prec M_\ell \in K_{\aleph_0}$, $\bar{a}_\ell \in M_\ell$, and p_ℓ, q_ℓ for $\ell = 1,2$:

(i) p_ℓ is a complete $L^2(P)$-type.

(ii) q_ℓ is a complete $L^0_{\omega_1,\omega}(P)$-type.

(iii) $\bar{a}_\ell$ materialize q_ℓ in (M_ℓ, N_ℓ).

(iv) $p_\ell \subseteq q_\ell$, $p_q = p_1$, $q_1 \neq q_2$.

So there are $M^+_\ell \in K_{\aleph_1}$, $M_\ell \prec M^+_\ell$ such that $\bar{a}_\ell$ realize q_ℓ in M^+_ℓ. W.l.o.g. the universe of M_ℓ is a set of countable ordinals, $|M_1| \cap |M_2| = \emptyset$. We can define a model $\mathfrak{A}$ with universe ω_1, with relation $<$ (the well ordering of ω_1), individual constants for the elements of M_1 and

M_2, the relations of M_1 and of M_2 etc. So as in [Sh1,] using $p_1 = p_2$, (using non well ordered models) we can find M_ℓ^*, $M_\ell \prec M_\ell^* \in K$, and $(M_1^*, N_1, \bar{a}_1) \cong (M_2^*, N_2, \bar{a}_\ell)$. But this easily contradict $q_1 \neq q_2$.

f) Suppose this fails.

The proof split to two cases.

Case A: $2^{\aleph_0} = 2^{\aleph_1}$.

We shall prove $I(\aleph_1, K) \geq 2^{\aleph_0}$, thus, (as $2^{\aleph_0} = 2^{\aleph_1}$) contradicting Hypothesis 4.5 (this will be the only use of the hypothesis).

Let $p_i (i < \omega_2)$ be distinct complete $L^0_{\omega_1,\omega}(P)$-types materialized in some (M, N) $(N \prec M \in K_{\aleph_0})$ (they exist by the assumption that (f) fail). For each i define $N_{i,\alpha}$, $\xi_{i,\alpha} (\alpha < \omega_1)$ and $\bar{a}_i$ such that:

(i) $N_{i,\alpha} \in K_{\aleph_0}$ has universe $\omega(1+\alpha)$

(ii) $\langle N_{i,\alpha} : \alpha < \omega_1 \rangle$ is $\prec$-increasing continuous.

(iii) $\bar{a}_{i,\alpha} \in N_{i,\alpha+1}$, $\bar{a}_{i,\alpha}$ materialize p_i in $(N_{i,\alpha+1}, N_{i,\alpha})$.

(iv) for every $\alpha < \beta < \omega_1$, $\bar{a} \in N_{i,\beta}$, $\bar{a}$ materialize in $(N_{i,\beta}, N_{i,\alpha})$ a complete $L^0_{\omega_1,\omega}(P)$-type.

(v) $\xi_{i,\alpha} < \omega_1$ is strictly increasing continuous in α.

(vi) for $\alpha < \beta$, N_β is almost $\underset{\sim\beta}{L}(N_\alpha)$-generic.

(vii) if $\alpha < \beta$, $\bar{a}, \bar{b} \in N_\beta$ materialize different $L^0_{\omega_1,\omega}(N_\alpha)$- types in N_β, *then* $\bar{a}, \bar{b}$ realize different $(L_{\omega_1,\omega} \cap \underset{\sim\xi_{i,\beta+1}}{L})(N_\alpha)$-types in N_β.

Let $\mathcal{A}_i^*$ be $(H(\aleph_2),\in)$ expanded by predicates for K, $\prec$, $\{\langle N_{i,\alpha} : \alpha < \omega_1\rangle\}$, $\{\bar{a}_i\}$, N_i and $\{i\}$.

Let $\mathcal{A}_i$ be a countable elementary submodel of $\mathcal{A}_i^*$ so $|\mathcal{A}_i| \cap \omega_1$ is an ordinal $\delta(i) < \omega_1$. It is also clear that $N_i^{\mathcal{A}_i}$ is $N_{i,\delta(i)}$. As $\mathcal{A}_i$ is defined for $i < \omega_2$, w.l.o.g. for some $\delta < \omega_1$, for every $i < \omega_2$, $\delta(i) = \delta$. Note that $(N_{i,\delta}, N_0)$ is $(D_i, \aleph_0)$-homogeneous for some D_i, and D_i is a set of complete $\underset{\sim\delta}{L}(P)$-types. Note that $(N_{i,\delta},N_{i,0},\bar{a}_{i,0}) \not\equiv (N_{j,\delta},N_{j,0},\bar{a}_{j,0})$ for $i \neq j$, hence $|\{j : D_j = D_i\}| \le \aleph_0$, hence w.l.o.g. $i \neq j \Rightarrow D_i \neq D_j$.

Let $\Gamma = \{D : D$ a set of complete $\underset{\sim\delta}{L}(P)$- types, such that for some model $\mathcal{A}_D$ of $\bigcap_{i<\omega_2} Th_{L_{\omega,\omega}}(\mathcal{A}_i)$, with $\{a : \mathcal{A}_D \models$ "a countable ordinal $\} = \delta$ (and the usual order) $D = \{\{\varphi(\bar{x}) : \varphi(\bar{x}) \in \underset{\sim\delta}{L}(P)$, and $\mathcal{A}_D \models$ "$(N_{i,\alpha},N_0) \Vdash \varphi[\bar{a}]$"$\} : \bar{a} \in \bigcup_{i<\delta} N_{i,\alpha}^{\mathcal{A}}\}$.

So $D_i \in \Gamma$ for $i < \omega_2$, hence Γ is uncountable.

By standard descriptive set theory Γ has power continuoum. So let $D(\zeta) \in \Gamma$ be distinct for $\zeta < 2^{\aleph_0}$. For each ζ, let $\mathcal{A}^0_{D(\zeta)}$ be as in the definition of Γ. We define by induction on $\alpha < \omega_1$, $\mathcal{A}^\alpha_{D(\zeta}$ such that

(a) $\mathcal{A}^\alpha_{D(\zeta)}$ is countable.

(b) $\alpha < \beta \Rightarrow \mathcal{A}^\alpha_{D(\zeta)} \prec_{L_{\omega,\omega}} \mathcal{A}^\beta_{D(\zeta)}$.

(c) for limit β, $\mathcal{A}^\alpha_{D(\zeta)} = \bigcup_{\beta<\alpha} \mathcal{A}^\beta_{D(\zeta)}$.

(d) if $d \in \mathcal{A}^{\alpha+1}_{D(\zeta)} - \mathcal{A}^\alpha_{D(\zeta)}$, $\mathcal{A}^{\alpha+1}_{D(\zeta)} \models$ "d a countable ordinal" then for $a \in \mathcal{A}^\alpha_{D(\zeta)}$, $\mathcal{A}^{\alpha+1}_{D(\zeta)} \models$ "if a is a countable ordinal then $a < a'$.

(e) for $\alpha = 0$ in (d) there is no minimal such d.

(f) for every α there is $d_{\zeta,\alpha} \in \mathcal{A}^{\alpha+1}_{D(\zeta)} - \mathcal{A}^\alpha_{D(\zeta)}$, $\mathcal{A}^{\alpha+1}_{D(\zeta)} \models$ "$d_{\zeta,\alpha}$ a countable ordinal and for $\alpha \neq 0$ it is minimal".

Let $M_{\zeta,\alpha}$ be the $d_{\zeta,\alpha}$-th member of the ω_1-sequence of models in $\mathcal{A}^{\beta}_{D(\zeta)}$ / $\beta > \alpha$ (remember $\{\langle N_{i,\alpha} : \alpha < \omega_1\rangle\}$ is a relation of $\mathcal{A}^*_i$, with name not depending on i). Let $M_\zeta = \bigcup_{\alpha<\omega_1} M_{\zeta,\alpha}$. It is easy to check that for $0 < \alpha < \beta$, $(M_{\zeta,\beta}, M_{\zeta,\alpha})$ is $(D(\zeta), \aleph_0)$-homogeneous.

So from the isomorphism type of M_ζ we can compute $D(\zeta)$. So $\zeta \neq \xi \Rightarrow M_\zeta \not\equiv M_\xi$. As $M_\zeta \in K_{\aleph_1}$ we finish.

Case B: $2^{\aleph_0} < 2^{\aleph_1}$.

By 3.5, K has the $\aleph_0$-amalgamation property. So clearly if $N \prec M \in K_{\aleph_0}$, $\bar{a} \in M$, *then* $\bar{a}$ materialize in (M, N) a complete $L^0_{\omega_1,\omega}(P)$-type. We want to use descriptive set theory.

We represent a complete $L^0_{\omega_1,\omega}(P)$-type materialize in some (M, N) by a real, by representing the isomorphism type of some $(M, N, \bar{a})$, $N \prec M \in K_{\aleph_0}$, $\bar{a} \in M$. The set of representatives is Borel, and the equivalence relation is Σ_1. [as $(M_1, N_1, \bar{a}_1)$, $(M_2, N_2, \bar{a}_2)$ represent the same type *if and only if* for some (M, N), $N \prec M \in K_{\aleph_0}$, there are K- embeddings $f_1 : M_1 \to M$, $f_2 : M_2 \to M$ such that $f_1(N_1) = f_2(N_2) = N$, $f_1(\bar{a}) = f_2(\bar{a})$].

By Burgess [B], (or see [Sh 8]) as there are $> \aleph_1$ equivalence classes, there is a perfect set of representation, pairwise representing different types.

From this we easily get that w.l.o.g. their restriction to some $L_{\tilde{\alpha}}(P)$ are distinct, contradicting part (a).

Remark: Note that in case (A) we get many types too but it was not clear whether we can make the N_ζ to be generic enough, to get the contradiction we got in case B.

(g) Easy.

Next we prove 4.8(e). As q_p does not exist there is a formula $\varphi(\bar{x}) \in L^1_{\omega_1,\omega}(P)$ and $N_\ell \prec M_\ell \in K_0$, $\bar{a}_\ell \in {}^{\omega>}|M_\ell|$ such that neither $(M_1, N_1) \Vdash^\lambda_K \varphi[\bar{a}_1]$ nor $(M_2, N_2) \Vdash^\lambda_K \neg\varphi(\bar{a}_1)$ holds.

So by the definition of the forcing, w.l.o.g. (possibly increasing M_1, M_2)

$$(M_1, N_1) \Vdash_K^\lambda \neg \varphi[\bar{a}_1]$$

$$(M_2, N_2) \Vdash_K^\lambda \varphi[\bar{a}_2]$$

We continue with $M_\ell^\dagger$ as there (forgetting the "realizing") and get the contradiction.

It now follows that using $L_{\omega_1,\omega}(P)$ would make little difference.

4.8B Remark: We may want to replace $L^0_{\omega_1,\omega}$ by $L^1_{\omega_1,\omega}$ in 4.6,4.7,4.8 (except that, for our benefit, in 4.8(e); we may retain the definition of $\underset{\sim}{L}^2(N)$). We lose the ability to build $\underset{\sim}{L}$-generic models in $K_{\aleph_1}$ (as the number of (even unary) relations on $N \in K_{\aleph_0}$ is $2^{\aleph_0}$, which may be $> \aleph_1$). However we can say "$\bar{a}$ materialize in $N \in K_{\aleph_0}$ the formula $\varphi(\bar{x}) \in L^1_{\omega_1,\omega}(N_n, N_{n-1}, \ldots, N_0;)$ where $N_0 < \cdots < N_n < N$, N_ℓ countable) if for every large enough countable $M < K_{\aleph_0}$, $\bar{a}$ materialize $\varphi(\bar{x})$ in M.

This suffcies for 4.6.

4.9 Concluding Remarks for Section 4.

We can get more information on the case $1 \leq I(\aleph_1, K) < 2^{\aleph_1}$.

1) As in 3.5,there is no difficulty in getting the results for the class of models of $\psi \in L_{\omega_1,\omega}(Q)$ as (using $(K, <)$ from the proof of 3.9(2)) in all constructions we get many non isomorphic models, we can make then to be in $K^F_{\aleph_1}$.

2) We can continue the analysis, e.g. deal with sequences $N_0 < N_1 < \cdots < N_k \in K_{\aleph_0}$ such that $N_{\ell+1}$ is almost $\underset{\sim}{L}_\alpha(N_\ell, N_{\ell-1}, \ldots, N_0)$-generic. We can get that for any countable $\underset{\sim}{L} \subseteq L_{\omega_1,\omega}(a)$ for some α, any strong $\underset{\sim}{L}$- generic $N \in K_{\aleph_1}$ is $\underset{\sim}{L}$-determined.

3) We can do the same for stronger logic.

Let us define a logic L^*. It has as variable

variables for elements $x_1, x_2, \ldots$ and variables for filters $E_1, E_2, \ldots.$

The atomic formula are:

(i) the usual ones

(ii) $x \in \mathrm{Dom}\ E$.

The logical operations are:

(a) $\wedge$ conjunction, $\neg$ negation

(b) $(\exists x)$ existential quantification x individual variable.

(c) $(\exists^f x)[\varphi, D]$.

(d) $(aaD)\varphi$

Now in $\exists^f x[\varphi, D]$, x is bounded but not D and in aaD D is bounded.

The satisfaction relation is defined as usual and

$M \models \exists^f x, D[\varphi(x, D), D]$ if and only if $\{x \in \mathrm{Dom}\ D : \models \varphi(x, D)\} \in D$

$M \models aaD\varphi(D)$ if and only if there is a function F from $\bigcup\limits_{n<\omega}{}^{n}(S_{<\aleph_1}(M)) \to S_{<\aleph_1}(M)$ such that:

if $A_n \subseteq M$, $|A_n| \le \aleph_0$, $A_n \subseteq A_{n+1}$ and $F(A_0, \ldots, A_{n+1}) \subsetneq A_{n+2}$ *then* $M \models \varphi[D_{\langle A_n : n<\omega\rangle}]$ where $D_{\langle A_n : n<\omega\rangle}$ is the filter on $\bigcup\limits_{n<\omega} A_n$, generated by $\{\bigcup\limits_{n<\omega} A_n - A_\ell : \ell < \omega\}$.

§5 There is a superlimit model in $\aleph_1$.

Here we make

5.1 Hypothesis : Like 4.5, but also $2^{\aleph_0} < 2^{\aleph_1}$.

This section is the deepest. The main difficulties are proving the facts which are obvious in the context of [Sh 1]. So while it was easy to show that every $p \in D^*(N)$ is definable over a finite set, it was not clear to me how to prove that if you extend the p to $q \in D^*(M)$, $(N \prec M \in K_{\aleph_0})$ by the same definition, then $q \vdash p$ (remember p,q are types materialize not realize, and at this point in the paper we still do not have the tools to replace the models by uncountable generic enough models). So we rather have to show that failure is a non-structure property i.e. implies existence of many models.

Also in stable amalgamation symmetry becomes much more complicated. We prove existence of stable amalgamation by four stages (5.15, 5.17(3), 5.20, 5.22). The symmetry is proved as a consequence of uniqueness of one sided amalgamation, (so it cannot be used in its proof). The culmination of the section is the existence of a superlimit models in $\aleph_1$ (5.24). This seems a natural stopping point as the next step should be phrasing the induction on n i.e. dealing with $\aleph_n$ and $\mathcal{P}(n-\ell)$- diagrams of models of power $\aleph_\ell$.

5.1 Definition: 1) For $N \in K_{\aleph_0}$ let

$D(N) = \{p :$ a complete $L^0_{\omega_1,\omega}(N)$- type over N such that for some $\bar{a} \in M \in K_{\aleph_0}$, $N \prec M$ and $\bar{a}$ materialize p in $(M, N)\}$, (i.e. the members of p have the form $\varphi(\bar{x},\bar{a})$, ($\bar{x}$ finite and fix for p) $\bar{a}$ a finite sequence from N and $\varphi \in L^0_{\omega_1,\omega}(N)$).

2) For $N \in K_{\aleph_0}$ let

$D^*(N) = \{p : p$ a complete $L^0_{\omega_1,\omega}(N;N)$-type such that for some $\bar{a} \in M \in K_{\aleph_0}$, $N \prec M$ and $\bar{a}$ materialize p in $(M, N; N)\}$.

Explanation: (so for every $\bar{b} \in N$ and $\varphi(\bar{x},\bar{y}) \in L^0_{\omega_1,\omega}(N;N)$ if $p(\bar{x}) \in D^*(N)$ then $\varphi(\bar{x},\bar{b}) \in p$ or $\neg\varphi(\bar{x},\bar{b}) \in p$ and if $p \in D(N)$, $\bar{b}$ finite then $\varphi(\bar{x},\bar{b}) \in p$ or $\neg\, \varphi(\bar{x},\bar{b}) \in p$.

[so a formula from $p \in D^*(N)$ may have all $c \in N$ as parameters]

5.2 Lemma: 1) K has the $\aleph_0$-amalgamation property.

2) If $N_0 < N_0 \in K_{\aleph_0}$, $A_i \subseteq N_0$ for $i \le n$ *then* for every sentence $\psi \in L^1_{\omega_1,\omega}(N_0, A_n, \ldots, A_1; A_0)$,

$$N \Vdash^{\aleph_1}_K \psi \quad or \quad N \Vdash^{\aleph_1}_K \neg\psi$$

3) If $N < M \in K_{\aleph_0}$, every $\bar{a} \in M$ materialize is (M, N) one and only one type in $D^*(N)$ (and also in $D(N)$). Also for every $q \in D^*(N)$ for some M', $M < M' \in K_{\aleph_0}$ some $\bar{b} \in M'$ materialize q in (M, N).

4) For every countable $\underset{\sim}{L} \subseteq \underset{\sim}{L}(P) \subseteq L^0_{\omega_1,\omega}$, and $N \in K_{\aleph_0}$, the number of complete $\underset{\sim}{L}(N;N)$-types p such that $N \Vdash^{\aleph_1}_K (\exists \bar{x}) \wedge p$ is countable.

5) There are countable $\underset{\sim\alpha}{L}(P) \subseteq L^0_{\omega_1,\omega}(P)$ for $\alpha < \omega_1$, increasing continuous in α, closed under finitary operations (and subformulas) such that:

(a) for each $N \in K_{\aleph_0}$ and complete $\underset{\sim\alpha}{L}(N;N)$ type p,

$$[N \Vdash^{\aleph_1}_L \exists \bar{x} \wedge p \Rightarrow \wedge p \in \underset{\sim\alpha+1}{L}(P)]$$

Hence for every $L^0_{\omega_1,\omega}(N)$ formula $\psi(\bar{x})$ for some $\varphi_n(\bar{x}) \in \bigcup_{\alpha<\omega} \underset{\sim\alpha}{L^0}(N)$ for every $N \in K_{\aleph_0}$

$$(N, N) \Vdash^{\aleph_1}_K (\forall \bar{x})[\psi(\bar{x}) \equiv \bigvee_{n<\omega} \varphi_n(\bar{x})]$$

6) For $N \in K_{\aleph_0}$, $|D^*(N)| \le \aleph_1$.

Proof: 1) By 3.5.

2) By 1)

3) By 2), (and (1)).

4) Like the proof of 4.7 (just easier)

5) Like 4.8(b)

6) Like 4.8(f).

5.4 Claim: 1) Each $p \in D(N)$ does not split (see [Sh 3] Ch I Definition 2.6, p. 11 or [Sh 1]) over a finite subset of N, hence is definable over it (that is: there is a function g_p, such that $g_p(\varphi(\bar{x},\bar{y}))$ is $\psi_{p,\varphi}(\bar{y},\bar{z}) \in \underset{\sim}{L}^0_{\omega_1,\omega}(P)$ such that for each $\varphi(\bar{x},\bar{y}) \in \underset{\sim}{L}^0_{\omega_1,\omega}(N)$, $\bar{a} \in N$, $[\varphi(\bar{x},\bar{a}) \in p \Leftrightarrow N \models \psi_{p,\varphi}(\bar{a},\bar{c})]$ where p does not split over $\bar{c}$).

2) Every automorphism of N maps $D(N)$ onto itself, and each $p \in D(N)$ has at most $\aleph_0$ possible images. If g is an isomorphism from $N_0 \in K_{\aleph_0}$ onto $N_1 \in K_{\aleph_0}$ then $g(D(N_0)) = D(N_1)$.

Proof: Easy.

5.5 Claim: Suppose $N_0 \prec N_1 \in K_{\aleph_0}$, and N_1 force that $\bar{a},\bar{b}$ $(\in N_1)$ materialize the same $L^0_{\omega_1,\omega}(N_0)$-type over N_0, *then* N_1 force they have the same $L^0_{\omega_1,\omega}(N_0;N_0)$-type.

So there is no essential difference between $D(N)$ and $D^*(N)$.

Remark: Note that in a formula of $L^0_{\omega_1,\omega}(N_0,N_0)$ all $c \in N_0$ may appear as individual constants.

Proof: We can assume N_1 is $(D_\alpha(N_0), \aleph_0)$-homogeneous for some α, (see 5.6 below) such that α is "big enough" (see the demand in the proof).

Now we shall prove there is an automorphism of N_1 over N_0 taking $\bar{a}$ to $\bar{b}$, and we do it, of course, by hence and forth argument. So by renaming and symmetry, it suffices to prove that for every $c \in N_1$, there is $d \in N_1$ such that $\bar{a}\char`^\langle c\rangle$, $\bar{b}\char`^\langle c\rangle$ have the same $L^0_{\omega_1,\omega}(N_0)$-generic-type over N_0. By the choice of α it suffices to find d in any N_2, $N_1 \prec N_2$. However by the previous claim this is easy. [as w.l.o.g. the $L^0_{\omega_1,\omega}(N_0)$- type over N_0 that $\bar{a}\char`^\langle c\rangle$ materialize in $(N_1,N_0)$ does not split over $\bar{a} \cap N_0$; so if $\bar{a}\char`^\langle c\rangle$, $\bar{b}\char`^\langle a\rangle$ materialize the same $L^0_{\omega_1,\omega}(N_0)$- type over N_0 then they materialize the same $L^0_{\omega_1,\omega}(N_0)$-type over N_0].

5.6 Fact: There are D_α, D^*_α $(\alpha < \omega_1)$ such that

(a) for $N \in K_{\aleph_0}$, $D_\alpha(N)[D^*_\alpha(N)]$ is a countable subset of $D(N)[D^*_\alpha(N)]$.

(b) for each $N \in K_{\aleph_0}$, $\langle D_\alpha(N) : \alpha<\omega_1\rangle$ as well as $\langle D^*_\alpha(N) : \alpha < \omega_1\rangle$ are increasing continuous.

(c) $D^*(N) = \bigcup_{\alpha<\omega_1} D^*(N), D(N) = \bigcup_{\alpha<\omega_1} D_\alpha(N)$.

(d) if $N_1, N_2 \in K_{\aleph_0}$, f an isomorphic from N_1 to N_2 *then* f maps $D_\alpha(N_1)$ onto $D_\alpha(N_2)$, and $D^*_\alpha(N_1)$ onto $D^*_\alpha(N_2)$.

(e) for every α and $N \in K_{\aleph_0}$ there is a $(D_\alpha(N), \aleph_0)$-*homogeneous model (see below 5.7) (obviously it is unique up to isomorphism over N.)

(f) if $N_0 < N_1 < N_0 \in K_{\aleph_0}$, N_2 is $(D_\alpha(N_1), \aleph_0)$-*homogeneous (see 5.7 below) and N_1 is $(D_\alpha(N_0), \aleph_0)$-*homogeneous or just every $\bar{a} \in N_1$ materialize in N_1 some $p_{\bar{a}} \in D_\alpha(N_0)$ *then* N_2 is $(D_\alpha(N_0), \aleph_0)$-*homogeneous.

(g) N_1 is $(D_\alpha(N_0), \aleph_0)$-*homogeneous, if and only if N_1 is $(D^*_\alpha(N_0)$-homogeneous where $N_0 < N_1 \in K_{\aleph_0}$.

Proof: Easy.

5.7 Definition: 1) We say that (N_1, N_0), or just N_1 is $(D_\alpha(N_0), \aleph_0)$-*homogeneous if:

a) every $\bar{a} \in N_1$ materialize in (N_1, N_0) over N_0 some $p \in D_\alpha(N_0)$ and every $q \in D_\alpha(N_0)$ is materialized by some $\bar{b} \in N_1$.

b) If $\bar{a},\bar{b} \in N_1$, $\bar{a},\bar{b}$ materialize in (N_1, N_0) the same type over N_0 and $c \in N_1$ *then* for some $d \in N_1$, $\bar{a}\frown\langle c\rangle, \bar{b}\frown\langle d\rangle$ materialize in (N_1, N_0) the same type over N_0.

2) Similarly for $(D^*_\alpha(N_0), \aleph_0)$-*homogeneity

5.7A Remark : 1) Now this is meaningful only for $N < M \in K_{\aleph_0}$, but later it becomes meaningful any $N < M \in K$.

2) Uniqueness for countable models hold in this context too. The two notions are equivalent.

Now by 5.5, 5.6

5.8 Conclusion: 1) If $N_0 \prec N_1 \prec N_2 \in K_{\aleph_0}$, and $\bar{a},\bar{b} \in N_2$, (remember N_2 determines the complete $L^0_{\omega_1,\omega}(N_1)$-generic type of $\bar{a},\bar{b}$) *then* from the $L^0_{\omega_1,\omega}(N_1)$-forcing-type of $\bar{a}$ over N_1 we can compute the $L^0_{\omega_1,\omega}(N_0)$-forcing type of $\bar{a}$ over N_0 (hence if the $L^0_{\omega_1,\omega}(N_1)$-forcing-types of $\bar{a},\bar{b}$ over N_1 are equal, then so are the $L^0_{\omega_1,\omega}(N_0)$-forcing types of $\bar{a},\bar{b}$ over N_0.

2) If (N_1, N_0) is $(D_\alpha(N_0), \aleph_0)$-*homogeneous *then* $(N_1, N_0, c)_{c \in N_0}$ is $(D^*_\alpha(N_0), \aleph_0)$-homogeneous.

5.9 Lemma: There is $N^* \in K_{\aleph_1}$ such that $N^* = \bigcup_{\alpha<\omega_1} N_0, N_\alpha \in K_{\aleph_0}$ is increasing continuous and $N_{\alpha+1}$ is $(D_{\alpha+1}(N_\alpha), \aleph_0)$-*homogeneous.

5.10 Theorem: The $N^* \in K_{\aleph_1}$ from 5.9, is unique, (even not depending on the choice of $D_\alpha(N)$'s) universal and model-homogeneous.

Proof:

Uniqueness: For $\ell = 0,1$ let N^ℓ_α, D^ℓ_α $(\alpha < \omega_1)$ be as in 5.6, 5.9, and we should prove $\bigcup_{\alpha<\omega_1} N^0_\alpha \cong \bigcup_{\alpha<\omega_1} N^1_\alpha$. As $D^\ell_\alpha(\alpha < \omega_1)$ is increasing and continuous, $|D^\ell_\alpha| \le \aleph_0$ and $\bigcup_{\alpha<\omega_1} D^\ell_\alpha = D$, clearly there is a closed unbounded $S \subseteq \omega_1$, such that $\alpha \in S \Rightarrow D^0_\alpha = D^1_\alpha$. Let $S = \{\alpha(i) : i < \omega_1\}$, $\alpha(i)$ increasing and continuous. Now we define by induction on $i < \omega_1$, an isomorphism f_i from $N^0_{\alpha(i)}$ on $N^1_{\alpha(i)}$, increasing with i. For $i = 0$ use the $\aleph_0$-categoricity of K, and for limit i, $f_i = \bigcup_{j<i} f_j$. Suppose f_i is defined, then by 5.4(2) f_i maps $D^0_{\alpha(i+1)} = D^0_{\alpha(i+1)}(N^0_{\alpha(i)})$ onto $D^0_{\alpha(i+1)}(N^1_{\alpha(i)})$ and by the choice of S, $D^0_{\alpha(i+1)} = D^1_{\alpha(i+1)}$. By the assumption on the N^ℓ_α, $N^\ell_{\alpha(i+1)}$ is $(D^\ell_{\alpha(i+1)}(N^\ell_{\alpha(i)}), \aleph_0$-*homogeneous. Summing up those facts and 5.6(e) we see that we can extend f_i to an isomorphism from $N^0_{\alpha(i+1)}$ onto $N^1_{\alpha(i+1)}$.

Now $\bigcup_{i<\omega_1} f_i$ is the required isomorphism.

Universality. Let $M \in K_{\aleph_1}$, so $M = \bigcup_{\alpha<\omega_1} M_\alpha$, M_α increasing, continuous and $|||M_\alpha||| \le \aleph_0$. We now define by induction $N_\alpha (\alpha < \omega_1)$ increasing and continuous, $|||N_\alpha||| = \aleph_0$, and for $\beta < \alpha$, N_α is $(D_\alpha(N_\beta), \aleph_0)$-*homogeneous, and $M_\alpha \prec N_\alpha$. The only novelity over 5.9 is the use of the $\aleph_0$-amalgamation property (which holds by 3.5, 0.2, 4.5). So the universality follows from the uniqueness.

Model-Homogeneity. So let N_α, D^*_α, N^* be as in 5.6, 5.9, and $M_\ell \prec N^*$ ($\ell = 0,1$) are countable, f an isomorphism from M_0 onto M_1. For some $\gamma < \omega_1$, $M_0, M_1 \prec N_\gamma$. Every type in $D(N_\gamma)$ is realized in N^* and N_β is $D_\beta(N_\gamma)$-*homogeneous for $\beta > \gamma$. For some $\alpha > \gamma$, every type from $D(M_\ell)$ realized in N_γ is from D_α (for $\ell = 0,1$) so by 5.6(f), N_α is $(D_\alpha(M_\ell), \aleph_0)$-*homogeneous for $\ell = 1,2$, so f can be extended to an automorphism of N_α, hence, as is the uniqueness part, to an automorphism of N^*.

5.11 Definition: 1) 1) If $N_0 \prec N_1 \in K_{\aleph_0}$, for $\ell = 1,2$, $p_\ell \in D(N_\ell)$, and they are definable in the same way (see 5.4, so both does not split over the same finite subset of N_0). Then we call p_1 the stationarization of p_0 over N_1.

2) If $N_0 \prec N_1 \in K_{\aleph_0}$, $\bar{a} \in N_1$, $gtp^*(\bar{a}, N_0, N_1)$ is the $p \in D^*(N_0)$ such that $N_1 \Vdash^{\aleph_1}_K \wedge p[\bar{a}]$. So $\bar{a}$ materialize (but not necessarily realize) $gtp(\bar{a}, N_0, N_1)$. We omit N_1 when clear from context.

3) We say $p = gtp^*(\bar{b}, N_0, N_1)$ is definable over $\bar{a} \in N_0$ if $gtp(\bar{b}, N_0, N_1) = p^- \stackrel{def}{=} p \cap L_{\omega_1,\omega}(N_0)$ is definable over $\bar{a}$ (note that $p \to p^-$ is a one-to-one mapping from $D^*(N_0)$ onto $D(N_0)$ by 5.5). So stationarization is defined for $p \in D^*(N_0)$ too.

5.11A Remark: 1) It is easy to justify the uniqueness implied by "the stationarization".

2) **Note:** if $N_0 \prec N_1 \prec N_2 \in K_{\aleph_0}$, $\bar{a} \in N_1$ then $gtp(\bar{a}, N_0, N_1) = gtp(\bar{a}, N_0, N_2)$.

5.12 Lemma: Suppose $N_0 \prec N_1 \in K_{\aleph_0}$, $p_\ell \in D^*(N_\ell)$, and p_1 is a stationarization of p_0 over N_1, *then* $p_1 \vdash p_0$ i.e. every sequences materializing p_1 materialize p_0.

Remark: 1) In [Sh1], [Sh2], [Sh3] the parallel proof of the claims were totally trivial, but here we need to invoke $I(\aleph_1, K) < 2^{\aleph_1}$.

2) A particular case can be proved in the context of §4.

Proof: So suppose N_0, N_1, p_0, p_1 contradict the claim. By 5.6(f) there is $N_2 \in K_{\aleph_0}$, and δ, $N_1 \prec N_2$ such that N_2 is $(D^*_\delta(N_0), \aleph_0)$-*homogeneous. We can find $p_2 \in D^*(N_2)$ which is the stationarization of p_0, p_1. So w.l.o.g. for some δ, N_1 is $(D^*_\delta(N_0), \aleph_0)$-*homogeneous δ, and w.l.o.g. p does not split over $\emptyset$ (by [Sh 3, VIII 1.3]). So for $N \in K_{\aleph_0}$, $N_0 \prec N$, let p_N be the stationarization of p over N and it can be defined really for any $N \in K_{\aleph_0}$. Now we define by induction on α a model $N_\alpha \in K_{\aleph_0}$ $(\alpha < \omega_1)$, $|N_\alpha| = \omega(1+\alpha)$, $[\beta < \alpha \Rightarrow N_\beta \prec N_\alpha]$; w.l.o.g. N_0, N_1 are the ones mentioned in the claim, and $\bar{a}_\alpha \in N_{\alpha+1}$ materialize the stationarization $p_\alpha \in D^*_\delta(N_\alpha)$ of p_0 over N_α, and for $\alpha < \beta$, N_β is $(D^*_\delta(N_\alpha), \aleph_0)$-homogeneous (see 5.6(f)). As for $\alpha > \beta$ $(N_\alpha, N_\beta) \equiv (N_1, N_0)$ clearly $\bar{a}_\alpha$ do not materialize p_β. Let $\mathcal{B}$ be $(H(\aleph_1), \in)$ expanded by N, $K \cap H(\aleph_1) \prec_K \restriction H(\aleph_1)$ and anything else which is necessary. For any $S \subseteq \omega_1$, let $\mathcal{B}_S$ be a model, satisfying some $\psi \in L_{\omega_1,\omega}(Q)$ which $\mathcal{B}$ satisfies and which "say" everything necessary, such that "the set of ordinals" of $\mathcal{B}_S$ is I, $I = \bigcup_{\alpha<\omega_1} I_\alpha$, even I_0 not well ordered, each I_α a countable initial segment of I, $\alpha < \beta \Rightarrow I_\alpha \subseteq I_\beta \wedge I_\alpha \neq I_\beta$, and $I - I_\alpha$ has a first element if and only if $\alpha \in S$, and then it is $s(\alpha)$. In particular ω and finite sets are standard in $\mathcal{B}_S$. For $s \in I$, $N^S_s = N_s^{\mathcal{B}_s}$ is defined naturally, and so is $N^S = N^{\mathcal{B}_s}$. Let $N^S_\alpha = \bigcup_{s \in I_\alpha} N^S_s$ (see [Sh4], [Sh1]). Let $s+1$ be the successor of s in I.

W.l.o.g. there is a countable $\mathcal{B}^- \prec \mathcal{B}$such that $\mathcal{B}^- \prec \mathcal{B}_S$, and there is no first s, $\mathcal{B}_S \models$ "s is a countable ordinal, $\alpha < s$" for every $\alpha \in \omega_1 \cap |\mathcal{B}_S|$. Let $\delta = \omega_1 \cap |sB'_S|$. So if $\mathcal{B}_S \models$ "$s < t$ are countable ordinals $> \alpha$" for $\alpha < \alpha(*)$, then (N^S_t, N^δ_s) is $D^*_\delta(N^S_s)$-*homogeneous.

So w.l.o.g. N_1 is $D^*_\delta(N_0)$-*homogeneous.

If $\alpha \in S$ then clearly the type $p = p_{N^S_\alpha}$ satisfies

(a) p is materialized in N^S (i.e. in N^S_β for a club of β's,)

but

(b) for a closed unbounded $C \subseteq \omega_1$ for no $\beta \in C, \beta > \alpha$ does a sequence from N^S materialize both p and its stationarization over N^S_β [remember $N^S_\alpha = N^S_{s_\alpha}$ because $\alpha \in S$]),

[Proof: As below or weaken (B) by restircting β to $\beta \in C \cap S$]

and (c) for a closed unbound set of $\beta > \alpha$, N^S_β is $(D^*_\delta(N^S_\alpha), \aleph_0)$-*homogeneous.

We shall prove that every $\alpha < \omega_1$, if $\alpha \notin S$ then α cannot satisfy the statements (a)-(c) above.

This is sufficient for if $f : N^{S(1)} \to N^{S(2)}$ (for $S(1), S(2) \subseteq \omega_1$) is an isomorphism, then for a closed unbounded set S^* of $\alpha < \omega_1$ f maps $N^{S(1)}_\alpha$ onto $N^{S(2)}_\alpha$, hence the property above is preserved, hence $S(1) \cap S^* = S(2) \cap S^*$. But there are $2^{\aleph_1}$, $S_i \subseteq \omega_1$ such that for $i \neq j$, $S_i - S_j$ is stationary.

So suppose $\alpha \notin S$, p are such that for even an unbounded set of β's N^S_β is $(D^*_\delta(N^S_\delta), \aleph_0)$-homogeneous, $p \in D^*_\delta(N^S_\alpha)$ is materialized by $\bar{a} \in N^S$ in N^S_β, and we shall get a contradiciton.

There are elements $0 = t(1) < t(1) < \cdots < t(k)$ of I, and $\bar{a}_{\ell+1} \in (N^S_{t(\ell)+1} - N^S_{t(\ell)}) \cup N^S_{t(\ell-1)}$ [stipulating $N^S_{t(a)}$ as N^S_0. such that $\bar{a} \subseteq \bar{a}_k$, $\bar{a}_\ell \subseteq a_{\ell+1}$, and $gtp(\bar{a}_{\ell+1}, N_{t(\ell)}, N_{t(\ell+1)})$ is definable over $\bar{a}_\ell$ [why they exist? because of the sentence saying that for every $\bar{a}$ we can find such $k, t(\ell)(\ell \leq k)$ $\bar{a}_\ell(\ell \leq k)$ as above is satisfied by $\mathcal{B}$ so we could have made $\mathcal{B}_S$ to inherit it by the choice of ψ above] It follows that $gtp(\bar{a}, N^S_{t(\ell)}, N^S_{t(k)})$ is definable over $\bar{a}_\ell$.

For some ℓ, $t(\ell) \in I_\alpha$, $t(\ell+1) \notin I_\alpha$. As $\alpha \neq S$ we can choose $t(*) \in I - I_\alpha$, $t(*) < s(\ell+1)$. As α, p satisfies (c), for some $\beta \in S$, $s_\beta > t(k)$ and N^S_β is $(D^*_\delta(N^S_\alpha), \aleph_0)$-*homogeneous. Now N^S_β, $N^S_{t(\ell+1)}$ are isomorphic over $N_{t(*)}$ (being $(D^*_\delta(N^S_{t(*)}), \aleph_0)$-*homogeneous by the choice of $\mathcal{B}_S$ as in proving (c). So $N^S_{t(\ell+1)}$ is $(D^*_\delta(N^S_\alpha), \aleph_0)$-*homogeneous too hence $(N^S_{t(\ell+1)}, N^S_\alpha) \equiv (N_1, N_0)$.

As clearly N^S_α, $N^S_{t(*)}$ are $(D^*_\delta(N^S_{t(\ell)+1})), \aleph_0)$-*homogeneous there is an isomorphism f_0 from N^S_α onto $N^S_{t(*)}$ over $N_{t(\ell)+1}$. As $N^S_{t(\ell+1)}$ is $(D^*_\delta(N^S_{t(*)}), \aleph_0)$-*homogeneous and $(D^*_\delta(N^S_\alpha), \aleph_0)$-*homogeneous we can extend f_0 to an automorphism f_1 of $N^S_{t(\ell+1)}$. Let γ satisfy $s(\gamma) \geq t(k)+1$. As $gtp(\bar{a}_k, N^S_{t(\ell+1)})$ is definable over $\bar{a}_\ell$, and $\bar{a}_\ell = f_0(\bar{a}_\ell) = f_1(\bar{a}_\ell)$ (as $\bar{a}_\ell \in N^S_{t(\ell)+1}$), and $N^S_{\gamma+1}$ is $(D^*_\delta(N^S_{t(\ell+1)}), \aleph_0)$-*homogeneous, we can extend f_1 to an automorphism f_2 of $N^S_{\gamma+1}$, $f_2(\bar{a}_k) = \bar{a}_k$.

So there is in N^S a sequence materializing both $gtp(\bar{a}, N^S_\alpha) = p_{N^S_\alpha}$ and its stationarization over $N^S_{t(\ell+1)}$: just $\bar{a}(\subseteq \bar{a}_k)$ (so use f_2).

This contradicts the assumption, as $(N_1, N_0) \cong (N^S_{t(\ell+1)}, N^S_\alpha)$.

Remark: 1) In (a),(b), (c) instead "$p_{N^s_\alpha}$" we could use: for $p \in D(N^S_\alpha)$.

2) Imitating the proof, we can show that (c) holds for any $\alpha < \omega_1$; and we can waive (b).

5.13 Claim: 1) If $\bar{a} \in N_0 \prec N_1 \prec N_2 \in K_{\aleph_0}$, $\bar{b} \in N_2$, $p_1 = gtp(\bar{b}, N_1, N_2)$ is definable over $\bar{a}$, then $p_0 = gtp(\bar{b}, N_0, N_2)$ is definable in the same way over $\bar{a}$, hence $gtp(\bar{b}, N_1, N_2)$ is its stationarization.

2) For a fixed countable $M \in K_{\aleph_0}$, to have a common stationarization is an equivalence relation over $\{p :$ for some $N \prec M, p \in D^*(N)\}$, (and we can choose the common stationarization in $D^*(M)$).

3) If $N_\alpha \in K_{\aleph_0}$ $(\alpha \leq \omega + 1)$ is $\prec$-increasing and continuous and $\bar{a} \in N_{\omega+1}$ then for some $n < \omega$, for every k, $n < k \leq \alpha \leq \omega$ implies $gtp(\bar{a}, N_\alpha, N_{\omega+1})$ is the stationarization of $gtp(\bar{a}, N_k, N_{\omega+1})$.

4) If $N \prec M \in K$, $N \in K_{\aleph_0}$, $\bar{a} \in M$, then for all $M' \in K_{\aleph_0}$, satisfying $\bar{a} \in M'$, $N \prec M' \prec M$, $gtp(\bar{a},N,M')$ is the same, we call it $gtp(\bar{a},N,M)$ (the new point is that M is not necessarily countable).

5) Suppose $N_0 \prec N_1$ (in K), $\bar{a} \in N_1$, then there is a countable $M \prec N_0$, such that for every countable M', $M \prec M' \prec N_0$, $gtp(\bar{a}, M', N_1)$ is the stationarization of $gtp(\bar{a}, M, N_1)$.

6) Part 3) holds for $N_\alpha \in K$ too and any limit ordinal instead of ω.

Proof: 1) W.l.o.g. for some α, N_2 is $(D^*_\alpha(N_0), \aleph_0)$-*homogeneous, and $(D^*_\alpha(N_1), \aleph_0)$-*homogeneous. Let $p_2 \in D^*(N_2)$ be the stationarization of p_1 for N_2.

By 5.5 we can use 5.10 for p_1,p_2.

So by 5.12 $p_2 \vdash p_1$. On the other hand, clearly there is an isomorphism f_0 from N_0 onto N_1, $f_0(\bar{a}) = \bar{a}$; and by the assumption above on N_2, f_0 can be extended to an automorphism f_1 of N_2.

Note that f_1 maps $gtp(\bar{b},N_0)$ to $gtp(f_1(\bar{b}),f_1(N_0))$ but $gtp(\bar{b}, N_0) = p_0$, $gtp(f_1(\bar{b}),f_1(N_0)) = gtp(\bar{b}, N_1) = p_1$. So f_1 maps p_0 to p_1 and p_2 to itself. So as p_2 is the stationarization of p_1, p_2 is a stationarization of p_0, so by 2) we shall finish.

2) Trivial

3) By 1)

4) Easy

5) By 3) and 4)

6) Easy by now.

5.14 Definition: By 5.13 we can define $D(N)$, $gtp(\bar{a}, N, M)$ and stationarization for not necessarily countable $N \in K$. Everything still holds, except that maybe some p's are not materialized.

More formally,

a) if $N \in K_{\aleph_0}$, $M \in K$, $N \prec M$, $p \in D^*(N)$ *then* the stationarization of p over M is defined as in 5.11.

b) $D^*(N)) = \{q :$ for some $N \prec M, N \in K_{\aleph_0}$ and $p \in D^*(N)$, q is the stationarization of p over $N\}$.

c) $gtp(\bar{a}, N, M)$ (where $\bar{a} \in M$, $N \prec M$ both is K) is the stationarization over N of $gtp(\bar{a}, N', M)$ for every large enough countable $N' \prec N$.

5.15 Claim: Suppose $N_0 \prec N_1, N_2$, $N_1 \in K_{\aleph_0}$, $N_2 \in K_{\aleph_0}$, $\bar{a} \in N_1$. Then we can find M; $N_0 \prec M$, and K-embeddings f_ℓ of N_ℓ into M over N_0 $(\ell = 1,2)$ such that $gtp(f_1(\bar{a}), f_2(N_2), M)$ is a stationarization of $p_0 = gtp(\bar{a}, N_0, N_1)$ (so $f_1(\bar{a}) \notin N_2$).

Remark: This strengthens 3.5.

Proof: Let $p_2 \in D(N_2)$ be the stationarization of p_0. Clearly we can find an $\alpha < \omega_1$ (in fact, a closed unbounded set of α's) and N'_1, N'_2 from $K_{\aleph_0}$ which are $(D_\alpha(N_0), \aleph_0)$- homogeneous $N_\ell \prec N'_\ell (\ell = 1,2)$ and some $\bar{b} \in N'_2$ materialize p_2. But by 5.10 $\bar{b}$ materialize p_0 hence there is an isomorphism f from N'_1 onto N'_2 over N_0, $f(\bar{a}) = \bar{b}$. Now let $M = N'_2$,

$f_1 = f \restriction N_1, f_2 = id$.

5.16 Claim : Suppose $N_0 < N_1 < N_2 \in K_{\aleph_0}$, $\bar{a}_i \in N_i$, $(i = 0,1,2)$ $\bar{a}_0 \subseteq \bar{a}_1 \subseteq \bar{a}_2$, $gtp(\bar{a}_1, N_0, N_2)$ is definable over $\bar{a}_0$, and $gtp(\bar{a}_2, N_1, N_2)$ is definable over $\bar{a}_1$. *Then* $gtp(\bar{a}_2, N_0, N_2)$ is definable over $\bar{a}_0$, ~Moreover the definition depends only on the definitions mentioned previously.

Proof: Suppose for $\ell = 0,1$, $N_i^\ell, \bar{a}_i^\ell (i = 0,1,2)$ are as above, and the corresponding definitions are the same and f_0 is an isomorphism from N_0^0 onto N_0^1, $f_0(\bar{a}_0^0) = \bar{a}_0^1$. We shall show that for some N_*^ℓ, $N_2^\ell < N_*^\ell$, f_0 can be extended to an isomorphism f_* from N_*^0 onto N_*^1, $f_*(\bar{a}_2^0) = \bar{a}_2^1$. It is easy to check that this is sufficient.

We can find an $\alpha < \beta$, and for $\ell = 0,1$, $N_3^\ell, N_4^\ell \in K_{\aleph_0}$, such that $N_2^\ell < N_3^\ell < N_4^\ell$ and N_3^ℓ is $(D_\alpha(N_i), \aleph_0)$-*homogeneous for $i = 0,1,2$ and N_4^ℓ is $(D_\beta(N_i^\ell), \aleph_0)$-*homogeneous for $i = 0,1,2,3$. Now there is $\bar{a}_4^\ell \in N_4^\ell$ materializing the stationarization of $gtp(\bar{a}_2^\ell, N_1^\ell)$ over N_3^ℓ. So by 5.12, a_4^ℓ materialize $gtp(\bar{a}_2^\ell, N_1^\ell)$ hence there is an automorphism f^ℓ of N_4^ℓ, $f^\ell \restriction N_1^\ell = id$ and $f^\ell(\bar{a}_2^\ell) = \bar{a}_4^\ell$. There is also an isomorphism f_1 extending f_0 from N_3^0 onto N_3^1 $f_1(\bar{a}_1^0) = \bar{a}_1^1$, and then there is an isomorphism f_2 extending f_1 from N_4^0 onto N_4^1, $f_2(\bar{a}_4^0) = a_4^1$. Now $f_* \stackrel{def}{=} (f^1)^{-1} f_2 f^0$ is an isomorphism from N_4^0 onto N_4^1 extending f_0 and $f_*(\bar{a}_2^0) = \bar{a}_2^1$, so we finish.

5.17 Conclusion : 1) For any $N_0 < N_1 \in K_{\aleph_1}$, there is N_2, $N_1 < N_2 \in K_{\aleph_1}$ and N_2 is $(D(N_0), \aleph_0)$-*homogeneous.

2) Also 5.15 holds for $N_2 \in K_{\aleph_1}$ (but still $N_1 \in K_{\aleph_0}$).

3) In fact we can demand (in 5.17(2) hence in 5.15 too) $gtp(f_1(\bar{c}), f_2(N_2), M)$ is a stationarization of $gtp(\bar{c}, N_0, N_1)$ for every $\bar{c} \in N_1$.

4) $K_{\aleph_2} \neq \emptyset$.

Proof: 1) It is enough to prove that: if $p(\bar{x}, \bar{y}) \in D(N_0)$, $\bar{a} \in N_1$ materialize $p(\bar{x}, \bar{y}) \restriction \bar{x}$ in (N_1, N_0) then for some $N_2 \in K_{\aleph_1}$, $N_1 < N_2$ and for some $\bar{b} \in N_2$, $\bar{a} {}^\wedge \bar{b}$ materialize $p(\bar{x}, \bar{y})$ in (N_2, N_0). Let $M_0 < N_0$ be countable and $q \in D(M_0)$ be such that $p(\bar{x}, \bar{y})$ is a stationarization of q. Define $M_i (0 < i < \omega_1)$ such that $M_i < N_1$, $N_1 = \bigcup_{i < \omega_1} M_i$, $\langle M_i : i < \omega_1 \rangle$ is increasing continuous sequence of countable models, $\bar{b} \in M_1$. $M_i \cap N_0 \prec N_0$, M_i and

(*) for every $\bar{c} \in M_i$, $gtp(\bar{c}, N_0, N_1)$ is a stationarization of $gtp(\bar{c}, N_0 \cap M_i, M_i)$.

We can find $M_1^* \in K_{\aleph_0}$, $M_1 \prec M_1^*$, (and $|M_1^*| \cap |N_1| = |M_1^*|$), and $\bar{a} \in M_1^*$ such that $q = gtp(\bar{a}\char`^\bar{b}, M_0, M_1^*)$ $\bar{a}_0 \in M_1 \cap N_0$, $\bar{a}_1 \in M_1$, $\bar{a}_2 \in M_1^*$, $\bar{b} \subseteq \bar{a}_2$, $\bar{a} \subseteq \bar{a}_1$, $gtp(\bar{a}_2, M_1, M_1^*)$, $gtp(\bar{a}_1, M_1 \cap N_0, M_1)$ are definable over $\bar{a}_1$,$\bar{a}_0$ respectively. Now we define M_i^*, $1 < i < \omega_1$ by induction on i such that:

(i) $\langle M_j^* : 1 \leq j \leq i \rangle$ is $\prec$- increasing continuous.

(ii) M_j^* is countable.

(iii) $|M_j^*| \cap |N_1| = |M_j|$.

(v) $gtp(\bar{a}_2, M_j, M_j^*)$ is a stationarization of $gtp(\bar{b}, M_1, M_1^*)$.

For $j = 1$ we have it.

For $j > 1$ successor: use 5.15.

For j limit: let $M_j^* = \bigcup_{1 \leq i < j} M_i^*$, condition (v) holds by 5.13(3).

By 5.16 (and (*)) for every j, $gtp(\bar{a}_2, N_0 \cap M_j, M_j^*)$ is a stationarization of $gtp(\bar{a}_2, N_0 \cap M_1, M_1^*)$. Hence easily $gtp(\bar{a}\char`^\bar{b}, N_0 \cap M_j, M_j^*)$ is a stationarization of $gtp(\bar{a}\char`^\bar{b}, N_0 \cap M_1, M_1^*)$.

So by 5.14 and the first sentence in the proof, we finish.

2) Similar proof (or use the proof of part (3)).

3) W.l.o.g. $N_2 \cong N^*$ from 5.9 (as we can replace N_1 by an extension so use 5.10).

Also (by 5.17(1)) there is M, $N_2 < M \in K_{\aleph_1}$, such that M is $(D(N_2), \aleph_0)$-*homogeneous. Let $\alpha < \omega_1$ be such that for every $\bar{a} \in N_1$, $gtp(\bar{a}, N_0, N_1) \in D_\alpha(N_0)$. Let $M = \bigcup_{i<\omega_1} M_i$, M_i $<$-increasing continuous, countable. So for some i, $\alpha < i < \omega_1$, $M_i \cap N_2 < M$ and for every $\bar{c} \in M_i$, $gtp(\bar{c}, N_2, M)$ is stationarization of $gtp(\bar{c}, N_2 \cap M_i, M_i)$ and M_i is $(D_i(N_2 \cap M_i), \aleph_0)$-*homogeneous. Now we can find an isomorphism f_0 from N_0 onto $N_2 \cap M_i$ (as K is $\aleph_0$-categorical) and extend is to an automorphism f_2 of N_2 (by 5.10-model homogeneity). Also w.l.o.g. there is N'_1, $N_1 < N'_1 \in K_{\aleph_0}$, N'_1 is $(D_i(N_1), \aleph_0)$-*homogeneous, hence is $(D_i(N_0), \aleph_0)$-*homogeneous (see 5.6(f)), hence there is an isomorphism f'_1 from N'_1 onto M_i extending f_0. Now $f_0, f'_1 \restriction N_1, f_2, M$ show that amalgamation as required exists (we just change names).

4) Immediate - use 1) or 2) or 3) ω_2 times.

Definition 5.18: For any $D_* = D_\alpha$, $\alpha < \omega_1$ (or just any reasonable such D_*) we define:

1) $M <_{D_*} N$ if $M <_K N$ and for every $\bar{a} \in N$,

$$gtp(\bar{a}, M, N) \in D_*(M)$$

2) K_{D_*} is the class of $M \in K$ which are the union of a family of countable submodels, which is directed by $<_{D_*}$.

5.19 Claim : 1) K_{D_*}, $<_{D_*}$ satisfies all the axioms from section 1 if the transitivity axiom (Ax II) is satisfied. Which means that D_* is closed under the operation implicit in 5.16. In this case K_{D_*} satisfies the conclusion of part 1 and K_{D_α} is $PC_{\aleph_0}$.

Proof: 1) By checking.

2) Easy, as in the proof of 5.12.

5.20 Claim : Suppose $N_0 < N_\ell \in K_{\aleph_0}$ ($\ell = 1,2$), $\bar{c} \in N_2$, *then* there is M, $N_0 < M$, and K-embeddings f_ℓ of N_ℓ into M, such that

(i) for every $\bar{a} \in N_1$, $gtp(f_1(\bar{a}), f_2(N_2), M)$ is a stationarization of $gtp(\bar{a}, N_0, N_1)$.

(ii) $gtp(f_2(\bar{c}),f_1(N_1),M)$ is a stationarization of $gtp(\bar{c}, N_0, N_2)$.

Remark: This is one more step toward stable amalgamation: in 5.15 we get it for one $\bar{a} \in N_1$, in 5.17(3) for every $\bar{a} \in N_1$.

Proof: W.l.o.g. $gtp(\bar{c}, N_0, N_2)$ is definable over ϕ. Clearly we can replace N_ℓ by any N'_ℓ, $N_\ell < N'_\ell \in K_{\aleph_0}$, and w.l.o.g. $N_0 = N_1 \cap N_2$. So we an assume that for some D_α as in 5.19; N_ℓ is $(D_\alpha(N_0), \aleph_0)$-homogeneous. Like the proof of 5.12, we can find a countable order I, such that every element $s \in I$ has an immediate successor $s+1$, 0 is first element, and $Q \subseteq I$ (Q-the rational order), and models $M_s \in K_{\aleph_0}$, $(s \in I)$ such that $s < t \Rightarrow M_s < M_t$ etc. So by 5.13(6) for every initial segment J of I, and $t \in I$ such that $J < t$, if J has no last element, and $I-J$ has no first element then M_t is $(D_\alpha(M_J), \aleph_0)$-*homogeneous, where $M_J = \bigcup_{s \in J} M_s = \bigcap_{t \in I-J} M_t$. We let $N_0^J = M_J$, $N_1^J = M_I$, and N_2^J be a $(D_\alpha(N_0^J), \aleph_0)$-homogeneous model, $N_0^J < N_2^J$ and w.l.o.g. $N_1^J \cap N_2^J = N_0^J$. Clearly the triples $(N_0, N_1,, N_2)$, (N_0^J, N_1^J, N_2^J) are isomorphic, and let f_0,f_1,f_2 be appropriate isomorphisms such that $f_0 \subset f_1,f_2$. Now by 5.17(3), by a proper choice of N_2^J, there is $M^J \in K_{\aleph_0}$, $N_\ell^J < M^J$ $(\ell = 0,1,2)$ such that for every $\bar{a} \in N_1^J$, $gtp(\bar{a}, N_2^J,M^J)$ is the stationarization of $gtp(\bar{a}, N_0^J, N_1^J)$.

Suppose our conclusion fails, then $gtp(f_2(\bar{c}), N_1^J, M^J)$, is not the stationarization of $gtp(f_2(\bar{c}), N_0^J, M^J)$, moreover we can replace N_1^J by N_t for any $t \in I-J$. Let $p_J = gtp(\bar{c}, N_1^J, M^J) = gtp(\bar{c}, M_I, M^J)$, then it is easy to check that $J_1 \neq J_2 \Rightarrow p_{J_1} \neq p_{J_2}$, but as $Q \subseteq I$, we have continuoum such p_J. Moreover, we can ensure that for $J_1 \neq J_2$ as above there is an automorphism of M_I taking p_{J_1} to p_{J_2}, contradiction (alternatively, repeat the proof of 5.12).

5.21 Definition : 1) K has the symmetry property, when the following holds. If $N_0 < N_\ell < N_3$ $(\ell = 1,2)$ and for every $\bar{a} \in N_1$, $gtp(\bar{a}, N_2, N_3)$ is a stationarization of $gtp(\bar{a}, N_0, N_3)$, *then* for every $\bar{b} \in N_2$, $gtp(\bar{a}, N_1, N_3)$ is a stationarization of $gtp(\bar{b}, N_0)$.

2) If $N_0, N_1, N_2 < N_3$ satisfies the assumption and conclusion of (1) we say N_1,N_2 are in stable amalgamation over N_0. If only the hypothesis of (1) hold we say they are in a one sided stable amalgamation (then the order of N_1, N_2) is important.

3) We say N_1, N_2 can be uniquely [one sidedly] amalgamated stably over N_0 provided that: *if* $N_0 < M^i$, f_1^i,f_2^i *are* K- embeddings of N_1, N_2 (resp.) into M^i over N_0,

$|||M^i||| \leq |||N_1||| + |||N_2||| + \lambda(K)$

such that $f_1^i(N_1), f_2^i(N_2)$ are in [one side] stable amalgamation over N_0, for $i = 1,2$, *then* there is $M_1, N_0 < M_1$ and K-embeddings g^1, g^2 of M^1, M^2 resp. into M over N_0, such that $g^1 f_1^1 = g^2 f_1^2$, $g^1 f_2^1 = g^2 f_2^2$.

We note

5.22 Claim: For any $N_0 < N_1, N_2$, all from $K_{\aleph_0}$, we can find M, $N_0 < M \in K_{\aleph_0}$, and K-embeddings f_1, f_2 of N_1, N_2 resp. over N_0 into N such that $N_0, f_1(N_1), f_2(N_1)$ are in stable amalgamation.

Proof: We define by induction on $\zeta < \omega_1$ $\langle M_\alpha^\zeta : \alpha < \omega_1 \rangle$ and $\bar{c}_\zeta$ such that:

(i) $\langle M_\alpha^\zeta : \alpha < \omega_1 \rangle$ is $<$-increasing continuous and $M_\alpha^\zeta \in K$.

(ii) for $\alpha < \zeta$, $M_\alpha^\zeta = M_\alpha^\alpha$ and for $\xi < \zeta$, $\alpha < \omega_1$, $M_\alpha^\xi < M_\alpha^\zeta$.

(iii) for ζ limit, $M_\alpha^\zeta = \bigcup_{\xi<\zeta} M_\alpha^\xi$.

(iv) for $\zeta < \omega_1$, $M_{\alpha+1}^\zeta$ is $(D_{\alpha+1}(M_\alpha^\zeta), \aleph_0)$-*homogeneous.

(v) For every $\bar{c} \in M_{\alpha+1}^\zeta$, $gtp(\bar{c}, M_\alpha^{\zeta+1}, M_{\alpha+1}^{\zeta+1})$ is a stationarization of $gtp(\bar{c}, M_\alpha^\zeta, M_{\alpha+1}^\zeta)$.

(iv) $\bar{c}_\zeta \in M_{\zeta+1}^{\zeta+1}$, and for $\zeta+1 < \alpha < \omega_1$, $gtp(\bar{c}_\zeta, M_\alpha^\zeta, M_\alpha^{\zeta+1})$ is a stationarization of $gtp(\bar{c}_\zeta, M_{\zeta+1}^\zeta, M_{\zeta+1}^{\zeta+1})$.

(v) for every $p \in D(M_\alpha^\zeta)$ for some $\zeta, \alpha < \zeta < \omega_1$, and $gtp(\bar{c}_\zeta, M_{\zeta+1}^\xi, M_{\zeta+1}^{\xi+1})$ is a stationariation of p.

There is no problem doing this (by 5.20).

Now easily, for a closed unbounded set of $\zeta < \omega_1$,

$(*)_\zeta$ (a) M^ζ_ξ is $(D_\zeta(M^0_\xi), \aleph_0)$-*homogeneous.

(b) for every $\bar{c} \in M^\zeta_\xi$, $gtp(\bar{c}, \bigcup_{\alpha<\omega_1} M^0_\alpha, \bigcup_{\xi<\omega_1} M^\xi_\xi)$ is a stationarization of $gtp(\bar{c}, M^\zeta_0, M^\zeta_\xi)$.

So as in the proof of 5.17(3) we can finish.

Theorem 5.23: 1) Suppose in addition to the hypothesis of this section that $2^{\aleph_1} < 2^{\aleph_2}$ and $I(\aleph_2,K) < \mu_0(\aleph_2)$ (usually this is $2^{\aleph_2}$, always $> 2^{\aleph_1}$, see [Sh2] 6.3).

Then K has the symmetry property, and stable amalgamation in $K_{\aleph_0}$ is unique and always exists (and really one sided amalgamation is unique).

Proof: The main point will be to prove that if one sided stable amalgamation in $K_{\aleph_0}$ is not unique (with the stable side fixed, of course) then $I(\aleph_2,K)$ is big, as then 5.22 stable amalgamation exists, hence by the uniqueness, also the symmetry property follows.

The proof of the main point is by imitating [Sh 2] 6.3. The problem is that we still have not proven the existence of a superlimit model of K of cardinality $\aleph_1$ though we have a candidate N^* from 5.9. So we use N^*, but to ensure we get it at limit ordinals, we have to take a stationary $S_0 \subseteq \omega_1$, with $\omega_1 - S_0$ not small, and devote it to ensure this, using 5.22.

The point of using S_0 is as follows:

5.23A Definition : On $\bar{K}_{<\aleph_1} = \{\bar{N} : N = \langle N_\alpha : \alpha < \omega_1\rangle$ is $<$-increasing continuous, $N_\alpha \in K_{\aleph_0}$, $N_{\alpha+1}$ is $D_\alpha(N_\alpha)$-*homogeneous$\}$ we define a two place relation $<^a_S$ (for $S \subseteq \omega_1$), $\bar{N}^1 <^a_S \bar{N}^2$ if and only if for some closed unbounded $C \subseteq \omega_1$ for every $\alpha \in S \cap C$, $N^2_{\alpha+1} \cap \bigcup_{\alpha<\omega_1} N^1_i = N^1_{\alpha+1}$ and $N^1_\alpha, N^1_{\alpha+1}, N^2_\alpha$ are in (one sided) stable amalgamation inside $N^2_{\alpha+1}$.

Now $\bar{N}^1 <^b_S \bar{N}^2$ is defined similarly except that we ask N^1_α, N^2_α, $N^1_{\alpha+1}$, are in (one sided) stable amalgamation.

Now in the construction we define $\bar{N}^\eta$ $(\eta \in {}^{\omega_2>}2)$ such that for $\nu \lneqq \eta$ $\bar{N}^\nu <^a_{S_0} \bar{N}^\eta$, $\bar{N}^\nu <^b_{\omega_1} N^\eta$ and we use:

5.23B Fact: 1) If $\bar{N}^n \leq^a_{S_0} \bar{N}^{n+1}$ and let C_n exemplify this (as in the definition) and let $C_\omega = \bigcap_{n<\omega} C_n$, $C'_\omega = \{\alpha,\alpha+1 : \alpha \in C_\omega\}$, and let $N^\omega_\alpha = \bigcup_{n<\omega} N^n_\beta$ when $\beta = Min[C'_\omega - \alpha]$. Then $\bigcup_\alpha N^\omega_\alpha = \bigcup_n (\bigcup_\alpha N^m_\alpha)$, $\bigcup_a N^n_\alpha < \bigcup_\alpha N^\omega_\alpha$, and $\bar{N}^n \leq_{S^a_0} \langle N^\omega_\alpha : \alpha < \omega_1 \rangle$.

2) The similar statement for $\leq^b_S$.

[**Proof:** Like 5.24.]

5.24 Theorem: Suppose the conclusions of 5.23 hold. Then K has a superlimit model in $\aleph_1$.

Proof: We have a candidate N^* from 5.9. So let $\langle N_i : i < \delta \rangle$ be $<$-increasing, $N_i \cong N^*$. If $\delta = \omega_1$ this is very easy. If $\delta = \omega$, let $N_\omega = \bigcup_{i<\omega} N_i$, and we can find $N^\alpha_i (i \leq \omega, \alpha < \omega_1)$ increasing (by $<$) countable and continuous, for $i < j \leq \omega$, $N^\alpha_i = N_i \cap N^\alpha_j$, and for any $\bar{a} \in N^\alpha_\omega$, $i < \omega$, $gtp(\bar{a}, N_1, N_\omega)$ is a stationarization of $gtp(\bar{a}, N^\alpha_i, N_\omega)$. Hence by 5.22 for $\bar{b} \in N_i$, $gtp(\bar{b}, N^\alpha_\omega, N_\omega)$ is a stationarization of $gtp(\bar{b}, N^\alpha_1, N_\omega)$. The rest is easy as $N_i \cong N^*$ for $i < \delta$, w.l.o.g. if $\alpha < \beta < \omega_1$, $i < \omega$, N^β_i is $D_j(N^\alpha_i)$-*homogeneous).

§6 Counterexamples.

In [Sh1] the statement of Conclusion 3.5 was proved for the first time where K is the class of atomic models of a first order theory assuming Jensen's diamond $\Diamond_{\aleph_1}$ (taking $\lambda = \aleph_0$). In [Sh2] the same theorem was proved using $2^{\aleph_0} < 2^{\aleph_1}$ only (in its form $\overset{2}{\underset{\aleph_1}{\Phi}}$ see [DSh]). Let us now concentrate on the case $\lambda = \aleph_0$. We asked whether the assumption $2^{\aleph_0} < 2^{\aleph_1}$ necessary to get Conclusion 3.5. In this section we construct three classes of models K^1, K^2, K^3 such that K^1 satisfy all the axioms needed in the proof of Conclusion 3.5 (but it is not an abstract elementary class - fails to satisfy Ax.IV).

K^2 is $PC_{\aleph_0}$ and is axiomatizable in $L_{\omega_1,\omega}(Q)$.

K^3 is $PC_{\aleph_0}$ and is axiomatizable in $L(Q)$.

Now the common phenomena to K^1, K^2, K^3 is that all of them satisfy the hypothesis of Conclusion 3.5, i.e. for $\ell = 1,2,3$, $I(\aleph_0, K^\ell) = 1$ and the $\aleph_0$-A.P. fails in K^ℓ, but assuming $\aleph_1 < 2^{\aleph_0}$ and $MA_{\aleph_1}$ for $\ell = 1,2,3$, $I(\aleph_1, K^\ell) = 1$.

Definition 6.1. Let P an infinite set. A family F of infinite subsets of P is called stochastically independent (s.i.) if for every $\eta \in {}^{\omega>}2$ (*notation:* for $X \in F$ denote $X^0 = X$ and $X^1 = P - X$) the following set $\bigcap_{k<\ell(\eta)} X_k^{\eta[k]}$ is infinite.

Definition 6.2. (1) The class of models K^1 is defined by

$$K^1 = \{M : M = \langle |M|, P, Q, \underset{\sim}{\in}\rangle, |M| = P \cup Q, \underset{\sim}{\in} \subseteq P \times Q, |P| = \aleph_0, P \cap Q = \emptyset.$$

(*Notation* let $A_y = \{x \in P : x \underset{\sim}{\in} y\}$ for every $y \in Q$).

the family $\{A_y : y \in Q\}$ is s.i. and for every disjoint $a, b \in S_{<\aleph_0}(P)$), $\|M\| = |\{y \in Q : (\forall q \in a)[q \underset{\sim}{\in} y] \wedge (\forall q \in b)[q \underset{\sim}{\notin} b]\}|\}$

(2) The notation of substructure $\prec_{K_1}$ is defined by: For $M_1, M_2 \in K^1$, $M_1 \prec_{K_1} M_2 \Rightarrow^{df} |M_1| \subseteq |M_2|$, $P^{M_1} = P^{M_2}$, and for all disjoint $a,b \in S_{<\aleph_0}(P)$, $|\{y \in Q^{M_2} - Q^{M_1} : (\forall p \in a)[p \underset{\sim}{\in} y] \wedge (\forall q \in b)[q \underset{\sim}{\notin} y\} | \geq \aleph_0$.

Lemma 6.3: The class (K^1, $\prec_{K^1}$) satisfy

(0) Ax 0.

(1) Ax. I.

(2) Ax. II.

(3) Ax. III.

(4) Ax. IV fails even for $\lambda = \aleph_0$.

(5) Ax. V fails for countable models.

(6) Ax. VI holds with $\lambda(K^1) = \aleph_0$.

(7) for every $M \in K^1$, $\|M\| \leq 2^{\aleph_0}$.

Proof: (0), (1), (2), follows trivially from the definition.

(3) To prove that $M = \bigcup_{i<\lambda} M_i \in K^1$,

it is enough to verify that for every disjoint $a,b \in S_{\aleph_0}(P^M)$ that $|\{y \in Q^M : (\forall p \in a)(\forall q \in b)[p \underset{\sim}{\in} y \wedge q \underset{\sim}{\notin} y)| = \|M\|$. By the assumption that $\{M_i : i < \lambda\}$ is increasing from the definition of $\prec_{K^1}$ it follows that M_{i+1} has a new y as above, i.e. $y \in M_{i+1} - M_i$ at least λ many; Also for each i there are at least $\|M_i\|$ many y's. Together there are at least $\|M\|$, y's.

(4) Let $\{M_n : n < \omega\} \subseteq K^1_{\aleph_0}$ be an increasing chain, let $M = \bigcup_{n<\omega} M_n$ by (3) $M \in K^1_{\aleph_0}$. Since $|Q^M| = \aleph_0$ by Claim 6.5(a) below there exists $A \subseteq P^M$ infinite such that $\{A_y : y \in Q^M\} \cup \{A\}$ is s.i. Now define $N \in K^1$ by $P^N = P^M$, let $y_0 \notin Q^M$, $Q^N = Q^M \cup \{y_0\}$, and finally let $\underset{\sim N}{\in} = \underset{\sim M}{\in} \cup \{\langle p, y_0\rangle : p \in P^N \wedge p \in A\}$, $Q^N = Q^M \cup \{y_0\}$. Clearly for every $n < \omega$, $M_n \prec_{K^1} M$ but N is not an $\prec_{K^1}$- extension of $M = \bigcup_{n<\omega} M_n$ because

the second part in Definition 6.2(2) is violated.

(5) Let $N_0 <_{K^1} M \in K_1$ be given, assume $\|\|M\|\| = \aleph_0$; as in (4) define $N_0 \subseteq N$, $|N_0| \subseteq |N|$ by adding a single element to Q^{N_0} (from the elements of Q^M) it is obvious that $N_0 <_{K^1} N$, $M_0 <_{K^1} N$ but $M_0 \nless_{K^1} N_1$.

(6) By closing the set under the second requirement in Definition 6.2(1).

(7) Let $y \neq y_2 \in Q^M$ we show that $A_{y_1} \neq A_{y_2}$; if $A_{y_1} \subseteq A_{y_2}$ then $A_{y_2} \cap (P^M - A_{y_1}) = \varnothing$ contradiction to the requirement that $\{A_y : y \in Q\}$ is s.i. hence $|Q^M| \leq 2^{|P^M|} = 2^{\aleph_0}$.

Theorem 6.4. $(K^1, <_{K^1})$ satisfy the hypothesis of Conclusion 3.5. Namely

(1) $I(\aleph_0, K^1) = 1$.

(2) Every $M \in K^1_{\aleph_0}$ has a proper $<_{K^1}$-extension in $K^1_{\aleph_0}$.

(3) K^1 is closed under chains of length $\leq \omega_1$.

(4) K^1 violates the $\aleph_0$-A.P.

Proof: (1) Let $M_1, M_2 \in K^1_{\aleph_1}$, pick the following enumerations $|M_1| = \{a_n : n < \omega\}$, and $|M_2| = \{b_n : n < \omega\}$. It is enough to define an increasing chain of finite partial isomorphisms from M_1 to $M_2 \{f_n : n < \omega\}$ such that for every $k < \omega$ let $n(k) < \omega$ satisfy $a_k \in \mathrm{Dom}\ f_{n(k)}$ and $b_k \in \mathrm{Rang}\ f_{n(k)}$ (finally take $f = \bigcup_{n<\omega} f_n$ and this will be an isomorphism from M_1 onto M_2).

Define the sequence $\{f_n : n < \omega\}$ by induction on $n < \omega$: let $f_0 = \varnothing$, if $n = 2m$ denote $k = \min\{k < \omega : a_k \notin \mathrm{Dom}\ f_n\}$. Distinguish between the following two alternatives:

If $a_k \in P^{M_1}$ let $\{a'_0, \ldots, a'_{j-1}\} = Q^{M_1} \cap \mathrm{Dom}\ f_n$. W.l.o.g. there exists $i \leq j-1$ such that for all $\ell < i$ $a_k \underset{\sim}{\in}^{M_1} a'_\ell$ and for all $i \leq \ell < j-1$, $a_k \underset{\sim}{\notin} a'_\ell$. By the first requirement in Definition

6.2(1) there exists $y \in P^{M_2}$ such that $y \underset{\sim}{\in}^{M_2} f_n(a'_\ell)$ for all $\ell < i$ and for all $i \le \ell < j-1 y \underset{\sim}{\in}^{M_2} f_n(a'_\ell)$. Finally $f_{n+1} = f_n \cup \{\langle a_k, y\rangle\}$.

If $a_k \in Q^{M_1}$ let $\{a'_0, \ldots, a'_{j-1}\} = P^{M_1} \cap \mathrm{Dom}\, f_n$ and as before we may assume that there exists $i \le j-1$ such that for all $\ell < i$ $a'_\ell \underset{\sim}{\in}^{M_1} a_k$ and for all $i \le \ell < j-1$ $(a'_\ell) \underset{\sim}{\in}^{M_1} a_k$. By the second requirement in Definition 6.2(2) there exists $y \in Q^{M_2}$ such that $(\forall \ell < i)[f_n(a'_\ell) \underset{\sim}{\in}^{M_2} y]$ and $(\forall \ell, i \le \ell < j-1)[f_n(a'_\ell) \underset{\sim}{\notin}^{M_2} y]$. Now define $f_{n+1} = f_n \cup \{\langle a_k, y\rangle\}$.

When $n = 2m+1$ act similarly on $b_{\min\{k<\omega : b_k \notin \mathrm{Rang}\, f_n\}}$.

(2) First we prove the following:

Claim 6.5. (a) Let P be countable. For every countable family F of infinite subsets of P if F is s.i. then there exists $A \subsetneq P$ infinite such that $F \cup \{A\}$ s.i.

(b) If A, F are as in (a) then for every infinite $B \subsetneq P$ satisfying $|A \Delta B| < \aleph_0$ also $F \cup \{B\}$ is s.i.

(c) Moreover in (a) we can require in addition that: for every $a, b \in S_{\aleph_0}(P)$ disjoint there exists $A \subsetneq P$ as in (a) satisfying $a \subsetneq A$ and $A \cap b = \emptyset$.

Proof of Claim 6.5.

(a) Let $F^* = \{X \subsetneq P : (\exists n < \omega)(\exists X_0 \in F) \cdots (\exists X_{n-1} \in F)(\exists k < n)[X = \bigcap_{i<n} X_i^{\text{if} i<k}]\}$.

Clearly $|F^*| = \aleph_0$ hence we may assume $\{S_n : n < \omega\} = F^*$ such that for every $k < \omega$ there exists $n > k$ such that $S_n = S_k$. Denote $P = \{p_n : n < \omega\}$.

Now define by induction $i : \omega \to \omega$:

Let $i(0) = 0$.

If $n = 2k+1$, let $i(n) = Min\{\ell < \omega : (\exists j < \omega)[i(n-1) < j \wedge j < \ell \wedge$

$$p_\ell \in (S_k - \{p_{i(0)}, \ldots, p_{i(n-1)}\}) \wedge p_j \in S_k \cap (P - \{p_{i(0)}, \ldots, p_{i(n-1)}\})]\}.$$

If $n = 2k+2$, let $i(n) = Min\}\ell < \omega : (\exists j < \omega)[i(n-1) < j \wedge j < \ell \wedge$

$$p_\ell \in ((P-S_k) - \{p_{i(0)}, \ldots, p_{i(n-1)}\}) \wedge p_j \in (P-S_k) \cap (P - \{p_{i(0)}, \ldots, p_{i(n-1)}\})]\}.$$

It is easy to verify that the construction is possible (use of Definition 6.2(1)). Directly from the construction it follows that $A = \{p_i(n) : n < \omega\}$ is a set as required.

(b) Easy.

(c) Let $a,b \in S_{\aleph_0}(P)$ disjoint, and F a countable family of s.i. sets.

Let $A' \subseteq P$ as proved by (A). According to (b) also $A = (A' \cup a) - b$ satisfies: the family $F \cup \{A\}$ is s.i.

Return to the proof of Theorem 6.4(2): Let $F = \{A_y \subseteq P^M : y \in Q^M\}$. Let $\{s_n : n < \omega\}$ an enumeration of $S_{\aleph_0}(P^M)$ with repetitions such that for ever disjoint $a,b \in S_{\aleph_0}(P^M)$ there exists $n < \omega$ such that $s_{2n} = a$, $s_{2n+1} = b$ and for all $k < \omega, s_{2k} \cap s_{2k+1} = \varnothing$.

It is enough to define $\{F_n : n < \omega\}$ increasing chain of s.i. families such that $F_0 = F$ and for all $k < \omega$ and every disjoint $a,b \in S_{\aleph_0}(P)$ $(\exists n < \omega)$ $(\exists A \in F_n - F_k)[a \subseteq A \wedge A \cap b = \varnothing]$, $\bigcup_{n<\omega} F_n$ enables us to define $N \in K^1_{\aleph_0}$ satisfying $N >_{\neq K^1} M$ as required. Assume F_n define; Define F_{n+1}; apply Claim 6.5(c) on F_n when substituting $a = s_{2n}, b = s_{2n+1}$ let $A \subseteq P$ be supplied by the Claim and define $F_{n+1} = F_n \cup \{A\}$. It is easy to check that $\{F_n : n < \omega\}$ satisfies our requirements.

(3) This is Ax III which we checked in Lemma 6.3(3).

(4) Let $M \in K^1_{\aleph_0}$ we shall find $M_\ell \in K^1_{\aleph_0}$ ($\ell = 0,1$), $M_\ell >_{K^1} M$, which cannot be amalgamated over M. Choose by Claim 6.5(1) $A \subseteq P$ infinite such that $\{A_y \subseteq P^M : y \in Q\} \cup \{A\}$ is s.i.; define $M_0 > M$ as the definition of N in part (2) choosing $F_0 = F \cup \{A\}$ and M_1 define as N and was constructed in (2) choosing $F_0 = F \cup \{P - A\}$. Clearly M_0, M_1 cannot be amalgamated over M (since the amalgam must contain a set and its complement.)

Theorem 6.6. Assume $MA_{\aleph_1} \wedge 2^{\aleph_0} > \aleph_1$. The class $(K^1, <_{K^1})$ is categorical in $\aleph_1$.

Proof: Let $M, N \in K^1_{\aleph_0}$. By repeated use of the idea in the proof of Lemma 6.3(6) for Ax. VI we get increasingly continuous chains $\{M_\alpha : \alpha < \omega_1\}, \{N_\alpha : \alpha < \omega_1\} \subseteq K^1_{\aleph_0}$ such that $M = \bigcup_{\alpha<\omega_1} M_\alpha$ and $N = \bigcup_{\alpha<\omega_1} N_\alpha$ such that for $\alpha < \beta$, $M_\alpha <_{K^1} M_\beta, N_\alpha <_{K^1} N_\beta$.

Now define a forcing notion which supplies an isomorphism $g : M \to N$.

$R = \{f \mid f$ is a partial finite isomorphism from M into N satisfying

$$(\forall \alpha < \omega_1)(\forall \alpha \in \text{Dom } f)[\alpha \in M_\alpha \Leftrightarrow f(\alpha) \in N_\alpha\},$$

the order is inclusion. It is easy to check that if $G \subseteq R$ is a generic then $g = \bigcup G$ is a *partial* isomorphisms from M to N, we show that Dom $g = |M|$: For every $\alpha \in |M|$ define $D_\alpha = \{f \in P : \alpha \in \text{Dom } f\}$, it is easy to show that for all $\alpha \in |M|$ the set D_α is dense. For $\alpha \in |M|$ let $\alpha(a) = Min\{\alpha < \omega_1 : \alpha \in M_\alpha\}$. Let $f \in P$ be a given condition, it is enough to find $h \in D_\alpha$ such that $f \subseteq h$ and $a \in$ Dom b. Let $A =$ Dom f, assume $B, C \subseteq A$ disjoint such that $B \cup C = A$ and $B =$ Dom $f \cap P^M, C =$ Dom $f \cap Q^M$. W.l.o.g. $b \notin B \cup C$. If $a \in P^M$ let $p(x,C) = tp(a,C)$. From the definition of K^1 there exists $b \in P^N$ such that $N \models p[b, f(C)]$. If $a \in Q^M$ we can find $b \in Q^{N_{\alpha(0)}} - \bigcup_{\beta<\alpha(0)} N_\beta$, realizing $f(tp(a, B))$, $b \notin f(C)$. Finally, let $h = f \cup \{\langle a,b \rangle\}$. If $a \in Q^M$ act as earlier but take the types of a over B (see the proof of Theorem 6.4(1)).

The proof that Rang $(g) = |N|$ is analogous to the proof that Dom $g = |M|$. Namely it suffices to prove existence of a directed subset G (in the universe) of R which is generic enough. This is the place we use $MA_{\aleph_1}$. In order to use MA we just have to show that R has the c.c.c. Let $\{f_\alpha : \alpha < \omega_1\} \subseteq R$ be given. it is enough to find $\alpha, \beta < \omega_1$ such that f_α, f_β has common extension. Without loss of generality we may assume $|M| \cap |N| = \emptyset$. By the finitary Δ-system lemma there exists $S \subseteq \omega_1$, $|S| = \aleph_1$ such that $\{\text{Dom } f_\alpha \cup \text{Range } f_\alpha : \alpha \in S\}$ is a Δ-system with kernel A. Let $B \subseteq |M|$, $C \subseteq |N|$ such that $A = B \cup C$, namely for all $\alpha \in S$ $f_\alpha \restriction B \to C$ but the number of possible functions from B to C is $|C|^{|B|} < \aleph_0$. Hence there exists $T \subseteq S, |T| = \aleph_1$ such that for all $\alpha, \beta \in T$, $f_\alpha \restriction B = f_\beta \restriction B$ but we by the choice of Δ-system for every $\alpha \in T$ we have $P^M \cap \text{Dom } f_\alpha \subseteq B$, $P^N \cap \text{Rang } f_a \subseteq C$, therefore for all $\alpha, \beta \in T$, $f_\alpha \cup f_\beta \in R$ and in particular then there exists $\alpha \neq \beta < \omega_1$ such that $f_\alpha \cup f_\beta \in R$.

In the terminology of [GSh1] Theorems 6.4 and 6.6 gives us together.

Conclusion 6.7. Assuming $2^{\aleph_0} > \aleph_1$ and $MA_{\aleph_1}$ there exists a nice category which has a universal object in $\aleph_1$, moreover it is categorical in $\aleph_1$.

Definition 6.8.

(1) $K^2 = \{M : M = \langle |M|, P, Q, \underset{\sim}{\in} \rangle,\ |P| = \aleph_0, |M| = P \cup Q, \in \subseteq P \times Q, P \cap Q = \emptyset$

$(\forall x \in Q) \forall a \in S_{<\aleph_0}(P)(\exists y \in Q)[A_x \,\Delta\, A_y = a]$ and

$(\forall k < \omega)(\forall y_0, \ldots, y_{k-1} \in Q)[\bigwedge\limits_{\ell < m < k} |A_{y_\ell} \,\Delta\, A_{y_m}| \geq \aleph_0 \rightarrow$

the set $\{A_{y_\ell} : \ell < k\}$ is s.i.] and

$Q(y) \wedge Q(z) \wedge (\forall x \in P)[x \underset{\sim}{\in} y \leftarrow\rightarrow x \underset{\sim}{\in} z] \rightarrow y = z,$

and for $k < \omega$ for some $y_0 \cdots y_k \in G$, $\bigwedge\limits_{\ell < m \leq k} |A_{y_\ell} \,\Delta\, A_{y_m}| \geq \aleph_0\}$.

(3) For $M_1, M_2 \in K^2$.

$$M_1 \prec_{K^2} M_2 \Leftrightarrow^{df} M_1 \subseteq M_2, P^{M_1} = P^{M_2}.$$

Theorem 6.9. $(K^2, \prec_{K^2})$ is an abstract elementary class which is categorical in $\aleph_0$ and the $\aleph_0$-A.P. fails. Assuming $2^{\aleph_0} > \aleph_1 + MA_{\aleph_1}$ we have $I(\aleph_1, K^2) = 1$.

It is defined by a sentence of $L_{\omega_1,\omega}$, $\prec$ is $L_{L_{\omega,\omega}}$ and $\prec_{\underset{\sim}{L}}$ whenever $L_{\omega,\omega} \subseteq \underset{\sim}{L} \subseteq L_{\omega_1,\omega}$.

Proof: Similar to Theorem 6.4 and 6.5.

Definition 6.10.

(1) $K^2 = \{M : M = \langle |M|, P, \underset{\sim}{Q}, \in, E\rangle, \langle |M|, P, \underset{\sim}{Q}, \in\rangle \in K^1$, E is

an equivalence relation on Q, every equivalence class countable and

$$(\forall x \in Q)(\forall a, b \in S_{<\aleph_0}(P))[a \cap b = \emptyset \rightarrow (\exists z)[zEx \wedge a \subseteq A_z \wedge b \cap A_z = \emptyset]\}.$$

(2) $M_1 \prec_{K^3} M_2 \Leftrightarrow^{df} M_1 \subseteq M_2$ and $\forall x \forall y [xEy \wedge x \in M_1 \rightarrow y \in M_2]$.

Theorem 6.11. (1) $K^1_{\aleph_1}$ has an axiomatization in $L(Q)$ and $\prec_{K^1}$ is $\prec^{**}$ from [Sh2].

(2) K^2 has an axiomatization in $L_{\omega_1,\omega}$ and $\prec_{K^2}$ is $\prec^*_{\omega_1,\omega}$ from [Sh2].

(3) K^3 has an axiomatization in $L(Q)$ and $\prec_{K^3}$ is $\prec^*$ from [Sh2].

(4) $(\forall \ell \in \{1,2,3\})[K^\ell$ is $PC_{\aleph_0}]$.

(5) If $MA_{\aleph_1}$ *then* K^ℓ is categorical in $\aleph_1$.

Proof: Easy.

6.12 Conclusion: Assuming $MA_{\aleph_1}$ there exists an abstract elementary class, which is $PC_{\aleph_0}$, categorical in $\aleph_0, \aleph_1$ but without the $\aleph_0$-amalgamated property.

Appendix: On stationary set

We represent the relevant facts from [Sh 6] (hopefully in a better way) and add slightly. This was written essentially by accident.

1. Definition: 1) For λ regular, a set $S \subseteq \lambda$ is called *good* if there is a sequence $\bar{a} = \langle a_i : i < \lambda \rangle$, a_i a subset of λ, such that for some closed unbounded $C \subseteq \lambda$:

$C \cap S \subseteq S^{*q}_{\lambda}[\bar{a}] = \{\gamma : (\exists a \subseteq \gamma)[\gamma = \sup a \wedge otp(a) < \gamma \wedge (\forall \alpha < \gamma)(\exists i < \gamma) a \cap \alpha = a_i] \ \ or \ \ \gamma = cf\,\gamma\}$

We say $\langle a_i : i < \lambda \rangle$ witness the goodness of S, and C exemplify this (p stands for positive, q for a variant n for negative.)

2) $I[\lambda]$ is the family of good subsets of α.

2. Lemma: 1) We can in 1.1 replace a_i by $\mathcal{P}_i$, $|\mathcal{P}_i| < \lambda$, $\mathcal{P}_i \subseteq \{a : a \subseteq \lambda$ is bounded, and "$a \cap \alpha = a_i$" by "$a \cap \alpha \in \mathcal{P}_i$" (and get an equivalent definition). [see 4) and 5) below]

2) we can demand in 1(1) that a has order type cf $(\gamma)$and $a_i \subseteq i$.

I.e. if for λ, $\bar{a}$ as in Definition 1(1) we let $S^{*p}_{\lambda}[\bar{a}] = \{\gamma < \lambda$: there is $a \subseteq \gamma$ of order type $cf\,\gamma$ such that $otp(a) = cf(\gamma)$, $\sup a = \gamma$ and $(\forall \alpha < \gamma)(\exists i < \gamma)[a \cap \alpha = a_i]\}$ we can use $S^{*q}_{\lambda}[\bar{a}]$ instead of $S^{*p}_{\lambda}[\bar{a}]$ in defining "a good set" (and hence $I[\lambda]$).

3) if $\langle a_i : i < \lambda \rangle$ witness the goodness of $S \subseteq \lambda$ and $\{a_i : i < \lambda\} \subseteq \{b_i : i < \lambda\} \subseteq \mathcal{P}(\lambda)$ *then* $\langle b_i : i < \lambda \rangle$ witness the goodness of S. In fact $S^{*p}_{\chi}(\langle b_i : i < \lambda \rangle) \subseteq S^{*p}_{\lambda}(\langle a_i : i < \lambda \rangle)$ *mod* D_{λ}.

4) $\langle a_i : i < \lambda \rangle$ witness that $S \subseteq \lambda$ is good iff $\langle \{a_i\} : i < \lambda \rangle$ witness that S is good.

5) If $\overline{\mathcal{P}}^{\ell} = \langle \mathcal{P}^{\ell}_i : i < \lambda \rangle$ are as in 2(1) for $\ell = 1,2$ and $\bigcup_i \mathcal{P}^1_i \subseteq \bigcup_i \mathcal{P}^2_i$ and $\overline{\mathcal{P}}^1$ witness that $S \subseteq \lambda$ is good *then* also $\overline{\mathcal{P}}^2$ witnesses it.

6) For λ uncountable regular, $\{\delta < \lambda : \delta$ a (weakly) inaccessible cardinal$\}$ belongs to $I[\lambda]$.

Proof: Trivial, e.g.

2) Let $\langle a_i : i < \lambda \rangle$ witness $S \subseteq \lambda$ is good.

For every limit $\delta < \lambda$ choose a closed unbounded subset C_δ of δ of order type $cf\,\delta$; let for $i < \lambda$, $\delta < \lambda$, $a_{i,\delta} = \{j \in a_i :$ the order type of $a_i \cap j$ belongs to $C_\delta\}$.

Let $\{a_{i,\delta} : i < \lambda, \delta < \lambda\} \cup \{\{i : i < \alpha\} : \alpha < \lambda\} = \{b_i : i < \delta\}$, let C exemplify $\langle a_i : i < \lambda\rangle$ witness the goodness of S.

Let $C_0 = \{\alpha \in C :$ for every $i < \alpha$ and limit $\delta < \alpha$ there is $\zeta < \delta$ such that $b_\zeta = a_{i,\delta}$ (if defined and α is a limit ordinal)$\}$.

Clearly $C_0 \subseteq C$ is closed unbounded in λ. Now for any $\gamma \in C_0 \cap S$ we know there is a set $a \subseteq \gamma$ such that $\sup(a) = \gamma$, $otp(a) < \gamma$, $\alpha \cap a = a_{i(\alpha)}$ for $\alpha \in a$ and $i(\alpha)$ is an ordinal $< \gamma$. Let $a^* = \{i \in a : otp(a \cap \alpha) \in C_{otp(a)}\}$. Now a^* is as required.

3. Lemma: 1) $I[\lambda]$ is a normal ideal, which include all non-stationary subsets of λ.

2) If $\lambda = \lambda^{<\lambda}$, *then* for some $S^{*n}_\lambda \subseteq \lambda$:

$I[\lambda] = \{S \subseteq \lambda : S \cap S_\lambda$ is not stationary$\} = \{S \subseteq \lambda : \langle a_i : i < \lambda\rangle$ witness S is good$\}$

for any $\langle a_i : i < \lambda\rangle$ enumerating $\{a \subseteq \lambda : |a| < \lambda\}$.

3) Always there is $S^{*n}_\lambda \subseteq T_\lambda \overset{def}{=} \{\delta < \lambda : \lambda^{<cf\,\delta} = \lambda\}$ such that $S \in I[\lambda] \wedge S \subseteq T_\lambda \Leftrightarrow S \subseteq \lambda \wedge S \cap S^{*n}_\lambda$ not stationary.

Proof: Easy.

4. Lemma: 1) If λ is regular, $\kappa < \lambda$, $(\forall \alpha < \lambda) |\alpha|^{<\kappa} < \lambda$ (e.g., $\lambda = \mu^+$, $\mu = \mu^{<\kappa}$) *then* $\{\delta < \lambda : cf(\delta) \le \kappa\} \in I[\lambda]$

2) Suppose $\lambda = \mu^+$, $cf(\mu) < \kappa < \mu$ and $(\forall \theta < \kappa)(\forall \chi < \mu)[\chi^\theta < \mu]$. Then there is $S \in I[\lambda]$ such that:

(*) *if* $\delta < \lambda, \theta < \kappa$, and $cf\,\delta = (2^\theta)^+$ or even just $(\forall \alpha < cf(\delta))$ $[|\alpha|^\theta < cf(\delta)]$ *then* for some closed unbounded $C_\delta \subseteq \delta$, $(\forall \alpha)[\alpha \in C_\delta \wedge cf(\alpha) \le \theta \rightarrow \alpha \in S]$.

3) For λ, μ, κ as in (2), there is a 2-place function c from λ to $cf\,\mu$ such that:

(a) for $\alpha < \beta < \gamma$, $c(\alpha,\gamma) \le Max\{c(\alpha,\beta), c(\beta,\gamma)\}$.

(b) $|\{\alpha < \beta : c(\alpha,\beta) = \gamma\}| < \mu$.

(c) $S^{*p}_{\lambda}[c] \overset{def}{=} \{\delta < \lambda$: δ has cofinality $\le \kappa$ and there is an unbounded $a \subseteq \delta$ such that $c \restriction a$ is bounded in $cf\,\mu$ (i.e. $(\exists\gamma < cf\,\mu)(\forall\alpha,\beta \in a)\,[\alpha < \beta \rightarrow c(\alpha,\beta) < \gamma]\}$ belongs to $I[\lambda]$.

Proof: Note that 4(1), is easy, and 4(2) follows from 8(1), 4(3). It is easy to satisfies (a), (b) of (4) and (c) follows [choose an increasing sequence $\langle \mu_i : i < cf\,\mu \rangle$ such that $\mu = \Sigma\{\mu_i : i < cf\,\mu\}$, and then define by induction on β, $\langle c(\alpha,\beta) : \alpha < \beta \rangle$such that (a) holds and

(b)$^+$ $\quad |\{\alpha < \beta : c(\alpha,\beta) = \gamma\}| = \mu_\gamma$.

Why (c) follows from (a) + (b)? Clearly for $\alpha < \lambda$, $i < cf(\mu)$, $\mathcal{P}_{\alpha,i} = \{a : a$ as a subset of $\{\beta : \beta < \alpha$ and $c(\beta,\alpha) < i\}$ of cardinality $< \kappa\}$ has cardinality $\le \mu$, so $\mathcal{P}_\alpha = \bigcup_{i<cf\,\mu} \mathcal{P}_{\alpha,i}$ has cardinality $\le \mu$. Now $S^{*p}_{\lambda}[\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle]$ is a subset of $S^{*p}_{\lambda}[c]$.

There are no problems].

5. Remark: 1) In 4(2), 4(3) we can replace $\lambda = \mu^+$ by $\lambda = \mu^{+\alpha}$, as α increases we get less information. See [Sh 6] xx.

2) In (3) really (a) + (b) implies (c) and note (7) below.

6. Definition: 1) A two place function c from an ordinal ζ to an ordinal ξ is called subadditive if:

for $\alpha < \beta < \gamma < \zeta \quad c(\alpha,\gamma) \le Max\{c(\alpha,\beta), c(\beta,\gamma)\}$

and $c(\alpha,\beta) = c(\beta,\alpha)$, $c(\alpha,\alpha) = 0$

2)$\lambda \rightarrow_p (S)^2_\theta$ mean: (for λ, θ regular cardinals, $S \subseteq \lambda$.)

Suppose

(*) *c is a two place function from* λ *to* θ, *c* subadditive.

Then for some closed unbounded $C \subseteq \lambda$, for every $\delta \in S \cap C$ of cofinality $> \theta$,

$(**)_\delta$ there is $A \subseteq \delta$, such that $\sup A = \delta$, and $\sup\{c(\alpha,\beta) : \alpha < \beta, \alpha \in A, \beta \in A\} < \theta$.

3) We say λ is θ-sawc (sub-additively weakly compact) *if:* for every subadditive two place function d from λ to θ, *there is* an unbounded subset A of λ such that $\sup\{c(\alpha,\beta) : \alpha < \beta,\ \alpha \in A,\ \beta \in A\} < \theta$.

7. Fact: In Definition 6(2) the following demand on $\delta \in S \cap C$ is equivalent to $(**)_\delta$ when $cf(\delta) > \theta$:

$(**)'_\delta$ there are $\alpha_i, \beta_i < \delta$ for $i < cf(\delta)$, $\delta = \bigcup_i \alpha_i = \bigcup_i \beta_i$ and $\sup\{c(\alpha_i,\beta_j) : i < j < cf(\delta)\} < \theta$.

Proof: If A is as in $(**)_\delta$ choose $\alpha_i, \beta_i \in A$ s.t. $\delta = \bigcup\{\alpha_i : i < cf\,\delta\}$, $\sup\{\beta_j : j < i\} < \alpha_i < \beta_i$, they are as required.

If $\alpha_i, \beta_i (i < cf(\delta))$ are as in $(**)'_\delta$, w.l.o.g. $[j < i \Rightarrow \alpha_i < \beta_i < \alpha_j < \beta_j]$, so as $cf(\delta) > \theta$ for some $\gamma_1 < \theta$

$$B \overset{def}{=} \{i : c(\beta_i, \alpha_{i+1}) = \gamma_1\}$$

is unbounded below $cf(\delta)$. Let

$$\gamma_0 = \sup\{c(\alpha_i,\beta_j) : i < j < cf(\delta)\} < \theta.$$

Now $A = \{\beta_i : i \in B\}$ is as required: for $j < i$ in B

$$c(\beta_j,\beta_i) \le Max\{c(\beta_j,\alpha_{j+1}), c(\alpha_{j+1},\beta_i)\} \le$$

$$Max\{\gamma_1,\gamma_0\}$$

8. Lemma: 1) Suppose λ,μ,κ are as in 4(2) (so 4(3)) and $\lambda \rightarrow_p (S)^2_{cf(\mu)}$, $S \subseteq \{\delta < \lambda : cf\,\delta < \kappa\}$ *then* $S \in I[\lambda]$.

2) If $(\forall\sigma)[\sigma^+ < \mu \rightarrow 2^\sigma < \lambda]$, $S \subseteq \{\delta < \lambda : cf\,\delta < \mu\}$, $S \in I[\lambda]$ and λ,θ are regular *then*

$\lambda \to_P (S)^2_\theta$.

3) Suppose λ, μ, κ are as in 4(2), c as in 4(3) (a),(b). Then for any $\delta < \lambda$ and $S \subseteq cf\,\delta$, there is an increasing continuous function $h : cf(\delta) \to \delta$, $\delta = \sup\{h(i) : i < cf(\delta)\}$, and a club $c \subseteq \delta$ such that

$$[cf\,\delta \to_P (S)^2_{cf\,\mu} \Rightarrow c \cap h''(S) \subseteq S^{*P}_\chi[c]]$$

9. Remark: Particularly assuming G.C.H. 4(3), 8(1), 8(2), 8(3) fits nicely.

Proof of 8: 1) Let c be a two place function satisfying 4(3) (a) + (b). By Definition 6, there is a closed unbounded $C \subseteq \lambda$ such that for $\delta \in C \cap S$ of cofinality $> cf(\mu)$, $(**)_\delta$ hold. Now $\{\delta < \lambda : cf\,\delta \le cf(\mu)\} \subseteq T_\lambda$ [by 4(2) as $\mu^{<cf\,\mu} \le \sum_{\substack{\chi<\mu \\ \theta<cf\,\mu}} \chi^\theta \le \sum_{\chi<\mu} \chi^{<\kappa}$ hence $\lambda^{<cf\,\mu} = (\mu^+)^{<cf\,\mu} = \mu^+ = \lambda$] so we can assume $cf(\delta) > cf(\mu)$. Now $(**)_\delta$ implies $\delta \in S^{*P}_\chi[c]$ (see 4(3)(c)), so by 4(3) we finish.

2) Let c be a two place function from λ to θ, subadditive. Let χ be regular large enough, and w.l.o.g. let $\langle a_i : i < \lambda \rangle$ exemplify $S \in I[\lambda]$ witnessed by C_0 with $otp(a_i) < \mu$ (see 2(2) above). Let $\langle N_i : i < \lambda \rangle$ be increasing continuous such that $N_i \prec (H(\chi), \in)$, $\langle a_i : i < \lambda \rangle \in N_0$, $N_i \cap \lambda$ is an ordinal, $\|N_i\| < \lambda$, and $\langle N_j : j \le i \rangle \in N_{i+1}$. Let $C = \{i < \lambda : N_i \cap \lambda = i$ and $i \in C_0\}$ (it is closed unbounded). Suppose $\delta \in C \cap S$, $cf(\delta) > \theta$, then there is $a \subseteq \delta = \sup \delta$ such that $(\forall \alpha \in a)[a \cap \alpha \in \{a_j : j < \delta\}]$ hence $(\forall \alpha \in a)$ $[a \cap \alpha \in N_\delta]$, and we also know $otp(a) = cf\,(\delta)$; and let $\{\alpha_i : i < cf(\delta)\}$ be an increasing enumeration of a. So there are $\alpha_i < \delta$, $[i < j \Rightarrow \alpha_i < \alpha_j]$, $\delta = \bigcup \{\alpha_i : i < cf\,\delta\}$ and for $i < cf(\delta)$, $\{\alpha_j : j < i\} \in N_{\beta_i}$ for some $\beta_i < \delta$. As for $i < cf\,\delta$, $|\{\alpha_j : j < i\}| < cf\,\delta < \mu$, so $2^{|\{\alpha_j\ j<i\}|} < \lambda$ hence $\{\zeta : \zeta < 2^{|\{\alpha_j : j<i\}|}\} \subseteq N_\delta$, so every subset of $\{\alpha_j : j < i\}$ belongs to $N_{\beta_{i+1}} \prec N_\delta$. As $cf(\delta) > \theta$ for some $\gamma < \theta$, $A = \{i < cf(\delta) \colon c(\alpha_i, \delta) < \gamma\}$ is unbounded below $cf(\delta)$, so by the previous sentence w.l.o.g. $A = cf(\delta)$. So $N_{\delta+1} \models (\exists x)(\forall y \in \{\alpha_j : j < i\})\ [c(y, x) \le \gamma \wedge \alpha_i < x]$ (as δ witness the $\exists x$) so there is such x in N_{β_i+1} call is β_i. So α_i, β_i are as required in $(**)'_\delta$ of Fact 7, so by 7 we finish.

3) Follows.

10. Lemma: 1) If $S \in I[\lambda]$ is stationary, and $(\forall \delta \in S)[cf(\delta) < \mu]$, and P is a μ-complete forcing notion ($\mu > \aleph_0$) then "$\Vdash_P$"S is a stationary subset of the ordinal λ"

2) If $S \subsetneq \{\delta < \lambda : cf(\delta) < \mu\}$ is stationary but included in S^{*n}_{λ} (see 3(3)), μ regular and $\lambda = \lambda^{<\mu}$ *then* for some μ-complete forcing notion P, $\Vdash_P$"S is not stationary" (in fact $P =$ Levi (μ, λ) is O.K.)

Remark: As for 10(2), it repeats Theorem 21, p. 366 of [Sh 6], Donder and Ben David note a defect: in the case $\lambda = \lambda^{<\lambda}$ (really $\lambda = \lambda^{<\mu}$) in the definition of the forcing P $(= \{\langle \alpha_i : i \le \zeta \rangle$: α_i increasing continuous $B_{\alpha_{i+1}} = \{\alpha_j : j \le i\}$) we forget to demand $\zeta < \mu$. [Note however that automatically $\zeta \le \mu$ as each B_i has cardinality $< \mu$, so we should just omit the maximal elements of P which make P totally trivial].

For the general case ($\lambda < \lambda^{<\mu}$) note that if some weak form of it fails, our definition of the set S^{*n}_{λ} make it empty. I.e. by Definition 8, p. 36 of [Sh 6], S^{*n}_{λ} make it empty. I.e. by Definition 2(1),2(1), p. 359 of [Sh 6] relaying on Definition 1, p. 358 of [Sh 6]. This demand "$S^{*n}_{\lambda} \subseteq gcf(x)$" is reasonable, as otherwise we cannot prove there is such a set. See here later. [18,19]

Proof: 1) Use $(\forall s)(s \in I[\lambda] \Rightarrow s \in I^{+}[\lambda])$ from 16(2) (see Definition 15)

2) Let $\langle a_i : i < \lambda \rangle$ list the subsets of λ of cardinality $< \mu$, each appearing λ times. If $P =$ Levi (μ,λ), in V^P λ has cofinality μ, so let $\langle \alpha_i : i < \mu \rangle$ be increasing, $\alpha_i < \lambda$, $\bigcup_{i<\kappa} \alpha_i = \lambda$. But forcing with P add no sequences of ordinals of length $<\mu$, so we can find inductively $j(i) < \lambda$, $j(i) > \bigcup \{j(\xi), \alpha_\xi : \xi < i\}$, $a_{j(i)} = \{\alpha_\xi : \xi < i\}$. Now $\{\delta < \lambda$: the set $\{j(i) : i < \mu\} \cap \delta$ is unbounded in $\delta\}$ is a club of λ in V^P, included in a good subset of λ from V.

10A Remark: It is natural to force with $Q(\langle a_i : i < \lambda \rangle) = \{\langle i_\zeta : \zeta < \xi^* \rangle : \zeta^* < \kappa, i_\zeta < \lambda,$ $[\zeta(1) < \zeta(2) \Rightarrow i_{\zeta(1)} < i_{\zeta(2)}]$, and $a_{i_\zeta} = \langle i_\xi : \xi < \zeta \}$.

* * *

In [Sh 6] we define S^{*n}_{λ} inside a larger set than $\{\delta < \lambda : \lambda^{<cf\,\delta} = \lambda\}$ (see 3(3)). We will present this addition, improved, i.e. $Gcf\,[\lambda]$, $gcf(\lambda)$ are bigger sets here than in [Sh 6, Definition 2].

11 Definition: 1) For a family F of subsets of θ let

$$tr(F) = \{A \cap \alpha : A \in F, \alpha < \theta\}$$

2) For θ regular uncountable let $club_{tr}(\theta) = Min\{\,|tr(F)| : F$ is a family of closed unbounded subsets of θ such that: every closed unbounded subset of θ contains some members of $F\}$.

Let $club_{tr}[\aleph_0] = \aleph_0$ and let F_θ exemplify $club_{tr}(\theta) = |F_\theta|$.

3) $Gcf[\lambda] = \{\theta : \theta$ is regular $\geq \aleph_0$ and, $\lambda = \lambda^{<\theta}$ or $club_{tr}(\theta) < \lambda\}$

4) $gcf[\lambda] = \{\delta < \kappa : cf\,\delta \in gcf[\lambda], cf(\delta) < \delta\} \cup \{\delta < \lambda : \delta$ a (weakly) inaccessible cardinal$\}$ $\cup \{\alpha < \lambda : \alpha = 0,$, or α successor ordinal$\}$

4) $gcf_{ac}[\lambda] = \{\delta \in gcf[\lambda] : cf\,\delta < \delta\}$

12 Fact: 1) If GCH, $\lambda > \aleph_0$ regular *then* $Gcf[\lambda] = \{\theta : \theta$ regular $< \lambda\}$, $gcf[\lambda] = \lambda$.

2) For regular uncountable θ, $\theta < club_{tr}(\theta) \leq 2^{<\theta} \leq 2^\theta$.

3) If $2^{<\theta} \leq \lambda$, (θ,λ regular) *then* $\theta \in Gcf(\lambda)$ [as this implies either $\lambda = 2^{<\theta}$ hence $\lambda = \lambda^{<\theta}$ or $\lambda > 2^{<\theta}$ hence $\lambda > club_{tr}(\theta)$].

13 Definition: 1) We call $\bar{a}$ an *enumeration* for λ if $\bar{a} = \langle a_i : i < \lambda\rangle$, each a_i a bounded subset of λ.

2) We call $\bar{a}$ a *rich enumeration* for λ if:

(i) $\bar{a}$ is an enumeration for λ

(ii) if $\lambda = \lambda^\theta$, (hence $\theta < \lambda$) *then* every subset of λ of cardinality $\leq \theta$ appears in a

(iii) if θ is an uncountable regular cardinal, and $club_{tr}(\theta) \leq \lambda$ *then* letting F_θ exemplify $club_{tr}(\theta) \leq \lambda$, for every limit ordinal $\delta < \lambda$ of cofinality θ, there is a closed unbounded subset $\langle \beta_i^\delta : i < \theta\rangle$ of δ (β_i^δ increasing continuous) such that

(*) for every $A \in F_\theta$ and $\zeta < \theta$, $\{\beta_i^\delta : i \in A \cap \zeta\}$ appear in $\bar{a}$.

3) In (1) (and (2)) we replace "enumeration" by $(<\mu)$-enumeration if we restrict ourselves to subsets of λ of cardinality $<\mu$ i.e. in (ii) $\theta \leq \mu$, in (iii) $\theta \leq \mu$.

4) For an unbounded subset S of λ, we say $\bar{a}$ is a rich enumeration for $(S, \theta)^{\ell}$ *if:*

(i) $\bar{a}$ is an enumeration for λ

(ii) if $\ell = 1$, $\lambda = \lambda^{<\theta}$ and every $b \subseteq \lambda$, $|b| < \theta$ appear in $\bar{a}$

(iii) if $\ell = 2$ $club_{tr}(\theta) \leq \lambda$, then for every $\delta \in S$ of cofinality θ the condition in (2) (iii) above holds.

14 Fact: 1) For every regular uncountable λ there is a rich enumeration;

2) For every $\lambda = cf\,\lambda > \mu$, λ has a rich μ-enumeration.

15 Definition: For λ regular uncountable

$I^+[\lambda] = \{S \subseteq \lambda$: for every cardinal $\chi > \lambda$ and $x \in H(\chi)$, for some closed $C \subseteq \lambda$, for every $\delta \in C \cap S$ there are a limit $\gamma < \delta$, and $N_i \prec (H(\chi), \in, x, \lambda)$, for $i < \gamma$, such that $\langle N_j : j \leq i \rangle \in N_i$, $N_i \cap \lambda$ is an ordinal $\alpha_i < \delta$ and $\delta = \bigcup_{i<\gamma} \alpha_i\}$.

16 Fact: 1) $I^+[\lambda]$ is a normal ideal on λ and in its definition w.l.o.g. $\gamma = cf\,\delta$,

2) $I[\lambda] \subseteq I^+[\lambda]$

3) If $S \subseteq gcf[\lambda]$ then: $S \in I[\lambda] \Leftrightarrow S \in I^+[\lambda]$

4) There is $S^{*n}_{\lambda} \subseteq gcf[\lambda]$, such that for every rich enumeration $\bar{a}$ for λ and $S \subseteq Gcf[\lambda]$: $S \in I[\lambda]$ *if and only if* $S \in I^+[\lambda]$ *if and only if* $S \cap S^{*n}_{\lambda} = \emptyset$ *mod* D_{λ} *if and only if* $S \subseteq S^{*p}_{\lambda}[\bar{a}]$ *mod* D_{λ}. We let $S^{*n}_{\lambda,\theta} = \{\delta < \lambda : cf(\delta) = \theta,\ \delta \in S^{*}_{\lambda}\}$ (this replace 3(3)) and

$$S^{*n}_{\lambda,<\theta} \stackrel{def}{=} \{\delta < \lambda : cf\,\delta < \theta,\ \delta \in S^{*n}_{\lambda}\}$$

5) for every rich enumeration $\bar{a}$ for λ, $gcd[\lambda] - S^{*n}_{\lambda}[\bar{a}] \equiv S^{*q}_{\lambda}$ *mod* D_{λ}.

6) for any $\theta < \lambda$, suppose (A) $\lambda = \lambda^{<\theta}$, $\{b \subseteq \lambda : |b| < \theta\} \subseteq \{a_i : i < \lambda\}$ (like 11(2)(ii) *or* (B) $club_{tr}(\theta) < \lambda$, F_{θ} exemplify it and $\bar{a}$ satisfies 11(2)(iii) for every $\delta \in S \subseteq \{\delta < \lambda : cf\,\delta > cf\,\theta\}$. *Then* $S \cap S^{*n}_{\lambda,\theta} = S - S^{*p}_{\lambda}[\bar{a}]$.

Proof: 1) The normality is easy, the "w.l.o.g. $\gamma = cf\,\delta$" is proved as in 2(2).

2) Let $S \in I[\lambda]$, so for some enumeration $\bar{a} = \langle a_i : i < \lambda\rangle$ for λ, $C \cap S \subseteq S^{*q}_{\lambda}[\bar{a}]$ for some closed unbounded $C \subseteq \lambda$. Let $\chi > \lambda$, $x \in H(\chi)$. We can find $\langle N_\zeta : \zeta < \lambda\rangle$ increasing continuous, $N_\zeta \prec (H(\chi), \in, x)$, such that $N_\zeta \cap \lambda$ is an ordinal $\|N_\zeta\| < \lambda$, $\langle N_j : j \leq \zeta\rangle \in N_{\zeta+1}$ and C, $\bar{a} \in N_0$. So $C' \stackrel{def}{=} \{\delta < \lambda : \delta \in C \text{ and } N_\delta \cap \lambda = \delta\}$ is a closed unbounded subset of λ.

Now for every $\delta \in C' \cap S$, there is a_i from $\bar{a}$, $otp(a_i) < \sup(a_i) = \delta$ and for $\alpha \in a_i$, $\alpha \cap a_i \in \{a_j : j < \delta\}$. As $\bar{a} \in N_0$ clearly $\{a_j : i < \delta\} \subseteq \bigcup_{\zeta<\delta} N_\zeta$. Let $a_i = \{\gamma_\varepsilon : \varepsilon < otp\ a_i\}$. Now we try to define by induction on $\varepsilon < otp(a_i)$ an ordinal $\zeta_\varepsilon < \delta$:

for $\varepsilon = 0$: $\zeta_\varepsilon = 0$

for ε *limit:* $\zeta_\varepsilon = \bigcup_{\beta<\varepsilon} \zeta_\beta$,

for ε *successor:* ζ_ε is the first ordinal ζ satisfying ζ is bigger than γ_ε and $\langle N_{\zeta_\beta} : \beta < \varepsilon\rangle$ belongs to N_{ζ_ε}.

The only reason for stopping is: ε limit $\bigcup_{\beta<\varepsilon} \zeta_\beta = \delta$; once this occurs at ε_0, $\langle N_{\zeta_\varepsilon} : \varepsilon < \varepsilon_0\rangle$ is as required [otherwise for limit and for zero there is no problem, and for ε successor, $\zeta_{\varepsilon-1}$ is defined and $< \delta$, so for some β, $\zeta_{\varepsilon-1} < \gamma_\beta < \delta$ [where $a_i = \{\gamma_\beta : \beta < otp\ a_i\}$) now $\langle \zeta_\beta : \beta < \varepsilon\rangle$ is definable inside the model $(H(\chi), \in)$ from the parameters $\langle N_j : j < \gamma_\beta\rangle$, $\langle \gamma_j : j < \beta\rangle$ only; as both are in $\bigcup_{j<\delta} N_j$, is $\langle \zeta_\beta : \beta < \alpha\rangle$, and similarly so is ζ_ε].

3) Fix $S \subseteq gcf[\lambda]$; by 16(2) it is enough to assume $S \in I^+[\lambda]$ and prove $S \in I[\lambda]$, we prove more in 16A below.

4) S^{*n}_{λ} is $gcf[\lambda] - S^{*p}_{\lambda}[\bar{a}]$ for any rich enumeration $\bar{a}$ for λ.

5), 6) Should be clear.

16A Subfact: If $S \subseteq gcf[\lambda]$, (λ regular uncountable) S belongs to $I^+[\lambda]$ and $\bar{a} = \langle a_i : i < \lambda\rangle$ is a rich enumeration for λ, *then* $S \subseteq S^{*p}_{\lambda}[\bar{a}]$ *mod* D_λ.

Proof of 16A: Let $x = \bar{a}$, $\chi = (2^\lambda)^+$, so as $S \in I^+[\lambda]$ (see Definition 15), there is a closed unbounded $C \subseteq \lambda$ such that (see 16(1)):

(*) for every $\delta \in C \cap \delta$ there is $\bar{N} = \langle N_i : i < cf(\delta)\rangle$ as in Definition 15.

Fix $\delta \in C \cap S$, and $\bar{N}$ and let $\theta = cf\,\delta$, $\alpha_i = N_i \cap \lambda$. Remember that $N_i \cap \{a_j : j < \lambda\} = \{a_j : j < \alpha_i\}$. We shall show that $\delta \in S^{*p}_\chi[\bar{a}]$, thus finishing.

Case 1: $\lambda^{<\theta} = \lambda$ (e.g. $cf\,\delta \le \aleph_0$).

In this case for each $i(*) < \theta$, $\{\alpha_i : i < i(*)\}$ belongs to $\{a_j : j < \lambda\}$ (as $\bar{a}$ is rich) and to $N_{i(*)+1}$ (as $\langle N_i : i < i(*)\rangle \in N_{i(*)+1}$, and $\lambda \in N_{i(*)+1}$); hence $\{\alpha_i : i < i(*)\}$ belongs to their intersection which is $\{a_j : j < \alpha_i\}$. So $\langle a_i : i < i(*)\rangle$ exemplify $\delta \in S^{*p}_\chi(\bar{a})$, as required.

Case 2: $cf\,\delta < \delta$, $club_{tr}(\theta) < \lambda$ where $\theta = cf\,\delta > \aleph_0$).

Let $F_{cf\,\delta}$ exemplify $club_{tr}(\theta) = |tr(F_\theta)|$, and let $\{\beta^\delta_i : i < \theta\}$ be as in Definition 13(1) (iii). So $A_0 = \{i < \theta : \beta^\delta_i = \alpha_i\}$ is a club of θ, hence for some club $A \in F_\theta$, $A \subseteq A_0$. By 13(1) (iii) for every $i(*) < \theta$, $\{\beta^\delta_i : i \in A,\ i < i(*)\}$ belongs to $\{a_i : i < \lambda\}$, but $A \cap i(*) \in \bigcup_{i<\theta} N_i$ [as $\theta < \delta$, hence w.l.o.g. $F_\theta \in \bigcup_{i<\theta} N_i$ hence $tr(F_\theta) \in \bigcup_{i<\theta} N_i$, but $|tr(F_\theta)| < \lambda$ hence $tr(F_\theta) \subseteq \bigcup_{i<\theta} N_i$]. Hence $\{\alpha_i : i \in A \cap i(*)\} \in \bigcup_{i<\theta} N_i$, so we finish.

Case 3: $\delta = cf\,\delta$.

Trivial.

17 Lemma: Suppose in V, $\lambda > \aleph_0$ is regular, $\theta \in Gcf[\lambda]$, so $S^{*n}_{\lambda,\theta}$ is defined.

Suppose further V^1 is an extension of the universe V (say same ordinals), $V^1 \models$ "$\lambda > \aleph_0$ is regular", and

$(*)_1$ $\quad V^1 \models$ "every subset of λ of cardinality $< \theta$ belongs to V", $\quad V \models$ "$\lambda = \lambda^{<\theta}$", or

$(*)_2$ $V^1 \models$ "F^V_θ satisfies for every club C of θ, there is $A \in F^V_\theta$, $A \subseteq C$" (but maybe $V^1 \models$ "θ not a cardinal") and $V \models$ "$|F^V_\theta| = club_{tr}(\theta) < \lambda$"

Then

(i) $V^1 \models$ "$cf\,\theta \in Gcf[\lambda]$ or $cf\,\theta = \aleph_0$"; and

(ii) $V^1 \models$ "$S^{*n}_{\lambda,cf\,\theta} \cap \{\delta : V \models cf\,\delta = \theta\} \equiv (S^{*n}_{\lambda,\theta})^V \bmod D_\lambda$" or equivalently: in V^1, $(S^{*n}_{\lambda,\theta})^V/D_\lambda$ is disjoint to every S/D_λ, $S \in I[\lambda]$.

Proof: Let $\bar{a}$ be a rich enumeration for λ in V.

By (*), $\bar{a}$ is still a rich enumeration in V^1 for $S = \{\delta < \lambda : V \models cf\,\delta = \theta\}$. By 16(6) we finish.

18 Lemma: If $\lambda > \aleph_0$ is regular, $S \in I^+[\lambda]$, $S \subseteq \{\delta < \lambda : cf\,\delta < \mu\}$, P is a μ-complete forcing notion *then*

$\Vdash_P$ "S is a stationary subset of λ (as an ordinal, λ may or may not be a cardinal)"

Complementary to 18 is

19 Lemma: Suppose $\theta \in Gcf(\lambda)$, $\aleph_0 < \theta \leq \mu = cf\,\mu < \lambda$ so $S^{*n}_{\lambda,\theta}$ is well defined.

1) If $\mu = \theta$, $\lambda = \lambda^{<\theta}$, $\Vdash_{Levi(\mu,\lambda)}$ "$(S^{*n}_{\lambda,\theta})^V$ is not stationary (as a subset of the ordinal λ, (remember $Levi(\theta,\lambda) = \{f : f$ a function from some $\alpha < \theta$ to $\lambda\}$, it is θ-complete).

2) If $S^{*n}_{\mu,\theta} = \emptyset$, $\lambda = \lambda^{<\theta}$, $\Vdash_{Levi(\mu,\lambda)}$ "$(S^{*n}_{\lambda,\theta})^V$ is not stationary".

3) In (1) and (2) we can replace Levi (θ,λ), by any forcing notion P which adds to λ no new subset of power $< \mu$ and $\Vdash_P$ "$cf\,\lambda = \mu$".

4) In (1),(2) we can replace "$\lambda = \lambda^{<\theta}$" by $club_{tr}(\theta) < \lambda$, if we replace Levi (μ,λ) by $Levi(\lambda,\lambda^{<\theta}) * \underset{\sim}{Levi}(\mu,\lambda)$.

Remark: A more general forcing is as follows: Let $\theta < \lambda$, $\kappa \leq \theta$, $\bar{b} = \langle b_i : i < \lambda\rangle$ exemplify that $S_\theta \in I[\theta]$ and $[\delta < \theta \wedge \delta \in S_\theta \Rightarrow cf\,\delta < \kappa]$ or just for some $\sigma = cf\,\sigma < \kappa$, $S_\theta = \{\delta < \theta : cf\,\delta = \sigma\}$, $S \subseteq \{\delta < \lambda : cf\,\delta = \sigma\}$ and $Q^{\bar{b},\theta} = \{\langle i_\zeta : \zeta < \zeta^*\rangle : \zeta^* < \theta,\ i_\zeta < \lambda,$

$[\zeta(1) < \zeta(2) \Rightarrow i_{\zeta(1)} < i_{\zeta(2)}]$, $a_{i_\zeta} = \{i_\xi : \xi \in b_\zeta\}\}$.

20 Claim: Suppose $\theta = cf\,\theta > \aleph_0$ for the regular cardinals λ and μ, $\lambda > \mu$ and $club_{tr}(\theta) < \mu$.

1) Given S^{*n}_λ, S^{*n}_μ, there is a club $C \subseteq \lambda$ such that: *for every* $\delta \in C$ of cofinality μ, *there is* an increasing continuous sequence $\langle \alpha_i : i < \mu \rangle$, $\bigcup_{i<\mu} \alpha_i = \delta$ and a club c of μ satisfying $[i \in c \wedge cf\, i = \theta \wedge i \notin S^{*n}_\mu \Rightarrow i \notin S^{*n}_\lambda]$.

2) If $\bar{a}$ is a rich enumeration for (λ,θ), then $\{\delta < \lambda : cf\,\delta = \mu$ implies that for some α_i, β_i $(i < \mu)$: $\langle \alpha_i : i < \mu \rangle$ is increasing continuous with limit δ, $\beta_i < \mu$ and defining for $i < \mu$, $b_i = \{j : a_j \in a_{\beta_i}\}$, $\langle b_i : i < \mu \rangle$ is a rich enumeration for $(\mu,\theta)\} \in D_\lambda$.

21. Lemma: 1) If κ is supercompact and e.g. $\lambda > \kappa > cf\,\lambda$, *then* $I[\lambda^+]$ is a proper ideal: $\lambda^+ \notin I[\lambda^+]$.

3) After suitable collapses, e.g. $cf\,\lambda = \aleph_0 < \lambda$ but still $\lambda^+ \notin I[\lambda^+]$.

22. Problem: 1) Is G.C.H. + $\{\delta < \aleph_{\omega+1} : cf\,\delta > \aleph_1\} \notin I[\aleph_{\omega+1}]$ consistent with ZFC.

2) Is

(*)$2^{\aleph_0} > \aleph_{\omega+1}$+ "there is no stationary $S \in I[\aleph_{\omega+1}]$"

consistent with ZFC?

3) Is

(*) $2^{\aleph_0} > \aleph_{\omega+1}$+ for no ultrafilter D on ω, $cf(\prod_n(\aleph_n, <)/D) = \aleph_{\omega+1}$

consistent with ZFC.

Remark: "$\aleph_{\omega+1}$ is a Jonson cardinal" implies (*) of (3) (see [Sh 9] which implies (*) of (2) (see [Sh]).

Having F cause slight inconvenience.

We define by induction on $\alpha < \lambda^+$, (M_α, N_α, a), and N^*_γ, N^*_γ, $M^*_\gamma, f_\gamma, g_\gamma$: for suitable γ's such that

(A) $M_\alpha, N_\alpha, M^*_\alpha, N^*_\alpha$ are isomorphic to M^*.

(B) $M_\beta (\beta \leq \alpha)$ is $\prec_K$- increasing continuous and similarly N_β, M^*_β, N^*_β.

(C) $F(M^*_{i+1}) = M_{i+2}$

(D) $F(N^*_{i+1}) = N^*_{i+2}$

(E) $(M_\beta, N_\beta, a) < (M_\alpha, N_\alpha, a)$ for $\beta < \alpha$.

(F) for γ limit or zero f_γ is an isomorphism from M^*_γ onto M_γ, g_γ is an isomorphism from N^*_γ onto N_γ.

(G) for γ limit or zero, $n > 0$: f_γ is an isomorphism from $N^*_{\gamma+n}$ onto $N_{\gamma+2n}$, g_γ is an isomorphism from $M^*_{\gamma+n}$ onto $M_{\gamma+2n-1}$.

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