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$\mathrm{ZF}+\mathrm{DC}+\mathrm{AX}_4$

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Abstract We consider mainly the following version of set theory: "ZF + DC and for every λ, $\lambda^{\aleph_0}$ is well ordered", our thesis is that this is a reasonable set theory, e.g. on the one hand it is much weaker than full choice, and on the other hand much can be said or at least this is what the present work tries to indicate. In particular, we prove that for a sequence $\bar{\delta} = \langle \delta_s : s \in Y \rangle$, $\mathrm{cf}(\delta_s)$ large enough compared to Y, we can prove the pcf theorem with minor changes (in particular, using true cofinalities not the pseudo ones). We then deduce the existence of covering numbers and define and prove existence of a class of true successor cardinals. Using this we give some diagonalization arguments (more specifically some black boxes and consequences) on Abelian groups, chosen as a characteristic case. We end by showing that some such consequences hold even in ZF above.

Keywords Set theory · Weak axiom of choice · pcf · Abelian groups

Dedicated to the memory of Richard Laver.

References to outside papers like [23, 2.13 = Ls.2] means to 2.13 where s.2 is the label used there, so intended only to help the author if more is added to [23].

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0 Introduction

0.1 Background and results

Everyone knows that the issue of weakening AC, the axiom of choice issue, is dead, settled, as naturally the axiom of choice is true, and its weakenings lead to bizarre universes on which there is not much to be proved, or assuming AC is irrelevant (as in inner models).

The works on determinacy are not a real exception: it e.g. replace Borel sets and projective sets by sets in $\mathbb{L}[\mathbb{R}]$, so have much to say on this inner model, for which the only choice missing is a well ordering of $\mathscr{P}(\mathbb{N})$. In [23] we suggest to consider several related axioms, the strongest of them being Ax_4, assuming ZF + DC of course. It is in a sense an anti-thesis to considering $\mathbb{L}[\mathbb{R}]$: it says we can well order (not all the subsets just) the countable subsets of any ordinal. This was continued in [25,27] and in Larson–Shelah [8]. We may wonder how to get natural models of ZF + DC + Ax_4. Such a natural model is gotten starting with $\mathbf{V} \models$ G.C.H. and forcing by the choiceless version of Easton forcing except for $\aleph_0$.

While [20] claims to prove that "the theory of pcf with weak choice is non-empty", [23] seems to us the true beginning of such set theory, proving (in ZFC + DC + Ax_4 or so): there is a class of successor regular cardinals, and for any set Y, ${}^Y\lambda$ can, in a suitable sense, be decomposed to "few" well order sets (see [23, 0.3] and more here in 2.19).

Much attention there was given to trying to get the results from weaker relatives of Ax_4. A major aim of this work is to try to justify:

Thesis 0.1 ZF + DC + Ax_4 is a reasonable set theory, for which much of combinatorial set theory can be generalized, but many times in a challenging way and even discover new phenomena.

In particular we consider diagonalization arguments, including in ZF alone. Returning to the original issue, i.e. the position that "set theory with weak choice is dead", which we had wholeheartedly supported, the paper's position here is that:

(a) AC is obviously true
(b) general set theory in ZF + DC + Ax_4 is a worthwhile endeavor
(c) an important reason for not adopting ZF + DC was the lack of something like (b), hence intellectual honesty urges you to investigate this direction
(d) this is just a way to look at strengthening existence results to existence by nicely definable sets.

Let us try to explain the results.

We assume ZF + DC. Consider a sequence $\bar{\delta} = \langle \delta_s : s \in Y \rangle$ of limit ordinals, when can we get a cofinal $<_I$-increasing sequence in $(\Pi\bar{\delta}, <_I)$ for I on ideal on Y? When can we get a parallel to the pcf-theorem?

In [26, §5], [27] we use $AC_{\mathscr{P}(Y)}$ (and DC) to deal with true pseudo cofinality, but here instead we continue [23] assuming Ax_4. In [23, 1.8=L6.1] we generalize the pcf-theorem [i.e. existence of $\langle \mathfrak{b}_{\mathfrak{a},\theta}, \bar{f}_{\mathfrak{a},\theta} : \theta \in \text{pcf}(\mathfrak{a})\rangle$] for countable index set Y. What about large Y, with each δ_s having cofinality large compared to Y? Here first we deal with D an $\aleph_1$-complete filter in 1.5; this continues the ideas of [23, 1.2=Lr.2]. We then can[1] choose $\langle A_\varepsilon, J_\varepsilon, \bar{f} : \varepsilon < \varepsilon(*)\rangle$, J_ε the $\aleph_1$-complete ideal on Y generated by $\{A_\zeta : \zeta < \varepsilon\}$, $\bar{f}$ cofinal in $(\Pi(\bar{\delta}\restriction A_\varepsilon), <_{I_\varepsilon})$. Can we waive "$\aleph_1$-complete"? For this in 1.7 we combine the above with a generalization of [23, 1.6=1p.4], i.e. above I_ε is the ideal on Y generated by $\{A_\zeta : \zeta < \varepsilon\}$. If I_ε is not $\aleph_1$-complete we deal essentially with all quotients of I_ε which are ideals on countable sets.

But in Theorem 1.7, what about $\Pi\bar{\delta}$ when $s \in Y \Rightarrow \text{cf}(\delta_s)$ small? With choice recalling [30], we cannot generalize the pcf theorem,[2] but here, even if each δ_s has countable cofinality this is not necessarily the case. This motivates the definition of the ideal $\text{cf} - \text{id}_{<\theta}(\bar{\delta})$ noting that in general it may well be that $s \in Y \Rightarrow \text{cf}(\delta_s) = \aleph_0$ but $\text{cf}(\Pi\bar{\delta})$ is large.

In our context, the set ${}^\kappa\lambda$ does not in general have a cardinality, i.e. its power is not a cardinal, i.e. an $\aleph$, equivalently the set is not well orderable. But surprisingly, by Theorem 2.34 in §2.4, relevant covering numbers exist, i.e. $\text{cov}(\lambda, \theta_3(\kappa), \kappa, \sigma)$ is a well defined $\aleph$ when the cardinality of the sets by which we cover $(<\theta_3(\kappa))$ is large enough compared to the ones we cover $(<\kappa)$. This is an additional witness for the covering number's naturality. This follows by moreover proving when $\kappa = \sigma = \aleph_1$, there is a cofinal subset which is well orderable. In particular here it gives us a way to circumvent the non-existence of well orders of ${}^\kappa\lambda$.

In Sects. 2.1, 2.2 we deal with relatives of Sect. 1: pcf system, eub and more. Also in 2.19 we give an improvement of the result of [23, §1].

Another issue is the "successor of a singular cardinal is regular" in Sect. 2.3. Recall that the consistency strength of two successive singular cardinal is large, but not for "a successor cardinal is singular". So a posteriori (i.e. after [23, §1]) it is natural to hope that if μ is singular large enough then μ^+ is regular. In [23, 2.13=Ls.2] we show that for many μ the answer is yes; here we get a stronger conclusion: μ^+ is a true successor cardinal; in fact $\alpha < \mu \Rightarrow |\alpha|^{\aleph_0} < \mu$ suffice; see 2.28(2).

Many proofs rely on diagonalizing so seemingly inherently use strong choice. Still we succeed to save some, see Sect. 3. As a test problem, we deal with constructing Abelian groups and with Black Boxes. We also note that [19] applies even in $ZF+AC_{\aleph_0}$ in 0.19.

A natural question is:

(∗) assume $\text{cf}(\mu) = \aleph_0$, $(\forall\alpha < \mu)(|\alpha|^{\aleph_0} < \mu)$
 (a) if $\mu \le \lambda < \mu^{\aleph_0}$ and λ is singular, is λ^+ a true successor? or at least
 (b) if $\mu \le \lambda < \text{pp}(\mu)$ and λ is singular is λ^+ is regular?

We may try to use $c\ell$ which is only $\aleph_1$-well founded, hence have to use $DC_{\aleph_1}$.

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1 We temporarily cheat a little, only $A_\varepsilon/I_\varepsilon$ is defined.

2 Still by [21], in ZFC, we can deal with $(\Pi\bar{\lambda}, <_I)$ if $\lambda_s > \theta$ and a relative of "$\mathscr{P}(I)/I$ satisfies the θ-c.c." hold.

Why do we concentrate on (∗)? We may try to prove that if $\mu > 2^{\aleph_0}$ is singular then $\lambda = \mu^+$ is regular improving [23, 2.13=Ls.2], where there are further restrictions on μ. A natural approach is letting $\chi \leq \mu$ be minimal such that $\chi^{\aleph_0} \geq \mu$, so $\chi > 2^{\aleph_0}$, so as there we can find $\bar{C}_1 = \langle C_\alpha : \alpha \in S^\lambda_{<\chi}\rangle$, $C_\alpha \subseteq \alpha = \sup(C_\alpha)$ and $|C_\alpha| < \chi$. But what about $S^\lambda_{\geq\chi}$? Assume $\lambda = \mathrm{pp}(\chi)$ so we can find $\langle \lambda_n : n < \omega\rangle$, each λ_n is $< \chi$, J ideal on ω, $\mathrm{tcf}(\Pi\lambda_n, <_J) = \lambda$ and $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ is $<_J$-increasing cofinal in $(\Pi\lambda_n, <_J)$. Without loss of generality $\mathrm{cf}(\alpha) > 2^{\aleph_0} \Rightarrow f_\alpha$ a $<_J$-eub of $\bar{f}\restriction\alpha$.

Another approach is to build an AD family $\mathscr{A} \subseteq [\lambda]^{\aleph_0}$ which induces a "good" function $c\ell_{\mathscr{A}} : \mathscr{P}(\lambda) \to \mathscr{P}(\lambda)$: where $c\ell_{\mathscr{A}}(u) = \cup\{A \in \mathscr{A} : A \cap u \text{ infinite}\}$, maybe let $\mathscr{A}_0$ be induced by $\bar{f}$.
Naturally we may ask (and deal with some, as mentioned).

Question 0.2 (1) Can we bound $\mathrm{hrtg}(\mathscr{P}(\mu))$ for μ singular? (recall Gitik–Koepke [3, pg.2]).
(2) Can we deduce $\mathrm{wlor}({}^Y\mu) = \mathrm{hrtg}({}^Y\mu)$ when μ is singular large enough? Maybe see [29, Ld21].
(3) In Sect. 1 we may replace θ by several θ_ℓ, defined by the proof (i.e. θ_ℓ is minimal satisfying some demands involving $\theta_0, \ldots, \theta_{\ell-1}$ and the pcf problem); but seemingly this does not make a serious gain, maybe see on this in [29, 5.2=Le4].
(4) Can we generalize RGCH (see [19], [22, §1]), see 0.19, 2.35.

We thank the referee for checking the paper very carefully discovering many things which should be mended much above the call of duty.

0.2 Preliminaries

Hypothesis 0.3 (1) We work in ZF + DC.
(2) Usually we assume $\mathrm{Ax}_{4,\partial}$, see Definition 0.4(5) relying on 0.5(3), 0.4(4), so a reader may assume it throughout; or even assume Ax_4, see 0.5(2),(1). Many times we use weaker relatives so we try to mention the case of $\mathrm{Ax}_{4,\lambda,\theta,\partial}$ actually used. So the case $\theta = \partial = \aleph_1$ means $\mathrm{Ax}_{4,\lambda}$ holds and note Ax_4 is stronger than $\mathrm{Ax}_{4,\aleph_1}$.
(3) So no such assumption means ZF + DC <u>but</u> still ∂ is a fixed cardinal $\geq \aleph_1$.

Definition 0.4 (1) $\mathrm{hrtg}(A) = \mathrm{Min}\{\alpha$: there is no function from A onto $\alpha\}$.
(2) $\mathrm{wlor}(A) = \mathrm{Min}\{\alpha$: there is no one-to-one function from α into A or $\alpha = 0 \wedge A = \emptyset\}$ so $\mathrm{wlor}(A) \leq \mathrm{hrtg}(A)$.

Definition 0.5 (1) Ax^4_λ means $[\lambda]^{\aleph_0}$ can be well ordered so $\lambda^{\aleph_0}$ is a well defined cardinal.
(2) Ax_4 means Ax^4_λ for every cardinality λ.
(3) $\mathrm{Ax}_{4,\lambda,\partial,\theta}$ means that ($\lambda \geq \partial \geq \theta \geq \aleph_1$ and): there is a witness $\mathscr{S}$ which means:

(a) $\mathscr{S} \subseteq ([\lambda]^{<\partial}, \subseteq)$
(b) for every $u_1 \in [\lambda]^{<\theta}$ there is $u_2 \in \mathscr{S}$ such that $u_1 \subseteq u_2$
(c) $\mathscr{S}$ is well-orderable
(d) for notational simplicity: $\mathscr{S}$ of minimal cardinality.

(3A) But we may use an ordinal β instead of λ above. So trivially $\mathrm{Ax}^4_\lambda \Rightarrow \mathrm{Ax}_{4,\lambda,\aleph_1,\aleph_1}$ because we can choose $\mathscr{S} = [\lambda]^{\le\aleph_0}$.
(3B) If $\mathrm{Ax}_{4,\lambda,\partial,\theta}$ then we let $\mathrm{cov}(\lambda,\partial,\theta,2)$ be the minimal $|\mathscr{S}|$ for $\mathscr{S}$ as in 0.5(3); necessarily it is $< \mathrm{wlor}([\lambda]^{<\partial})$ which is $\le \mathrm{hrtg}([\lambda]^{<\partial})$; so if $\neg\mathrm{Ax}_{4,\lambda,\partial,\theta}$ then it is not well defined.
(3C) We say $(\mathscr{S}_*, <_*)$ witness $\mathrm{Ax}_{4,\lambda,\partial,\theta}$ <u>when</u> $\mathscr{S}_*$ is as in part (3) and $<_*$ is a well ordering of $\mathscr{S}_*$.
(4) Let $\mathrm{Ax}_{4,\lambda,\partial}$ mean $\mathrm{Ax}_{4,\lambda,\partial,\aleph_1}$; note that even if $\partial = \aleph_1$, $\mathrm{Ax}_{4,\lambda,\partial}$ is not Ax^4_λ.
(5) Let $\mathrm{Ax}_{4,\partial}$ mean $\mathrm{Ax}_{4,\lambda,\partial}$ for every λ, so $\mathrm{Ax}_{4,\partial}$ is not the same as Ax^4_∂.
(6) We may write $\le \theta$ instead of θ^+, and writing an ordinal α instead of ∂ means $\mathrm{otp}(u_1) < \alpha$ in clause (b) of part (3); similarly for the other parameters.

We try to make the paper reasonably self-contained. Still we assume knowledge of [23, §(0B)], the preliminaries, in particular, recall:

Claim 0.6 (1) *For every λ, ∂ such that $\mathrm{Ax}_{4,\lambda,\partial}$ there is a function $c\ell$, moreover one which is (we may use α instead of λ) definable from $(\mathscr{S}_*, <_*)$ where $(\mathscr{S}_*, <_*)$ witness $\mathrm{Ax}_{4,\lambda,\partial}$, see 0.5(3),(3B), even uniformly such that:*

(a) $c\ell: \mathscr{P}(\lambda) \to \mathscr{P}(\lambda)$
(b) $u \subseteq c\ell(u) \subseteq \lambda$, *(but we do not require $c\ell(c\ell(u)) = c\ell(u)$)*
(c) $|c\ell(u)| < \mathrm{hrtg}([u]^{\aleph_0} \times \partial)$, *and if* Ax_4 *even* $\le |u|^{\aleph_0}$ *for* $u \subseteq \lambda$
(d) *there is no sequence $\langle u_n : n < \omega\rangle$ such that $u_{n+1} \subseteq u_n \nsubseteq c\ell(u_{n+1})$.*

(2) *We can above replace $\mathrm{Ax}_{4,\lambda,\partial}$ by: there is a well orderable $\mathscr{S}_* \subseteq [\lambda]^{<\partial}$ such that there is no $u \in [\lambda]^{\aleph_0}$ satisfying $v \in \mathscr{S}_* \Rightarrow \aleph_0 > |v \cap u|$.*

Proof (1) Recall $\mathscr{S}_* \subseteq [\lambda]^{<\partial}$ and $u_1 \in [\lambda]^{\le\aleph_0} \Rightarrow (\exists u_2 \in \mathscr{S}_*)(u_1 \subseteq u_2)$ and $<_*$ is a well ordering of $\mathscr{S}_*$ and let $\langle w^*_i : i < \mathrm{otp}(\mathscr{S}_*, <_*)\rangle$ list $\mathscr{S}_*$ in $<_*$-increasing order; if Ax_4 we can use $\mathscr{S}_* = [\lambda]^{\aleph_0}$. For $v \in [\lambda]^{\le\aleph_0}$ let $\mathbf{i}(v) = \mathbf{i}(v, \mathscr{S}_*, <_*) = \min\{i : v \backslash w^*_i$ is finite$\}$.

For $u \subseteq \lambda$ let $c\ell(u) = \cup\{w^*_i :$ for some $v \in [u]^{\aleph_0}$ we have $i = \mathbf{i}(v)\} \cup u \cup \{0\}$.

So clearly clauses (a), (b) of the conclusion hold.

For clause (c) define $F : [u]^{\aleph_0} \times \partial \to \lambda$ by $F(v,\alpha) =$ the αth member of $w^*_{\mathbf{i}(v)}$ when $\mathrm{otp}(w^*_{\mathbf{i}(v)}) > \alpha$, and 0 otherwise; clearly F is a function from $[u]^{\aleph_0} \times \partial$ to λ and its range is included in $c\ell(u)$ and includes $c\ell(u)\backslash u$; we like F to be onto $c\ell(u)$, but clearly $u\backslash\mathrm{Rang}(F)$ is finite, hence this last part can be corrected easily hence $c\ell(u)$ has cardinality $< \mathrm{hrtg}([u]^{\aleph_0} \times \partial)$ so we are done with clause (c).

Lastly, to prove clause (d), toward contradiction assume $\bar{u} = \langle u_n : n < \omega\rangle$ and $u_{n+1} \subseteq u_n \nsubseteq c\ell(u_{n+1})$ for every n; by DC or just $\mathrm{AC}_{\aleph_0}$ choose $\bar{\alpha} = \langle \alpha_n : n < \omega\rangle$ such that $\alpha_n \in u_n \backslash c\ell(u_{n+1})$. Now let $v = \{\alpha_n : n < \omega\}$ and $i = \mathbf{i}(v)$, so for every n, $v\backslash(v \cap u_n)$ is finite hence $\mathbf{i}(v) = \mathbf{i}(v \cap u_n)$ and let n be such that $v\backslash w^*_i \subseteq \{\alpha_0, \ldots, \alpha_{n-1}\}$, so $\alpha_n \in w^*_i \subseteq c\ell(u_{n+1})$, contradicting the choice of α_n.
(2) Similarly but first for any infinite $v \subseteq \lambda$ let $\mathbf{i}(v) = \mathbf{i}(v, \mathscr{S}_*, <_*) := \min\{i : v \cap w^*_i$ is infinite$\}$. Second, $F(v,\alpha)$ is:

- the αth member of $w^*_{\mathbf{i}(v)}$ if $\alpha < \mathrm{otp}(w^*_{\mathbf{i}(v)})$
- 0 otherwise.

Third, note:

- if $u \subseteq \lambda$ then $u \backslash \{F(v, \alpha) : v \in [u]^{\aleph_0}$ and $\alpha < \partial\}$ is finite.

[Why? If not, let v be a subset of the difference of cardinality $\aleph_0$, (exist by our assumption), hence $\{F(v, \alpha) : \alpha < \lambda\}$ is not disjoint to v, contradiction.]

Fourth, in the end, instead of "let n be such that $v \backslash w^*_i \subseteq \{\alpha_0, \ldots, \alpha_{n-1}\}$" we choose n such that $\alpha_n \in w^*_{\mathbf{i}(v)} \cap v$; possible as $w^*_{\mathbf{i}(v)} \cap v = w^*_{\mathbf{i}(v)} \cap \{\alpha_n : n < \omega\}$ is infinite and $n < \omega \Rightarrow \mathbf{i}(v) = \mathbf{i}(\{\alpha_k : k > n\})$. $\square$

Observation 0.7 (1) *For any set* Y*, if* μ *a cardinal and* $\theta := \mathrm{hrtg}(Y)$ *then* $\mathrm{hrtg}(Y \times \mu) \leq (\theta + \mu)^+$.
(2) *In* 0.6 *we can replace clause* (c) *by:*

$(c)'$ $|c\ell(u)| < \max\{\partial^+, \mathrm{hrtg}([u]^{\aleph_0})\}$.

Proof (1) Assume F is a function from $Y \times \mu$ onto an ordinal γ.

For $\beta < \mu$ let $v_\beta = \{F(y, \beta) : y \in Y\}$, so $\langle v_\beta : \beta < \mu\rangle$ is a well defined sequence of subsets of the ordinal γ with union γ, and clearly $\beta < \mu \Rightarrow |v_\beta| < \mathrm{hrtg}(Y) = \theta$. Really we can use $v'_\beta = v_\beta \backslash \cup \{v_\alpha : \alpha < \beta\}$, in this case clearly $\langle v'_\beta : \beta < \mu\rangle$ is a partition of γ. Hence easily $|\gamma| = |\bigcup_{\beta<\mu} v_\beta| = |\bigcup_{\beta<\mu} v'_\beta| \leq \theta + \mu$, so the desired result follows.
(2) Let $\theta = \mathrm{hrtg}([u]^{\aleph_0})$, if $\theta \leq \partial$ then applying part (1), $\mathrm{hrtg}([u]^{\aleph_0} \times \partial) \leq (\theta + \partial)^+ = \partial^+$ so we are done. If $\theta > \partial$, then $\mathrm{hrtg}([u]^{\aleph_0} \times \partial) \leq \mathrm{hrtg}([u]^{\aleph_0} \times [u]^{\aleph_0})$ and if $|u| \geq \aleph_0$ we have $|[u]^{\aleph_0} \times [u]^{\aleph_0}| = |u|^{\aleph_0}$ hence we are done.

Lastly, if $\neg(|u| \geq \aleph_0)$ then (as $u \subseteq \lambda$) necessarily u is finite and so $c\ell(u) = u \cup \{0\}$ hence $|c\ell(u)| < \partial$, so having covered all cases we are done. $\square$

Convention 0.8 (1) Let "there is y satisfying $\psi(y, a)$, ∂-uniformly definable (or uniformly ∂-definable) for $a \in A$" means that there is a formula $\varphi(x, y, z)$ such that:

- for every μ large enough if $a \in A$ and $\mathrm{Ax}_{4,\mu,\partial}$ holds and $<_*$ well orders some $\mathscr{S}_* \subseteq [\mu]^{<\partial}$ as in 0.5(3) then $(\exists! y)[\varphi(y, a, <_*) \wedge \psi(y, a)]$.

(1A) Note that it follows that there is a definable function $A \mapsto \mu_A \in$ card such that above, $\mu \geq \mu_A$ suffice.
(2) Similarly with (∂, θ)-uniformly definable <u>when</u> we use $\mathrm{Ax}_{4,\mu,\partial,\theta}$; and (μ, ∂, θ)-uniformly definable when we fix μ.
(3) If the parameter (∂) or (∂, θ) or (μ, ∂, θ) is clear we may omit it. We may not always remember to state this.
(4) δ denotes an ordinal, limit one if not said otherwise.

Definition 0.9 Let D be a filter on a set Y.
(1) For $\bar\delta \in {}^Y\mathrm{Ord}$ let $\lambda = \mathrm{tcf}(\Pi\bar\delta, <_D)$ means that $(\Pi\bar\delta, <_D)$ has true cofinality λ, i.e. λ is a regular cardinal and there is a witness that is a $<_D$-increasing sequence $\langle f_\alpha : \alpha < \lambda\rangle$ of members of $\Pi\bar\delta$ which is cofinal in $(\Pi\bar\delta, <_D)$; but sometimes we allow λ to be an ordinal so not unique. [Why helpful? See part (2)].
(2) We say that $\bigwedge_{i\in I}\lambda_i = \mathrm{tcf}(\Pi\bar\delta_i, <_D)$ <u>when</u> $\bar\delta_i \in {}^Y\mathrm{Ord}$ for $i \in I$ and there is a sequence $\big\langle\langle f^i_\alpha : \alpha < \lambda_i\rangle : i \in I\big\rangle$ such that $\langle f^i_\alpha : \alpha < \lambda_i\rangle$ is as above for $\lambda_i = \mathrm{tcf}(\Pi\bar\delta_i, <_D)$,
but λ_i may be any ordinal hence is not unique; so $\bigwedge_{i\in I}\lambda_i = \mathrm{tcf}(\Pi\bar\delta_2, <_D)$ and $i \in I \Rightarrow \lambda_i = \mathrm{tcf}(\Pi\bar\delta_i, <_D)$ has a different meaning.
(3) Assume $\bar f = \langle f_\alpha : \alpha < \delta\rangle$ and $\alpha < \delta \Rightarrow f_\alpha \in {}^Y\mathrm{Ord}$ and D is a filter on Y. We say $f \in {}^Y\mathrm{Ord}$ is a $<_D$ $-$eub of $\bar f$ <u>when</u>:

(a) $\alpha < \delta \Rightarrow f_\alpha \le f \mod D$
(b) if $g \in {}^Y\mathrm{Ord}$ and $(\forall s \in Y)(g(s) < f(s) \vee g(s) = 0)$ then $(\exists\alpha < \delta)(g \le f_\alpha \mod D)$.

Definition 0.10 (1) Let Y be the set and let κ be an infinite cardinal.

(a) $\mathrm{Fil}^1_\kappa(Y)$ is the set of κ-complete filters on Y, (so Y is defined from D as $\cup\{X : X \in D\}$)
(b) $\mathrm{Fil}^2_\kappa(Y) = \{(D_1, D_2) : D_1 \subseteq D_2$ are κ-complete filters on Y, ($\emptyset \notin D_2$, of course)$\}$; in this context $Z \in \bar D$ means $Z \in D_2$
(c) $\mathrm{Fil}^3_\kappa(Y, \mu) = \{(D_1, D_2, h) : (D_1, D_2) \in \mathrm{Fil}^2_\kappa(Y)$ and $h : Y \to \alpha$ for some $\alpha < \mu\}$, if we omit μ we mean $\mu = \mathrm{hrtg}([Y]^{\le\aleph_0} \times \partial) \cup \omega$, recalling 0.3
(d) $\mathrm{Fil}^4_\kappa(Y, \mu) = \{(D_1, D_2, h, Z) : (D_1, D_2, h) \in \mathrm{Fil}^3_\kappa(Y, \mu)$ and $Z \in D_2\}$; omitting μ means as above.

(2) For $\mathfrak{y} \in \mathrm{Fil}^4_\kappa(Y, \mu)$ let $Y = Y^{\mathfrak{y}} = Y_{\mathfrak{y}}$, $\mathfrak{y} = (D^{\mathfrak{y}}_1, D^{\mathfrak{y}}_2, h^{\mathfrak{y}}, Z^{\mathfrak{y}}) = (D_{\mathfrak{y},1}, D_{\mathfrak{y},2}, h_{\mathfrak{y}}, Z_{\mathfrak{y}}) = (D_1[\mathfrak{y}], D_2[\mathfrak{y}], h[\mathfrak{y}], Z[\mathfrak{y}])$; similarly for the others and let $D^{\mathfrak{y}} = D[\mathfrak{y}]$ be $D^{\mathfrak{y}}_1 + Z^{\mathfrak{y}}$ recalling $D + Z$ is the filter generated by $D \cup \{Z\}$.
(3) If $\kappa = \aleph_1$ we may omit it.

We now repeat to a large extent [23,26].

Definition/Claim 0.11 Assume δ is a limit ordinal (or zero for some parts), $D = D_1 \in \mathrm{Fil}^1_{\aleph_1}(Y)$, $\bar f = \langle f_\alpha : \alpha < \delta\rangle$ is a sequence of members[3] of ${}^Y\mathrm{Ord}$, usually $<_{D_1}$-increasing in ${}^Y\mathrm{Ord}$, f is a $\le_D$-upper bound of $\bar f$ but there is no such $g <_D f$; necessarily there is such f (using DC).
(1) [Definition] Let $J = J[f, \bar f, D] := \{A \subseteq Y$: either $A = \emptyset \mod D$ <u>or</u> $A \in D^+$ but there is a $\le_{D+A}$-upper bound $g <_{D+A} f$ of $\bar f\}$.
(2) $J[f, \bar f, D]$ is an $\aleph_1$-complete ideal on Y disjoint to D.
(3) [Definition] Recalling $D_1 = D$, let $D_2 = D_2(f, \bar f, D_1) = \mathrm{dual}(J[f, \bar f, D_1]) := \{A \subseteq Y : Y \backslash A \in J[f, \bar f, D_1]\}$; note that, e.g. as D_1 is $\aleph_1$-complete then D_2 is an $\aleph_1$-complete filter on Y extending D_1.

[3] We can use any index set instead of δ (in particular the empty one), <u>except</u> in part (5); this applies also to Definition 0.9.

(4) In (3), f is a unique modulo D_2, i.e. if also $g \in {}^Y\mathrm{Ord}$, is a $\leq_{D_1}$-upper bound of $\bar{f}$ and $J[g, \bar{f}, D_1] = J[f, \bar{f}, D_1]$ then $g = f \mod D_2$, equivalently $\mod J[f, \bar{f}, D_1]$.
(5) If ($\bar{f}$ is $\leq_{D_1}$-increasing, and) $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ then f from above is a $<_{D_2}$-eub of $\bar{f}$, see Definition 0.9(3).

Definition 0.12 Assume $f \in {}^Y\mathrm{Ord}$, $D_2 \supseteq D_1$ are $\aleph_1$-complete filters on Y, $c\ell$ is as in 0.6 for $\alpha(*)$ and $\mathrm{Rang}(f) \subseteq \alpha(*)$.
(0) For some $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$, $D^{\mathfrak{y}}_1 = D_1$, $D^{\mathfrak{y}}_2 = D_2$ and the function f satisfies $\mathfrak{y}$, see below.
(1) We say $f: Y \to \mathrm{Ord}$ satisfies $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_0}(Y)$ when:

(a) if $Z \in D^{\mathfrak{y}}_2$ and $Z \subseteq Z_{\mathfrak{y}}$ then $c\ell(\{f(t): t \in Z\}) = c\ell(\{f(t): t \in Z_{\mathfrak{y}}\}$
(b) $y \in Z_{\mathfrak{y}} \Rightarrow h_{\mathfrak{y}}(y) = \mathrm{otp}(f(y) \cap c\ell(\mathrm{Rang}(f \restriction Z_{\mathfrak{y}})))$
(c) if $t \in Y$ and $f(t) \in c\ell\{f(s): s \in Z_{\mathfrak{y}}\}$ then $t \in Z_{\mathfrak{y}}$
(d) $y \in Y \backslash Z_{\mathfrak{y}} \Rightarrow f(y) = 0$.

(2) "Semi satisfies" mean we omit clause (d).
(3) Let "weakly satisfies" means we omit clauses (c),(d).

Definition 0.13 Let $Y, f, \bar{f}, D$ be as in 0.11 and $Y, \alpha(*), c\ell$ as in 0.12.
(1) We say f is the $(\mathfrak{y}, c\ell) - eub$ of $\bar{f}$ or $\mathfrak{y} - eub$ of $\bar{f}$ or canonical $\bar{f}$-eub for $\mathfrak{y}$ (and $c\ell$) when:

(a) $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$
(b) $\bar{f} = \langle f_\alpha: \alpha < \alpha_* \rangle$
(c) f_α, f are from ${}^Y\alpha(*)$
(d) $f_\alpha \leq_{D_{\mathfrak{y},1}} f$
(e) $D_{\mathfrak{y},1} = D$ and $D_{\mathfrak{y},2} \supseteq \mathrm{dual}(J[f, \bar{f}, D_{\mathfrak{y},q}])$
(f) f satisfies $\mathfrak{y}$ (for $c\ell$).

Claim 0.14 *Let $Y, f, \bar{f}, D$ as in 0.11, $f, \alpha(*), c\ell$ as in 0.12.*
(1) *The "the" is 0.13 is justified, that is, f is unique given $c\ell$ (so $\alpha(*), \bar{f}, \mathfrak{y}$).*
(2) *There is one and only one $\mathfrak{y}$ such that*

(a) $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$
(b) $D_{\mathfrak{y},1} = D$
(c) $D_{\mathfrak{y},2} = \mathrm{dual}(J[f, \bar{f}, D])$
(d) f *semi satisfies* $\mathfrak{y}$.

(3) *For the $\mathfrak{y}$ from part* (2), *letting $g = (f \restriction Z_{\mathfrak{y}}) \cup (0_{Y \backslash Z_{\mathfrak{y}}})$ we have g is the canonical $\bar{f}$ − eub for $\mathfrak{y}$ (and $c\ell$), in particular it satisfies y.*

Proof Should be clear. □

Recall the related (not really used)

Definition/Claim 0.15 Assume $D \in F^1_{\aleph_1}(Y)$ and $f: Y \to \mathrm{Ord}$.
(1) [Definition] $J[f, D] = \{A \subseteq Y: A = \emptyset \mod D$ or $A \in D^+$ and $\mathrm{rk}_{D+A}(f) > \mathrm{rk}_D(f)\}$.
(2) J is an $\aleph_1$-complete filter disjoint to D.
(3) If $f_1, f_1: Y \to \mathrm{Ord}$ and $J[f_1, D] = J[f_2, D]$.
(4) There is one and only $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$ such that f semi satisfies $\mathfrak{y}$, $D_{\mathfrak{y},1} = D$ and $D_{\mathfrak{y},2} = \mathrm{dual}(J[f, D])$.
(5) In (4) there is a unique f' which satisfies $\mathfrak{y}$ and $f' \restriction Z_{\mathfrak{y}} = f \restriction Z_{\mathfrak{y}}$.

Notation 0.16 Let $A \leq_{\mathrm{qu}} B$ means that $A = \emptyset$ or there is a function from B onto A.

Observation 0.17 *Assume $\partial \leq |Y|$ and even $\partial \subseteq Y$ for transparency.*
(1) $\mathrm{Fil}^4_{\aleph_1}(Y) \leq_{\mathrm{qu}} |\mathscr{P}(\mathscr{P}(3 \times Y))|$.
(2) *Also* ${}^{\omega}(\mathrm{Fil}^4_{\aleph_1}(Y)) \leq_{\mathrm{qu}} \mathscr{P}(\mathscr{P}(Y))$.
(3) *If* $\theta = \mathrm{hrtg}(\mathscr{P}(\mathscr{P}(Y))$ *<u>then</u>* θ *satisfies:*

- *if* $\alpha < \theta$ *then* $\mathrm{hrtg}(\mathscr{P}([\alpha]^{\aleph_0} \times \partial)) \leq \theta$
- *so if* Ax_4 *then* $|\alpha|^{\aleph_0} \times \partial < \theta$.

(4) *Assume* Ax_4. *If* $\alpha < \mathrm{hrtg}(\mathscr{P}(Y))$ *then* $|\alpha|^{\aleph_0} < \mathrm{hrtg}(\mathscr{P}(Y))$*; hence if* $\partial \leq |Y|$ *and* $\alpha < \mathrm{hrtg}(\mathscr{P}(Y))$ *then* $|\alpha|^{\aleph_0} \times \partial < \mathrm{hrtg}(\mathscr{P}(Y))$.

Remark 0.18 If Y is a set of ordinals, infinite to avoid trivialities <u>then</u> $|Y \times 3| = |Y|$, justifying this see 2.13.

Proof (1) Let $Y_0 = Y$, $Y_{\ell+1} = \mathscr{P}(Y_\ell)$ for $\ell = 0, 1$ and let $Y_1^* = [Y_1]^{\leq \aleph_0}$, $Y_2^* = \mathscr{P}(Y_1^*)$, $Y_0' = 3 \times Y$ and $Y_{\ell+1}' = \mathscr{P}(Y_\ell')$ for $\ell = 0, 1$

$(*)_1$ $|Y_0| + 1 = |Y_0|$ and even $|Y_0| + \partial = |Y_0|$.

[Why? As $\partial \leq |Y|$ is an infinite cardinal.]

$(*)_2$ $|Y_1| = \partial \times |Y_1|$ and $\partial \times |Y_1^*| = |Y_1^*|$ and $|Y_1'| = |Y_1' \times \partial| = \partial \times |Y_1'|$.

[Why? Both follow by $(*)_1$.]

$(*)_3$ $|Y_2| \times |Y_2| = |Y_2|$ and $|Y_0| \leq |Y_1| \leq |Y_2|$ and $|Y_2'| \times |Y_2'| = |Y_2'|$; moreover [for part (2)] $|{}^{\omega}(Y_2)| = |Y_2|$ and $|{}^{\omega}(Y_2')| = |Y_2'|$

[Why? Follows by $(*)_2$.]

$(*)_4$ $\{D_{\mathfrak{y},\ell}: \mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)\}$ has power $\leq |Y_2|$ for $\ell = 1, 2$.

[Why? By the definition each $D_{\mathfrak{y},\ell}$ is a subset of $\mathscr{P}(Y) = \mathscr{P}(Y_0) = Y_1$.]

$(*)_5$ $\{Z_{\mathfrak{y}}: \mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)\}$ has power $\leq |Y_1|$.

[Why? As $Z_{\mathfrak{y}} \subseteq Y = Y_0$ so $Z_{\mathfrak{y}} \in Y_1$.]

$(*)_6$ $[Y]^{\aleph_0} \times \partial$ has the same power as $[Y]^{\leq\aleph_0}$.

[Why? Let Z be a set of ordinals disjoint to Y of order type ∂; by $(*)_1$ we have $|Y| = |Y \cup Z|$ hence $|[Y]^{\leq\aleph_0} = |[Y \cup Z]^{\leq\aleph_0}| \geq |[Y]^{\leq\aleph_0} \times [\partial]^{\leq\aleph_0}| \geq |[Y]^{\leq\aleph_0} \times \partial| \geq [Y]^{\leq\aleph_0}$.]

$(*)_7$ $|Y \times [Y]^{\aleph_0} \times [Y]^{\aleph_0}| \leq |\mathscr{P}(3 \times Y)| \leq |Y_2|$.

[Why? The mapping $(y, u_1, u_2) \mapsto \{(0, y), (1, z_1), (2, z_2): z_1 \in u_1, z_2 \in u_2\}$ from $Y \times [Y]^{\aleph_0} \times [Y]^{\aleph_0}$ into $\mathscr{P}(3 \times Y)$ prove the first inequality, the second inequality follows from $|3 \times Y| = |3 \times Y_0| \leq |Y'_1| = |Y_1|$.]

$(*)_8$ $\mathcal{H} := \{h_{\mathfrak{y}}: \mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)\} \leq_{\mathrm{qu}} |Y'_2|$.

[Why? Recalling $(*)_6$ clearly $|\mathcal{H}| \leq |\{h: h$ a function, $\mathrm{Dom}(h) = Y$ and $\mathrm{Rang}(h)$ a bounded subset of $\mathrm{hrtg}([Y]^{\leq\aleph_0} \times \partial)\}| \leq |\{h: h$ a function from Y into some $\alpha < \mathrm{hrtg}([Y]^{\leq\aleph_0})\}| \leq_{\mathrm{qu}} |X_1|$ where

$$X_1 := \Big\{(h, g)\text{: for some ordinal } \alpha, g \text{ is a partial function from } [Y]^{\leq\aleph_0} \text{ onto } \alpha, \text{ so necessarily } \alpha < \mathrm{hrtg}([Y]^{\leq\aleph_0}) \text{ and } h \text{ is a function from } Y \text{ into } \alpha\Big\}.$$

Clearly $|\mathcal{H}| \leq |X_1|$. Let $t \notin Y$ and for $(h, g) \in X_1$ let $\mathrm{set}(h, g) := \{(y, u_1, u_2) : y = t \wedge g(u_1) \leq g(u_2)$ or $y \in Y$ and $u_1, u_2 \in [Y]^{\leq\aleph_0}$ satisfies $h(y) = g(u_1)$ and $g(u_2) = g(u_1)\}$. Easily $(h, g) \mapsto \mathrm{set}(h, g)$ is a one-to-one function from X_1 into $X_2 := \mathscr{P}(X_3)$ where $X_3 := (Y \cup \{t\}) \times [Y]^{\leq\aleph_0} \times [Y]^{\leq\aleph_0}$ and by $(*)_7$ we have $|X_3| = |\mathscr{P}(3 \times Y)|$. Hence $|X_1| \leq |X_2| = |\mathscr{P}(X_3)| \leq |\mathscr{P}(\mathscr{P}(3 \times Y))|$. Recalling $|\mathcal{H}| \leq |X_1|$ we are done proving $(*)_8$.]

Now $|\mathrm{Fil}^4_{\aleph_1}(Y)| \leq |\mathrm{Fil}^1_{\aleph_1}(Y) \times \mathrm{Fil}^1_{\aleph_1}(Y) \times \mathcal{H} \times \mathscr{P}(Y)|$ by the definition of $\mathrm{Fil}^4_{\aleph_1}$ and this is, by the inequalities above $\leq_{\mathrm{qu}} |Y'_2| \times |Y'_2| \times |Y'_2| \times |Y'_1| \leq_{\mathrm{qu}} |Y'_2|^4 = |Y'_2|$.

(2), (3), (4) Should be clear. □

Note also we may wonder about the RGCH, see [19], we note (not using any version of Ax_4), that we can get such a result using only $\mathrm{AC}_{\aleph_0}$. From the results of Sect. 1 we can deduce more. See 2.35.

Theorem 0.19 [ZF + $AC_{\aleph_0}$] *Assume that* $\mu > \aleph_0$ *and* $\chi < \mu \Rightarrow \mathrm{hrtg}(\mathscr{P}(\chi)) < \mu$. *Then for every* $\lambda > \mu$ *for some* $\kappa < \mu$ *we have:*

$(*)_{\lambda,\mu,\kappa}$ *if* $\theta \in (\kappa, \mu)$ *and* D *is a* κ*-complete filter on* θ *then there is no* $<_D$*-increasing sequence* $\langle f_\alpha : \alpha < \lambda^+\rangle$ *of members of* ${}^\theta\lambda$.

Remark 0.20 In 0.19 we can replace "$\chi < \mu \Rightarrow \mathrm{hrtg}(\mathscr{P}(\chi)) < \mu$" by $\chi < \mu \Rightarrow \mathrm{wlor}(\mathscr{P}(\chi)) < \mu$; this holds by the proof.

Proof Assume that this fails for a given λ. We choose $\kappa_n < \theta_n < \mu$ by induction on n. Let $\kappa_0 = \aleph_0$, so $\kappa_0 = \aleph_0 < \mu$ as required. Assume $\kappa_n < \mu$ has been chosen, note that it cannot be as required so there is $\theta \in [\kappa_n, \mu)$ such that it exemplifies $\neg(*)_{\lambda,\mu,\kappa_n}$ and let θ_n be the first such θ.

Given θ_n let $\kappa_{n+1} := \mathrm{wlor}(\mathscr{P}(\theta_n))$ so $\kappa_{n+1} \in (\theta_n, \mu) \subseteq (\kappa_n, \mu)$. So $\langle \kappa_n : n < \omega\rangle$ is well defined increasing and $\mu_* = \sum_n \kappa_n \le \mu$. Let $X_n = \{(\theta, D, \bar f) : \theta \in [\kappa_n, \kappa_{n+1})$, D is a κ_n-complete filter on θ, $\bar f = \langle f_\alpha : \alpha < \lambda^+\rangle$ is a $<_D$-increasing sequence of members of ${}^\theta\lambda\}$, so by the construction we have $X_n \ne \emptyset$ and $\langle X_n : n < \omega\rangle$ exist being well defined. As we are assuming $AC_{\aleph_0}$ there is a sequence $\langle(\theta_n, D_n, \bar f_n) : n < \omega\rangle$ from $\prod_n X_n$.

We can consider $\bar f = \langle \bar f_n : n < \omega\rangle$ (and also $\bar\kappa = \langle \kappa_n : n < \omega\rangle$) as a set of ordinals (using a pairing function on the ordinals) hence $\mathbf{V}_* = \mathbf{L}[\bar f, \bar\kappa]$ is a model of ZFC and a transitive class. In $\mathbf{V}_*$ we can define D'_n as the minimal κ_n-complete filter on θ_n such that $\bar f_n$ is $<_{D'_n}$-increasing. Clearly $(2^{\theta_n})^{\mathbf{V}_*} < \mathrm{wlor}(\mathscr{P}^{\mathbf{V}}(\theta_n)) < \mu$ hence $\mathbf{V}_* \models$ "μ_* is strong limit". By [19] or see [22, §1,1.13 = Lg.8] where $\lambda^{[\partial,\theta]}$ is defined we get a contradiction. $\square$

1 The pcf theorem again

We prove a version of the pcf theorem; weaker than [11, Ch.I,II] as we do not assume just $\min\{\mathrm{cf}(\alpha_y) : y \in Y\} > \mathrm{hrtg}(Y)$ but a stronger inequality. Still we gain in a point which disappears under AC: dealing with a sequence of possibly singular ordinals (and the ideal $\mathrm{cf}-\mathrm{id}_{<\theta}(\bar\delta)$, see below). In addition we gain in having the scales being uniformly definable. Also the result is stronger than in [27], as we use functions rather than sets of functions; (i.e. true cofinality rather than pseudo true cofinality; of course, the axioms of set theory used are different accordingly; full choice in [11], ZF+DC+$AC_{\mathscr{P}(Y)}$ in [27] and ZF + DC + Ax_4 here).

It seems natural in our context instead of looking at $\{\mathrm{cf}(\delta_s) : s \in Y\}$ we should look at:

Definition 1.1 (1) For a sequence $\bar\delta = \langle \delta_s : s \in Y\rangle$ of limit ordinals and a cardinal θ let $\mathrm{cf}-\mathrm{id}_{<\theta}(\bar\delta) = \{X \subseteq Y$: there is a sequence $\bar u = \langle u_s : s \in Y\rangle$ such that $s \in X \Rightarrow u_s \subseteq \delta_s = \sup(u_s)$ and $s \in X \Rightarrow \mathrm{otp}(u_s) < \theta\}$.
(2) Let $\mathrm{cf}-\mathrm{fil}_{<\theta}(\bar\delta)$ be the filter dual to the ideal $\mathrm{cf}-\mathrm{id}_{<\theta}(\bar\delta)$.
(3) We may replace $\bar\delta$ by a set of ordinals, i.e. instead of $\langle \alpha : \alpha \in u\rangle$ we may write u.

(4) For $\bar{\delta} = \langle \delta_s : s \in Y \rangle$ and $\bar{\theta} = \langle \theta_s : s \in Y \rangle$ we define $\text{cf} - \text{id}_{<\bar{\theta}}(\bar{\delta})$ similarly to part (1); similarly in the other cases.
(5) For $\bar{\theta}$ a sequence of infinite cardinals, let $\text{cf} - \text{fil}_{<\bar{\theta}}(\bar{\delta})$ be the dual filter; similarly in the other cases.

Observation 1.2 (1) *In* 1.1, $\text{cf} - \text{id}_{<\theta}(\bar{\delta})$, $\text{cf} - \text{id}_{<\bar{\theta}}(\bar{\delta})$ *are ideals on* Y *or equal to* $\mathscr{P}(Y)$.
(1A) *Moreover* $\aleph_1$*-complete ideals.*
(2) *Similarly for the filters.*

Proof Should be clear, e.g. use the definitions recalling we are assuming $\text{AC}_{\aleph_0}$. □

Observation 1.3 *Assume*

(a) $D = \text{cf} - \text{fil}_{<\bar{\theta}}(\bar{\delta})$ *is a well defined filter (that is* $\emptyset \notin D$*), so* $\bar{\delta} \in {}^Y\text{Ord}$ *is a sequence of limit ordinals,* $\bar{\theta} = \langle \theta_s : s \in Y \rangle \in {}^Y\text{Car}$*, e.g.* $\bigwedge_s \theta_s = \theta$
(b) $\bar{\mathscr{U}} = \langle \mathscr{U}_s : s \in Y \rangle$ *satisfies* $\mathscr{U}_s \subseteq \delta_s$, $\text{otp}(\mathscr{U}_s) < \theta_s$ *for* $s \in Y$,
(c) $g \in \Pi\bar{\delta}$ *is defined by*
  - $g(s)$ *is* $\sup\{\alpha + 1 : \alpha \in \mathscr{U}_s\}$ *if this value is* $< \delta_s$
  - $g(s)$ *is zero otherwise.*

Then

(α) g *belongs to* $\Pi\bar{\delta}$ *indeed*
(β) *if* $f \in \prod_{s \in Y} \mathscr{U}_s \subseteq \Pi\bar{\delta}$ *then* $f < g \mod D$.

Remark 1.4 Clause (b) of 1.3 holds, e.g. if $\mathscr{U} \subseteq \text{Ord}, \text{otp}(\mathscr{U}) < \min\{\theta_s : s \in Y\}, \mathscr{U}_s = \mathscr{U} \cap \delta_s$.

Proof Clause (α) is obvious by the choice of the function g; for clause (β) let $f \in \prod_{s \in Y} \mathscr{U}_s$ and let $X = \{s \in Y : f(s) \geq g(s)\}$. Necessarily $s \in X$ implies (by the assumption on f and the definition of X) that $(\exists\alpha)(\alpha \in \mathscr{U}_s \wedge g(s) \leq \alpha)$ which implies (by clause (c), the definition of g) that $g(s) = 0 \wedge \sup(u_s) = \delta_s$. So by the definition of $\text{cf} - \text{fil}_{<\bar{\theta}}(\bar{\delta})$ we have $X \in \text{cf} - \text{fil}_{<\bar{\theta}}(\bar{\delta})$ hence we are done. □

Claim 1.5 *Assume* $\text{Ax}_{4,\partial}$*, see Definition* 0.5(3)*; if* (A) *then* (B) *where:*

(A) *we are given* Y*, an arbitrary set,* $\bar{\delta}$*, a sequence of limit ordinals and* μ*, an infinite cardinal (or just a limit ordinal) such that:*
  (a) $\bar{\delta} = \langle \delta_s : s \in Y \rangle$ *and* $\mu = \sup\{\delta_s : s \in Y\}$
  (b) D_* *is an* $\aleph_1$*-complete filter on* Y*, it may be* $\{Y\}$
  (c) θ *is any cardinal satisfying:*
    (α) $\text{cf} - \text{id}_{<\theta}(\bar{\delta}) \subseteq \text{dual}(D_*)$,

(β) $\alpha < \theta \Rightarrow \text{hrtg}([\alpha]^{\aleph_0} \times \partial) \le \theta$ *so* $\partial < \theta$

(γ) $\text{hrtg}(\mathscr{P}(Y)) \le \theta$

(δ) $\text{hrtg}(\text{Fil}^4_{\aleph_1}(Y)) \le \theta$

(B) *there are* $\alpha_*, f, \bar{f}, A_*/D_*$ ∂*-uniformly defined from the triple* $(Y, \bar{\delta}, D_*)$*, see* 0.8 *such that* (*see more in the proof*)*:*

- (a) α_* *is a limit ordinal of cofinality* $\ge \theta$
- (b) $\bar{f} = \langle f_\alpha : \alpha < \alpha_* \rangle$
- (c) $f_\alpha \in \Pi\bar{\delta}$ *and* $f \in \Pi\bar{\delta}$
- (d) $\bar{f}$ *is* $<_{D_*}$*-increasing*
- (e) $A_* \in D_*^+$
- (f) $\bar{f}$ *is cofinal in* $(\Pi\bar{\delta}, <_{D_*+A_*})$
- (g) *if* $Y \backslash A_* \in D_*^+$ *then* f *is a* $<_{D_*+(Y\backslash A_*)}$*-ub of the sequence* $\bar{f}$.

Remark 1.6 (1) Note that we do not use $\text{AC}_{\mathscr{P}(Y)}$ and even not AC_Y which would simplify.
(2) Note that θ is not necessarily regular.
(3) In $(A)(c)(\delta)$, we can restrict ourselves to $\aleph_1$-complete filters on Y extending D_*.
(4) Originally we use several θ's to get best results but not clear if worth it.
(5) Why for a given Y there is θ as in 1.5(A)(c)(β), (γ), (δ)? see 0.17(3).
(6) In 1.5 we can replace the assumption $\text{Ax}_{4,\partial}$ by $\text{Ax}_{4,\text{hrtg}({}^Y\mu),\partial}$, see 0.5(4),(5).
(7) Concerning $(A)(c)(\alpha)$ note that this holds when each δ_s is an ordinal $\le \mu$ of cofinality $\ge \theta$.
(7A) In $(A)(c)(\beta)$, if Ax_4 then the demand is equivalent to "$\partial < \theta$ and $\alpha < \theta \Rightarrow |\alpha|^{\aleph_0} < \theta$", see 0.17(4).

Proof We can define μ by $(A)(a)$ and θ as minimal such that $(A)(c)$ holds and recall ∂ is given. Let

$(*)_1$ (a) $\lambda_* = \text{hrtg}({}^Y\mu)$

(b) $\mathscr{S}_{\lambda_*} \subseteq [\lambda_*]^{<\partial}$ is as in 0.5(3)

(c) $<_{\lambda_*}$ be a well ordering of $\mathscr{S}_{\lambda_*}$

(d) $\bar{w}^* = \langle w_i^* : i < \text{otp}(\mathscr{S}_{\lambda_*}, <_{\lambda_*}) \rangle$ list $\mathscr{S}_{\lambda_*}$ in $<_{\lambda_*}$-increasing order

$(*)_2$ $c\ell$ be as in 0.6 for λ_*

$(*)_3$ $\Omega = \{\alpha < \lambda_* : \aleph_0 \le \text{cf}(\alpha) < \theta\}$.

$(*)_4$ There is a sequence $\bar{e}$ (in fact, ∂-uniformly definable one) such that:

- (a) $\bar{e} = \langle e_\alpha : \alpha \in \Omega \rangle$
- (b) $e_\alpha \subseteq \alpha = \sup(e_\alpha)$
- (c) e_α has order type $< \theta$;

and we can add

$(c)_1$ e_α has order type $< \partial$ if $\mathrm{cf}(\alpha) = \aleph_0$

$(c)_2$ e_α has cardinality $< \mathrm{hrtg}([\mathrm{cf}(\alpha)]^{\aleph_0} \times \partial)$.

[How?

- If $\mathrm{cf}(\alpha) = \aleph_0$ let $\mathbf{i}(\alpha) = \min\{i : w_i^* \cap \alpha$ is unbounded in $\alpha\}$ and $e_\alpha = w^*_{\mathbf{i}(\alpha)} \cap \alpha$.
- If $\mathrm{cf}(\alpha) > \aleph_0$ let $e_\alpha = c\ell(e)$ where e is any club of α of order type $\mathrm{cf}(\alpha)$ such that $(\forall e')[e' \subseteq e$ a club of $\alpha \Rightarrow c\ell(e') = c\ell(e)]$.

[Why? Such e exists by the choice of $c\ell$ in 0.6 and if e'_*, e''_* are two such clubs then $e'_* \cap e''_*$ is a club of α of order type $\mathrm{cf}(\alpha)$ and $c\ell(e') = c\ell(e' \cap e'') = c\ell(e'')$ by the assumption on e' and on e'' respectively, so e_α is well defined.]

Lastly, the cardinality is as required by the clause $(A)(e)(\beta)$ and 0.6(c); similarly to [23, 2.11=Lr.9].

So $(*)_4$ holds indeed.]

Now we try to choose $f_\alpha \in \Pi\bar{\delta}$ by induction on α such that $\beta < \alpha \Rightarrow f_\beta < f_\alpha \mod D_*$.

Case 1: $\alpha = 0$

Let f_α be constantly zero, i.e. $s \in Y \Rightarrow f_\alpha(s) = 0$, clearly $f_\alpha \in \Pi\bar{\delta}$ as each δ_s is a limit ordinal.

Case 2: $\alpha = \beta + 1$

Let $f_\alpha(s) = f_\beta(s) + 1$ for $s \in Y$, so $f_\alpha \in \Pi\bar{\delta}$ as $f_\beta \in \Pi\bar{\delta}$ and each δ_s is a limit ordinal and $\gamma < \alpha \Rightarrow f_\gamma < f_\alpha \mod D_*$ as $f_\gamma \le f_\beta < f_\alpha \mod D_*$.

Case 3: α is a limit ordinal of cofinality $< \theta$.

So e_α is well defined and we define $f_\alpha : Y \to \mathrm{Ord}$ as follows: $f_\alpha(s)$ is equal to $\sup\{f_\beta(s) + 1 : \beta \in e_\alpha\}$ if this is $< \delta_s$ and is zero otherwise.

$(*)_5$ $f_\alpha \in \Pi\bar{\delta}$.

[Why? Obvious.]

Let $\mathscr{U}_{\alpha,s} = \{f_\beta(s) + 1 : \beta \in e_\alpha\}$, so clearly $\langle \mathscr{U}_{\alpha,s} : s \in Y\rangle$ is well defined and $\sup(\mathscr{U}_{\alpha,s})$ is an ordinal, it is $\le \delta_s$ as $\beta \in e_\alpha \Rightarrow f_\beta \in \Pi\bar{\delta}$. Let $X = \{s \in Y : f_\alpha(s) > 0$ equivalently $\delta_s > \sup(\mathscr{U}_{\alpha,s})\}$

$(*)_6$ $X \in D_*$, i.e. $X = Y \mod D_*$.

[Why? For $s \in Y \backslash X$ note that $|\mathscr{U}_{\alpha,s}| \le_{\mathrm{qu}} |e_\alpha|$ and $|e_\alpha| < \theta$ by $(*)_4(c)$, hence $|\mathscr{U}_{\alpha,s}| < \theta$. By the choice of X and Definition 1.1 we have $Y \backslash X \in \mathrm{cf} - \mathrm{id}_{<\theta}(\bar{\delta})$ hence by the clause (A)(c)(α) of the assumption of the claim, $X = Y \mod D_*$ as promised.]

$(*)_7$ if $\beta < \alpha$ then $f_\beta < f_\alpha \mod D_*$.

[Why? Clearly e_α has no last element so we can choose $\gamma \in e_\alpha \backslash (\beta + 1)$ and let $X' = \{s \in Y : f_\beta(s) < f_\gamma(s)\}$. Necessarily $X' \in D_*$ hence $X' \cap X \in D_*$ but clearly $s \in X' \cap X \Rightarrow f_\beta(s) < f_\gamma(s) < f_\alpha(s)$ so $(*)_7$ holds.]

We arrive to the main case.

Case 4: α a limit ordinal of cofinality $\geq \theta$

Let

- $\bar{f}^{\alpha} = \langle f_\beta : \beta < \alpha \rangle$
- $\mathbf{D} = \{D : D$ is an $\aleph_1$-complete filter on Y extending $D_*\}$
- $\mathbf{D}^1_\alpha = \{D \in \mathbf{D}$: $\bar{f}^{\alpha}$ is not cofinal in $(\Pi\bar{\delta}, <_D)\}$
- $\mathbf{D}^2_\alpha = \{D \in \mathbf{D}^1_\alpha$: $\bar{f}^{\alpha}$ has a $<_D$ -upper bound $f \in \Pi\bar{\delta}\}$
- $\mathbf{D}^3_\alpha = \{D \in \mathbf{D}^2_\alpha$: $\bar{f}^{\alpha}$ has a $<_D$ -eub $f \in \Pi\bar{\delta}\}$.

For every $D \in \mathbf{D}^3_\alpha$ let

- $\mathscr{F}^3_{\alpha,D} = \big\{f \in \Pi\bar{\delta}$: f is a $<_D$ -eub of $\langle f_\beta : \beta < \alpha\rangle\big\}$.

Note

$\odot_1$ if $D_1 \in \mathbf{D}^1_\alpha$ and f exemplifies this then for some D_2, $D_1 \subseteq D_2 \in \mathbf{D}$ and f is a $<_{D_2}$-upper bound of $\bar{f}$, i.e. f exemplifies $D_2 \in \mathbf{D}^2_\alpha$; in fact D_2 is uniformly definable from f (and $\bar{f}^{\alpha}$, D_1).

[Why? Let $\bar{A} = \langle A_\gamma : \gamma < \alpha\rangle$ be defined by $A_\gamma := \{s \in Y : f(s) \leq f_\gamma(s)\}$. So $\langle A_\gamma/D_1 : \gamma < \alpha\rangle$ is increasing (in the Boolean algebra $\mathscr{P}(Y)/D_1$, of course), but clearly $|\{A/D_1 : A \subseteq Y\}| \leq_{\rm qu} |\mathscr{P}(Y)|$ and $\mathrm{hrtg}(\mathscr{P}(Y)) \leq \theta$ by clause $(A)(c)(\gamma)$ of the assumption. Let $\mathscr{U} = \{\gamma < \alpha$: for no $\beta < \gamma$ do we have $A_\gamma = A_\beta \mod D\}$, so clearly $|\mathscr{U}| < \mathrm{hrtg}(\mathscr{P}(Y)) \leq \theta$ by $(A)(c)(\gamma)$ but by the present case assumption, $\mathrm{cf}(\alpha) \geq \theta$ so $\langle A_\gamma/D_1 : \gamma < \alpha\rangle$ is necessarily eventually constant. Let $\alpha(*) = \min\{\gamma$: if $\beta \in (\gamma, \alpha)$ then $A_\beta = A_\gamma \mod D_1\}$; it is well defined (and $< \alpha$). Now $A_{\alpha(*)} \notin D_1$ as otherwise $f \leq f_{\alpha(*)} < f_{\alpha(*)+1} \mod D_1$ contradicting the assumption on f. Let $D_2 := D_1 + (Y \backslash A_{\alpha(*)})$. Clearly D_2 is as required.]

$\odot_2$ if $D \in \mathbf{D}^2_\alpha$ and f exemplifies it then for some g we have:

(a) $g \in \Pi\bar{\delta}$

(b) $g \leq_D f$

(c) g is a $<_D$-upper bound of $\langle f_\gamma : \gamma < \alpha\rangle$

(d) there is no $h \in \Pi\bar{\delta}$ which is an $<_D$-upper bound of $\langle f_\gamma : \gamma < \alpha\rangle$ such that $h <_D g$.

[Why? Use DC and D being $\aleph_1$-complete.]

$\odot_3$ if $D_1 \in \mathbf{D}^2_\alpha$ and g is as in $\odot_2$ then for a unique pair $(\mathfrak{y}, f)$ we have

(a) $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$

(b) $D_{\mathfrak{y},1} = D_1$

(c) $D_{\mathfrak{y},2} = \mathrm{dual}(J[g, \bar{f}^{\alpha}, D_1])$ from 0.11(1)

(d) $Z_{\mathfrak{y}}$ satisfies:

(α) $Z_{\mathfrak{y}} \in D_{\mathfrak{y},2}$

(β) $Z \in D_{\mathfrak{y},2} \wedge Z \subseteq Z_{\mathfrak{y}} \Rightarrow c\ell((\mathrm{Rang}(g \restriction Z_{\mathfrak{y}}) = c\ell(\mathrm{Rang}(g \restriction Z)$,

(γ) if $t \in Y$ and $g(t) \in c\ell(\mathrm{Rang}(g \restriction Z_{\mathfrak{y}})$ then $t \in Z_{\mathfrak{y}}$

(e) $h_{\mathfrak{y}} : Z_{\mathfrak{y}} \to \mathrm{Ord}$ (really into some $\alpha < \mathrm{hrtg}(\mathscr{P}(Y))$ is defined by $g(s) =$ the $h_{\mathfrak{y}}(s)$th member of $c\ell(\mathrm{Rang}(g \restriction Z_{\mathfrak{y}}))$ if $s \in Z_{\mathfrak{y}}$ and

(f) $f : Y \to \mathrm{Ord}$ is defined by $f \restriction Z_{\mathfrak{y}} = g \restriction Z_{\mathfrak{y}}$ and $f(s) = 0$ for $s \in Y \backslash Z_{\mathfrak{y}}$.

[Why? We apply 0.14(2) with $g, \langle f_\gamma : \gamma < \alpha \rangle$ here standing for $f, \bar{f}$ there to define $\mathfrak{y}$ and then let $f = (g \restriction Z_{\mathfrak{y}}, 0_{Y \backslash Z_{\mathfrak{y}}})$ as in 0.14(3).]

In particular, the "unique" in $\odot_3$ is justified by:

$\odot_3'$ if $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$ and f', f'' are $\mathfrak{y}$ – eub of $\bar{f}^\alpha$ <u>then</u> $f' = f''$, i.e. 0.14(3).

Also, (recalling $\mathrm{dom}(f') = \mathrm{dom}(f'') = Z_{\mathfrak{y}}$ by $\odot_3$, (e), see 0.11(4))

$\odot_3''$ if $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$ and f', f'' satisfy $\odot_3(e)$ <u>then</u> $f' = f'' \mod D_{\mathfrak{y},2}$.

Recalling 0.11(5), for $D \in \mathbf{D}$ let

$\odot_4$ $\mathfrak{Y}^2_{\alpha,D} = \{\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y) : D \subseteq D_{\mathfrak{y},1}$ and some $f \in {}^{Z[\mathfrak{y}]}\mathrm{Ord}$ is a $\mathfrak{y}$-eub of $\bar{f}^\alpha\}$

$\odot_5$ for each $\mathfrak{y} \in \mathfrak{Y}^2_{\alpha,D}$, let $f_{\mathfrak{y}} = f^2_{\alpha,D,\mathfrak{y}}$ be the unique function $f \in \Pi(\bar{\delta} \restriction Z_{\mathfrak{y}})$ which is the canonical $\mathfrak{y}$-eub of $\langle f_\gamma : \gamma < \alpha \rangle$.

Now let

$\odot_6$ for $s \in Y$ let $\mathscr{U}^*_{\alpha,s} = \{f_{\mathfrak{y}}(s) : \mathfrak{y} \in \mathfrak{Y}^2_{\alpha,D}$ *for some* $D \in \mathbf{D}\}$.

Clearly

$\odot_7$ (a) $\langle \mathscr{U}^*_{\alpha,s} : s \in Y \rangle$ is well defined

(b) $\mathscr{U}^*_{\alpha,s} \subseteq \delta_s$

(c) if $s \in Y$ then $|\mathscr{U}^*_{\alpha,s}| < \theta$.

[Why? Clause (a) holds by $\odot_6$ and clause (b) by $\odot_5 + \odot_6$. As for clause (c) by $\odot_6$, $\mathscr{U}^*_{\alpha,s}$ is the range of the function $\mathfrak{y} \mapsto f_{\mathfrak{y}}(s)$ for $\mathfrak{y} \in \mathfrak{Y}^2_{\alpha,D}, s \in Z_{\mathfrak{y}}$, so clearly $|\mathscr{U}^*_{\alpha,s}| \leq_{\mathrm{qu}} |\mathfrak{Y}^2_{\alpha,D}| \leq_{\mathrm{qu}} |\mathrm{Fil}^4_{\aleph_1}(Y)|$ hence $|\mathscr{U}^*_{\alpha,s}| < \mathrm{hrtg}(\mathrm{Fil}^4_{\aleph_1}(Y))$ which is $\leq \theta$ by (A)(c)(δ) of the claim.]

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$\odot_8$ $X := \{s \in Y: \sup(\mathscr{U}^*_{\alpha,s}) < \delta_s\} = Y \mod \text{cf} - \text{id}_{<\theta}(\bar\delta)$ hence $X \in D_*$.

[Why? By $\odot_7(a), (b), (c)$ and Definition 1.3 we have $X = Y \mod \text{cf} - \text{id}_{<\theta}(\bar\delta)$ but by (A)(c)(α), this implies $X \in D_*$.]

So define $f_\alpha \in \Pi\bar\delta$ by:

$\odot_9$ $f_\alpha(s) \begin{cases} \text{is } \sup(\mathscr{U}^*_{\alpha,s}) & \text{if } s \in X \\ \text{is } 0 & \text{if } s \in Y\backslash X \end{cases}$

Also clearly

$\odot_{10}$ $f_\alpha \in \Pi\bar\delta$

and also

$\odot_{11}$ if $\mathfrak{y} \in \mathfrak{Y}^2_{\alpha,D}$, $D \in \mathbf{D}$ and $\beta < \alpha$ <u>then</u> $f_\beta < f_\alpha \mod D_{\mathfrak{y},2}$.

For $\beta < \alpha$ let $A^\alpha_\beta = \{s \in Y : f_\beta(s) < f_\alpha(s)\}$ so $\bar{A}^\alpha = \langle A^\alpha_\beta : \beta < \alpha\rangle$ is well defined and $\langle A^\alpha_\beta/D_* : \beta < \alpha\rangle$ is decreasing (in the Boolean Algebra $\mathscr{P}(Y)/D_*$) and is eventually constant as $\text{hrtg}(\mathscr{P}(Y)/D_*) \le \text{hrtg}(\mathscr{P}(Y)) \le \theta$ by clause (A)(c)(γ) of the assumption so let $\gamma(\alpha) = \min\{\gamma < \alpha$: for every $\beta \in (\gamma, \alpha)$ we have $A^\alpha_\beta/D_* = A^\alpha_\gamma/D_*\}$.

If $A^\alpha_{\gamma(\alpha)} \in D_*$ then $\beta < \alpha \Rightarrow A^\alpha_\beta \supseteq A^\alpha_{\max\{\beta,\gamma(\alpha)\}} = A^\alpha_{\gamma(\alpha)} \mod D_* \Rightarrow f_\beta < f_\alpha \mod D_*$, so f_α is as required. Otherwise, $A^\alpha_{\gamma(\alpha)} \notin D_*$, so $A_* := Y \backslash A^\alpha_{\gamma(\alpha)} \in D^+_*$ so $D_1 = D_* + A_* \in \mathbf{D}$. Now if $D_1 \in \mathbf{D}^1_\alpha$ then by $\odot_1$ there is D_2 such that $D_1 \subseteq D_2 \in \mathbf{D}^2_\alpha$ hence there is $g \in \Pi\bar\delta$ as in $\odot_2$ for D_2 hence there is $\mathfrak{y} \in \text{Fil}^4_{\aleph_1}(Y)$ as in $\odot_3$ hence $f_\mathfrak{y} \in \Pi(\bar\delta)$ as in $\odot_5$, so $Z_\mathfrak{y} \in D_{\mathfrak{y},2}$, and by the choice of $\mathscr{U}^*_{\alpha,s}(s \in Y)$ and f_α we have $f_\mathfrak{y} \le f_\alpha \mod D_{\mathfrak{y},2}$ hence $\beta < \alpha \Rightarrow f_\beta < f_\alpha \mod D_{\mathfrak{y},2}$ so $f_{\gamma(\alpha)} < f_\alpha \mod D_{\mathfrak{y},2}$. But $A_* \in D_1 = D_{\mathfrak{y},1} \subseteq D_{\mathfrak{y},2}$ and by the choice of $A^\alpha_{\gamma(\alpha)}$ and A_* we have $f_\alpha \restriction A_* \le f_{\gamma(\alpha)} \restriction A_*$ contradicting the previous sentence.

So necessarily ($A_* \in D^+_*$ and) $D_1 = D_* + A_* \in \mathbf{D}$ does not belong to $\mathbf{D}^1_\alpha$ which means $\bar{f}^\alpha$ is cofinal in $(\Pi\bar\delta, <_{D_*+A_*})$ hence letting the desired $(\alpha_*, f, \bar{f}, A_*/D_*)$ in (B) of 1.5 be $(\alpha, f_\alpha, \bar{f}^\alpha, A_*/D_*)$ we are done. $\square$

Theorem 1.7 <u>*The pcf Theorem:*</u> $[\text{Ax}^4_{4,\theta,\partial}, \theta = \text{hrtg}({}^Y\mu) + \text{DC}]$

If (A) *then* $(\text{B})^+$ *where:*

(A) *we*[4] *are given* Y, *an arbitrary set*, $\bar\delta$, *a sequence of limit ordinals and* μ, *an infinite cardinal* (*or just a limit ordinal*) *such that*

(a) $\bar\delta = \langle \delta_s : s \in Y\rangle$ *and* $\mu = \sup\{\delta_s : s \in Y\}$

4 Clause (A) here is as in 1.5(A) but D_* is just a filter on Y, not necessarily $\aleph_1$-complete filter on Y [i.e. we weaken clause (b)], noting that possibly $D_* = \{Y\}$, still we require $\text{cf} - \text{fil}_{<\theta}(\bar\delta) \subseteq D_*$.

(b) D_* *is an* $\aleph_1$*-complete filter*[5] *on* Y*, it may be* $\{Y\}$

(c) θ *is any cardinal satisfying:*

(α) $\mathrm{cf}-\mathrm{id}_{<\theta}(\bar\delta) \subseteq \mathrm{dual}(D_*)$*, note that this holds when each* δ_s *is an ordinal* $\le \mu$ *of cofinality* $\ge \theta$*, see below*

(β) $\alpha < \theta \Rightarrow \mathrm{hrtg}([\alpha]^{\aleph_0} \times \partial) \le \theta$ *so* $\partial < \theta$

(γ) $\mathrm{hrtg}(\mathscr{P}(Y)) \le \theta$

(δ) $\mathrm{hrtg}(\mathrm{Fil}^4_{\aleph_1}(Y)) \le \theta$

(B)$^+$ *there are* $\varepsilon(*)$, $\bar D^*$, $\bar A^*$, $\bar E^*$, $\bar\alpha^*$, $\bar g$*, in fact* ∂*-uniformly definable from* $(Y, \bar\delta, D_*)$ *such that:*

(a) $\varepsilon(*) < \mathrm{hrtg}(\mathscr{P}(Y))$

(b) $\bar D^* = \langle D^*_\varepsilon : \varepsilon \le \varepsilon(*)\rangle$ *and* $\bar E^* = \langle E^*_\varepsilon : \varepsilon < \varepsilon(*)\rangle$

(c) $\bar D^*$ *is a* $\subset$*-increasing continuous sequence of filters on* Y

(d) *if* $\varepsilon = \zeta + 1$ *<u>then</u>* D^*_ε *is a filter on* Y *generated by* $D_\zeta \cup \{A\}$ *for some* $A \subseteq Y$ *such that* $A \in D^+_\zeta$

(e) $D^*_0 = D_*$

(f) D^*_ε *is a filter on* Y *for* $\varepsilon < \varepsilon(*)$ *but* $D^*_{\varepsilon(*)} = \mathscr{P}(Y)$,

(g) (α) $\bar\alpha^* = \langle \alpha^*_\varepsilon : \varepsilon \le \varepsilon(*)\rangle$

(β) $\bar\alpha^*$ *is an increasing continuous sequence of ordinals*

(γ) $\alpha^*_0 = 0, \mathrm{cf}(\alpha^*_{\varepsilon+1}) \ge \theta$

(δ) $\varepsilon(*)$ *is a successor ordinal*

(h) $\bar g = \langle g_\alpha : \alpha < \alpha^*_{\varepsilon(*)}\rangle$ *is a sequence of members of* $\Pi\bar\delta$*, so of functions from* Y *into the ordinals*

(i) *if* $\beta < \alpha < \alpha_{\varepsilon+1}$ *<u>then</u>* $g_\beta < g_\alpha \mod D^*_\varepsilon$

(j) $\bar A^* = \langle A^*_\varepsilon / D^*_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ *where* $A^*_\varepsilon \subseteq Y$*, so only* $A^*_\varepsilon / D^*_\varepsilon$ *is computed*[6] *not* A^*_ε*, still* $(Y \backslash A^*_\varepsilon)/D^*_\varepsilon$ *and* $D^*_\varepsilon + (Y \backslash A^*_\varepsilon)$ *are well defined*

(k) $D^*_{\varepsilon+1} = D^*_\varepsilon + A^*_\varepsilon$ *and* $E^*_\varepsilon = D^*_\varepsilon + (Y \backslash A^*_\varepsilon)$ *if* ε *is a successor ordinal and* D_ε *if otherwise*

(l) $\langle g_\alpha : \alpha \in [\alpha^*_\varepsilon, \alpha^*_{\varepsilon+1})\rangle$ *is increasing and cofinal in* $(\Pi\bar\delta, <_{E_\varepsilon})$ *so also* $\bar g \restriction \alpha^*_{\varepsilon+1}$ *is.*

Remark 1.8 (1) Note that unlike the ZFC case, the $\alpha^*_{\varepsilon+1}$'s (and even $\alpha^*_{\varepsilon+1} - \alpha^*_\varepsilon$) are ordinals rather than regular cardinals and we do not exclude here $\varepsilon < \zeta \wedge \mathrm{cf}(\alpha^*_{\varepsilon+1}) = \mathrm{cf}(\alpha^*_{\zeta+1})$. Also we do not know that $\langle \mathrm{cf}(\alpha^*_\varepsilon) : \varepsilon < \varepsilon(*)\rangle$ is increasing or even non-decreasing.
(2) We may get $\langle \alpha^*_{\varepsilon+1} - \alpha^*_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ non-decreasing but this is of unclear value. [For this we proceed as below but when we arrive to $\varepsilon + 1$ and there is $\zeta < \varepsilon$ such that $\alpha^*_{\varepsilon+1} - \alpha^*_\varepsilon < \alpha^*_\zeta - \alpha^*_{\zeta+1}$, choose the first one, we go back, retaining only $\bar g \restriction \alpha^*_\zeta$.

[5] This is reasonable as we normally use $D_* = \mathrm{dual}(\mathrm{cf}-\mathrm{id}_{<\theta}(\bar\delta))$ which is $\aleph_1$-complete by 1.3(1A).

[6] But see 2.16.

Now we try again to choose g'_α for $\alpha \geq \alpha^*_\zeta$ but demanding $g'_{\alpha^*_\zeta+\beta} \geq g_{\alpha^*_\varepsilon+\beta}, g_{\alpha^*_\zeta+\beta}$. This process converges.]
(3) However 2.11(5) below is a simpler way. Working harder we get $\langle \alpha^*_{\varepsilon+1} - \alpha^*_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ is (strictly) increasing (using increasing rectangles of functions).
(4) As in $(*)_4$ of the proof of 1.5, without loss of generality $\alpha^*_{\varepsilon+1} - \alpha^*_\varepsilon < \text{hrtg}([\text{cf}(\alpha^*_{\varepsilon+1})]^{\aleph_0}) = ([\text{cf}(\alpha^*_{\varepsilon+1})]^{\aleph_0})^+$.
[Why? As have first chosen $\langle g'_\alpha : \alpha \in (\alpha^*_\varepsilon, \alpha'_{\varepsilon+1}]\rangle$ and just as $\langle g_\alpha : \alpha \in (\alpha^*_\varepsilon, \alpha^*_{\varepsilon+1}]\rangle$ was chosen before we choose $\langle g_\alpha : \alpha \in (\alpha^*_\varepsilon, \alpha^*_{\varepsilon+1}]\rangle$ by (e_α is as in $(*)_4$)

- $\alpha^*_{\varepsilon+1} = \alpha^*_\varepsilon + \text{otp}(e_{\alpha'_{\varepsilon+1}} \backslash (\alpha^*_\varepsilon + 1))$
- if $\beta \in e_{\alpha'_{\varepsilon+1}} \backslash (\alpha^*_\varepsilon + 1)$ and $\gamma = \text{otp}(e_\alpha \cap \beta) \backslash (\alpha^*_\varepsilon + 1)$ then $g_\gamma = g'_\beta$
- if $\beta = \alpha^*_{\varepsilon+1}$ then $g_\beta = g'_{\alpha'_{\varepsilon+1}}$.

So we are done.
(5) Concerning (β) of 1.7$(B)^+(e)$, recall that D_* include $\text{cf} - \text{fil}_{<\theta}(\bar{\delta})$.
(6) Concerning 1.7$(B)^+(f)$, if $D^*_{\varepsilon(*)} = \mathscr{P}(Y)$ then it is not really a filter.
(7) Concerning 1.7 $(B)^+(i)$, note that using this clause in Definition 2.1(2) we mean only $\leq !$, that is we may have

$(B)^+$ $(i)'$ if $\beta < \alpha < \alpha_{\varepsilon+1}$ then $g_\beta \leq g_\alpha \mod D^*_\varepsilon$.

Proof Let $\mathscr{S}_{\lambda_*}, <_{\lambda_*}$ $\langle w^*_i : i < \text{otp}(\mathscr{S}_{\lambda_*}, <_{\lambda_*})\rangle$ as well as $\bar{e}$ be as in the proof of 1.5.

We try to choose $(\alpha^*_\varepsilon, \bar{g} \restriction (\alpha^*_\varepsilon + 1), \bar{D}^\varepsilon)$, $\bar{D}^\varepsilon = \langle D^*_\xi : \xi < \varepsilon\rangle$ by induction on $\varepsilon < \text{hrtg}(\mathscr{P}(Y))$ such that the relevant parts of $(B)^+$ holds, but if $\emptyset \in D^*_\varepsilon$ then $g_{\alpha^*_\varepsilon}$ is not well defined, so $\bar{g}^\varepsilon = \bar{g} \restriction \alpha^*_\varepsilon = \langle g_\alpha : \alpha < \alpha^*_\varepsilon\rangle$ and $\langle A^*_\zeta / D^*_\zeta : \zeta < \varepsilon\rangle$ are determined. Clearly the induction has to stop before $\text{hrtg}(\mathscr{P}(Y))$, otherwise the sequence $\langle A_\zeta / D^*_\zeta : \zeta < \text{hrtg}(\mathscr{P}(Y))\rangle$ gives a contradiction to the definition of $\text{hrtg}(\mathscr{P}(Y))$.

<u>Case A</u>: $\varepsilon = 0$

Let $\alpha^*_\varepsilon = 0$, $D^*_\varepsilon = D_*$ and g_0 is constantly zero.

<u>Case B</u>: ε a limit ordinal

Let $\alpha^*_\varepsilon = \cup\{\alpha^*_\zeta : \zeta < \varepsilon\}$, $D^*_\varepsilon = \cup\{D^*_\zeta : \zeta < \varepsilon\}$ and $\bar{g} \restriction \alpha^*_\varepsilon$ is naturally defined and define $g_{\alpha^*_\varepsilon} \in \Pi\bar{\delta}$ by, for $s \in Y$ letting $g_{\alpha^*_\varepsilon}(s) = \cup\{g_{\alpha^*_\zeta}(s) + 1 : \zeta < \varepsilon\}$ if it is $< \delta_s$ and 0 otherwise. As in Case 3 of the proof of 1.5, clause $(B)^+(i)$ is satisfied, because $\text{hrtg}(\mathscr{P}(Y)) > \varepsilon$.

<u>Case C</u>: $\varepsilon = \zeta + 1$ and $\emptyset \notin D^*_\zeta$.

Let (note that $A_{\mathbf{a},n}$ in (b) below is almost equal to $Y \backslash A^*_{\xi_n}$ but we know only $A^*_{\xi_n} / D^*_{\xi_n}$):

$(*)_1$ (a) $\mathbf{J}_{\zeta,1} = \{A \subseteq Y : A \in (D^*_\zeta)^+$ and $D^*_\zeta + A$ is $\aleph_1$-complete$\}$

(b) $\mathbf{U}_\zeta = \{\mathbf{a}: \mathbf{a} = \langle (A_n, \xi_n): n < \omega\rangle = \langle (A_{\mathbf{a},n}, \xi_{\mathbf{a},n}): n < \omega\rangle$,
for every $n < \omega$ we have
$\xi_n < \zeta$ and $D^*_{\xi_n+1} = D^*_{\xi_n} + (Y \backslash A_n)$ and
$A_\mathbf{a} := \cup\{A_n : n < \omega\} \neq \emptyset \mod D^*_\zeta\}$;
so this concerns witnesses to D^*_ζ being not $\aleph_1$-complete and
$A_\mathbf{a} \in D^+_\zeta \subseteq \mathscr{P}(Y)$

(c) $\mathbf{J}_{\zeta,2} = \{A \subseteq Y : A \in (D^*_\zeta)^+$ and for some $\mathbf{a} \in \mathbf{U}_\zeta$ we have $A \subseteq A_\mathbf{a}\}$.

Note

$(*)_2$ (a) $\mathbf{J}_{\zeta,1} \cup \mathbf{J}_{\zeta,2} \subseteq (D^*_\zeta)^+$ is dense, i.e. if $A \in (D^*_\zeta)^+$ then for some $B \subseteq A$, we have $B \in \mathbf{J}_{\zeta,1} \cup \mathbf{J}_{\zeta,2}$

(b) if $\ell \in \{1, 2\}$, $A \in \mathbf{J}_{\zeta,\ell}$, $B \subseteq A$ and $B \in D^+_\zeta$ <u>then</u> $B \in \mathbf{J}_{\zeta,\ell}$.

[Why Clause (a)? Because we are assuming that D_* is $\aleph_1$-complete in (A)(b). For clause (b), just read the definition of $\mathbf{J}_{\zeta,\ell}$].

Now we try to choose f_α (or pedantically f^ε_α if you like) by induction on α such that:

$(*)_3$ (a) $f_\alpha \in \Pi\bar{\delta}$

(b) $\beta < \alpha^*_\zeta \Rightarrow g_\beta < f_\alpha \mod D^*_\zeta$; follows by (c) + (d)

(c) $\beta < \alpha \Rightarrow f_\beta < f_\alpha \mod D^*_\zeta$

(d) $f_0 = g_{\alpha^*_\zeta}$.

Arriving to α, $\bar{f} = \langle f_\beta : \beta < \alpha\rangle$ has been defined. Let $\mathbf{J}^*_{\zeta,\alpha} = \{A \subseteq Y : A \in (D^*_\zeta)^+$ and $\bar{f}$ has an upper bound in $(\Pi\bar{\delta}, <_{D^*_\zeta+A})\}$.

<u>Sub-case C1</u>: $(\mathbf{J}_{\zeta,1} \cup \mathbf{J}_{\zeta,2}) \cap \mathbf{J}^*_{\zeta,\alpha}$ is dense in $((D^*_\zeta)^+, \supseteq)$.

First, as in the proof of 1.5, (that is, choosing f_α in the inductive step in the proof) we can define $\bar{f}^1_{\zeta,\alpha}$ such that:

$(*)_4$ (a) $\bar{f}^1_{\zeta,\alpha} = \langle f^1_{\zeta,\alpha,A} : A \in \mathbf{J}_{\zeta,1} \cap \mathbf{J}^*_{\zeta,\alpha}\rangle$

(b) $f^1_{\zeta,\alpha,A} \in \Pi\bar{\delta}$

(c) $f^1_{\zeta,\alpha,A}$ is a $<_{D^*_\zeta+A}$-upper bound of $\{g_{\alpha^*_\zeta}\} \cup \{f_\beta : \beta < \alpha\}$.

Second, we consider $\mathbf{a} \in \mathbf{U}_\zeta$ hence $A_\mathbf{a} \in \mathbf{J}_{\zeta,2}$.

Let

- for $u \subseteq \alpha^*_\zeta$ let $g^{[u]} \in \Pi\bar{\delta}$ be defined by $g^{[u]}(s) = \sup\{g_\beta(s) + 1 : \beta \in u\}$ if this supremum is $< \delta_s$ and 0 otherwise.

Note that

$(*)_5$ (a) if $A \subseteq Y$, $A = \emptyset \mod D^*_\zeta$ then for every $f \in \Pi\bar{\delta}$ for some finite $v \subseteq \alpha^*_\zeta$ we have $\{s \in A: \neg(\exists \beta \in v)(f(s) < g_\beta(s))\} = \emptyset \mod D_*$

(b) if $u_1 \subseteq u_2$ are from $[\alpha^*_\zeta]^{<\partial}$ then $g^{[u_1]} \leq g^{[u_2]} \mod D_*$.

[Why? By induction on ζ using $(B)^+(k)$, (l) recalling $D^*_0 = D_*$ or see the proof of 2.13. Clause (b) is proved by $\mathrm{cf} - \mathrm{id}_{<\partial}(\bar{\delta}) \subseteq \mathrm{cf} - \mathrm{id}_{<\theta}(\bar{\delta}) \subseteq \mathrm{dual}(D_*)$.]

$(*)_6$ if $f \in \Pi\bar{\delta}$ then f has a $<_{D^*_\zeta + A_\mathbf{a}}$-upper bound and even a $<_{D_* + A_\mathbf{a}}$-upper bound of the form $g^{[u]}$ for some countable $u \subseteq \alpha^*_\zeta$.

[Why? Let $f \in \Pi\bar{\delta}$, now for each n there is $\alpha_n < \alpha^*_{\xi_{\mathbf{a},n}+1}$ such that $f < g_{\alpha_n} \mod (D^*_{\xi_{\mathbf{a},n}} + A_{\mathbf{a},n})$, moreover, see $(*)_5(a)$, there is a finite set $v_n \subseteq \alpha^*_{\xi_{\mathbf{a},n}+1}$ such that $(\forall s \in A_{\mathbf{a},n})(\exists \beta \in v)(f(s) < g_\beta(s))$. As those are finite sets of ordinals (or use $\mathrm{AC}_{\aleph_0}$) there is such a sequence $\langle v_n : n < \omega\rangle$, so $u = \cup\{v_n : n < \omega\}$ is as required, recalling $\mathrm{cf} - \mathrm{id}_{<\aleph_1}(\bar{\delta}) \subseteq \mathrm{dual}(D_*)$ as in earlier cases so we have proved (a) of $(*)_5$.]

Lastly (well defined by $(*)_5(b) + (*)_6$ recalling our sub-case assumption):

$(*)_7$ let $\bar{f}^2_{\zeta,\alpha} = \langle f^2_{\zeta,\alpha,\mathbf{a}} : \mathbf{a} \in \mathbf{U}_\zeta\rangle$ be defined by: $f^2_{\zeta,\alpha,\mathbf{a}}$ is $g^{[u]}$ where $u = u_\mathbf{a} \in \mathscr{S}_{\lambda_*}$ is the $<_{\lambda_*}$-first $u \in \mathscr{S}_{\lambda_*}$ for which $g^{[u]} \in \Pi\bar{\delta}$ is a $<_{D^*_\zeta + A_\mathbf{a}}$-common upper bound of $\{g_{\alpha^*_\zeta}\} \cup \{f_\beta : \beta < \alpha\}$.

Note that

$(*)_8$ if $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{U}_\zeta$ and if $A_{\mathbf{a}_1}/D^*_\zeta = A_{\mathbf{a}_2}/D^*_\zeta$ then $f^2_{\zeta,\mathbf{a}_1,\alpha} = f^2_{\zeta,\mathbf{a}_2,\alpha}$.

Having defined $\langle f^1_{\zeta,\alpha,A} : A \in \mathbf{J}_{\zeta,1} \cap \mathbf{J}^*_{\zeta,\alpha}\rangle$ and $\langle f^2_{\zeta,\alpha,\mathbf{a}} : \mathbf{a} \in \mathbf{U}_\zeta \cap \mathbf{J}^*_{\zeta,\alpha}\rangle$, of course, they all depend on ζ; we define $f_\alpha \in {}^Y\mathrm{Ord}$ by

$(*)_9$ $f_\alpha(s)$ is: the supremum below if it is $< \delta_s$ and zero otherwise. where the supremum is $\sup(\{f^1_{\alpha,\zeta,A}(s) + 1 : A \in \mathbf{J}_{\zeta,1} \cap \mathbf{J}^*_{\zeta,\alpha}\} \cup \{f^2_{\zeta,\alpha,\mathbf{a}}(s) + 1 : \mathbf{a} \in \mathbf{U}_\zeta\})$.

So indeed $f_\alpha \in \Pi\bar{\delta}$ as in the end of the proof of 1.5 and is as required for α as $\mathrm{hrtg}(\mathbf{J}_{\zeta,1} \cap \mathbf{J}^*_{\zeta,\alpha}) \leq \mathrm{hrtg}(\mathscr{P}(Y)/D_*) \leq \theta$ and $\mathrm{hrtg}(\{f_{\zeta,\alpha,\mathbf{a}} : \mathbf{a} \in \mathbf{U}_\zeta\}) \leq \mathrm{hrtg}(\{A_\mathbf{a} : \mathbf{a} \in \mathbf{U}_\zeta\}) \leq \mathrm{hrtg}(\mathscr{P}(Y)) \leq \theta$ because of $(*)_8$ (so even $\mathrm{hrtg}(\mathscr{P}(Y)/D^*_\zeta)$ suffice); note that we have used $(A)(c)(\beta)$.

<u>Sub-case C2</u>: $(\mathbf{J}_{\zeta,1} \cap \mathbf{J}_{\zeta,2}) \cap \mathbf{J}^*_{\zeta,\alpha}$ is not dense in $((D^*_\zeta)^+, \supseteq)$.

Let $A_* \in (D^*_\zeta)^+$ be such that $A \subseteq A_* \wedge A \in (D^*_\zeta)^+ \Rightarrow A \notin (\mathbf{J}_{\zeta,1} \cup \mathbf{J}_{\zeta,2}) \cap \mathbf{J}^*_{\zeta,\alpha}$. Wlog for some $\ell \in \{1, 2\}$ we have $A_* \in \mathbf{J}_{\zeta,\ell}$.

As in the proof of 1.5, necessarily α is a limit ordinal of cofinality $\geq \theta$. Now as in Sub-Case C1 we define $\bar{f}^1_{\zeta,\alpha} = \langle f^1_{\zeta,\alpha,A} : A \in (\mathbf{J}_{\zeta,1} \cup \mathbf{J}_{\zeta,2}) \cap \mathbf{J}^*_{\zeta,\alpha}\rangle$ satisfying: $f^1_{\zeta,\alpha,A}$ is a $<_{D^*_\zeta + A}$-upper bound of $\langle f_\beta : \beta < \alpha\rangle$. Let $f^* \in \Pi\bar{\delta}$ be defined by

- $f_*(s)$ the supremum below if it is $< \delta_s$ and is zero otherwise, where $\sup\{f^1_{\zeta,\alpha,A}(s) + 1 : A \in (\mathbf{J}_{\zeta,1} \cup \mathbf{J}_{\zeta,2}) \cap J^*_{\zeta,\alpha}\}$.

As in the proof of 1.5 there is $\beta < \alpha$ such that $\gamma \in [\beta, \alpha) \Rightarrow \{s \in Y : f_\gamma(s) < f_*(s)\} = \{s \in y : f_\beta(s) < f_*(s)\} \mod D^*_\zeta$.

Let β_* be the minimal such β. Lastly, let $A_\zeta = \{s \in y : f_{\beta_*}(s) \geq f_*(s)\}$ and

- $E_\varepsilon = E_\zeta + A_\zeta$
- $D^*_\varepsilon = D^*_\zeta + (Y \backslash A_\zeta)$
- $\alpha^*_\varepsilon = \alpha^*_\varepsilon + \alpha$
- $g_\beta = f_\beta$ for $\beta \in (\alpha^*_\zeta, \alpha^*_\varepsilon)$
- $g_{\alpha^*_\varepsilon} = f_*$.

<u>Case D</u>: None of the above.

So $Y \in D^*_\varepsilon$ and we are done. □

Discussion 1.9 In the results above, is $\langle \mathrm{cf}(\alpha^*_{\varepsilon+1}) : \varepsilon < \varepsilon(*)\rangle$ without repetitions? Certainly this is not obviously so and it seems we can maneuver $\bar{\delta}$ and the closure operation to be otherwise. But can we replace $\bar{\alpha}^*$ and $\bar{g}$ to take care of this? Clearly if $\mathscr{U} \subseteq \alpha^*_{\varepsilon(*)}$ satisfies $\varepsilon < \varepsilon(*) \Rightarrow \alpha^*_{\varepsilon+1} = \sup(\mathscr{U} \cap \alpha^*_{\varepsilon+1})$ then we can replace $\bar{g}$ by $\bar{g} \restriction \mathscr{U}$ so by renaming get $\bar{\alpha}' = \langle \mathrm{otp}(\mathscr{U} \cap \alpha^*_\varepsilon) : \varepsilon \leq \varepsilon(*)\rangle$. So $\mathrm{cf}(\alpha^*_\varepsilon) = \mathrm{cf}(\alpha^*_\zeta) \Leftrightarrow \mathrm{cf}(\alpha'_\varepsilon) = \mathrm{cf}(\alpha'_\zeta)$ and if we have $\mathrm{cf}(\alpha'_\varepsilon) = \mathrm{cf}(\alpha'_\zeta) \Rightarrow \alpha'_{\varepsilon+1} - \alpha'_\varepsilon = \alpha'_{\zeta+1} - \alpha'_\zeta$ we can change $\bar{g}$ to get desired implication. So if $\mathrm{AC}_{\varepsilon(*)}$ holds we are done but we are not assuming it. In this case we also get $\langle \alpha'_{\varepsilon+1} \backslash \alpha'_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ is a sequence of regular cardinals.

2 More on the pcf theorem

2.1 When the cofinalities are smaller

Definition 2.1 (1) We say $\mathbf{x} = (Y, \bar{\delta}, \theta, \varepsilon(*), \bar{\alpha}^*, \bar{D}^*, \bar{E}^*, \bar{f}) = (Y_\mathbf{x}, \bar{\delta}_\mathbf{x}, \theta_\mathbf{x}, \varepsilon_\mathbf{x}, \bar{\alpha}_\mathbf{x}, \bar{D}_\mathbf{x}, \bar{E}_\mathbf{x}, \bar{f}_\mathbf{x})$ is a pcf-system or a pcf-system for $\bar{\delta}$ or for $(\Pi\bar{\delta}, <_D)$ <u>when</u> they are as in $(B)^+$ of 1.7, with $\bar{f}$ here standing for $\bar{g}$ there; so $\bar{\delta} = \langle \delta_s : s \in Y\rangle$, δ_s a limit ordinal; now 2.3 below apply, we will use $\bar{D}_\mathbf{x} = \langle D^\mathbf{x}_\varepsilon : \varepsilon < \varepsilon_\mathbf{x}\rangle = \langle D_{\mathbf{x},\varepsilon} : \varepsilon < \varepsilon_\mathbf{x}\rangle$, similarly for $\bar{f}$, $D_\mathbf{x} = D^\mathbf{x}_0$; let $\varepsilon(\mathbf{x}) = \varepsilon_\mathbf{x}$.
(2) Above we say is "almost a pcf-system" if we demand $\bar{f} \restriction [\alpha_{\mathbf{x},\varepsilon}, \alpha_{\mathbf{x},\varepsilon+1})$ is only $\leq_{D_{\mathbf{x},\varepsilon}}$-increasing (still cofinal) so using $(B)^+(i)'$ instead of $(B)^+(i)$, see 1.7, 1.8(7).
(3) Above we say $\mathbf{x}$ is "weakly a pcf-system" <u>when</u> in 1.7(B)$^+$—we weaken clause (i) as in part (2) and we omit $\bar{E}^*$, i.e. omit clauses (j),(k) but retain (l) which means: if $X_0 \in D^*_{\varepsilon+1} \backslash D^*_\varepsilon$, $X_1 = Y \backslash X_0$ then $\bar{f} \restriction [\alpha^*_\varepsilon, \alpha^*_\varepsilon)$ is $\leq_{D^*_\varepsilon}$-increasing and cofinal in $(\Pi\bar{\delta}, <_{D^*_\varepsilon + X_1})$ and $\bar{f}$ is $\leq_{D^*_\varepsilon}$-increasing.

Observation 2.2 *If θ, Y, D and $\bar{\delta} = \langle \delta_s : s \in Y\rangle$ satisfies clause (A) of 1.7, <u>then</u> there is a pcf-system $\mathbf{x}$ for $(\Pi\bar{\delta}, <_D)$ with $\theta_\mathbf{x} = \theta$.*

Proof By 1.7. □

Observation 2.3 *Let* $\mathbf{x} = (Y, \bar\delta, \theta, \varepsilon(*), \bar\alpha^*, \bar D^*, \bar E^*, \bar f)$ *be as in* 1.7 (*with* $\bar f$ *instead of* $\bar g$) *or Definition* 2.1(2).
(1) $(\Pi\bar\delta, <_{D_{\mathbf{x}}})$ *has a cofinal well orderable set, in fact, of cardinality* $|\alpha^*_{\varepsilon(*)}|$.
(2) *Assume* $f \in \Pi\bar\delta$ *and for* $\varepsilon < \varepsilon(*)$ *we let* $\beta_\varepsilon = \min\{\beta : \beta \in [\alpha^*_\varepsilon, \alpha^*_{\varepsilon+1})$ *satisfy* $f < f_\beta \mod (E^*_{\varepsilon+1})\}$, *then:*

(a) $\beta_\varepsilon \in [\alpha^*_\varepsilon, \alpha^*_{\varepsilon+1})$ *is well defined hence* $\langle \beta_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ *is well defined*
(b) *for some finite* $u \subseteq \varepsilon(*)$ *we have* $f < \sup\{f_{\beta_\varepsilon} : \varepsilon \in u\}$
(b)$^+$ *moreover* $\langle f_{\beta_\varepsilon} : \varepsilon \in u\rangle$ *is* ∂*-uniformly definable from* f *and* $\bar\delta$ *and* D^*_0 (*equivalently,* f *and* $\mathbf{x}$).

Proof (1) By (2).
(2) Easy; e.g.
Clause (b): Let $\varepsilon \le \varepsilon_{\mathbf{x}}$ be minimal such that

(∗) $\varepsilon = \varepsilon_*$ for some finite $u \subseteq [\varepsilon, \varepsilon_{\mathbf{x}})$ we have $f < \max\{f_{\beta_\zeta} : \zeta \in u\} \mod D_{\mathbf{x},\varepsilon}$.

Now ε is well defined because $\varepsilon_{\mathbf{x}}$ is a successor ordinal and $\langle f_\beta : \beta < \alpha^*_{\varepsilon(\mathbf{x})}\rangle$ is cofinal in $\Pi\bar\delta, <_{D_{\mathbf{x},\varepsilon(\mathbf{x})-1}}$ and so $u = \{\beta_{\varepsilon(\mathbf{x})-1}\}$ is as required.

If $\varepsilon = \zeta+1, < \varepsilon_{\mathbf{x}}$ and u is as in (∗) the set $Z = \{s \in Y : f(s) < \max\{f_{\beta_\zeta}(s) : \zeta \in u\}$ is $= \emptyset \mod E_{\zeta+1}$ and repeat the argument for $\varepsilon = \varepsilon_{\mathbf{x}} - 1$.

If ε is a limit ordinal, this leads to contradiction as $D_{\mathbf{x},\varepsilon} = \cup\{D_{\mathbf{x},\zeta} : \zeta < \varepsilon\}$.

Lastly, if $\varepsilon = 0$ then we are done. □

Discussion 2.4 (1) In 2.3, we may restrict ourselves to $\aleph_1$-complete filters only, so replace ε_* by $\{\varepsilon < \varepsilon_* : E^*_\varepsilon$ is $\aleph_1$-complete$\}$ but use countable u.
(2) Similarly for θ-complete.
(3) Recall that with choice or just AC_Y, the ideal $\mathrm{cf} - \mathrm{id}_{<\theta}(\bar\delta)$ is degenerate: if, for transparency, θ is regular, then $\mathrm{cf}-\mathrm{id}_{<\theta}(\bar\delta) = \{X \subseteq Y : (\forall s \in X)[\mathrm{cf}(\delta_s) < \theta]$ and $|X| < \theta\}$.

We have dealt with $(\prod_s \delta_s, <_D)$ when $D \supseteq \mathrm{cf} - \mathrm{fil}_{<\theta}(\bar\delta)$ and $\theta \ge \mathrm{hrtg}(\mathrm{Fil}^4_{\aleph_1}(Y))$; we try to lower the restriction on the cardinal θ with some price.

Definition 2.5 Assume D is a filter on Y, $\alpha(*)$ an ordinal and $\bar f = \langle f_\alpha : \alpha < \alpha(*)\rangle$ is a $\le_D$-increasing sequence of members of ${}^Y\mathrm{Ord}$ and $f \in {}^Y\mathrm{Ord}$ is not $<_D$-below any f_α. We define

$$\mathrm{id}(f, \bar f, D) = \{Z \subseteq Y : \text{there is } \alpha < \alpha(*) \text{ such that } Z \subseteq \{s \in Y : f(s) < f_\alpha(s)\} \mod D\}.$$

Claim 2.6 *For* $Y, D, \bar f, f$ *as in Definition* 2.5 *above.*
(1) $\mathrm{id}(f, \bar f, D)$ *is an ideal on* Y *extending* $\mathrm{dual}(D)$.
(2) f *is a* $\le_{\mathrm{id}(f,\bar f,D)}$*-upper bound of* $\bar f$.
(3) *For* $A \in D^+$ *we have:* $\mathscr{P}(A) \cap \mathrm{id}(f, \bar f, D) \subseteq \mathrm{dual}(D)$ <u>*iff*</u> f *is a* $\le_{D+A}$*-upper bound of* $\bar f$.

(4) *If* $A \in D^+ \cap \mathrm{id}(f, \bar{f}, D)$ <u>*then*</u> *for every* $\alpha < \alpha(*)$ *large enough,* $f < f_\alpha$ mod $(D + A)$.
(5) $\mathrm{id}(f, \bar{f}, D) = \mathrm{id}(f', \bar{f}, D)$ *when* $f' \in {}^Y\mathrm{Ord}$ *and* $f' =_D f$.

Proof Straightforward. □

Notation 2.7 (1) Given $\bar{\delta} = \langle \delta_s : s \in Y \rangle$ and set u of ordinals let $h_{[u,\bar{\delta}]}$ be the function h with domain Y such that: $h(s)$ is $\sup(u \cap \delta_s)$ <u>when</u> it is $< \delta_s$, is 0 <u>when</u> otherwise.
(2) For $\bar{u} = \langle u_s : s \in Y \rangle$ we define $h_{[\bar{u},\bar{\delta}]}$ similarly.

Claim 2.8 *If we assume* $\oplus$ *below and* $(A) + (B)$ <u>*then*</u> (C) *where:*

$\oplus$ (a) $\mathrm{Ax}_{4,\theta} \wedge |Y| \leq \aleph_0$

(b)$_{\kappa,\theta}$ *the union of any sequence of length* $\leq \kappa$ *of sets of ordinals each of cardinality* $< \theta$ *is of cardinality* $< \theta$

(c) $\kappa \leq \theta$

(A) (a) $\bar{\delta} = \langle \delta_s : s \in Y \rangle$ *is a sequence of limit ordinals*

(b) D *is a filter on* Y

(c) $D \supseteq \mathrm{cf} - \mathrm{fil}_{<\theta}(\bar{\delta})$

(d) $\mu = \cup\{\delta_s : s \in Y\}$

(B) δ_* *is an ordinal and*

(a) $f_\alpha \in \prod_{s \in Y} \delta_s$ *for* $\alpha < \delta_*$

(b) *if* $\alpha < \beta < \delta_*$ *then* $f_\alpha < f_\beta \mod D$

(c) $\bar{f} = \langle f_\alpha : \alpha < \delta_* \rangle$ *is not cofinal in* $(\prod_{s \in Y} \delta_s, <_D)$

(d) $\mathrm{cf}(\delta_*) > \kappa$

(C) *we can* θ*-uniformly define (or* (θ, κ)*-uniformly define)* g *such that:*

(*a*) $g \in \prod_{s \in Y} \delta_s$ *is not* $<_D$*-below any* f_α

(b) *if* $g \leq_D g' \in \prod_{s \in Y} \delta_s$ *then* $\mathrm{id}(g', \bar{f}, D) = \mathrm{id}(g, \bar{f}, D)$.

Remark 2.9 (1) See more in 2.13.
(2) Do we uniformly have the parallel of: some stationary $S \subseteq S^\lambda_{\kappa^+}$ belongs to $\check{I}_{\kappa^+}[\lambda]$? See later.
(3) We can weaken 2.8 $\oplus(a)$ to $\mathrm{Ax}_{4,\mu,\theta,\kappa} \wedge \mathrm{hrtg}(Y) \leq \kappa$, [see 0.5(3)] the proof is written for this.

Proof <u>Stage A:</u>
Let $(\mathscr{S}_*, <_*)$ witness $\mathrm{Ax}_{4,\mu,\theta,\kappa}$.
We try to choose $g_\varepsilon, u_\varepsilon, Y_\varepsilon$ by induction on $\varepsilon < \kappa$ such that:

⊞ (a) $g_\varepsilon \in \prod_{s \in Y} \delta_s$

(b) $u_\varepsilon \subseteq \mu$ has cardinality $< \theta$ and $\zeta < \varepsilon \Rightarrow u_\zeta \subseteq u_\varepsilon$

(c) $Y_\varepsilon = \{s \in Y : \delta_s = \sup(\delta_\varepsilon \cap u_\varepsilon)\} = \emptyset \mod D$

(d) if $s \in Y \backslash Y_\varepsilon$ and $\zeta < \varepsilon$ then $g_\zeta(s) < g_\varepsilon(s)$

(e) $g_\varepsilon = h_{[u_\varepsilon, \bar{\delta}]}$, see 2.7

(f) if ε is a limit ordinal then:

- $u_\varepsilon = \cup\{u_\zeta : \zeta < \varepsilon\}$
- $g_\varepsilon(s)$ is $\cup\{g_\zeta(s) : \zeta < \varepsilon\}$ when it is $< \delta_s$
  is 0 when otherwise

(g) if $\varepsilon = \zeta + 1$ then

(α) g_ζ is not as required on g in clause (C)

(β) u_ε is the $<_*$-first $u \in \mathscr{S}_*$ extending u_ζ such that if we define g_ε as $h_{[u,\bar{\delta}]}$ then it is a counterexample like g' there

(h) if $\varepsilon = 0$, g_ε is defined from u_ε similarly.

Now we shall finish by proving in stages B,C below that:

$(*)_1$ if we have defined g_ε but g_ε is as required on g in clause (C)(b), then we are done; this is obvious

$(*)_2$ we can choose g_ε if $\varepsilon = 0$

$(*)_3$ if $\langle g_\zeta : \zeta < \varepsilon \rangle$ was defined we can define g_ε if ε is a limit ordinal $< \kappa$

$(*)_4$ if $\varepsilon = \zeta + 1$ and $\langle g_\xi : \xi \leq \zeta \rangle$ has been defined and g_ζ fail (C), then we can define g_ε

$(*)_5$ we cannot succeed to choose $\langle g_\varepsilon : \varepsilon < \kappa \rangle$.

Stage B:
Proof of $(*)_5$:

Toward contradiction assume $\langle g_\varepsilon : \varepsilon < \kappa \rangle$ is well defined.

For $\varepsilon < \kappa$ and $\alpha < \delta_*$ let $Z_{\varepsilon,\alpha} = \{s \in Y : g_\varepsilon(s) \geq f_\alpha(s)\}$ and let $Y_\varepsilon = \{s \in Y : \sup(u_\varepsilon \cap \delta_s) = \delta_s\}$, it belongs. By clauses (b), (c), (e) of $\boxplus$ we have $Z_{\varepsilon_1,\alpha} \backslash Y_{\varepsilon_1} \subseteq Z_{\varepsilon_2,\alpha} \backslash Y_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2, \alpha < \delta_*$.

Now by clause $(g)(\beta)$ of $\boxplus$, if $\varepsilon = \zeta + 1$ then for some $\alpha < \delta_*$, $Z_{\varepsilon,\alpha} \notin \mathrm{id}(g_\zeta, \bar{f}, D)$ and let α_ζ be the minimal such α. As $\mathrm{cf}(\delta_*) > \kappa$ by Clause (B)(d) of the assumption, $\gamma := \cup\{\alpha_\zeta : \zeta < \kappa\}$ is $< \delta_*$.

Now the sequence $\langle Y_\varepsilon : \varepsilon < \kappa \rangle$ is $\subseteq$-increasing sequence of subsets of Y because $\langle u_\varepsilon : \varepsilon < \kappa \rangle$ is by $\boxplus(b)$ and the choice of Y_ε. By $\oplus(a)$ we have $\mathrm{hrtg}(Y) \leq \kappa$.

Also clearly

$\bullet_2$ $Z_{\varepsilon+1,\gamma} \nsubseteq Z_{\varepsilon,\gamma} \mod D$ and Y_ε.

Together $\langle Z_{\varepsilon+1,\gamma} \backslash Z_{\varepsilon,\gamma} \backslash Y_\varepsilon : \varepsilon < \kappa \rangle$ is a sequence pairwise distinct non-empty of subsets of Y, so recalling $\mathrm{hrtg}(Y) \leq \kappa$, this is contradiction to the first paragraph.

Stage C:

Obviously $(*)_1$ holds.

Proof of $(*)_2$: we can choose g_ε for $\varepsilon = 0$

- $\bullet_1$ there is $g'' \in \prod_{s \in Y} \delta_s$ such that $\alpha < \delta_* \Rightarrow g'' \nleq f_\alpha \mod D$.

[Why? By clause (B)(c) of the claim. For such a g'' there is $u \in \mathscr{S}_*$ such that $\text{Rang}(g'') \subseteq u$ because $\text{hrtg}(Y) \leq \kappa$ and $\mathscr{S}_*$ witness $\text{Ax}_{4,\mu,\theta,\kappa}$. We choose $u \in \mathscr{S}_*$ as the $<_*$-first such $u \in \mathscr{S}_*$ and choose $g \in \prod_{s \in Y} \delta_s$ as $h_{[u,\delta]}$.]

So

- $\bullet_2$ $g \in \Pi \delta_s$
- $\bullet_3$ $g'' \leq g \mod D$.

[Why? Recall $\text{cf}-\text{fil}_{<\theta}(\bar{\delta}) \subseteq D$ by the assumption (A)(c), hence $\{s \in Y : \sup(u \cap \delta_2) \leq g(s)\}$ as $|u| < \theta$ being a member of $\mathscr{S}_*$. So as $(\forall s \in Y)(g''(s) \in \delta_s \cap u)$ we have $g'' \leq g \mod D$ by the choice of u.]

- $\bullet_4$ $\alpha < \delta_* \Rightarrow g \nleq f_\alpha \mod D$.

[Why? By $\bullet_3$ and by the choice of g'' in $\bullet_1$.]

<u>Proof of $(*)_3$</u>: limit ε

We define g_ε as in $\boxplus(f)$, as it is as required because $D \supseteq \text{cf} - \text{fil}_{<\theta}(\bar{\delta})$ by clause (A)(c) of the assumption recalling $\oplus(b)_{\kappa,\theta}$ of the assumption.

<u>Proof of $(*)_4$</u>:

So we are assuming g_ζ is well defined but fail (C)(b) as exemplified by g, let $u \in \mathscr{S}_*$ be $<_*$-minimal such that $\text{Rang}(g) \subseteq u$ and let $h = h^*_{[u,\bar{\delta}]} + 1$, that is $s \in Y \Rightarrow h(s) = h_{[u,\bar{\delta}]}(s) + 1 < \delta_s$ hence $g <_J h_{[u,\bar{\delta}]} \mod D$ and we can finish easily as in the proof of $(*)_2$. □

Observation 2.10 $\text{cf}(\alpha(*)) \geq \theta$ <u>*when*</u>

(a) *D is a filter on Y*
(b) *$\bar{\delta} = \langle \delta_s : s \in Y \rangle$ is a sequence of limit ordinals*
(c) $D \supseteq \text{cf} - \text{fil}_{<\theta}(\bar{\delta})$
(d) *$\bar{f} = \langle f_\alpha : \alpha < \alpha(*) \rangle$ is $<_D$-increasing sequence of members of $\prod_{s \in Y} \delta_s$*
(e) *$\bar{f}$ has no $<_D$-upper bound in $\prod_{s \in Y} \delta_s$.*

Proof The proof splits into cases proving the existence of a $<_D$-upper bound $g \in \prod_{s \in Y} \delta_s$.

<u>Case 1</u>: $\alpha(*) = 0$

The constantly zero function $g : Y \to \{0\}$ can serve.

<u>Case 2</u>: $\alpha(*)$ is a successor ordinal

Let $\alpha(*) = \beta + 1$ and g be defined by $g(s) = f_\beta(s) + 1$. As each δ_s is a limit ordinal, $g \in \prod_{s \in Y} \delta_s$.

<u>Case 3</u>: $\text{cf}(\alpha(*)) \in [\aleph_0, \theta)$

Let $w \subseteq \alpha(*)$ be cofinal of order type $\mathrm{cf}(\alpha(*))$, let $u_s = \{f_\alpha(s) : \alpha \in w\}$ for $s \in Y$ so $\bar{u} := \langle u_s : s \in Y \rangle$ is well defined and $s \in Y \Rightarrow |u_s| < \theta$, hence $g = h_{[\bar{u},\bar{\delta}]}$ is as required. □

Claim 2.11 *If* $\boxplus$ *below holds then* $\oplus_1 \Rightarrow \oplus_2 \Rightarrow \oplus_3$ *<u>where</u>*

$\oplus_1$ $\mathrm{Ax}_{4,\mu,\theta,\kappa}$

$\oplus_2$ *there is a well orderable set cofinal in* $(\Pi\bar{\delta}, <_D)$, *defined* (μ, θ, κ)*-uniformly*

$\oplus_3$ *we can* (θ, κ)*-uniformly define a* $<_D$*-increasing sequence* $\bar{f} = \langle f_\alpha : \alpha < \alpha(*) \rangle$ *in* $(\prod_{s \in Y} \delta_s, <_D)$ *with no upper bound*

where

$\boxplus$ (a) *D a filter on Y*
(b) $\bar{\delta} = \langle \delta_s : s \in Y \rangle$ *is a sequence of limit ordinals*
(c) $D \supseteq \mathrm{cf} - \mathrm{fil}_{<\theta}(\bar{\delta})$
(d) $\mathrm{hrtg}(Y) \le \kappa \le \theta$
(e) $\mu = \sup\{\delta_s : s \in Y\}$.

Proof <u>$\oplus_1 \Rightarrow \oplus_2$</u>
Let $(\mathscr{S}_*, <_*)$ witness $\mathrm{Ax}_{4,\mu,\theta,\kappa}$.

For every $g \in \Pi\bar{\delta}$, $\mathrm{Rang}(g)$ is a subset of $\sup\{\delta_s : s \in Y\} = \mu$ of cardinality $< \mathrm{hrtg}(Y) \le \kappa$ hence there is $u \in \mathscr{S}_*$ such that $\mathrm{Rang}(g) \subseteq u$, so $|u| < \theta$ hence easily $g \le h_{[u,\bar{\delta}]} \mod D$, see 2.7. Hence $\mathscr{F} = \{h_{[u,\bar{\delta}]} : u \in \mathscr{S}_*\}$ is a cofinal subset of $(\Pi\bar{\delta}, <_D)$ and being $\le_{\mathrm{qu}} \mathscr{S}_*$ it is well orderable. Recall $h_{[u,\bar{\delta}]} \in \Pi\bar{\delta}$ is defined by $h_{[u,\bar{\delta}]}(s)$ is $\sup(\delta_s \cap u)$ if $\sup(\delta_2 \cap u) < \delta_s$ and is zero otherwise.

Now $\mathscr{F} \subseteq \Pi\bar{\delta}$ being cofinal in $(\Pi\bar{\delta}, <_D)$ follows from $D \supseteq \mathrm{cf} - \mathrm{fil}_{<\theta}(\bar{\delta})$ that is $\boxplus(c)$.

<u>$\oplus_2 \Rightarrow \oplus_3$</u>

Let $\mathscr{F} \subseteq \Pi\delta$ be cofinal in $(\Pi\bar{\delta}, <_D)$ and $<_*$ well order $\mathscr{F}$. We try to choose f_α by induction on the ordinal α. If $\bar{f}^\alpha = \langle f_\beta : \beta < \alpha \rangle$ has no $<_D$-upper bound we are done so assume $g \in \prod_{s \in Y} \delta_s$ is a $<_D$-upper bound of $\bar{f}^\alpha$ so there is $h \in \mathscr{F}$ such that $g <_D h$, so h is a $<_D$-lub of $\bar{f}$ and let $f_\alpha \in \mathscr{F}$ be the $<_*$-minimal such h. Necessarily for some α we cannot continue so $\bar{f}^\alpha$ is as promised. □

Conclusion 2.12 *In clause* (C) *of* 2.8, *assuming* $\bar{f}$ *has no* $\le_D$ *−ub in* $\Pi\bar{\delta}$ *and letting*

- $Z_\alpha = \{s \in Y : g(s) < f_\alpha(s)\}$ *for* $\alpha < \delta_*$
- $\mathscr{W} = \{\alpha < \delta_* : Z_\beta \ne Z_\alpha \mod D$ *for every* $\beta < \alpha\}$
- $D_\alpha = D + Z_\alpha$ *for* $\alpha < \delta_*$
- $\alpha_* = \min\{\alpha \le \delta_*$: *if* $\alpha < \delta_*$ *then* $Z_\alpha \in D^+\}$

we can add:

(c) $\langle Z_\alpha/D : \alpha \in \mathscr{W}\rangle$ *is* $\subseteq$*-increasing and* $\alpha_* < \delta_*$
(d) *for* $\alpha \in \mathscr{W}, \alpha \geq \alpha_*, D_\alpha$ *is a filter on* Y *and* $\langle f_{\alpha+\gamma} : \gamma < \delta_* - \alpha$ *and* $\alpha + \gamma \in \mathscr{W}\rangle$ *is* $<_{D_\alpha}$*-increasing and cofinal in* $\Pi\bar{\delta}$
(e) $\langle D_\alpha : \alpha \in \mathscr{W}\backslash\alpha_*\rangle$ *is a strictly* $\subseteq$*-increasing sequence of filters of* Y *and* $0 \in \mathscr{W}$
(f) $\bar{f}$ *is* $<_{D_\alpha}$*-increasing and* $<_{D_*}$*-cofinal in* $\Pi\bar{\delta}$ *if* $\alpha \in \mathscr{W}\backslash\alpha_*$
(g) *if* $\mathrm{cf}(\delta_*) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ *then* $\mathscr{W}$ *has a last member.*

Proof Easy or see [11, Ch.II,§2]; but we elaborate.
Clause (c): The sequence is $\subseteq$-increasing as $\bar{f}$ is $<_D$-increasing and $\alpha_* < \delta_*$ as otherwise $\alpha < \delta \Rightarrow B_\alpha \leq g \mod D$ hence $\bar{f}$ has a $\leq_D -ub$ in $\Pi\bar{\delta}$, contradiction.
Clause (d): D_α is a filter as by clause (c), $\alpha \geq \alpha_* \Rightarrow Z_\alpha \in D^+$ and obviously $Z_\alpha \in D^+ \Rightarrow (D_\alpha$ is a filter).
Clause (e): By the definition of $\mathscr{W}$.
Clause (f): By (C)(a),(b) and clause (d).
Clause (g): Obvious. □

Theorem 2.13 *Assume* $\boxplus(a) - (e)$ *of* 2.11.
(1) *If* $\mathrm{cf}(\theta) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ *and* $\mathrm{Ax}4_{,\mu,\theta,\kappa}$*, then the conclusion* $(B)^+$ *of Theorem* 1.7 *holds, i.e. there is a* pcf*-system* $\mathbf{x}$ *such that* $Y_{\mathbf{x}} = Y, \bar{\delta}_{\mathbf{x}} = \bar{\delta}, \theta_{\mathbf{x}} = \theta$.
(2) *Without the extra assumption* $\mathrm{cf}(\theta) \geq \mathrm{hrtg}(\mathscr{P}(Y))$*, we get only a weakly pcf-system* [*see* 2.1(3)] $\mathbf{x}$ *with* $\theta = \mathrm{hrtg}(\mathscr{P}(Y))$.
(3) *If there is a weak* pcf*-system* $\mathbf{x}$ *for* $\bar{\delta}$ *then* $\Pi\bar{\delta}$ *has a subset which is a well-orderable and is cofinal in* $(\Pi\bar{\delta}, <_{D_{\mathbf{x}}})$.
(4) *If* $(\Pi\bar{\delta}, <_D)$ *has a well-orderable cofinal subset and* $\mathrm{hrtg}(\mathscr{P}(Y)) \leq \theta$ *then there is a* pcf*-system* $\mathbf{x}$ *for* $\bar{\delta}$ *with* $D_{\mathbf{x}} = D$.
(5) *If* $(\Pi\bar{\delta}, <_D)$ *has a well-ordered cofinal subset and* $\theta \geq \mathrm{hrtg}(Y)$ *then there is a* pcf*-system* $\mathbf{x}$ *for* $\bar{\delta}$ *with* $D_{\mathbf{x}} = D, \alpha_{\mathbf{x},\varepsilon+1} - \alpha_{\mathbf{x},\varepsilon}$ *increasing.*

Remark 2.14 Note that later parts of 2.13 supersede earlier ones. One reason for this is that it may be better to avoid using inner models, developing the set theory of $ZF + DC + Ax4$ per se.

Proof (1) We repeat the proof of 1.7, but using 2.8, 2.10, 2.13, i.e. in case (c) after $(*)_3$ we use [12]. But a simpler argument is that by 2.11 we know that there is a $<_D$-cofinal subset $\mathscr{F}$ of $\Pi\bar{\delta}$ which is well orderable, say by $<_*$.
(2) Like part (1).
(3) Let $\mathbf{x}$ be a weak pcf-system for $(\Pi\bar{\delta}, <_D)$, clearly $\{f_{\mathbf{x},\alpha} : \alpha < \alpha_{\mathbf{x},\varepsilon(\mathbf{x})}\}$ is a well orderable subset of $\Pi\bar{\delta}$ and so is $\mathscr{F} = \{\max\{f_{\mathbf{x},\alpha_\ell} : \ell < n\} : \bar{\alpha} = \langle\alpha_\ell : \ell < n\rangle$ is a finite sequence of ordinals $\langle\alpha_{\mathbf{x},\varepsilon(\mathbf{x})}\rangle\}$. Hence it suffices to prove that the set $\mathscr{F}$ is cofinal in $(\Pi\bar{\delta}, <_{D_{\mathbf{x}}})$.

This means to show that

$(*)$ for every $g \in \Pi\bar{\delta}$ there are n and $\alpha_\ell < \alpha_{\mathbf{x},\varepsilon(\mathbf{x})}$ for $\ell < n$ such that $g < \max\{f_{\mathbf{x},\alpha_\ell}(s) : \ell < n\} \mod D_{\mathbf{x}}$.

For this we prove by induction on $\varepsilon \leq \varepsilon_{\mathbf{x}}$ that

$(*)_\varepsilon$ if $X \in D_{\mathbf{x},\varepsilon}$ and $g \in \Pi\bar{\delta}$ then we can find $Z \in D_{\mathbf{x}}$ and n and $\alpha_\ell < \alpha_{\mathbf{x},\varepsilon}$ for $\ell < n$ such that $s \in Z \backslash X \Rightarrow g(s) < \max\{f_{\mathbf{x},\alpha_\ell}(s) : \ell < n\}$.

This suffices as for $\varepsilon = \varepsilon_{\mathbf{x}}$ we can use $X = \emptyset$.

For $\varepsilon = 0$ necessarily $Z := X$ is as required because $X \in D_{\mathbf{x},\varepsilon} = D_{\mathbf{x}}$.

For ε a limit ordinal, if $X \in D_{\mathbf{x},\varepsilon}$ then for some $\zeta < \varepsilon$, $X \in D_{\mathbf{x},\zeta}$ and use the induction hypothesis for ζ.

For $\varepsilon = \zeta + 1$, we are given $X \in D_{\mathbf{x},\varepsilon}$ and $g \in \Pi\bar{\delta}$. By clause $(B)^+(\ell)$ of 1.7 if $X \in D_{\mathbf{x},\zeta}$ use the induction hypothesis so without loss of generality $X \notin D_{\mathbf{x},\zeta}$ hence $D_{\mathbf{x},\zeta} + (Y \backslash X)$ is a filter on $Y_{\mathbf{x}}$ and it is $\supseteq E_{\mathbf{x},\zeta}$. So by clause $(B)^+(l)$ of Theorem 1.7 there is $\alpha \in [\alpha_{\mathbf{x},\zeta}, \alpha_{\mathbf{x},\zeta+1})$ such that $g < f_{\mathbf{x},\alpha} \mod (D_{\mathbf{x},\zeta} + (Y_{\mathbf{x}} \backslash X))$.

Let $X_1 = \{s \in Y : s \notin X \text{ and } g(s) < f_{\mathbf{x},\alpha}(s)\}$, so $X_1 \in D_{\mathbf{x},\zeta} + (Y_{\mathbf{x}} \backslash X)$ hence $X_2 := X \cup X_1 \in D_{\mathbf{x},\zeta}$ so by the induction hypothesis there are n_1 and $\beta_\ell < \alpha_{\mathbf{x},\zeta}$ for $\ell < n_1$ and $Z \in D_{\mathbf{x}}$ such that $s \in Z \backslash X_2 \Rightarrow g(s) < \max\{f_{\mathbf{x},\beta_\ell}(s) : \ell < n_1\}$. Let $n = n_1 + 1$ and let α_ℓ be β_ℓ if $\ell < n_1$, α_ℓ be α if $\ell = n_1$, so Z, $\langle \alpha_\ell : \ell < n \rangle$ witness the desired conclusion in $(*)_\varepsilon$. So we can carry the induction and as said above this suffices.

(4) Let $\mathscr{F} \subseteq \Pi\bar{\delta}$ be well orderable $<_D$-cofinal subset so let $\bar{g} = \langle g_\alpha : \alpha < \alpha(*) \rangle$ list $\mathscr{F}$.

Case 1: $Y \subseteq \text{Ord}$

Let $\mathbf{V}_1 = \mathbf{L}[\bar{g}]$ and $\mathbf{V}_2 = \mathbf{V}_1[D]$, using D as a predicate so $\mathbf{V}_1, \mathbf{V}_2$ are transitive models of ZFC and let $D_2 = D \cap \mathbf{V}_2 \in \mathbf{V}_2$, of course, also $\mathbf{V}_2 \models$ "θ a cardinal $> |Y|$".

In $\mathbf{V}_2$ we let $\bar{\lambda} = \langle \lambda_s : s \in Y \rangle$ be defined by $\lambda_s = \text{cf}(\delta_s)^{\mathbf{V}_2}$. Now if $u \in \mathbf{V}_2$ is a set of ordinals of cardinality $< \theta$ then the set $\{s : \delta_s > \sup(u \cap \delta_s)\}$ belongs to D hence to $D \cap \mathbf{V}_2$; this implies that $Y_* = \{s \in Y : \lambda_s \geq \theta\}$ belong to D. Now apply the pcf theorem in $\mathbf{V}_2$ on $\langle \lambda_s : s \in Y_* \rangle$ getting $\langle J_{<\mu}, Y_\mu : \mu \in \mathfrak{b} \rangle$ and $\langle g_{\lambda,\alpha} : \lambda \in \mathfrak{b}, \alpha < \lambda \rangle$ where $\mathfrak{a} = \{\lambda_s : s \in Y_*\}$, $\mathfrak{b} = \text{pcf}(\mathfrak{a})^{\mathbf{V}_2}$, in particular such that

- $\mathfrak{b} = \text{pcf}\{\lambda_s : s \in Y\}$
- $Y_\mu \subseteq Y$
- $J_{<\mu}$ is the ideal on Y generated by $\{Y_\lambda : \lambda \in b \cap \mu\}$
- $\langle g_{\lambda,\alpha} : \alpha < \lambda \rangle$ is a sequence of members of $\prod_{s \in Y_*} \lambda_s$, $<_{J^+_{<\mu}(Y_* \backslash Y_\mu)}$-increasing and cofinal.

We can translate this to get a pcf-system for $(\Pi\bar{\delta}, <_D)$ in $\mathbf{V}_2$ hence in $\mathbf{V}$.

Case 2: $Y \nsubseteq \text{Ord}$

We shall show that it essentially suffices to deal with $\bar{\delta}$ without repetitions. Note that each $f \in \mathscr{F}$ or just f a function from Y into Ord induces an equivalence relation eq_f on $Y_{\mathbf{x}}$: $s_1(\text{eq}_f)s_2 \Leftrightarrow f(s_1) = f(s_2) \wedge \delta_{s_1} = \delta_{s_2}$. For any such equivalence relation e on $Y_{\mathbf{x}}$, the set $\mathscr{F}_e = \{f \in \mathscr{F} : \text{eq}_f = e\}$ can be translated to one as in Case 1, and if for some such e, $\mathscr{F}_e$ is cofinal in $(\Pi\bar{\delta}, <_{D_{\mathbf{x}}})$ then we are done, but in general this is

not clear. Without loss of generality $\mathscr{E} = \{e_f : f \in \mathscr{F}\}$ is closed under intersection and assume there is no e as above. We can define a function F from $\mathscr{E}$ into $\alpha(*)$ by $F(e) = \min\{\alpha\colon \text{there is no } f \in \mathscr{F} \text{ such that } e_f = e \wedge g_\alpha \leq f\}$, it is well defined by the present assumption and let $u = \text{Rang}(F)$, so $|u| < \text{hrtg}(\mathscr{E}) \leq \text{hrtg}(\mathscr{P}(Y \times Y)) \leq \theta$, and we can finish easily.
(5) Let $u_\alpha := \text{Rang}(g_\alpha)$, $v := \{\delta_s : s \in Y\}$ so all are subsets of μ of cardinality $< \text{hrtg}(Y)$, so $\bar{u} = \langle u_\alpha : \alpha < \alpha(*)\rangle$ is well defined and let $\mathbf{V}'_1 = \mathbf{L}[\bar{u}, v]$ is a well defined universe, a model of ZFC. In $\mathbf{V}'_1$ w can define $\bar{\delta}'$, listing v in increasing order and $\bar{g}' = \langle g'_\alpha : \alpha < \alpha(*)\rangle$ where $g'_\alpha = h_{[u_\alpha, \bar{\delta}']}$. In $\mathbf{V}$ define $\bar{f}''_\alpha = \langle g''_\alpha : \alpha < \alpha(*)\rangle$ where $g''_\alpha = h_{[u_\alpha, \bar{\delta}]}$. As $\theta \geq \text{hrtg}(Y)$ clearly $g_\alpha \leq g''_\alpha \mod \text{cf} - \text{fil}_{<\theta}(\bar{\delta})$ hence $g_\alpha \leq g''_\alpha$ mod D hence without loss of generality $\bar{g}' = \bar{g}$. As there is no real difference between $\bar{\delta}$ and $\bar{\delta}'$ and we can deal with $\bar{g}', \bar{\delta}'$ via $\mathbf{L}[\bar{g}', \bar{\delta}']$ as in Case 1 of the proof of part (4) and finish easily. □

Discussion 2.15 Alternate proof: suppose we can uniformly choose $\bar{f} = \langle f_\alpha : \alpha < \delta_*\rangle$ which is $<_D$-increasing and cofinal in $(\Pi\bar{\delta}, <_D)$.

We define an equivalence relation E on $|\mathscr{F}|$ by: $\alpha E \beta$ iff $e_{g_\alpha} = e_{g_\beta}$; let $\bar{\beta} = \langle \beta_\zeta = \beta(\zeta) : \zeta < \zeta(*)\rangle$ list $\{\alpha < |\mathscr{F}| : \alpha = \min(\alpha/E)\}$ in increasing order and let $\zeta : |\mathscr{F}| \to \zeta(*)$ be $\zeta(\alpha) = \min\{\zeta : \alpha \in \beta_\zeta/E\}$.

Let $\bar{\xi}^* = \langle \xi^*_\zeta : \zeta < \zeta(*)\rangle$ where[7] $\xi^*_\zeta = \text{pr}(\text{otp}(\text{Rang}(f_{\alpha_\zeta})), \text{otp}(\text{Rang}(\bar{\delta}))$ and for $\alpha < |\mathscr{F}|$ let $\hat{g}_\alpha$ be the function from $\xi^*_{\zeta(\alpha)}$ to Ord defined by $\hat{g}_\alpha(\xi) = \gamma$ iff for some $s \in Y_{\mathbf{x}}$ we have $f_\alpha(s) = \gamma \wedge \xi = \text{pr}(\text{otp}(\text{Rang}(g_{\beta_{\zeta(\alpha)}}) \cap g_\alpha(s)), \text{otp}(\text{Rang}(\bar{\delta}) \cap \delta_s))$.

Lastly, let $R = \{(\zeta_1, \zeta_2, \xi_1, \xi_2) : \text{for some } s \in Y \text{ for } \ell = 1, 2 \text{ we have } \zeta_\ell < \zeta(*), \xi_\ell < \xi^*_{\zeta_\ell}, \xi_\ell = \text{pr}(\text{otp}(\text{Rang}(g_{\alpha_{\zeta_\ell}}) \cap g_{\alpha_{\zeta_\ell}}(s), \text{otp}(\text{Rang}(\bar{\delta}) \cap \delta_s))\}$. Now we use $\mathbf{V}_1 = \mathbf{L}[\bar{\delta}, \bar{g}, E, R, \bar{\xi}^*]$ let $\bar{D} = \langle D_\zeta : \zeta < \zeta(*)\rangle$, $D_\zeta = D_{\mathbf{x}}(e_{g_{\alpha_\zeta}})$, $\mathbf{V}_2 = \mathbf{V}_1[D_{\mathbf{x}}]$ and for $\zeta < \zeta(*)$ let $\bar{\lambda}_\zeta = \langle \lambda_{\zeta,\xi} : \xi < \xi_\zeta\rangle$, $\lambda_{\zeta,\xi} = \text{cf}(\delta_s)$ when $\xi = \text{pr}(\xi, s)$ for some appropriate ε.

Clearly $\zeta < \zeta(*) \Rightarrow \xi_\zeta < \theta$, as before without loss of generality $\lambda_{\zeta,f} = \text{cf}(\lambda_{\zeta,\xi}) \geq \theta$ and $\theta > \text{hrtg}(Y)$ by an assumption hence the pcf analysis in $\mathbf{V}_2$ of $\Pi\bar{\lambda}_\zeta$ is O.K.; moreover and $\{\lambda_{\eta,\xi} : \xi < \xi_\zeta\}$ does not depend on.

Now the analysis for $\bar{\lambda}_0$ recalling $\text{eq}_{\bar{\delta}} = e_{g_0} = e_{g_{\alpha_0}}$ is enough.

Claim 2.16 *If* $\mathbf{x}$ *is a pcf-system* *then* *there is* $\bar{Y}$ *defined uniformly such that (so may write* $\bar{Y}^{\mathbf{x}} = \langle Y^{\mathbf{x}}_\varepsilon : \varepsilon < \varepsilon_{\mathbf{x}}\rangle$*):*

(a) $\bar{Y} = \langle Y_\varepsilon : \varepsilon < \varepsilon(*)\rangle$
(b) $Y_\varepsilon \subseteq Y_{\mathbf{x}}$
(c) $D^{\mathbf{x}}_{\varepsilon+1} = D^{\mathbf{x}}_\varepsilon + Y_\varepsilon$.

Proof Fix $\varepsilon < \varepsilon_{\mathbf{x}}$, if $\varepsilon_{\mathbf{x}} = \varepsilon + 1$ let $Y_\varepsilon = Y$, so assume $\varepsilon + 1 < \varepsilon_{\mathbf{x}}$. So for some $Y \subseteq Y_{\mathbf{x}}$ we have $D_{\mathbf{x},\varepsilon+1} = D_{\mathbf{x},\alpha} + Y$ hence $E_{\mathbf{x},\varepsilon} = D_{\mathbf{x},\varepsilon} + (Y_{\mathbf{x}} \backslash Y)$; and $f_{\mathbf{x},\alpha_{\mathbf{x},\varepsilon}}$ is a $<_{D_{\mathbf{x},\varepsilon+1}}$-upper bound of $\bar{f}_{\mathbf{x}} \restriction [\alpha_{\mathbf{x},\varepsilon}, \alpha_{\mathbf{x},\varepsilon+1})$. But $\bar{f}_{\mathbf{x}} \restriction [\alpha_{\mathbf{x},\varepsilon}, \alpha_{\mathbf{x},\varepsilon+1})$ is cofinal in $(\Pi\bar{\delta}_{\mathbf{x}}, <_{E_{\mathbf{x},\varepsilon}})$ hence we can find $\beta \in [\alpha_{\mathbf{x},\varepsilon}, \alpha_{\mathbf{x},\varepsilon+1})$ such that $f_{\mathbf{x},\alpha_{\mathbf{x},\varepsilon+1}} < f_{\mathbf{x},\beta} \mod E_{\mathbf{x},\varepsilon}$.

[7] Recall pr is a one-to-one function from Ord × Ord onto Ord such that $(\alpha_1 \leq \alpha \wedge \beta_1 \leq \beta) \Rightarrow \text{pr}(\alpha_1, \beta_1) \leq \text{pr}(\alpha, \beta)$.

Let β_* be the minimal such β and easily $Y_\varepsilon := \{s \in Y_{\mathbf{x}} : f_{\mathbf{x},\beta_*}(s) < g_{\mathbf{x}}, \alpha_{\mathbf{x},\varepsilon+1}(s)\}$ is as required. □

2.2 Elaborations

Claim 2.17 *Assume* $\mathrm{Ax}_{4,\lambda,\partial}$.

For any λ *we can* ∂*-uniformly define the following.*

(1) *For* $\delta < \lambda$ *of cofinality* $\aleph_0$*, an unbounded subset* e_δ *of* δ *of order type* $< \partial$.

(2) *For* $\theta = \mathrm{hrtg}(Y)$, $\bar\delta = \langle \delta_s : s \in Y \rangle$ *a sequence of limit ordinals* $< \lambda$ *of uncountable cofinality satisfying* $Y \in \mathrm{cf} - \mathrm{id}_{<\theta}(\bar\delta)$, *(see* 1.1*) a closed* $u_* \subseteq \sup\{\delta_s : s \in Y\}$, *unbounded in each* δ_s *of cardinality* $< \mathrm{hrtg}([\theta_1]^{<\partial})$ *where*

- $\theta_1 = \min\{|u| : (\forall s)[s \in Y \to \delta_s = \sup(u \cap \delta_s)\}$ *is necessarily* $< \theta$.

(3) *For* $\delta < \lambda$*, an unbounded subset* e_δ *of cardinality* $< \mathrm{hrtg}([\mathrm{cf}(\delta)]^{\aleph_0})$.

Proof (1) See [23] or as in the proof of $(*)_4$ inside the proof of 1.5.

(2) Let $\mathbf{U}_{\bar\delta} = \{u : u \subseteq \sup\{\delta_s : s \in Y\}$ of cardinality $< \theta$ and $u \cap \delta_s$ an unbounded subset of δ_s for every $s \in Y\}$. By the assumption "$Y \in \mathrm{cf} - \mathrm{id}_{<\theta}(\bar\delta)$" clearly $\mathbf{U}_{\bar\delta} \neq \emptyset$, hence $\mathbf{U}'_{\bar\delta} = \{u \in \mathbf{U}_{\bar\delta} : u$ is closed$\}$ is non-empty. Using $c\ell$ from 0.6, the set $u_* = \cap\{c\ell(u) : u \in \mathbf{U}'_{\bar\delta}\}$ has cardinality $< \mathrm{hrtg}([\min\{|u| : u \in \mathbf{U}_\delta\}]^{<\partial})$. It suffices to prove

- $\bullet_1$ if $u_n \in \mathbf{U}'_{\bar\delta}$ for $n < \omega$ <u>then</u> $u := \cap\{u_n : n < \omega\}$ belongs to $\mathbf{U}'_{\bar\delta}$.

[Why? Clearly it is a subset of μ of cardinality $< \theta$, being $\subseteq u_0$ and it is closed because each u_n is. But for any $s \in Y$, why is u unbounded in δ_s? Because δ_s has uncountable cofinality and the members of $\mathbf{U}'_{\bar\delta}$ are closed.

- $\bullet_2$ for some $u \in \mathbf{U}'_{\bar\delta}$, $|u| \leq \theta_1$ and without loss of generality u is closed, so $|u_*| \leq |c\ell(u)| \leq \mathrm{hrtg}([\theta_1]^{\leq\aleph_0})$ as promised.

(3) By the proof of $(*)_4$ inside the proof of 1.5. □

We give a sufficient condition for $<_D$-eub existence, try to write such that we get the trichotomy.

Claim 2.18 <u>*The eub-existence claim:*</u>

Assume $\mathrm{Ax}_{4,\partial}$ *or just* $\mathrm{Ax}_{4,\mathrm{hrtg}({}^Y\mu),\partial}$. *The sequence* $\bar f$ *has a* $<_D$*-eub* [*see Definition* 0.11(5)], *even one* ∂*-uniformly definable from* $(Y, D, \bar f)$ <u>*when*</u>*:*

⊞ (a) (θ, Y) *satisfies clauses* $(A)(c)(\beta), (\gamma), (\delta)$ *of* 1.5

(b) D *is a filter on* Y*, so not necessarily* $\aleph_1$*-complete*

(c) $\bar{f} = \langle f_\alpha : \alpha < \delta \rangle$

(d) $f_\alpha \in {}^Y\mathrm{Ord}$ *is* $\leq_D$*-increasing*

(e) $\mathrm{cf}(\delta) \geq \theta$ *and* $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\prod_{s\in Y} \zeta_s)$ *when* $\zeta_s < \mathrm{hrtg}(\mathscr{P}(Y))$ *for* $s \in Y$.

Proof Toward contradiction assume that the desired conclusion fails. Let $\alpha^*_s = \cup\{f_\alpha(s) : \alpha < \delta\}$ for $s \in Y$ and $\alpha_* = \sup\{\alpha^*_s + 1 : s \in Y\}$.

We try to choose g_ζ and $\beta_\zeta < \delta$ by induction on $\zeta < \mathrm{hrtg}(\mathscr{P}(Y)/D) \leq \mathrm{hrtg}(\mathscr{P}(Y))$ such that:

$\oplus$ (a) $g_\zeta \in \prod_{s\in Y}(\alpha^*_s + 1)$

(b) if $\alpha < \delta$ then $f_\alpha < g_\zeta \mod D$

(c) if $\varepsilon < \zeta$ then $g_\zeta \leq g_\varepsilon \mod D$ and $g_\zeta/D \neq g_\varepsilon/D$

(d) g_ζ and $\beta_\zeta < \delta$ are defined as below.

Clearly impossible as $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\mathscr{P}(Y))$ by assumption $\boxplus(d)$, so we shall get stuck somewhere. If $\bar{g}^\zeta = \langle g_\varepsilon : \varepsilon < \zeta \rangle$ is well defined, we let $\bar{u}_\zeta = \langle u_{\zeta,s} : s \in Y \rangle$ be defined by $u_{\zeta,s} = \{\gamma : \text{for some } \beta < \zeta \text{ and } n \text{ we have } \gamma + n = g_\beta(s) \text{ or } \gamma + n = \alpha^*_s\}$, so $u_{\zeta,\alpha} \subseteq \alpha^*_s + 1$ and $|u_{\zeta,\alpha}| \leq \aleph_0 + |\zeta|$ even uniformly. Next for $\alpha < \delta$ we let $f^{\zeta,1}_\alpha \in \prod_{s\in Y}(\alpha_s + 1)$ be defined by $f^{\zeta,1}_\alpha(s) = \min(u_{\zeta,s} \backslash f_\alpha(s))$, clearly well defined and belongs to $\prod_{s\in Y}(\alpha^*_s + 1)$ and is $\leq_D$-increasing. Now $\{f^{\zeta,1}_\alpha : \alpha < \delta\} \subseteq \prod_{s\in Y} u_{\zeta,s}$ so as $\mathrm{cf}(\delta) \geq \mathrm{hrtg}({}^Y(1+\zeta)) \geq \mathrm{hrtg}(\prod_s u_{\zeta,s})$, necessarily $\langle f^{\zeta,1}_\alpha/D : \alpha < \delta \rangle$ is eventually constant. Let $\beta_{\zeta,1} = \min\{\beta < \delta : \text{if } \alpha \in (\beta, \delta) \text{ then } f^{\zeta,1}_\alpha = f^{\zeta,1}_\beta \mod D\}$ so $\alpha < \delta \Rightarrow f_\alpha \leq_D f^{\zeta,1}_{\beta_{\zeta,1}} \mod D$ and let $g_{\zeta,1} = f^{\zeta,1}_{\beta_{\zeta,1}}$. If $g_{\zeta,1}$ is a $<_D$-eub of $\bar{f}$ we are done, otherwise the construction will split to cases.

Let $Y_0 = \{s \in Y : f^{\zeta,1}_{\beta_{\zeta,1}}(s) = 0\}$, $Y_1 = \{s \in Y : f^{\zeta,1}_{\beta_{\zeta,1}}(s) \text{ is a successor ordinal}\}$ and $Y_2 = \{s \in Y : f^{\zeta,1}_{\beta_{\zeta,1}}(s) \text{ is a limit ordinal of cofinality} < \theta\}$ and $Y_3 = \{s \in Y : f^{\zeta,1}_{\beta_{\zeta,1}}(s)$ is a limit ordinal of cofinality $\geq \theta\}$, so $\langle Y_0, Y_1, Y_2, Y_3 \rangle$ is a partition of Y.

$(*)$ without loss of generality $Y_\ell \in D$, $g_{\zeta,1}$ is not an lub and even $Y_\ell = Y$ from some $\ell < 4$.

[Why? For each $\ell < 4$ such that $Y_\ell \in D^+$, clearly we can replace D by $D + Y_\ell$ hence (by the present assumption) a $<_{D+Y_\ell}$-eub g'_ℓ exists; if $Y_\ell \notin D^+$ let g_ℓ be constantly zero. Lastly, $\cup\{g^*_\ell \restriction Y_\ell : \ell < 4\}$ is as required.]

<u>Case 0</u>: $Y_0 \in D$ so $Y_0 = Y$

Trivial.

<u>Case 1</u>: $Y_1 \in D$ so $Y_1 = Y$

Define $g_\zeta \in \prod_{s\in y}(\alpha_s + 1)$ by: $g_\zeta(s) = g_{\zeta,1}(s) - 1$. Clearly it is still a $\leq_D$-upper bound of $\bar{f}$ as $\bar{f}$ is $<_D$-increasing, and $g_\zeta < g_\varepsilon \mod D$ for every $\varepsilon < \zeta$. Lastly, let $\beta_\zeta = \beta_{\zeta,1}$.

<u>Case 2</u>: $Y_2 \in D$

Let $\langle e_\alpha : \alpha < \alpha_* \rangle$ be as in 2.17(1),(3) for $\alpha < \delta$, then we define $f^{\zeta,2}_\alpha \in \prod_{s \in Y_2}(\alpha_s + 1)$ by $f^{\zeta,2}_\alpha(s) = \min(e_{g_{\zeta,1}(s)} \backslash f_\alpha(s))$ and let $\zeta_s = \mathrm{otp}(e_{g_{\zeta,1}(s)}) < \theta$, this holds by 1.5(A)(c)(β) which in turn holds by $\boxplus(a)$ of the assumption of the claim.

Now as $\mathrm{cf}(\delta) \geq \mathrm{hrtg}(\prod_{s \in Y_2} \zeta_s) = \mathrm{hrtg}(\prod_{s \in Y_2} e_{g_{\zeta,1}(s)})$ clearly $\langle f^{\zeta,2}_\alpha / D : \alpha < \delta \rangle$ is eventually constant, so $\beta_{\zeta,2} = \min\{\beta < \delta : \text{if } \alpha \in (\beta, \delta) \text{ then } f^{\zeta,2}_\alpha / D = f^{\zeta,2}_\beta / D\}$ is well defined. Let $\beta_\zeta = \sup(\{\beta_{\zeta,1}, \beta_{\zeta,2}\} \cup \{\beta_\varepsilon + 1 : \varepsilon < \zeta\})$ it is $< \delta$, $\mathrm{cf}(\delta) > |\zeta|$ and let $g_\zeta = f^{\zeta,2}_{\beta_\zeta}$. Clearly $\varepsilon < \zeta \Rightarrow g_\zeta = f^{\zeta,7}_{\beta_\zeta} < f^{\zeta,1}_{\beta_{\zeta,1}} \leq g_\varepsilon \mod D$, so (g_ζ, β_ζ) are as required.

<u>Case 3</u>: $Y_3 = Y$

Let $\bar{f}' = \langle f'_\alpha : \alpha < \delta \rangle$, $f'_\alpha \in \prod_s g_{\zeta,1}(s)$ defined as $f_\alpha(s)$ if $< g_{\zeta,1}(s)$, zero otherwise.

Now $g_{\zeta,1}$ is not a $<_D$-eub of $\bar{f}$ hence there is $h \in {}^Y\mathrm{Ord}$ such that $h < g_{\zeta,1} \mod D$ and for no $\alpha < \delta$ do we have $h < f_\alpha \mod D$. But h was not canonically chosen. Clearly the assumption of 2.2, i.e. 1.7 holds with $Y, \theta, g_{\zeta,1}, \bar{f}'$ here standing for $Y, \theta, \bar{\delta}, \bar{f}$ here. So there is a pcf-system $\mathbf{x}$ with $Y_\mathbf{x} = Y, \theta_\mathbf{x} = \theta, D_\mathbf{x} = D, \bar{f}_\mathbf{x} = \bar{f}'$ and $\bar{\delta}_\mathbf{x} = g_{\zeta,1}$.

Hence by 2.3(1) we can define a pair $(\mathscr{F}, <_*)$ such that $\mathscr{F} \subseteq \prod_{s \in Y_2} g_{\zeta,1}(s)$ is cofinal and $<_*$ a well ordering of $\mathscr{F}$.

So as $g_{\zeta,1}$ is not a $<_D$-eub of $\bar{f}$ there is $h \in \mathscr{F}$ witnessing this and let $h_* \in \mathscr{F}$ be the $<_*$-first one.

Let

$$\begin{aligned}\beta_{\zeta,3} = \min \Big\{ \beta < \alpha \colon \text{ if } \alpha \in (\beta, \delta) \text{ then } & \Big\{ s \in Y : f_\alpha(s) \leq g_\zeta(h_*((s))) \Big\} \\ &= \Big\{ s \in Y : f_\beta(s) \leq h_*(s) \Big\} \mod D + Y_2 \Big\},\end{aligned}$$

well defined as before. Lastly, let $g_\zeta \in {}^Y\mathrm{Ord}$ be defined as follows: $g_\zeta(s)$ is

- $h_*(s)$ if $f_{\beta_{\zeta,3}}(s) \leq h_*(s)$
- $f_{\beta_{\zeta,3}}(s)$ if $f_{\beta_{\zeta,3}}(s) > h_*(s)$. □

Now we give a version of the main theorem of [23, §1]. From this we may try to understand better ${}^\kappa\lambda$ and use it in constructions, i.e. to diagonalize.

Theorem 2.19 $[\mathrm{Ax}_{4,\lambda,\partial}]$ *For $\kappa < \lambda$ letting $X_\kappa = {}^\omega(\mathrm{Fil}^4_{\aleph_1}(\kappa))$, we can ∂-uniformly define $\langle (\mathscr{S}_t, <_t) : t \in X_\kappa \rangle$ such that:*

(a) $\cup\{\mathscr{S}_t : t \in X_\kappa\} = {}^\kappa\lambda$

(b) *$<_t$ is a well ordering of $\mathscr{S}_t$*

(c) *there is an equivalence relation E on ${}^\kappa\lambda$ such that:*

  (α) *$({}^\kappa\lambda)/E$ is well ordered*

  (β) *each equivalence class is of power $\leq X_\kappa$*

(d) *moreover for some $\bar{g} = \langle g_{\mathfrak{y},\alpha} : \mathfrak{y} \in X_\kappa, \alpha \in S_\mathfrak{y} \rangle$ and $\bar{S} = \langle S_\mathfrak{y} : \mathfrak{y} \in X_\kappa \rangle$ and $\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \beta(*) \rangle$ we have*

(α) $\beta(*) < \mathrm{hrtg}(\alpha(*)]^{\aleph_0})$ *where* $\alpha(*) = \sup\{\mathrm{rk}_D(\lambda) : D \in \mathrm{Fil}^1_{\aleph_1}(Y)\}$

(β) $\beta(*) = \cup\{S_{\bar{\mathfrak{y}}} : \bar{\mathfrak{y}} \in X_\kappa\}$

(γ) $\{g_{\bar{\mathfrak{y}},\alpha} : \bar{\mathfrak{y}} \in X_\kappa, \alpha \in S_{\bar{\mathfrak{y}}}\}$ *is equal to* ${}^\kappa\lambda$

(δ) $g_{\bar{\mathfrak{y}}_1,\alpha_1} = g_{\bar{\mathfrak{y}}_2,\alpha_2}$ *implies* $\alpha_1 = \alpha_2$

(ε) $\bar{\mathscr{F}} = \langle \mathscr{F}_\beta : \beta < \beta(*)\rangle$ *is a partition of* ${}^\kappa\lambda$

(ζ) $|\mathscr{F}_\beta| \le_{\mathrm{qu}} |X_\kappa|$.

Remark 2.20 (1) We may compare with [23, §1].
(2) Recall 0.17(2).

Proof Fix a witness $c\ell$ of $\mathrm{Ax}_{4,\lambda,\partial}$. For every $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$ and ordinal α there is at most one $f \in {}^Y(\lambda+1)$ such that f satisfies $\mathfrak{y}$ so $f{\restriction}(Y\backslash Z_{\mathfrak{y}})$ is constantly zero and $D^{\mathfrak{y}}_2 = \mathrm{dual}(J[f, D^{\mathfrak{y}}_1])$, see 0.12, 0.15 and $\alpha = \mathrm{rk}_D(f)$; in this case call it $f_{\mathfrak{y},\alpha}$ and let $S_{\mathfrak{y},\lambda}$ be a set of α such that $f_{\mathfrak{y},\alpha}$ is well defined.

So $\langle f_{\mathfrak{y},\alpha} : \mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y), \alpha \in S_{\mathfrak{y},\alpha}\rangle$ is well defined. For every $f \in \lambda$ and $\aleph_1$-complete filter D_1 on Y for some $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$ satisfying $D_{\mathfrak{y},1} = D_1$ and ordinal α we have $f = f_{\mathfrak{y},\alpha} \mod D_{\mathfrak{y},2}$ (in fact $\alpha = \mathrm{rk}_{D_1}(f) < \mathrm{rk}_D(\lambda) \le \alpha(*)$, $\alpha(*)$ from $(d)(\alpha)$ of the Theorem).

Now

$(*)_1$ for every ${}^Y(\lambda+1)$ there is a countable set $\mathfrak{Y} \subseteq \mathrm{Fil}^4_{\aleph_1}(Y)$ such that

(α) f semi-satisfies each $\mathfrak{y} \in \mathfrak{Y}$

(β) $Y = \cup\{Z_{\mathfrak{y}} : \mathfrak{y} \in \mathfrak{Y}\}$

(γ) for each $\mathfrak{y} \in \mathfrak{Y}$, for some α we have $f{\restriction}Z_{\mathfrak{y}} = f_{\mathfrak{y},\alpha}{\restriction}Z_{\mathfrak{y}}$.

[Why? Let $\mathscr{Z} = \{Z_{\mathfrak{y}} : \mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$ and for some $\alpha \in S_{\mathfrak{y},\lambda}$ we have $f{\restriction}Z_{\mathfrak{y}} = f_{\mathfrak{y},\alpha}{\restriction}Z_{\mathfrak{y}}\}$. If Y is the union of a countable subset of $\mathscr{Z}$ then $Y = \cup\{Z_{\mathfrak{y}_n} : n\}$ for some $\{\mathfrak{y}_n : n < \omega\} \subseteq \mathrm{Fil}^4_{\aleph_1}(Y)$ and we are easily done. If not, $D_1 := \{Z \subseteq Y : Z$ includes $(Y\backslash\bigcup_n Z_{\mathfrak{y}}$: for some $\langle \mathfrak{y}_n : n < \omega\rangle \in {}^\omega(\mathrm{Fil}^4_{\aleph_0}(Y))$ we have $Z_{\mathfrak{y}_n} \in \mathscr{Z}$ for $n < \omega\}$ is an $\aleph_1$-complete filter and we easily get a contradiction.]

Recall $S_{\mathfrak{y},\lambda} = \{\alpha < \alpha(*) : f_{\mathfrak{y},\alpha}$ well defined$\}$ and by Ax_4 we can find a list $\langle \eta_\beta : \beta < \beta(*)\rangle$ of $\{\eta : \eta \in {}^\omega\alpha(*)\}$, $\beta(*) < \mathrm{hrtg}({}^\omega\beta(*))$ and even $\beta(*) = |\beta(*)|^{\aleph_0}\}$.

Now for every $\bar{\mathfrak{y}} \in X_\kappa = {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$, let $W_{\bar{\mathfrak{y}}} = \{\beta < \beta(*) : \eta_\beta(n) \in S_{\mathfrak{y}_n,\lambda}$ for each n and $\cup\{f_{\mathfrak{y}_n,\eta_n(\alpha)}{\restriction}Z_{\mathfrak{y}_n} : n < \omega\}$ is a function, in fact one from Y to $\lambda+1\}$. For $\beta \in W_{\bar{\mathfrak{y}}}$ let $g_{\bar{\mathfrak{y}},\beta}$ be $\cup\{f_{\mathfrak{y}_n,\eta_\beta(n)} : n < \omega\}$ and let $S_{\bar{\mathfrak{y}}} = \{\beta \in W_{\bar{\mathfrak{y}}} : g_{\bar{\mathfrak{y}},\beta} \notin \{g_{\bar{\mathfrak{z}},\gamma} : \bar{\mathfrak{z}} \in X_\kappa$ and $\gamma < \beta\}\}$.

Note that

$(*)_2$ (a) $\langle S_{\bar{\mathfrak{y}}} : \bar{\mathfrak{y}} \in X_\kappa\rangle$ exist

(b) $\bigcup_{\mathfrak{y}} S_{\bar{\mathfrak{y}}} \subseteq \beta(*)$

(c) $\langle S_{\bar{\mathfrak{y}}} : \bar{\mathfrak{y}} \in X_\kappa\rangle$ has union $\beta(*)$.

Note also that clause (d) of the theorem implies clauses (a),(b); (let $\mathscr{S}_{\bar{\mathfrak{y}}} = \{g_{\bar{\mathfrak{y}},\alpha} : \alpha \in S_{\bar{\mathfrak{y}}}\}$ and $<_{\bar{\mathfrak{y}}} = \{(g_{\bar{\mathfrak{y}},\alpha}, g_{\bar{\mathfrak{y}},\alpha}, g_{\bar{\mathfrak{y}},\beta}) : \alpha < \beta$ are from the set $S_{\bar{\mathfrak{y}}}$ of ordinals).

Also clause (d) implies clause (c) letting $E = \{g_{\bar{\mathfrak{y}}_1,\alpha_1}, g_{\bar{\mathfrak{y}}_2,\alpha_2}) : \bar{\mathfrak{y}}_\ell \in X - \kappa, \alpha_\ell \notin S_{\bar{\mathfrak{y}}_\ell}$ for $\ell = 1, 2$ and $\alpha_1 = \alpha_1\}$.

So it is enough to prove clause (d).
Now

- clause $(d)(\alpha)$ holds by the choices of $\alpha(*)$, $\beta(*)$
- clause $(d)(\beta)$: we have only $\beta(*) \supseteq \cup\{S_{\bar{\mathfrak{y}}} : \bar{\mathfrak{y}} \in X_\kappa\}$, but we can replace $\beta(*)$ by $\mathrm{otp}(\cup\{S_{\bar{\mathfrak{y}}} : \bar{\mathfrak{y}} \in X_\kappa\}$
- clause $(d)(\gamma)$: $g_{\bar{\mathfrak{y}},\beta}$ is defined above but why ${}^\kappa\lambda = \{g_{\bar{\mathfrak{y}},\alpha} : \bar{\mathfrak{y}} \in X_\kappa, \alpha \in S_{\bar{\mathfrak{y}}}\}$? As said above, if $f \in {}^\kappa\lambda$ by $(*)_1$ there is a countable $\mathfrak{Y} \subseteq \mathrm{FIL}^4_{\aleph_1}(Y)$ as there, hence for some sequence $\langle(\mathfrak{y}_n, \alpha_n) : n < \omega\rangle$ we have $\mathfrak{Y} = \{\mathfrak{y}_n : n < \omega\}$ and $f \restriction Z_{\mathfrak{y}_n} = f_{\mathfrak{y}_n,\alpha_n} \restriction Z_{\mathfrak{y}}$. Hence $\bar{\mathfrak{y}} := \langle \mathfrak{y}_n : n < \omega\rangle \in X_\kappa$ and for some $\gamma < \beta(*)$ we have $\eta_\gamma = \langle \alpha_n : n < \omega\rangle$. So $f = \cup\{f_{\mathfrak{y}_n,\eta_\gamma(n)} \restriction Z_{\mathfrak{y}_n} : n < \omega\} = g_{\bar{\mathfrak{y}},\beta}$ so $f \in W_{\bar{\mathfrak{y}}}$, and $f = g_{\bar{\mathfrak{y}},\gamma}$, hence by the choice of $S_{\bar{\mathfrak{y}}}$ there are $\bar{\mathfrak{z}} \in X_\kappa$ and $\beta^* \leq \gamma$ such that $\beta \in w'_{\bar{\mathfrak{z}}}$ and $f = g_{\bar{z},\beta}$, so we are done
- clause $(d)(\delta)$: look again at the choice of $S_{\bar{\mathfrak{y}}}$.
- clauses (d) (ε), (ζ): Follows □

Conclusion 2.21 *Assume* $\mathrm{Ax}_{4,\partial}$. *If* $\partial \leq \kappa < \mu$ *and* $\mathrm{hrtg}(\mathrm{Fil}^4_{\aleph_1}(\kappa)) < \mu$ *<u>then</u> the following cardinals are almost equal* (*as in* [27, §(3A)])*:*

(a) $\mathrm{hrtg}({}^\kappa\mu)$
(b) $\mathrm{wlor}({}^\kappa\mu)$
(c) *o-Depth*$^+_\kappa({}^\kappa\mu) = \sup\{\mathrm{o-Depth}^+_D(\mu) : D$ *a filter*$\}$.

Proof By 2.19. □

A drawback of the pcf theorem is the demand $\theta \geq \mathrm{hrtg}(\mathrm{Fil}^4_{\aleph_1}(Y))$ rather than just $\theta \geq \mathrm{hrtg}(\mathscr{P}(Y))$ or even $\theta \geq \mathrm{hrtg}(Y)$. Note: in [10, Ch.XII,§5] we work to assume just the parallel of $\theta \geq \mathrm{hrtg}(\mathscr{P}(Y))$, i.e. $\mathrm{Min}(\mathfrak{a}) > 2^{|\mathfrak{a}|}$ rather than the parallel of $\theta \geq \mathrm{hrtg}(\mathscr{P}(\mathscr{P}(Y))$, i.e. $\mathrm{Min}(\mathfrak{a}) > 2^{2^{|\mathfrak{a}|}}$ and only in [16] we succeed to use just the parallel of $\theta \geq \mathrm{hrtg}(Y)$.

We may try to analyze not $\Pi\bar{\delta}$, $\bar{\delta} = \langle \delta_s : s \in Y\rangle$ but rather all $\Pi(\bar{\delta} \restriction Z)$, $Z \in \mathscr{A}$ simultaneously where $\mathscr{A} \subseteq \mathscr{P}(Y)$, demanding $Z \in \mathscr{A} \Rightarrow \theta \geq \mathrm{hrtg}(\mathrm{Fil}^4_{\aleph_1}(Z))$ but less on $|Y|$; hopefully see [29].
We may consider

Definition 2.22 Let $\mathrm{Ax}_{5,F}$ say: if $Y = \kappa \in \mathrm{Card}$ then $\mathrm{Ax}_{5,\kappa,F(\kappa)}$ where $\mathrm{Ax}_{5,Y,\theta}$ means that: if $\bar{\delta} = \langle \delta_s : s \in Y\rangle$ is a sequence of limit ordinals and $D = \mathrm{cf}-\mathrm{fil}_{<\theta}(\bar{\delta})$ <u>then</u> there is a pcf-system $\mathbf{x}_{\bar{\delta}}$ for $(\Pi\bar{\delta}, <_D)$, see 2.13. Moreover, the choice of $\mathbf{x}_\delta$ is ∂-uniform.

Definition 2.23 (1) We say $\mathbf{p}$ is a pcf-problem <u>when</u> it consists of:

(a) $\bar{\delta} = \langle \delta_s : s \in Y\rangle$ and $\mu = \sup\{\delta_s : s \in Y\}$ and $\mathscr{A} \subseteq \mathscr{P}(Y)$
(b) D_* is a filter on Y, it may be $\{Y\}$
(c) $\theta = \theta[Y, \bar{\delta}, D_*] = \theta[Y, \delta, D_*, \partial]$ is any cardinal satisfying:
  (α) $\mathrm{cf}-\mathrm{id}_{<\theta}(\bar{\delta}) \subseteq \mathrm{dual}(D_*)$, note that this holds when each δ_s is an ordinal $\leq \mu$ of cofinality $\geq \theta$, see below

(β) $\alpha < \theta \Rightarrow \mathrm{hrtg}([\alpha]^{\aleph_0} \times \partial) \le \theta$ so $\partial < \theta$ and so if Ax_4 then the demand is equivalent to "$\partial < \theta$ and $\alpha < \theta \Rightarrow |\alpha|^{\aleph_0} < \theta$"

(γ) $\mathrm{hrtg}(\mathrm{Fil}^4_{\aleph_1}(Z)) \le \theta$ for every $Z \in \mathscr{A}$.

(2) For $\mathbf{p}$ a pcf-problem let $\bar{\delta}_{\mathbf{p}} = \bar{\delta}, \delta_{\mathbf{p},s} = \delta_s$, etc., if clear from the context $\mathbf{p}$ is omitted.
(3) For D a filter on $Y_{\mathbf{p}}$ extending $D_{\mathbf{p}}$ let $c\ell_{\mathbf{p}}(D) = c\ell(D, \mathbf{p}) = \{A \subseteq Y_{\mathbf{p}}$: if $Z \in \mathscr{A}_{\mathbf{p}}$ then $A \cup (Y_{\mathbf{p}} \backslash Z) \in D$.
(4) $\mathbf{p}$ is nice if $\mathrm{hrtg}(\mathscr{P}(Y)) \le \theta_{\mathbf{p}}$.

Definition 2.24 We say $\mathbf{x}$ is a wide pcf system <u>when</u> $\mathbf{x}$ consists of (if we omit $(g)(\alpha)$, (β), i.e. A_ε we say "almost wide"):

(a) $\mathbf{p}$, a pcf-problem let $D_{\mathbf{x}} = D_{\mathbf{p}}, \theta = \theta_{\mathbf{p}}$, etc.
(b) an ordinal $\varepsilon_{\mathbf{x}} = \varepsilon(\mathbf{x})$
(c) $\bar{\alpha}^* = \langle \alpha^*_\varepsilon : \varepsilon \le \varepsilon_{\mathbf{x}} \rangle$ is increasing continuous
(d) (α) $\bar{D} = \langle D_\varepsilon : \varepsilon \le \varepsilon_{\mathbf{x}} \rangle$ is a continuous sequence of filters on Y except that possibly $D_{\varepsilon_{\mathbf{x}}} = \mathscr{P}(Y)$
  (β) $D_\varepsilon = c\ell_{\mathbf{p}}(D_\varepsilon)$
  (γ) for limit ε, $D_\varepsilon = c\ell_{\mathbf{p}}(\bigcup_{\zeta<\varepsilon} D_\zeta)$
(e) $D_0 = D_{\mathbf{x}}$ is $\mathrm{cf} - \mathrm{fil}_\theta(\bar{\delta})$
(f) $\bar{E} = \langle E_\varepsilon : \varepsilon < \varepsilon_{\mathbf{x}} \rangle$
(g) for each $\varepsilon < \varepsilon_{\mathbf{x}} < \theta$ there is $A_\varepsilon \in D^+_\varepsilon$ such that
  (α) $D_{\varepsilon+1} = D_\varepsilon + A_\varepsilon$
  (β) $E_\varepsilon = D_\varepsilon + (u \backslash A_\varepsilon)$
  (γ) there are $a_\varepsilon \subseteq \kappa$ and $h_\varepsilon \in \prod_{i \in \varepsilon} u_i$ such that $\{(i, h_\varepsilon(i)) : i \in a_\varepsilon\} \notin D_\varepsilon$
  - but A_ε is not necessarily unique, only $A_\varepsilon / D_\varepsilon$ is, and of course, also $a_\varepsilon, h_\varepsilon$ are not necessarily unique
  (δ) there is $Z \in \mathscr{A}$ such that $Z \in \mathrm{dual}(D_{\varepsilon+1}) \backslash \mathrm{dual}(D_\varepsilon)$
(h) $\bar{f} = \langle f_\alpha : \alpha < \varepsilon_{\mathbf{x}} \rangle$, $f_\alpha \in \Pi \bar{\delta}$
(i) $\bar{f} \restriction \alpha_{\varepsilon+1}$ is $\le_{D_\varepsilon}$-increasing
(j) $\bar{f} \restriction [\alpha_\varepsilon, \alpha_{\varepsilon+1})$ is $<_{E_\varepsilon + Z}$-cofinal for some $Z \in D^+_\varepsilon$.

Theorem 2.25 *Assume* $\mathrm{Ax}_{4,\partial}$. *Assume* $\mathbf{p}$ *is a pcf-problem and* $\mathrm{hrtg}(\mathscr{A}_{\mathbf{p}}) \le \theta_{\mathbf{p}}, \partial < \theta_{\mathbf{p}}$. <u>*Then*</u> *there is a wide pcf-system* $\mathbf{x}$ *such that* $\mathbf{p}_{\mathbf{x}} = \mathbf{p}$.

Proof As in Sect. 1 we try to choose α_ε and $\langle f_\alpha : \alpha \le \alpha_\varepsilon \rangle, D_\varepsilon, E_\varepsilon$ by induction on ε satisfying the relevant demands. The main point is having chosen $\langle \alpha_\xi, D_\xi : \xi \le \zeta \rangle, \langle f_\alpha : \alpha \le \alpha_\zeta \rangle$, we try to choose for $\varepsilon = \zeta + 1$. So we try to choose f_α for $\alpha > \alpha_\zeta$ by induction on α satisfying the relevant conditions. Arriving to limit α let $\mathscr{A}^1_\alpha := \{Z \in \mathscr{A} : Z \notin \mathrm{dual}(D_\varepsilon)\}$ and $\mathscr{A}^2_\alpha = \{Z \in \mathscr{A}^1_\alpha : \langle f_\beta : \beta < \alpha \rangle$ has a $<_{D_\varepsilon + Z}$-upper bound in $\Pi \bar{\delta}\}$. If $\mathscr{A}^1_\alpha = \emptyset$ we are done. If $\mathscr{A}^2_\alpha \neq \emptyset$ by Sect. 1 we can define

$\langle f_{\alpha,Z}: Z \in \mathscr{A}^2_\alpha\rangle$ such that $f_{\alpha,Z} \in \Pi\bar{\delta}$ is an $<_{D_\varepsilon+Z}$-upper bound of $\langle f_\beta: \beta < \alpha\rangle$ and let $f_\alpha \in \Pi\bar{\delta}$ be defined by $f_\alpha(s) = \sup\{f_{\alpha,Z}(s) : Z \in \mathscr{A}^2_\alpha\}$ if $< \delta_s$ and zero otherwise. As $\theta \geq \text{hrtg}(\mathscr{A}_\mathbf{p}) \geq \text{hrtg}(\mathscr{A}^2_\alpha)$, clearly $\beta < \alpha \wedge Z \in \mathscr{A}^2_\alpha \Rightarrow f_\beta < f_\alpha \mod (D_\varepsilon + Z)$. If $\mathscr{A}^2_\alpha = \mathscr{A}^1_\alpha \neq \emptyset$, then f_α is as required as we are assuming $D_\varepsilon = c\ell_\mathbf{p}(D_\varepsilon)$. If $\mathscr{A}^2_\alpha \neq \mathscr{A}^1_\alpha$, let $\alpha_{\varepsilon+1} = \alpha$ and f_α is as required. □

2.3 True successor cardinals

Contrary to our ZFC intuition, without full choice successor cardinals, may be singular. On history we may start with Levy proving ZF + "$\aleph_1$ is singular" is consistent and end with Gitik proving ZF + $(\forall\lambda)(\text{cf}(\lambda) = \aleph_0)$ is consistent, using large cardinals. Note: for two successive cardinals are singular" has quite high consistency strength.

A major open question is whether ZF + DC + $(\forall\lambda)(\text{cf}(\lambda) \leq \aleph_1)$ is consistent. But when ZF + DC + Ax_4 holds the situation is very different. Also contrary to our ZFC intuition, successor cardinals may be measurable.

For a cardinal to be a true successor is saying it fits our ZFC intuition. In particular, it avoid the two anomalies mentioned above, and eventually it will enable us to carry various constructions; all this motivates Question 2.27.

We continue the investigation in [23] of successor of singulars, not relying on [23].

Definition 2.26 (1) We say λ is a true successor cardinal <u>when</u> for some cardinal μ, $\lambda = \mu^+$ and we have a witness $\bar{f}$, which means $\bar{f} = \langle f_\alpha: \alpha \in [\mu, \lambda)\rangle$ and f_α is a one-to-one function from α into μ.
(1A) We say $\bar{f}$ is an onto-witness when each f_α is onto μ, see 2.28(1) below.
(2) We say a set $\mathscr{U} \subseteq \text{Ord}$ is a smooth set <u>when</u> there is a witness $\bar{f}$ which means that $\bar{f} = \langle f_\alpha: \alpha \in \mathscr{U}\rangle$, f_α is a one-to-one function from α onto $|\alpha|$.

We may naturally ask

Question 2.27 (1) Is there a class of successor of regular cardinals which are true successor cardinal? See 2.28(2).
(2) Assume μ is strong limit (i.e. $\alpha < \mu \Rightarrow \text{hrtg}(\mathscr{P}(\mu)) < \mu$) of cofinality $\aleph_1$, so μ^+ is regular, but assume in addition that μ^{++} is regular $< \text{pp}(\mu)$, see[8] [11, Ch.II]. Is μ^{++} truely successor?
(3) Assume μ is strong limit of cofinality $\aleph_0$ and μ^{+2} is singular, is μ^{+3} a true successor cardinal?

Claim 2.28 (1) *If λ is true successor <u>then</u> λ is regular and has an onto-witness (computed uniformly from a witness).*
(2) [$\text{Ax}^4_{\mu^+}$ *or just* $\text{Ax}_{4,\mu^+,\partial}$] *Assume μ is singular and* $(\forall\alpha < \mu)(\text{hrtg}([\alpha]^{\aleph_0} \times \partial) < \mu)$. *<u>Then</u> μ^+ is a true successor cardinal.*
(3) [$Ax_{4,\partial}$ *or just* $\text{Ax}_{4,\lambda,\partial}$] *The set $\mathscr{U}$ of ordinals $\alpha < \lambda$ such that $|\alpha|$ is singular and* $(\forall\beta < |\alpha|)[\text{hrtg}([\beta]^{\aleph_0} \times \partial) \leq |\alpha|]\}$ *is a smooth set of ordinals.*
(4) *For every ordinal α_*, $\alpha_* \in \text{cf} - \text{id}_{\langle(\text{hrtg}([\text{cf}(\alpha)]^{\aleph_0} \times \partial): \alpha < \alpha_*\rangle}(\langle\alpha: \alpha < \alpha_*\rangle)$.*

[8] Generality with weak choice there is a choice to be made, but assuming Ax_4 or so and $\text{cf}(\mu) = \aleph_0$, there is no problem.

Proof Let pr be the classical one-to-one function from Ord × Ord onto Ord such that $\mathrm{pr}(\alpha,\beta) < (\max\{\alpha,\beta\})^2$ and $\mathrm{pr}_\mu = \mathrm{pr}\restriction(\mu\times\mu)$.
(1) Let $\bar f = \langle f_\alpha : \alpha \in [\mu,\mu^+)\rangle$ witness λ is truely a successor. First define, for $\alpha \in [\mu,\mu^+)$ a function $f'_\alpha : \alpha \to \mu$ by $f'_\alpha(\beta) = \mathrm{otp}(\mathrm{Rang}(f_\alpha) \cap f_\alpha(\beta))$; obviously it is a one-to-one function from α into μ with range an initial segment; but $|\mathrm{Rang}(f'_\alpha)| = |\alpha| = \mu$ so $\mathrm{Range}(f'_\alpha) = \mu$, $\langle f'_\alpha : \alpha \in [\mu,\mu^+)\rangle$ is as promised.

Second proving λ is regular, toward contradiction let $\mathscr{U}$ be such that $\mathscr{U} \subseteq \lambda = \sup(\mathscr{U})$, $\mathscr{U} \cap \mu = \emptyset$ and $\mathrm{otp}(\mathscr{U}) < \lambda$, so without loss of generality $\le \mu$. Now we shall combine $\langle f_\alpha : \alpha \in \mathscr{U}\rangle$ to get $|\lambda| \le \mu$ by getting a one to one function f from λ into $\mu\times\mu$; for $i < \lambda$ let $\alpha_i = \min\{\alpha \in \mathscr{U} : \alpha > i\}$ and define $f(i) = \mathrm{pr}(\mathrm{otp}(\mathscr{U} \cap \alpha_i), f_{\alpha_i}(\alpha))$. So f exemplifies $|\lambda| \le |\mu\times\mu|$ but the latter is μ, contradiction.
(2) By part (3) applied to $\mathscr{U} = [\mu,\mu^+)$.
(3) Let $\mathscr{S} \subseteq [\lambda]^{<\partial}$ witness $\mathrm{Ax}_{4,\lambda,\partial}$ and $<_*$ a well ordering of $\mathscr{S}$. Let $\alpha_* = \cup\{\alpha+1 : \alpha \in \mathscr{U}\}$ let $c\ell : [\alpha_*]^{\aleph_0} \to \alpha_*$ be as in 0.6, let $<_*$ be a well order $\mathscr{S}$ and let u_β for $\beta < \alpha_*$ be defined by

- if $\beta = 0$ then $u_0 = \emptyset$
- if $\beta = \gamma+1$ then $u_\beta = \{\gamma\}$
- if $\mathrm{cf}(\beta) > \aleph_0$ then $u_\beta = \cap\{\cup\{c\ell(v) : v \in [u]^{\aleph_0}\} : u \text{ a club of } \beta\}$
- if $\mathrm{cf}(\beta) = \aleph_0$ the $u_\beta = v_\beta \cap \beta$ where v_β is the $<_*$-first $v \in \mathscr{S}$ such that $\beta = \sup(v \cap \beta)$.

Now choose f_α for $\alpha \in \mathscr{U} \cap \alpha_*$ by induction on α using $\mathrm{pr}_{|\alpha|}$.
(4) By $(*)_4$ in the proof of 1.5, in particular, $(c)_2$ there. □

Claim 2.29 (1) *If* $\lambda = \mu^+$ *then* λ *is a true successor iff* $\lambda \in \mathrm{cf}-\mathrm{id}_{<(\mu+1)}(\lambda)$, (*which means* $\lambda \in \mathrm{cf}-\mathrm{id}_{<(\mu+1)}(\langle\alpha:\alpha<\lambda\rangle)$) *iff* $\lambda \in \mathrm{cf}-\mathrm{id}_{<\gamma}(\langle\alpha:\alpha<\lambda\rangle)$ *for some* $\gamma < \lambda$.
(2) *When* μ *is singular, we can add: iff* $\lambda \in \mathrm{cf}_{<\mu}(\langle\alpha:\alpha<\lambda\rangle)$.

Proof (1) First condition implies second condition:

So assume λ is a true successor, let $\langle f_\alpha : \alpha \in [\mu,\mu^+)\rangle$ witness it. For each $\alpha < \mu^+ = \lambda$ we choose u_α as follows:

Case 1: $u_\alpha = \alpha$ if $\alpha < \mu$

Case 2: $\alpha \ge \mu$

For any $j < \mu$ let $\mathscr{U}_{\alpha,j} = \{\beta < \alpha : f_\alpha(\beta) < j\}$, so $\langle \mathscr{U}_{\alpha,j} : j < \mu\rangle$ is $\subseteq$-increasing with union α and $|\mathscr{U}_{\alpha,j}| \le |j| < \mu$. If for some j the set $\mathscr{U}_{\alpha,j}$ is unbounded in α let $j(\alpha)$ be the minimal such j and $u_\alpha = \mathscr{U}_{\alpha,j(\alpha)}$.

If for every j, $\mathscr{U}_{\alpha,j}$ is bounded in α let $u_\alpha = \{\sup(\mathscr{U}_{\alpha,j}) : j < \mu\}$, so easily $\mathrm{otp}(u_\alpha) \le \mu$. So $\langle u_\alpha : \alpha < \lambda\rangle$ witness $\lambda \in \mathrm{cf}-\mathrm{id}_{<(\mu+1)}(\lambda)$, i.e. the second condition holds.

Second condition implies third condition:

Trivial.

Third condition implies first condition:

Let $\gamma < \lambda$ and let $\bar{u} = \langle u_\alpha : \alpha < \lambda \rangle$ witness $\lambda \in \mathrm{cf} - \mathrm{id}_{<\gamma}(\langle \alpha : \alpha < \lambda \rangle)$; let $f_* : \gamma \to \mu$ be one-to-one. Defined a one-to-one function $f_\alpha : \alpha \to \mu$ by induction on $\alpha \in [\mu, \lambda)$, the induction step as in the proof of 2.28(1).
(2) Assume μ is singular; obviously the fourth condition implies the third.

Second condition implies the fourth condition:
Let $\langle u_\alpha : \alpha < \lambda \rangle$ witness $\lambda \in \mathrm{cf} - \mathrm{id}_{<(\mu+1)}(\langle \alpha : \alpha < \lambda \rangle)$, let f_α be the unique order preserving function from u_α onto $\mathrm{otp}(u_\alpha)$. Let $u \subseteq \mu = \sup(u)$ has order type $\mathrm{cf}(\mu)$ or just $< \mu$. Let u'_α be u_α if $\mathrm{otp}(u_\alpha) < \mu$ and be $\{\beta \in u_\alpha : f_\alpha(\beta) \in u\}$ if $\mathrm{otp}(u_\alpha) = u$. □

The next claim says that quite many partial squares on $\lambda = \mu^+$ exists.

Claim 2.30 $[\mathrm{Ax}_{4,\partial}]$ *Assume λ is the true successor of μ, $\theta \leq \kappa = \mathrm{cf}(\mu)$, $\theta \leq \theta_1 < \mu$, $\partial < \theta$ and $\alpha < \mu \Rightarrow \mathrm{hrtg}({}^{\theta>}\alpha) < \mu$ and $\alpha < \theta = \mathrm{hrtg}([\alpha]^{<\partial}) < \theta_1$.*
Then we can find $\bar{C} = \langle C_{\varepsilon,\alpha} : \varepsilon < \mu, \alpha \in S_\varepsilon \rangle$ such that:

(a) $S_\varepsilon \subseteq S^\lambda_{<\theta_1} := \{\delta < \lambda : \mathrm{cf}(\delta) < \theta_1\}$
(b) $S^\lambda_{<\theta} \subseteq \cup\{S_\varepsilon : \varepsilon < \mu\}$
(c) $C_{\varepsilon,\alpha} \subseteq \alpha$ *and* $C_{\varepsilon,\alpha}$ *is closed unbounded in* α
(d) $\beta \in C_{\varepsilon,\alpha} \Rightarrow C_{\varepsilon,\beta} = C_{\varepsilon,\alpha} \cap \beta$
(e) $\mathrm{otp}(C_{\varepsilon,\alpha}) < \theta_1$.

Proof Let $X \subseteq \lambda$ code:

- a witness to "λ is the true successor of μ"
- the set $S^*_0 := S^\lambda_{<\theta}$, $S^*_1 = S^\lambda_{<\theta_1}$
- a witness to $\mathrm{cf}(\mu) = \kappa$
- $\langle e_\alpha : e < \lambda \rangle$ as in $(*)_4$ of the proof of 1.5 so $\alpha \in S^*_0 \Rightarrow |e_\alpha| < \theta_1$.

So $\mathbf{L}[X] \models$ "$\lambda = \mu^+, \mathrm{cf}(\mu) = \kappa \geq \theta$" and $\chi < \mu \Rightarrow \chi^{<\theta} < \mu$. If $\mathbf{L}[X] \models$ "μ is regular", by [17, §4] and if $\mathbf{L}[X] \models$ "μ is singular" by Dzamonja–Shelah [2] we get the result in $\mathbf{L}[X]$ and the same $\bar{C}$ works in $\mathbf{V}$. □

For more on successor, see [27, §(3A)] and in [29, 0x=Ls3].

2.4 Covering number

Definition 2.31 (1) Let $\mathrm{cov}(\lambda, \theta, \leq Y, \sigma)$ be the minimal cardinal χ such that [if no such χ exists, it is ∞ (or not well defined)]: there is a set $\mathscr{P}$ of cardinality χ such that:

(a) $\mathscr{P} \subseteq [\lambda]^{<\theta}$
(b) if $f \in {}^Y\lambda$ then there is $\mathscr{P}' \subseteq \mathscr{P}$ of cardinality $< \sigma$ such that $\mathrm{Rang}(f) \subseteq \cup\{u : u \in \mathscr{P}'\}$.

(1A) Writing κ instead "$\leq Y$" means $f \in \bigcup_{\alpha<\kappa} {}^{\alpha}\lambda$.
(2) If $\sigma = 2$ we may omit it.
(3) Writing "$\leq \theta$" instead of θ means θ^+, i.e. $\mathscr{P} \in [\lambda]^{\leq\theta}$.

Definition 2.32 (1) We say $([\gamma]^\theta, \subseteq)$ strongly[9] has cofinality $\leq \chi$ <u>when</u> there is $\bar{f} = \langle f_\alpha : \alpha < \alpha_* \rangle$ such that $|\alpha_*| = \chi$ and $f_\alpha : \theta \to \mu$ and for every $u \in [\gamma]^\theta$ there is α such that $u \subseteq \mathrm{Rang}(f_\alpha)$.
(2) We replace "$\leq \chi$" by "χ" <u>when</u> in addition $([\gamma]^\theta, \subseteq)$ has cofinality χ.

Claim 2.33 *If $([\gamma]^\theta, \subseteq)$ has cofinality χ and θ^+ is a truely successor <u>then</u> $([\gamma]^\theta, \subseteq)$ strongly has cofinality χ.*

Proof Easy. □

Theorem 2.34 *Assume $\mathrm{Ax}_{4,\partial}$, $\partial < \theta_*$, $\langle \theta_Y = \theta(Y) : Y \in \theta_* \rangle$ is such that (θ_Y, Y) satisfies the demands on (θ, Y) in 1.5 and $\theta_Y < \theta_*$ and so θ_* is strong limit in the sense $Y \in \theta_* \Rightarrow \mathrm{hrtg}(\mathrm{Fil}^4_{\aleph_1}(Y)) < \theta_*$, equivalently $\kappa < \theta_* \Rightarrow \mathrm{hrtg}(\mathscr{P}(\mathscr{P}(\kappa)) < \theta_*$ (and $\theta_* > \partial$; see 0.17).*
(1) *For all cardinals $\lambda \geq \theta_*$ we have $\mathrm{cov}(\lambda, \leq \theta_*, < \theta_*, 2)$ is well defined (i.e. $< \infty$).*
(2) *Even ∂-uniformly and in some inner model $\mathbf{L}[X]$, $X \subseteq \mathrm{Ord}$ we have witness for those covering numbers.*

Proof Let $\lambda_* = \cup\{\mathrm{hrtg}({}^\kappa\lambda) : \kappa < \theta_*\}$

$\boxplus_1$ (a) let $(\mathscr{S}_*, <_*)$ be such that $\mathscr{S}_* \subseteq [\lambda_*]^{<\partial}$ satisfy $(\forall u \in [\lambda_*]^{\aleph_0})$ $(\exists v \in \mathscr{S}_*)[u \subseteq v]$ and $<_*$ is a well ordering of $\mathscr{S}_*$
(b) we define $c\ell$ and $\mathscr{S}_{\lambda_*,\kappa} \subseteq [\lambda_*]^{<\partial}$, $<_{\lambda_*,\kappa}$, $\langle w^*_{\kappa,i}, i < \mathrm{otp}(\mathscr{S}_{\lambda_*}, <_*) \rangle$, Ω_κ, $\bar{e}_\kappa$ as in $(*)_1 - (*)_4$ in the proof of 1.5 with κ here standing for Y there, from $(\mathscr{S}_*, <_*)$.

So we can choose $\bar{F} = \langle F^1_\kappa : \kappa < \theta_* \rangle$ where

$\boxplus_2$ (a) F^1_κ is a function
(b) $\mathrm{Dom}(F^1_\kappa) = \{f : f \in {}^\kappa(\lambda + 1)$ and $i < \kappa \Rightarrow \mathrm{cf}(f(i) \geq \theta_\kappa\}$
(c) $F^1_\kappa(f)$ is a pair $(\mathscr{F}^1_f, <^1_f)$ such that
(α) $\mathscr{F}^1_f \subseteq \prod_{i<\kappa} f(i)$ is cofinal, i.e. modulo the filter $\{\kappa\}$
(β) $<^1_f$ is a well ordering of $\mathscr{F}^1_f$.

[Why possible? By 2.2 and 2.3(2).]
Let $(\theta_{n+1}(\kappa))$ exist and is $< \theta_*$, see [23, 0.14] where

[9] Without "strongly" we have only $f_\alpha : \gamma_\alpha \to \mu$ where $\gamma_\alpha < \theta^+$.

$\boxplus_3$ for $\kappa < \theta_*$, let $\theta_0(\kappa) = \theta_\kappa$ and $\theta_{n+1}(\kappa) := \min\{\sigma$: if $\langle u_i : i < \kappa\rangle$ is a sequence of sets of ordinals each of cardinality $< \theta_n(\kappa)$ then $\sigma > |\bigcup_{i<\kappa} u_i|\}$.

Choose $\langle(\mathscr{F}^2_{\kappa,n}, <^2_{\kappa,n}) : \kappa < \theta_*\rangle$ by induction on n, so $\langle(\mathscr{F}^2_{\kappa,n}, <^2_{\kappa,n}) : n < \omega$ and ordinal $\kappa < \theta_*\rangle$ exists, such that

$\boxplus_4$ (a) if $n = 0$ then $\mathscr{F}^2_{\kappa,n} = \{f^2_*\}$, $f^2_* \in {}^\kappa(\lambda + 1)$ is constantly λ

(b) if $f \in \mathscr{F}^2_{\kappa,n}$ then f is a function from κ into $\{u \subseteq \lambda + 1 : |u| \leq \theta_n(\kappa)\}$

(c) $<^2_{\kappa,n}$ well orders $\mathscr{F}^2_{\kappa,n}$

(d) if $f \in \mathscr{F}^2_{\kappa,n}$ then for $\ell < 4$ we let g^ℓ_f be the following function; its domain is κ and for $i < \kappa$ we let:

$\underline{\ell = 0}$: $g^\ell_f(i) = \{\alpha \in f(i) : \alpha = 0\}$

$\underline{\ell = 1}$: $g^\ell_f(i) = \{\alpha \in f(i) : \alpha$ is a successor ordinal$\}$

$\underline{\ell = 2}$: $g^\ell_f(i) = \{\alpha \in f(i) : \alpha$ is a limit ordinal of cofinality $< \theta_\kappa\}$

$\underline{\ell = 3}$: $g^\ell_f(i) = \{\alpha \in f(i) : \mathrm{cf}(\alpha) \geq \theta_\kappa\}$

(d)(α) if $f_1 \in \mathscr{F}^2_{\kappa,n}$ <u>then</u> for some $f_2 \in \mathscr{F}^2_{\kappa,n+1}$, $f_2(i) = \{\beta : \beta + 1 \in g^1_{f_1}(i)\}$

(β) if $f_1 \in \mathscr{F}^2_{\kappa,n}$ then for some $f_2 \in \mathscr{F}^2_{\kappa,n+1}$ we have $f_2(i) = \cup\{e_{\kappa,\alpha} : \alpha \in g^2_{f_1}(i)$ and $\mathrm{cf}(\alpha) < \theta\}$,

(γ) if $f_1 \in \mathscr{F}^2_{\kappa,n}$ letting $u := \mathrm{otp}(\cup\{g^3_{f_1}(i) : i < \kappa\})$, i.e. $\zeta = \zeta_f = \mathrm{otp}(u) < \theta_*$, $\bar{\delta}_{f_1} = \langle\delta_{f_1,\iota} : \iota < \zeta\rangle$ increasing $\delta_{f_1,\iota} \in u$ and $\mathrm{otp}(\delta_{f_1,\iota} \cap u) = \iota$ then $F^1_{\mathrm{otp}(u)}(\bar{\delta}_{f_1}) \subseteq \mathscr{F}^2$

(e)(α) $\mathscr{F}^2_{\kappa,n+1}$ is minimal under the conditions above

(β) $<^2_{\kappa,n+1}$ is chosen naturally.

We can choose a set X_2 of ordinals such that $\langle\mathscr{F}^2_{\kappa,n} : \kappa \in \theta_*, n < \omega\rangle$ belongs to $\mathbf{L}[X_2]$ hence a list $\langle w^*_\alpha : \alpha < \alpha_2(*)\rangle \in \mathbf{L}[X_2]$ of $\{\mathrm{Rang}(f) : f \in \mathscr{F}^2_{\kappa,n}$ for some $\kappa < \theta_*, n < \omega\}$ and a list $\bar{u} = \langle u_\alpha : \alpha < \alpha_3(*)\rangle$ of a cofinal subset of $[\alpha_2(*)]^{\aleph_0}$ and X_3 such that $X_2, \bar{u} \in \mathbf{L}[X_3]$.

Now for any ordinal $\kappa < \theta_*$ and $f \in {}^\kappa\lambda$ we can choose finite $v_n \subseteq \alpha_2(*)$ by induction on n such that:

$(*)_n$ (a) $\lambda \in \cup\{w^*_\alpha : \alpha \in v_n\}$ for $n = 0$

(b) if $i < \kappa$, $f(i) \notin \cup\{w^*_\alpha : \alpha \in v_n\}$ then $\min(\bigcup_{\alpha\in v_n} w^*_\alpha \backslash f(i)) > \min(\bigcup_{\alpha\in v_n} w^*_\alpha \backslash f(i))$.

So $\langle v_n : n < \omega\rangle$ exists hence $v = \bigcup_n v_n \in \mathbf{L}[X_3]$, hence $w = \bigcup_{\alpha\in v} w^*_\alpha \in \mathbf{L}[X_3]$ has cardinality $\leq \theta_*$ and includes $\mathrm{Rang}(f)$ because if $i < \kappa \wedge f(i) \notin \cup\{w^*_\alpha : \alpha \in v\}$ then $\langle\min(\bigcup_{\alpha\in v_n} w^*_\alpha \backslash f(i)) : n < \omega\rangle$ is a strictly decreasing sequence of ordinals. So we should just let $\mathscr{P} = \{u \subseteq \lambda : u \in \mathbf{L}[X_3]$ and $\mathbf{L}[X_3] \models$ "$|u| \leq \theta_*\}$ witness the desired conclusion. □

Now (like [27, §(3A)] see definitions there)

Conclusion 2.35 *Assume* Ax_4. *If* μ *is a singular cardinal such that* $\kappa < \mu \Rightarrow \theta_\kappa := \mathrm{hrtg}(\mathscr{P}(\mathscr{P}(\kappa))^+ < \mu$ *and* $\lambda > \mu$ *then for some* $\kappa < \mu$ *we have:* $\mathrm{cov}(\lambda, \mu, \mu, \kappa) = \lambda$.

Proof Use [19] in $\mathbf{L}[X]$ where $X \subseteq \mathrm{Ord}$ is as in 2.34(2). □

Discussion 2.36 (0) From 2.34, 2.35 we can get also smooth closed generating sequence (see [18, §6], [13].
(1) We would like to get better bounds. A natural way is to fix κ, consider $\theta_1 > \kappa$ and $\mathbf{f}: \kappa \to [\lambda]^{<\theta_1}$ and ask for $\mathscr{F} \subseteq \{f: \kappa \to [\lambda]^{<\theta_2}\}$ such that for every $g \in \prod_{i<\kappa}(\mathbf{f}(i) \cup \{1\} \backslash \{0\})$ and $g_i \in \prod_{i<\kappa} g_*(i)$ there is $f \in \mathscr{F}$ such that $(\forall i < \kappa)(f(i) \cap [g_1(i), g_*(i)) \neq \emptyset)$.
(2) We can get also strong covering, see [11, Ch.VII].
(3) Can we get something better on μ singular strong limit? a BB?, (BB means black box, see [15] and in Sect. 3, possibly see more in [28].
(4) We like to improve 2.34, in particular Sect. 2.3, for this we have to improve Sect. 2.1. We would like to replace $\mathrm{Fil}^4_{\aleph_1}(Y)$, i.e. $\mathrm{hrtg}(\mathrm{Fil}^4_{\aleph_1}(Y))$ by $\mathrm{hrtg}(\mathscr{P}(Y))$ and even $\mathrm{hrtg}(Y)$, as done in ZFC in [16]. We do not know to do this but we try a more modest aim: suppose we deal only with $[Y]^{\leq\kappa}$ or so. So hopefully in [29], we still have $\mathrm{hrtg}(\mathrm{Fil}^4_{\aleph_1}(\kappa))$ but $\mathrm{hrtg}(\mathscr{P}(Y))$ only.

3 Black Boxes

There are many proofs in ZFC using diagonalization of various kinds so they seem to depend heavily on choice. Using Ax_4 we succeed to generalize one such method—one of the black boxes from [15], it seems particularly helpful in constructing Abelian groups and modules; see on applications in the books Eklof–Mekler [1] and Göbel–Trlifaj [4].

The proof specifically uses countable models and Ax_4. Naturally we would like to assume we have only $\mathrm{Ax}_{4,\partial}$. But existing versions implies $\mathscr{P}(\mathbb{N})$ is well ordered and more, whereas $\mathrm{Ax}_{4,\partial}$ does not imply this.

3.1 Existence proof

Hypothesis 3.1 $\mathrm{ZF} + \mathrm{DC} + \mathrm{Ax}_4$ [so $\partial = \aleph_1$]

The following is like [15, 3.24(3)], the relevant cardinals provably exists but may be less common than there: conceivably true successor are only successor of singular strong limit cardinals.

Theorem 3.2 *If* (*A*) *then* (*B*) *where:*

(A) (a) $\lambda = \mu^+$ *is a true successor*
(b) $\mu = \mu^{\aleph_0}$
(c) $S = \{\delta < \lambda: \mathrm{cf}(\delta) = \aleph_0$ *and* μ *divides* $\delta\}$ *or just* S *is a stationary subset of* λ *such that* $\delta \in S \Rightarrow \mathrm{cf}(\delta) = \aleph_0 \wedge \mu < \delta \wedge (\mu | \delta)$

(d) $\bar{\gamma}^* = \langle \bar{\gamma}^*_\delta : \delta \in S\rangle$ *with* $\bar{\gamma}^*_\delta = \langle \gamma^*_{\delta,n} : n < \omega\rangle$ *an increasing* ω*-sequence of ordinals with limit* δ, *(exist, see* $(*)_7$*)*

(B) *we can find* $\mathbf{w} = (\alpha, \mathbf{W}, \dot{\zeta}, h, \bar{\mathbf{k}}) = (\alpha_{\mathbf{w}}, \mathbf{W}_{\mathbf{w}}, \dot{\zeta}_{\mathbf{w}}, h_{\mathbf{w}}, \bar{\mathbf{k}}_{\mathbf{w}})$ *such that (we may denote* $\alpha_{\mathbf{w}}$ *by* $\ell g(\mathbf{w})$ *and may omit it):*

(a) (α) $\mathbf{W} = \langle \bar{N}_\alpha : \alpha < \alpha_{\mathbf{w}}\rangle$

(β) $\bar{N}_\alpha = \langle N_{\alpha,n} : n < \omega\rangle$ *is* $\prec$*-increasing sequence of models*

(γ) $\tau(N_{\alpha,n}) \subseteq \mathcal{H}(\aleph_0)$ *and* $\tau(N_{\alpha,n}) \subseteq \tau(N_{\alpha,n+1})$

(δ) $\mathbf{k} = \langle \bar{k}_\alpha : \alpha < \alpha_{\mathbf{w}}\rangle$, $\bar{k}_\alpha = \langle k_{\alpha,n} : n < \omega\rangle$ *is increasing, let* $k_{\mathbf{w}}(\alpha, n) = k(\alpha, n) = k_{\alpha,n}$

(ε) $|N_{\alpha,n}| \subseteq |N_{\alpha,n+1}| \subseteq \lambda$ *but* $N_{\alpha,n} \neq N_{\alpha,n+1}$

(ζ) *let* $N_\alpha = N_{\alpha,\omega} = \lim(\bar{N}_\alpha)$, *that is,* $\tau(N_{\alpha,\omega}) = \cup\{\tau(N_{\alpha,n}) : n < \omega\}$ *and* $(N_{\alpha,\omega} \restriction \tau(N_{\alpha,n})) \supseteq N_{\alpha,n}$

(η) *the universe of* $N_{\alpha,n}$ *is a countable subset of* λ

(b) (α) $\dot{\zeta}$ *is a function from* $\alpha_{\mathbf{w}}$ *into S, non-decreasing*

(β) *if* $\dot{\zeta}(\alpha) = \delta$ *then* $\delta = \sup\{\gamma^*_{\delta,n} : n < \omega\} = \sup(N_\alpha)$

(γ) *if* $\alpha < \alpha_{\mathbf{w}}$ *and* $\dot{\zeta}(\alpha) = \delta \in S$ *and* $n < \omega$ *then* $N_{\alpha,n+1} \backslash N_{\alpha,n} \subseteq (\gamma^*_{\delta,k(\alpha,n)}, \gamma^*_{\delta,k(\alpha,n)+1})$ and $|N_{\alpha,n}| \subseteq \gamma^*_{\delta,k(\alpha,n)}$

(c) *if M is a model with universe* λ *and vocabulary* $\subseteq \mathcal{H}(\aleph_0)$ *then for stationarily many* $\delta \in S$, *there is* α *such that* $\dot{\zeta}(\alpha) = \delta$, $N_\alpha \prec M$.

(d) (α) *if* $\dot{\zeta}(\alpha) = \delta = \dot{\zeta}(\beta)$ *then* $N_\alpha \cong N_\beta$, $|N_\alpha| \cap \mu = |N_\beta| \cap \mu$, $\bar{k}_\alpha = \bar{k}_\beta$; *moreover,* $\text{otp}(|N_\alpha|) = \text{otp}(|N_\beta|)$ *and the unique order preserving mapping is an isomorphism from* $N_{\alpha,n}$ *onto* $N_{\beta,n}$ *for every n and is the identity on* $|N_\alpha| \cap \mu$ *and on* $N_\alpha \cap N_\beta$ *and so maps* $N_\alpha \cap \gamma^*_{\delta,k(\alpha,n)}$ *onto* $N_\beta \cap \gamma^*_{\delta,k(\beta,n)}$

(β) *if* $\dot{\zeta}(\alpha) = \delta = \dot{\zeta}(\beta)$ *but* $\alpha \neq \beta$ *then*

- $N_\alpha \cap N_\beta$ *is an initial segment of both* N_α *and of* N_β
- $N_\alpha \cap N_\beta \subseteq N_{\alpha,n+1} \cap N_{\beta,n+1}$ *and* $N_\alpha \cap N_\beta \supseteq N_{\alpha,n} = N_{\beta,n}$ *for some n.*

Remark 3.3 (1) The existence proof is uniform (that is, **w** can be defined from $(<_*, \bar{f})$ where: $<_*$ is a well ordering of $[\chi]^{\aleph_0}$ for χ large enough and $\bar{f}$ is a witness for λ being a true successor. Moreover, also $\bar{\gamma}^*$ can be chosen uniformly (as well as the witness for λ-being a true successor.
(2) We would like to add (A)(e) to the assumption and add (B)(e) to the conclusion of 3.2 where:

(A)(e) (α) $\bar{C} = \langle C_\delta : \delta \in S\rangle$

(β) $C_\delta \subseteq \delta = \sup(C_\delta)$

(γ) $\text{otp}(C_\delta) = \omega$ and let $\bar{\gamma}^*_\delta = \langle \gamma^*_{\delta,n} : n < \omega\rangle$ list C_δ in increasing order

(δ) $\bar{C}$ weakly guess clubs, i.e. for every club E of λ for stationarily many $\delta \in S$ we have $(\forall n)(E \cap (\gamma^*_{\delta,n}, \gamma^*_{\delta,n+1}) \neq \emptyset)$, moreover

(ε) $\langle S_\varepsilon : \varepsilon < \lambda \rangle$ is a partition of S such that $\bar{C} \restriction S_\varepsilon$ weakly guess clubs for each ε

(B) (e) $N_{\alpha,n+1} \backslash N_{\alpha,n}$ is included in $[\gamma^*_{\delta,n}, \gamma^*_{\delta,n+1})$, that is $k_{\mathbf{w}}(\alpha, n) = n$.

But not clear if (A) is provable in our context. Still, repeating the ZFC proof works in ZF + $\mathrm{DC}_{\aleph_1}$ and gives even "$\bar{C}$ guess clubs", i.e. "$\{\gamma_{\delta,n} : n < \omega\} \subseteq C_\delta$". But we ask only for "weakly guess", see 3.3(2), $(A)(e)(\delta)$ so using Ax_4 just adding $\mathrm{AC}_{\mathscr{P}(\mathbb{N})}$ suffice.[10] However, clause $(B)(d)(\beta)$ is a reasonable substitute.
(2) We may strengthen clause (B)(d) by adding:

(γ) if $\dot{\zeta}(\alpha) = \delta = \zeta(\beta)$ then $|N_\alpha| \cap \gamma(\delta, 0) = |N_\beta| \cap \gamma(\delta, 0)$ call it u_δ.

For this in $(*)_6$ the partition should be $\langle S_\varepsilon : \varepsilon < \lambda \rangle$ as ε should determine also N_δ, etc.
(3) The use of κ possibly $> \aleph_1$ in 3.4 is not necessary for 3.2.
(4) Note that in proof we need $\mu = \mu^{\aleph_0}$ for proving $(*)_3$. Note that for $(*)_6(a), (b), (c)$ we need just "λ is a true successor of μ". To get clause (d) too, it suffices to have $\mu = \mu^{\aleph_0}$.
(5) We may prove also 3.7 inside the proof of 3.2.

Proof Now

$\boxplus_1$ there are g^0, g^1 such that

(a) g^0, g^1 are two-place functions from λ to λ which are zero on μ

(b) (α) if $\alpha \in [\mu, \lambda)$ then $\langle g^0(\alpha, i) : i < \mu \rangle$ enumerate $\{j : j < \alpha\}$ without repetitions

(β) if $\alpha, i < \lambda$ and $\alpha < \mu \vee i \geq \mu$ then $g^0(\alpha, i) = 0$

(c) (α) $g^1(\alpha, g^0(\alpha, i)) = i$ <u>when</u> $i < \mu \leq \alpha < \lambda$

(β) if $\alpha < \mu$ and $i < \lambda$ then $g^1(\alpha, i) = 0$

(d) there is $\gamma_* \in (\mu, \lambda)$ such that for every countable $u \subseteq \lambda$ closed under g^0, g^1 there is v such that:

(α) $v \subseteq \gamma_*$ is countable

(β) $\mathrm{otp}(v) = \mathrm{otp}(u)$

(γ) $v \cap \mu = u \cap \mu$

(δ) v is closed under g^0, g^1

(ε) the (unique) order preserving function from u onto v commute with g^0, g^1

(ζ) we can arrange that $\gamma_* = \mu + \mu$.

[10] That is, having $\bar{S} = \langle S_\varepsilon : \varepsilon < \mu \rangle$ for each ε choose the first increasing function $f \in {}^\omega\omega$ such that $\langle \gamma^*_{\delta, f(n)} : \delta \in S_\varepsilon \rangle$ weakly guess clubs.

[Why? As λ is truely successor there is no problem to choose g^0, g^1 satisfying clauses (a), (b), (c). On $\mathscr{U} = \{u \subseteq \mu^+: u$ countable closed under $g^0, g^1\}$ we define an equivalence relation E by $(d)(\beta), (\gamma), (\varepsilon)$. Now as $\mu = \mu^{\aleph_0}$, $\mathscr{U}/E$ has cardinality μ hence recalling λ is regular we can prove that γ_* as required in $(d)(\alpha) - (\varepsilon)$ exists. In fact, ∂-uniformly we have a well ordering $<_{\mathscr{U}}$ of $\mathscr{U}$; without loss of generality $u_1 <_{\mathscr{U}} u_2 \Rightarrow \sup(u_1) \leq \sup(u_2)$.

To have $\gamma_* = \mu + \mu$, let τ_* be the vocabulary $\{F_0, F_1\}$ with F_0, F_2 binary function and let $\mathbf{M} = \{M: M$ is a τ_*-model with universe $|M|$ a countable subset of $\mu + \mu$ such that $\alpha, \beta \in M \cap \mu \Rightarrow F_0(\alpha, \beta) = 0 = F_1(\alpha, \beta)$ and the functions F_0^M, F_1^M satisfies the relevant cases of the demands $(a), (b), (c)$ on $(g^0, g^1)\}$.

Clearly $\mathbf{M}$ has cardinality μ and moreover we can (uniformly) define a list $\langle M_\varepsilon: \varepsilon < \mu\rangle$ of $\mathbf{M}$.

Let $i_\varepsilon = \text{otp}(|M_\varepsilon| \backslash \mu)$ and by induction on $\varepsilon < \mu$ we choose $(h_\varepsilon, \gamma_\varepsilon)$ such that:

$\boxplus_{1.2}$ (a) $\gamma_0 = \mu$

(b) $\langle \gamma_\zeta: \zeta \leq \varepsilon\rangle$ is increasing continuous

(c) h_ε is an order preserving function from $|M_\varepsilon| \backslash \mu$ onto $[\gamma_\varepsilon, \gamma_{\varepsilon+1})$.

Next let $N_\varepsilon \in \mathbf{M}$ be such that $h_\varepsilon \cup \text{id}_{|M_\varepsilon| \cap \mu}$ is an isomorphism from M_ε onto N_ε.

Now we define the two-place function g_0^*, g_1^* from λ to λ as follows

$\boxplus_{1.3}$ (a) if $\varepsilon < \mu$ and $\gamma_\varepsilon \leq \alpha < \gamma_{\varepsilon+1}$ then

- if $i \in N_\varepsilon \cap \mu$ then $g_0^*(\alpha, i) = F_0^{N_\varepsilon}(\alpha, i)$
- $\langle g_0^*(\alpha, i): i \in \mu \backslash N_\varepsilon\rangle$ lists $\alpha \backslash N_\varepsilon$ without repetition and is derived from $\langle g^0(\alpha, i): i < \mu\rangle$ and N_ε as in the proof of the Cantor–Bendixon theorem (that $|A| \leq |B| \wedge |B| \leq |A| \Rightarrow |A| = |B|$):

(b) if $\alpha \in [\mu + \mu, \lambda)$ then $i < \mu \Rightarrow g_0^*(\alpha, i) = g^0(\alpha, i)$

(c) if $\alpha \in [\mu, \lambda)$ and $j < \alpha$ then $g_1^*(\alpha, j)$ is defined as the unique $i < \mu$ such that $g_0^*(\alpha, i) = j$

(d) in all other cases the value is zero.

Now g_0^*, g_1^* are well defined, just recall $\boxplus_1(a), (b), (c)$. So $\boxplus_1$ holds indeed.]

Clearly

$(*)_1$ if $u_1, u_2 \subseteq \lambda$ are closed under g^0, g^1 and $u_1 \cap \mu = \mu_2 \cap \mu$ <u>then</u> $u_1 \cap u_2$ is an initial segment of u_1 and of u_2.

Let $\mathbf{N}$ be the set of tuples $(\bar{N}, \bar{\gamma})$ satisfying

$(*)_2$ (a) $\bar{N} = \langle N_n: n < \omega\rangle$

(b) N_n is a model with vocabulary $\tau(N_n) \subseteq \mathcal{H}(\aleph_0)$

(c) $N := \cup\{N_n : n < \omega\}$ is countable with universe $\subseteq \gamma_*$

(d) $\tau(N_n) \subseteq \tau(N_{n+1})$ with $N_n \subseteq N_{n+1} \restriction \tau_n$

(e) $\bar{\gamma} = \langle \gamma_n : n < \omega \rangle$ is an increasing sequence of ordinals satisfying $\cup\{\gamma_n : n < \omega\} = \cup\{\alpha + 1 : \alpha \in \cup\{N_n : n < \omega\}\} < \gamma_*$

(f) $N_n = (N_{n+1} \restriction \tau(N_n)) \restriction \gamma_n$

(g) $\sup(N_n) < \gamma_n = \min(N_{n+1} \backslash N_n)$

(h) N_n is closed under g_0, g_1.

Recalling $\mathcal{H}_{<\aleph_1}(\gamma) = \{u : u$ a countable set such that $u \cap \mathrm{Ord} \subseteq \gamma$ and $y \in u \backslash \gamma \Rightarrow |y| < \aleph_1$. Clearly $\mathbf{N} \subseteq \mathcal{H}_{<\aleph_1}(\gamma_*)$ so as $\mu^{\aleph_0} = \mu = |\gamma_*|$, clearly $\mathbf{N}$ is well orderable so (and using parameter witnessing, $\mathrm{Ax}^4_\lambda +$ "λ is a true successor cardinal" to uniformize) let

$(*)_3$ (a) $\langle (\bar{N}_\varepsilon, \bar{\gamma}_\varepsilon) : \varepsilon < \mu \rangle$ list $\mathbf{N}$

(b) $\bar{N}_\varepsilon = \langle N_{\varepsilon,n} : n < \omega \rangle$, $\bar{\gamma}_\varepsilon = \langle \gamma_{\varepsilon,n} : n < \omega \rangle$

(c) $N_\varepsilon = N_{\varepsilon,\omega} := \cup\{N_{\varepsilon,n} : n < \omega\}$, i.e. $N_\varepsilon = \lim(\bar{N}_\varepsilon)$.

Next

$(*)_4$ for each $\varepsilon < \mu$ let $\mathbf{N}_\varepsilon$ be the set of pairs $(\bar{N}, \bar{\gamma})$ such that:

(a) $\bar{N} = \langle N_n : n < \omega \rangle$

(b) $N = \cup\{N_n : n < \omega\}$ is a $\tau(N_\varepsilon)$-model

(c) N_n is a $\tau(N_{\varepsilon,n})$-model with universe $\subseteq \lambda$

(d) there is h, an order preserving function from $N_{\varepsilon,\omega}$ onto N commuting with g^0, g^1 mapping $N_{\varepsilon,n}$ onto N_n, (i.e. $h \restriction N_{\varepsilon,n}$ is an isomorphism from $N_{\varepsilon,n}$ onto N_n) and being the identity on $N_\varepsilon \cap \mu$ and mapping $\gamma_{\varepsilon,n}$ to γ_n

$(*)_5$ for $\delta \in S$ and $\varepsilon < \mu$ let $\mathbf{N}_{\varepsilon,\delta}$ be the set of pairs $(N, \bar{\gamma}) \in \mathbf{N}_\varepsilon$ such that $\sup\{\gamma_n : n < \omega\} = \delta$ and for clause $(B)(b)(\gamma)$ for every n for some k, $N_{n+1} \backslash N_n \subseteq (\gamma^*_{\delta,k}, \gamma^*_{\delta,k+1})$

$(*)_6$ there is a partition $\bar{S} = \langle S_\varepsilon : \varepsilon < \mu \rangle$ of S to stationary sets.

[Why? By Larson–Shelah [8].]

$(*)_7$ there is $\langle \bar{\gamma}^*_\delta : \delta \in S \rangle$ such that each $\bar{\gamma}^*_\delta$ is an increasing ω-sequence with limit δ.

[Why? By Ax_4.]

$(*)_8$ there is, (in fact as in all cases in this proof, uniformly definable), a sequence $\langle (\bar{N}_\alpha, \bar{\gamma}_\alpha, u_\alpha) : \alpha < \alpha(*) \rangle$ and function $\dot{\zeta} : \alpha(*) \to S$ such that:

(a) $\dot{\zeta}$ is non-decreasing

(b) $(\bar{N}_\alpha, \bar{\gamma}_\alpha) \in \mathbf{N}_{\varepsilon,\dot{\zeta}(\alpha)}$ when $\dot{\zeta}(\alpha) \in S_\varepsilon$, moreover

(b)′ if $\varepsilon < \mu$ and $\delta \in S_\varepsilon$ then $\{(\bar{N}_\alpha, \bar{\gamma}_\alpha): \alpha < \alpha(*)$ satisfies $\dot{\zeta}(\alpha) = \delta\}$ list $\mathbf{N}_{\varepsilon,\delta}$

$(*)_9$ let $N_{\alpha,\omega} = \cup\{N_{\alpha,n}: n < \omega\}$.

[Why? By $(*)_5$, $(*)_6$ and using a well ordering of $[\lambda]^{\aleph_0}$.]

Now ignoring clause (c), clauses of (B) should be clear. Lastly, clause (c) holds by the following Theorem 3.4, in our case $\kappa = \aleph_1$. □

Theorem 3.4 *If* (*A*) *then* (*B*) *where:*

(A) (a)(α) $\lambda > \kappa$ *are regular uncountable cardinals*

(β) $\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$

(b)(α) *if* $\alpha < \lambda$ *then* $\mathrm{cf}([\lambda]^{<\kappa}, \subseteq)$ *is* $< \lambda$

(β) $\mathbf{U}_* \subseteq [\lambda]^{<\kappa}$ *is well orderable and cofinal (under* $\subseteq$*)*

(γ) $|\mathbf{U}_* \cap [\alpha]^{<\kappa}| < \lambda$ *for* $\alpha < \lambda$

(c) *M is a model with universe* λ *and vocabulary* τ, τ *not necessarily well orderable*

(d) *if* $\alpha < \kappa$ *then* $\lambda > \mathrm{hrtg}(\{N : N$ *a* τ*-model with universe* α*; may add that some order preserving mapping is an elementary embedding of N into M*$\})$

(B) *there is* $\bar{N}$*, uniformly defined from witnesses to (A) such that:*

(a) $\bar{N} = \langle N_\eta : \eta \in {}^{\omega>}\lambda\rangle$

(b) $\tau(N_\eta) = \tau$

(c) N_η *has cardinality* $< \kappa$ *and* $N_\eta \cap \kappa$ *is an ordinal* $< \kappa$

(d) N_η *is an elementary submodel of M*

(e) *if* $\nu \triangleleft \eta$ *then* N_ν *is a (proper) initial segment of* N_η

(f) *if* $n < \omega$ *and* $\eta, \nu \in {}^n\lambda$ *then there is an order preserving function from* N_η *onto* N_ν *which is an isomorphism*

(g) *if* $n < \omega$, $\eta \in {}^n\lambda$ *and* $\gamma < \lambda$ *then there is* ν *such that* $\eta \triangleleft \nu \in {}^{n+1}\lambda$ *and* $\min(N_\nu \backslash N_\eta) > \gamma$.

Remark 3.5 (1) We may consider adding: $N_\eta(\eta \in {}^\omega\lambda)$ has Σ_1-property and use: hrtg(the set of expansions of $\bar{N}^*$) $< \lambda$.
(2) The ZFC version of 3.4 is from Rubin–Shelah [9].
(3) Note that in 3.4 the vocabulary is constant whereas in 3.2 it is not. But the difference is not serious as in 3.2 the vocabulary is $\subseteq \mathcal{H}(\aleph_0)$ so there is one vocabulary which is enough to code any other.
(4) We may continue in [29, 8.2=Lg19].

Proof Now

$(*)_0$ without loss of generality $\mathbf{U}_* \subseteq [\lambda]^{<\kappa}$ is closed under countable unions and initial segments.

[Why? By (A)(a),(b), the point is that the closure retains the properties.]

$(*)_1$ let $\mathbf{N}$ be the set of $\bar{N}$ such that

(a) $\bar{N} = \langle N_n : n < \omega \rangle$

(b) (α) $N_n \prec M$ has cardinality $< \kappa$

(β) moreover, $|N_n| \in \mathbf{U}_*$

(c) $|N_n|$ is an initial segment of $|N_{n+1}|$

(d) N_n has cardinality $< \kappa$ and $N_0 \cap \kappa$ is an ordinal $< \kappa$

(e) $\tau(N_n) = \tau$

Now

$(*)_2$ $\mathbf{N}$ is well orderable

[Why? Recall $\mathbf{U}_*$ is well orderable so let $\langle u^*_\alpha : \alpha < \alpha_* \rangle$ list it. Now N_n is determined by $|N_n|$ (because $N_n \prec M$) and $|\alpha_*|^{\aleph_0}$ is well orderable so we are done.]

$(*)_3$ let $\langle \bar{N}_\alpha : \alpha < \alpha_* \rangle$ list $\mathbf{N}$ and let $\langle u^*_\alpha : \alpha < \alpha_* \rangle$ list $\mathbf{U}_*$.

[Why exists? By $(*)_2$ and $(A)(b)(\beta)$ of the theorem assumption.]

$(*)_4$ (a) we say $\bar{N}', \bar{N}'' \in \mathbf{N}$ are equivalent and write $\bar{N}' \mathscr{E} \bar{N}''$ <u>when</u> for every n, $\text{otp}(|N'_n|) = \text{otp}(N''_n)$ and the order preserving function from $|N'_n|$ onto $|N''_n|$ is an isomorphism and $N'_0 = N''_0$

(b) let $\mathbf{N}' = \{\bar{N} : \bar{N} = \langle N_\ell : \ell \le n \rangle = \bar{N}' \restriction (n+1)$ for some $\bar{N}' \in \mathbf{N}, n \in \mathbb{N}\}$

(c) we define the equivalence relation $\mathscr{E}'$ on $\mathbf{N}'$ by $\bar{N}^1 \mathscr{E}' \bar{N}^2$ if $\bar{N}^1, \bar{N}^2$ has the same length and the parallel of clause (a) holds

(d) $\mathscr{E}$ and $\mathscr{E}'$ have $\le \mu$ equivalence classes.

[Why? E.g. clause (d) by clause $(A)(d)$ of the theorem's assumption.]

$(*)_5$ E_1 is a club of λ where $E_1 := \{\delta < \lambda : \delta$ is a limit ordinal such that $M \restriction \delta \prec M$ and if $\bar{N} \in \mathbf{N}$ and $\sup(N_0) < \delta$ then there is $\bar{N}' \in \mathbf{N}$ which is $\mathscr{E}$-equivalent to $\bar{N}$ with $N'_0 = N_0$ and $\sup(\cup\{N'_n : n < \omega\}) < \delta\}$.

[Why? Think, noting that we can consider only $\{\bar{N}_\alpha : \alpha < \alpha_{**}$ and $\bar{N}_\alpha$ is not $\mathscr{E}$-equivalent to $\bar{N}_\beta$ when $\beta < \alpha\}$.]

$(*)_6$ for $\bar{N}^* \in \mathbf{N}$ and $\bar{N} \in \mathbf{N}'$ such that $N_0 = N_0^*$ we define $\mathrm{rk}(\bar{N}, \bar{N}^*) \in \mathrm{Ord} \cup \{-1, \infty\}$ by defining when $\mathrm{rk}(\bar{N}, \bar{N}^*) \geq \alpha$ by induction on the ordinal α as follows:

(a) $\underline{\alpha = 0}$: $\mathrm{rk}(\bar{N}, \bar{N}^*) \geq \alpha$ iff $\bar{N}\mathscr{E}'(\bar{N}^* \restriction \ell g(\bar{N}))$

(b) $\underline{\alpha \text{ limit}}$: $\mathrm{rk}(\bar{N}, \bar{N}^*) \geq \alpha$ iff $\beta < \alpha \Rightarrow \mathrm{rk}(\bar{N}, \bar{N}^*) \geq \beta$

(c) $\underline{\alpha = \beta + 1}$: $\mathrm{rk}(\bar{N}, \bar{N}^*) \geq \alpha$ iff for every $\gamma < \lambda$ there is $\bar{N}^+$ such that
- $\bar{N} \lhd \bar{N}^+ \in \mathbf{N}'$
- $\mathrm{rk}(\bar{N}^+, \bar{N}^*) \geq \beta$
- $\ell g(\bar{N}^+) = \ell g(\bar{N}) + 1$
- if $n = \ell g(\bar{N})$ then $\gamma < \min(N_n^+ \backslash N_{n-1})$.

Consider the statement

$\boxtimes$ for some $\bar{N}^* \in \mathbf{N}$, $\mathrm{rk}(\langle N_0^* \rangle, \bar{N}^*) = \infty$.

Why enough? Reflect. Why true? First

$\boxplus_1$ E_2 is a club of λ where

$$E_2 = \{\delta \in E_1\colon \text{ if } \bar{N}^* \in \mathbf{N}, \sup(\cup\{N_n^*\colon n < \omega\}) < \delta, \bar{N} \in \mathbf{N}', \sup(\cup\{\bar{N}_\ell\colon \ell < \ell g(\bar{N})\}) < \delta \text{ and } 0 \leq \mathrm{rk}(\bar{N}, \bar{N}^*) < \infty, \underline{\text{then}} \text{ there is no } \bar{N}' \text{ such that } \bar{N} \lhd \bar{N}' \in \mathbf{N}', \mathrm{rk}(\bar{N}', \bar{N}^*) = \mathrm{rk}(\bar{N}, \bar{N}^*) \text{ and } \ell g(\bar{N}') = \ell g(\bar{N}) + 1 \text{ such that letting } n = \ell g(\bar{N}) \text{ we have } \min(N_n' \backslash N_{n-1}) \geq \delta\}$$

[Why? Reflect.]

Now choose

$\boxplus_2$ there is an increasing sequence $\langle \delta_n\colon n < \omega \rangle$ of members of E_2 with limit $\delta \in E_2$ (in fact can do this uniformly; e.g. let δ_n be the nth member of E_2).

Lastly, choose $\langle u_{n,\ell}\colon n < \omega \rangle$ by induction on n such that

(a) $u_{n,\ell} \in \mathbf{U}_* \cap [\delta_n]^{<\kappa}$

(b) $u_{n,\ell+1}$ is u_α^* for the minimal α such that $u_\alpha^* \subseteq \delta_n$ and it includes $u_{n,\ell+1}^* \cap \delta_n$ where $u_{n,\ell+1}^*$ is the M-Skolem hull of the set
- $(\cup\{u_{m,k} \cup \{\delta_m\}\colon m < \omega, k < \ell\} \cup \{\alpha\colon \alpha \leq \sup(u_{n,\ell} \cap \kappa)\}$,

(the Skolem functions are just "the first example"; note that the $\sup(u_{n,\ell} \cap \kappa)$ may be zero).

Let $u_n = \cup\{u_{n,\ell}\colon \ell < \omega\}$, $N_n^* = M \restriction u_n$. Now we are done by $(*)_0(a)$ so $\boxplus$ is indeed true and said above is enough. $\square$

Conclusion 3.6 *Assume $\lambda = \mu^+$ is a true successor and $\mu = \mu^{\aleph_0}$. Then there is an $\aleph_1$-free Abelian group of cardinality λ such that* $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$.

Proof Straight by Theorem 3.2 as in [14] or see in Sect. 3.2. □

Theorem 3.7 (1) *We can strengthen the conclusion of* 3.2 *by replacing (B)(c) to*

(B) $(c)^+$ *if* $\langle \bar{N}'_\eta : \eta \in {}^{\omega>}\lambda \rangle$ *is as in* 3.4 *(B) (a), (c)–(f) for* $\kappa = \aleph_1$, *replacing (B)(b) by "*$\tau(N_\eta) \subseteq \mathcal{H}(\aleph_0)$, $|N'_\eta| \in [\lambda]^{\le \aleph_0}$*"* *then for stationarily many* $\delta \in S$ *for some* α *and* $\eta \in {}^{\omega}\lambda$ *we have* $\dot{\zeta}(\alpha) = \delta$ *and* $\bar{N}_\alpha = \langle N'_{\eta \restriction n} : n < \omega \rangle$.

(2) *In* 3.2, *if* $\kappa < \lambda$ *as in* 3.4 *and we can replace* $(\bar{N}, \bar{\gamma})$ *by* $\langle N_\eta : \eta \in {}^{\omega>}\kappa \rangle$.

Discussion 3.8 There is a recent BB helpful in constructing $\aleph_n$-free Abelian groups, (usually is the product of n BB's); in [24] it is proved to exist, and using it construct $\aleph_n$-free Abelian group G such that $\mathrm{Hom}(G, \mathbb{Z}) = 0$. This is continued, Göbel–Shelah [5], Göbel–Shelah–Strüngman [7] use it to deal with modules and in Göbel–Herden–Shelah [6] use it to construct $\aleph_n$-free Abelian group with endomorphism ring isomorphic to a given suitable ring. See [28] for later work.

We try to generalize a version of it but note that we cannot use BB for λ_{n+1} with $\|N_\eta\| = \lambda_n$ as in the ZFC-proof. But instead we can use 3.7! See Sect. 3.2 below and maybe more in [29].

3.2 Black Boxes with no choice

Context 3.9 We assume ZF only (for this sub-section).

Here we try to deal with ZF-proofs.

We now define a black box, BB suitable without choice (even weak ones).

Definition 3.10 (1) For a natural number $\mathbf{k}$ we say $\mathbf{x}$ is a $\mathbf{k}$-g.c.p. (general combinatorial parameter) when $\mathbf{x}$ consists of (so $Y = Y_{\mathbf{x}}$, etc.):

(a) the set Y and the sets X_m for $m < \mathbf{k}$ are pairwise disjoint
(b) $\Lambda \subseteq \{\bar{\eta} : \bar{\eta} = \langle \eta_m : m < \mathbf{k} \rangle$ and $\eta_m \in {}^{\omega}(X_m)$ for $m < \mathbf{k}\}$
(c) $|Y| \le |X_0|$ and moreover
$(c)^+$ $f_0 : Y \to X_0$ is one to one
(d) if $m \in (0, \mathbf{k})$ then $|X_m| \ge {}^{(X_{<m})}Y$ where $X_{<m} = \prod_{\ell<m} {}^{\omega}(X_\ell)$, moreover
$(d)^+$ $f_m : \{t : t$ a function from $X_{<m}$ to $Y\} \to X_m$ is one to one.

(1A) We say a $\mathbf{k}$-g.c.p. $\mathbf{x}$ is standard when $f_{\mathbf{x},m}$ is the identity for every $m < \mathbf{k}$ and we fix $y_* \in Y$.
(2) For $\mathbf{x}$ a $\mathbf{k}$-g.c.p. (as above) we say $\mathbf{w}$ is a $\mathbf{x}$-BB, i.e. an $\mathbf{x}$-black box when $\mathbf{w}$ consists of ($\mathbf{x} = \mathbf{x}_{\mathbf{w}}$ and):

(a) $\Lambda = \Lambda_{\mathbf{w}} \subseteq \Lambda_{\mathbf{x}}$; (if $\Lambda = \Lambda_{\mathbf{x}}$ we may omit it)

(b) (α) $h: \Lambda \to {}^{(\mathbf{k}+1)\times\omega}Y$, so we write $h(\bar\eta) = \langle h_{m,n}(\bar\eta): m \le \mathbf{k}, n < \omega\rangle$ so $h_{m,n}$ is a function from Λ into Y

(β) for every $g: \Omega \to Y$, see below for some $\bar\eta \in \Lambda$ we have $(\forall m < \mathbf{k})(\forall n)(h_{m,n}(\bar\eta) = g(\bar\eta \upharpoonright (m,n))$

(c) notation:

(α) $\bar\nu = \bar\eta \upharpoonright (m,n)$ when $\bar\nu = \langle \nu_\ell: \ell < \mathbf{k}\rangle$ and ν_ℓ is n_ℓ if $\ell < \mathbf{k} \wedge \ell \neq m$ and is $\nu_\ell = \eta_\ell \restriction n$ if $\ell = m$

(β) $\Omega_m = \{\bar\eta \upharpoonright (m,n): n < \omega$ and $\eta \in \Lambda_{\mathbf{w}}\}$ so $\Omega_m \subseteq \{\bar\eta: \bar\eta = \langle \eta_\ell: \ell < \mathbf{k}\rangle$ and for $\ell < \mathbf{k}$, $[\ell \neq m \Rightarrow \eta_\ell \in {}^{\omega}(X_\ell)]$ and $[\ell = m \Rightarrow \eta_\ell \in {}^{\omega>}X_\ell]\}$

(γ) $\Omega = \bigcup_{m<\mathbf{k}} \Omega_m$.

(3) Above $\mathbf{k}_{\mathbf{x}} = \mathbf{k}(\mathbf{x}) = \mathbf{k}$, $\Omega_{\mathbf{w}} = \Omega$, $\Omega_{\mathbf{w},m} = \Omega_m$, etc.
(4) In Claim 3.13 below we call $\bar z$ simple when it has the form $\langle a_{\bar\eta,n} z: \bar\eta \in \Lambda_{\mathbf{x}}, n < \omega\rangle$ where $a_{\bar\eta,n} \in \mathbb{Z}$.

Claim 3.11 (1) *For every Y, $y_* \in Y$ and $\mathbf{k}$ there is, moreover we can define a standard $\mathbf{k}$-g.c.p. $\mathbf{x}_{\mathbf{k}}$ (with witnesses $f_{\mathbf{x},m}$ = identity).*
(2) *For every such $\mathbf{x}_{\mathbf{k}}$ we can define an $\mathbf{x}$-BB $\mathbf{w} = \mathbf{w}_{\mathbf{x}_{\mathbf{k}}}$.*

Remark 3.12 Why we do not choose $\Lambda_{\mathbf{w}} = \Lambda_{\mathbf{x}}$? We can have $\Lambda_{\mathbf{w}} = \Lambda_{\mathbf{x}}$ using a constant value $\in Y$ for the additional cases, so for definability choose a fixed $y_* \in Y$ in 3.10(1), see 3.10(2).

Proof (1) By induction $m < \mathbf{k}$ we define (X_m, f_m) by:

- $X_m = Y$ if $m = 0$
- $X_m = \{t: t$ is a function from $X_{<m} = \prod_{\ell<m}(X_\ell)$ to $Y\}$ if $m > 0$
- $f_m = \mathrm{id}_{X_m}$ (so is one to one onto).

Now check.
(2) Case 1: $\mathbf{k} = 1$

Let $\Lambda_{\mathbf{w}} = {}^{\omega}(\mathrm{Rang}(f_0))$, so $\Omega_{\mathbf{w}} = \Omega_{\mathbf{w},0} = {}^{\omega>}(\mathrm{Rang}(f_0))$, $h_{\mathbf{w},n}$ or pedantically $h_{\mathbf{w},0,n}$ is a function from $\Omega_{\mathbf{w},0} = \Lambda_{\mathbf{w}} = \{\langle\eta\rangle: \eta \in {}^{\omega}(\mathrm{Rang}(f_0))\}$ and $\Omega_{\mathbf{w}} = \{\langle\eta\rangle: \eta \in {}^{\omega>}(\mathrm{Rang}(f_0))\}$ and $\langle\eta\rangle \in \Lambda_{\mathbf{w}} \Rightarrow \langle\eta\rangle \upharpoonright (0,n) = \langle\eta\restriction n\rangle$.

Now for $n < \omega$ we let $h_{\mathbf{w},0,n}: \Lambda_{\mathbf{w}} \to Y$ be defined by

- $h_{\mathbf{w},0,n}(\langle\eta\rangle) = \eta(n) \in Y$ for $\eta \in \Lambda_{\mathbf{w}}$.

Obviously clauses (a),(b)(α) from 3.10(2) holds but what about clause (b)(β) of 3.10?

Now for any $g: \Omega_{\mathbf{w}} \to Y$ we choose $y_n \in Y$ by induction on n as follows: $y_n = g(\langle f_0(y_\ell): \ell < n\rangle) = g(\langle y_\ell: \ell < n\rangle)$. So $\eta := \langle y_\ell: \ell < \omega\rangle \in {}^{\omega}(\mathrm{Rang}(f_0))$ is as required.

Case 2: $\mathbf{k} > 1$

Let $\Lambda_{\mathbf{w}} = \{\bar\eta: \bar\eta = \langle \eta_m : m < \mathbf{k}\rangle$ and $\eta_m \in {}^{\omega}(\mathrm{Rang}(f_m))$ for $m < \mathbf{k}\}$ hence $\Omega_m = \Omega_{\mathbf{w},m}$ and $\Omega_* = \Omega_{\mathbf{w}}$ are well defined.

We now define $h_{m,n} = h_{\mathbf{w},m,n}$ for $m < \mathbf{k}$, $n < \omega$

$(*)_1$ for $\bar\eta \in \Lambda_m = \{\bar\eta \upharpoonleft (m,n) : \bar\eta \in \Lambda_{\mathbf{w}}$ and $n < \omega\}$ we let $h_{m,n}(\bar\eta) = \big(f_m^{-1}(\eta_m(n))\big)(\bar\eta\restriction m)$ if $m > 0$ and $h_{m,n}(\bar\eta) = f_m^{-1}(\eta_m(n))$ if $m = 0$.

Why well defined and $\in Y$? Clearly if $m = 0$ then $h_{m,n}(\bar\eta) = f_m^{-1}(\eta_m(n)) \in Y$ as $\eta_m \in {}^{\omega}(X_0) = {}^{\omega}Y$ and if $m > 0$ then $\eta_m(n) \in X_m$ hence $f_m^{-1}(\eta_m(n)) \in {}^{(X_{<m})}Y$ so is a function from $X_{<m} := \prod_{\ell<m} {}^{\omega}X_\ell$ into Y so $\bar\eta\restriction m \in X_{<m}$ hence $\big(f_m^{-1}(\eta_m(n))\big)(\bar\eta\restriction m) \in Y$. So clause $(b)(\alpha)$ of Definition 3.10 is satisfied. What about clause $(b)(\beta)$ of Definition 3.10(2)? So let a function $g : \Omega \to Y$ be given and we shall prove that there is $\bar\eta \in \Lambda_{\mathbf{w}}$ as required, in fact define it. Toward this we choose $\eta_m \in {}^{\omega}(\mathrm{Rang}(f_m)) \subseteq {}^{\omega}(X_m)$ by downward induction on m, and we shall let $\eta_m = \langle f_m(t_{m,n}) : n < \omega\rangle$, where we choose $t_{m,n} \in \mathrm{Dom}(f_m)$ by induction on $n < \omega$ as follows:

$(*)_2$ if $m > 0$ then $t_{m,n}$ is the following function from $\{\bar\eta\restriction m : \bar\eta \in \Lambda_{\mathbf{w}}\} = X_{<m} = \prod_{\ell<\omega} {}^{\omega}(X_\ell)$ to Y: if $\bar\nu = \langle \nu_\ell : \ell < m\rangle \in \mathrm{Dom}(t_{m,n})$ then $t_{m,n}(\bar\nu)$ is $g(\bar\rho) \in Y$ where $\bar\rho = \langle \rho_\ell : \ell < \mathbf{k}\rangle$ is defined by:
- if $\ell > m$ then $\rho_\ell = \eta_\ell$, is well defined by the induction hypothesis on m
- if $\ell = m$ then $\rho_\ell = \langle f_m(t_{m,0}), \ldots, f_m(t_{m,n-1})\rangle$, well defined by the induction hypothesis on n
- if $\ell < m$ then $\rho_\ell = \nu_\ell$, given

$(*)_3$ if $m = 0$ then $t_{m,n} = g(\bar\rho)$ where $\bar\rho$ is chosen as above except that there is no $\bar\nu$.

Now check. □

Claim 3.13 *Let* $\mathbf{x}$ *be a* $\mathbf{k}$*-g.c.p. see* 3.10(1) *and* $\mathbf{w}$ *an* $\mathbf{x}$*-BB, see* 3.10(2) *and* $\Lambda = \Lambda_{\mathbf{w}}, \Omega = \Omega_{\mathbf{w}}$*, etc. Then* $G \in \mathscr{G}_{\mathbf{x}} \Rightarrow G_{\mathbf{x},0} \subseteq G \subseteq_{\text{purely}} G_{\mathbf{x},1}$ *where* $\subseteq_{\text{purely}}$ *is from* 3.16(0) *and* $G \in \mathscr{G}_{\mathbf{x}}$ *<u>iff</u> some* $\bar z$, $G = G_{\mathbf{x},\bar z}$*, which means:*

(a) $G_0 = G_{\mathbf{x},0} = \oplus\{\mathbb{Z}x_\rho : \rho \in \Omega\} \oplus \mathbb{Z}z$

(b) $G_1 = G_{\mathbf{x},1} = \oplus\{\mathbb{Q}x_\rho : \rho \in \Omega\} \oplus \mathbb{Q}z \oplus \{\mathbb{Q}y_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\}$

(c) $\bar z = \langle z_{\bar\eta,n} : \eta \in \Omega_{\mathbf{w}}\rangle$ *is a sequence of members of* $G_{\mathbf{x},1}$

(d) *for* $\bar\eta \in \Lambda$ *we define* $y_{\bar\eta,n} = y_{\bar z,\bar\eta,n}$ *by induction on* n:
- $y_{\bar\eta,0} = y_{\bar\eta}$,
- $n!y_{\bar\eta,n+1} = y_{\bar\eta,n} - \sum_{m<\mathbf{k}} x_{\bar\eta\upharpoonleft(m,n+1)} - z_{\bar\eta,n}$

(e) *G is the (Abelian) subgroup of G_1 generated by* $\{x_{\bar\eta} : \bar\eta \in \Omega\} \cup \{y_{\bar\eta,m} : \bar\eta \in \Lambda, n < \omega\} \cup \{z\}$.

Proof Straightforward. □

Claim 3.14 *Let* $\mathbf{k}, \mathbf{x}, \mathbf{w}, \bar z$ *be as in* 3.10, 3.10(2), 3.13.
(1) $G_{\mathbf{x},\bar z}$ *is almost* $\aleph_{k(\mathbf{x})}$*-free* (*see below Definitions* 3.16 *and* 3.15) *provided that* $\bar z$ *has the form* $\langle a_{\bar\eta,n}z : \bar\eta \in \Lambda_{\mathbf{x}}, n < \omega\rangle$ *where* $a_{\bar\eta,n} \in \mathbb{Z}$ (*or less as in* [24]).
(2) *In Claim* 3.13 *above,* $G_{\mathbf{x},\bar z}$ *is definable* (*in ZF!*) *from* $(\mathbf{x}, \bar z)$.
(3) *For* $\mathbf{x}$ *a* $\mathbf{k}$*-g.c.p. and* $\mathbf{w}$ *an* $\mathbf{x}$*-BB such that* $Z \subseteq Y_{\mathbf{x}}$ *we can define* $\bar z = \bar z_{\mathbf{w}}$ *such that* $G_{\mathbf{x},\bar z}$ (*is well defined and*) *satisfies* $h \in \mathrm{Hom}(G_{\mathbf{x},\bar z}, \mathbb{Z}) \Rightarrow h(z) = 0$.

(4) *For* $\mathbf{x}$ *a* $\mathbf{k}$*-g.c.p. and* $\mathbf{w}$ *an* $\mathbf{x}$*-BB we can define an* $\aleph_{\mathbf{k}(\mathbf{x})}$*-free Abelian group* G *such that* $\mathrm{Hom}(G, \mathbb{Z}) = \{0\}$.

Discussion 3.15 (1) Assume $H \subseteq G = G_{\mathbf{x},\bar{z}}$ is a subgroup of cardinality $< \aleph_{\mathbf{k}(\mathbf{x})}$. For each $t \in G$ let Y_t be the minimal $Y \subseteq Y_{\mathbf{x}} = \{x_\rho : \rho \in \Omega_{\mathbf{x}}\} \cup \{z\} \cup \{y_{\bar{\eta}} : \eta \in \Lambda_{\mathbf{x}}\}$ such that $t \in \oplus\{\mathbb{Q}x : x \in Y\}$. If $\Omega_{\mathbf{x}} \cup \Lambda_{\mathbf{x}}$ is linearly ordered then $\cup\{Y_t : t \in H\}$ has cardinality $< \aleph_{\mathbf{k}(\mathbf{x})}$ but in general this explains the "weakly" or "almost" in 3.14. However, it may occur that this holds for the "wrong" reason say $\aleph_0 \not\leq |A|$ in Definition 3.16(2). But the proof of 3.11, 3.14 gives A-s with "many" such subsets. Note that if $\mathbf{k} \geq 2$, then in 3.14 (1), $G_{\mathbf{x},\bar{z}}$ is strongly $\aleph_1$-free.
(2) For proving 3.14(1) note that in the definition of $\mathscr{G}_{\mathbf{x}}$ in [24] there is a use of choice: dividing the stationary set $S_m \subseteq \lambda_m$ to λ_m pairwise disjoint sets or just the choice of $\bar{z} = \langle z_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{w}} \rangle$. However, we can just "glue together" copies of the G constructed above; i.e. start with G and for every non-zero pure $z \in G$, add G_z of $h_z : G \to G_z$ identify $x_{<>}$ with z, etc.

Definition 3.16 Let G be a torsion free Abelian group (the torsion free means $G \models$ "$nx = 0$", $n \in \mathbb{Z}$, $x \in G$ implies $n = 0 \vee x = 0_{\mathbb{Z}}$).
(0) $H \subseteq G$ if H is a subgroup; $H \subseteq_{\text{purely}} G$, H a pure subgroup of G, means $H \subseteq G$ and $n \in \mathbb{Z}\backslash\{0\}$, $nx \in G$, $nx \in H \Rightarrow x \in H$.
(1) We say G is a weakly κ-free <u>when</u>: there is a set A such that the pair (G, A) is κ-free, see part (2).
(2) We say (G, A) is κ-free when: $A \subseteq G$ and $\mathrm{PC}_G(A) = G$ and if $B \subseteq A$ has cardinality $< \kappa$ then $\mathrm{PC}_{G(B)} \subseteq G$ is a free Abelian group recalling $\mathrm{PC}_G(A) =$ the minimal pure subgroup of G which includes A.
(3) We say G is almost κ-free <u>when</u> there is a set A such that the pair (G, A) is almost κ-free, see part (4).
(4) The pair (G, A) is almost κ-free when: (G, A) is κ-free and A is independent in G (i.e. $\sum_{\ell<n} a_\ell x_\ell = 0 \Rightarrow \bigwedge_{\ell<n} a_\ell = 0$ when $x_0, \ldots, x_n \in A$ without repetition).

Proof <u>Proof of 3.14</u>: (1) Let $A = \{x_\rho : \rho \in \Omega_{\mathbf{x}}\} \cup \{z\} \cup \{y_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{x}}\}$. It is easy to check that A is independent in G [see 3.16(4)] and $\mathrm{PC}_G(A) = G$ so for any $t \in G$ there is a unique finite $Y_t \subseteq A$ such that $t \in \mathrm{PC}_G(Y_t)$, Y_t of minimal cardinality.

Now if $B \subseteq A$ has cardinality $< \aleph_{\mathbf{k}(\mathbf{x})}$, then also $Y_B := \{\rho : x_\rho \in B\} \cup \{\bar{\eta} \restriction (m, n) : y_{\bar{\eta}} \in B, m < \mathbf{k}(\mathbf{x}) \text{ and } n < \omega\}$ has cardinality $< \aleph_{\mathbf{k}(\mathbf{x})}$.

For some $Y \subseteq \mathrm{Ord}$ in $\mathbf{L}[Y]$ there is a $\mathbf{k}$-c.p. $\mathbf{x}'_1$ and $\bar{z}_1$ such that $G_{\mathbf{x}_1,\bar{z}_1} \in \mathbf{L}[Y]$ is isomorphic (in $\mathbf{V}$) to $\mathrm{PC}_G(B)$. So by [24] we are done.
(2) Should be clear.
(3) We shall define uniformly (in ZF) from $\mathbf{k}$-g.c.p. $\mathbf{x}$ and $\mathbf{w}$ an $\mathbf{x}$-BB a sequence $\bar{z}$ such that the Abelian group $G = G_{\mathbf{x},z_{\mathbf{w}}}$ satisfies $h \in \mathrm{Hom}(G, \mathbb{Z}) \Rightarrow h(z) = 0$.

For each $\bar{\eta} \in \Lambda$ let $\bar{a} = \langle a_{\mathbf{w},\bar{\eta},n} : n < \omega \rangle \in {}^\omega\mathbb{Z}$ be defined by:

$(*)$ $a_{\mathbf{w},\bar{\eta},n}$ is

- $\sum_{m<\mathbf{k}} h_{m,n+1}(\bar{\eta})$ when $\{h_{m,n}(\bar{\eta}) : m < \mathbf{k}\} \subseteq \mathbb{Z}$
- 0 when otherwise.

We shall choose $b_{\mathbf{w},\bar\eta,n} \in \mathbb{Z}$ for $n < \omega$ such that

(∗) if $a_{\mathbf{w},\bar\eta,0} \neq 0$ then there are no $t_n \in \mathbb{Z}$ for $n < \omega$ such that for every n (eq_n) $n!t_{n+1} = t_n - a_{\mathbf{w},\bar\eta,n+1} = b_{\mathbf{w},\bar\eta,n} \cdot a_{\mathbf{w}_{\eta,0}}$.

Why then can we choose? We choose $b_{\mathbf{w},\bar\eta,n} \in \mathbb{N} \subseteq \mathbb{Z}$ as minimal such that we cannot find $t_0, \dots, t_n \in \mathbb{Z}$ such that $t_0 = \{-n, -n-1, \dots, -1, 0, 1, m \dots, n\}$ and for every $m < n+$ we have $\mathbb{Z} \models$ "$n!t_{m+1} = t_n - a_{\mathbf{w},\bar\eta,m+1} - b_{\mathbf{w},\bar\eta,m} - a_{\mathbf{w}_{\eta,0}}$.

Now we define

(∗) $\bar z = \bar z_{\mathbf{w}} = \langle b_{\mathbf{w},\bar\eta,n} \cdot z : \bar\eta \in \Lambda_{\mathbf{x}}, n < \omega\rangle$.

So

(∗) (a) $G_{\mathbf{x},\bar z}$ is well defined

(b) if $g \in H(G_{\mathbf{x},\bar z}, \mathbb{Z})$ then $h(z) = 0_{\mathbb{Z}}$.

[Why? Clause (a) is obvious. For clause (b) if g is a counterexample by the choice of $\mathbf{w}$ there is $\bar\eta \in \Lambda_{\mathbf{w}}$ such that $m < \mathbf{k} \wedge n < \omega \Rightarrow g(x_{\bar\eta\upharpoonleft(m,n)}) = h_{m,n}(\bar\eta)$ that is $n < \omega \Rightarrow \sum_{m<\mathbf{k}} g(x_{\bar\eta\upharpoonleft(m,n+1)} = a_{\mathbf{w},\bar\eta,n}$. Now use the choice of $\langle b_{\mathbf{w},\bar\eta,n} : n < \omega\rangle$ to get a contradiction.]

(4) We derive an example from $G_{\mathbf{w}}$ from part (3).

Let $\Omega' = \Omega'_{\mathbf{x}} = \{\rho : \rho$ a finite sequence of members of $\Omega\}$ and for $\rho \in \Omega'$ let

(∗) (a) $X_\rho = X_{\mathbf{x},\rho} = \{x_{\rho,\bar\eta} : \bar\eta \in \Omega_{\mathbf{w}}\}$

(b) $Y_\rho = Y_{\mathbf{x},\rho} = \{y_{\rho,\bar\eta} : \bar\eta \in \Lambda_{\mathbf{w}}\}$

(∗) (a) $G'_0 = G'_{\mathbf{x},0} = G'_{0,0} \otimes G'_{0,1}$ where

(b) $G'_{0,0} = G'_{\mathbf{x},0,0} = \oplus\{\mathbb{Z}_{\rho,\bar\eta} : \rho \in \Omega'_{\mathbf{w}}, \bar\eta \in \Omega_{\mathbf{w}}\}$

(c) $G'_{0,1} = G'_{\mathbf{x},0,1} = \mathbb{Z}z$

(∗) (a) $G'_1 = G'_{\mathbf{x},1} \oplus G_{\mathbf{w},2,1} \oplus G_{\mathbf{w},2,1} \oplus G_{\mathbf{w},1,2}$ where

(b) $G'_{1,0} = G'_{\mathbf{w},1,0} = \oplus\{\mathbb{Q}x_{\rho,\bar\eta} : \rho \in \Omega'_{\mathbf{x}}$ and $\bar\eta \in \Omega_{\mathbf{x}}\} \supseteq G'_{0,1}$

(c) $G'_{1,1} = G'_{\mathbf{w},1,1} = \mathbb{Q}z \supseteq G'_{0,1}$

(d) $G'_{1,2} = G'_{\mathbf{w},1,2} = \oplus\{\mathbb{Q}y_{\rho,\bar\eta} : \rho \in \Omega'_{\mathbf{x}}$ and $\bar\eta \in \Lambda_{\mathbf{x}}\}$.

Let

(∗) (a) z_ρ be z if $\rho = \langle\rangle$ and $x_{\rho\restriction\ell,\rho(\ell)}$ if $\beta \in \Omega'_{\mathbf{x}} \backslash \{<>\}$

(b) let $y_{\rho,\bar\eta,0} = y_{\rho,\bar\eta}$

(c) for $\rho \in \Omega'_{\mathbf{w}}$ and $\bar\eta \in \Lambda_{\mathbf{x}}$ we define $y_{\rho,\bar\eta,n}$ by induction on $n > 0$

- $y_{\rho,\bar\eta,n+1} = (y_{\rho,\bar\eta,n} + \sum_{m\leq k} x_{\rho\bar\eta\upharpoonleft(m,n)} + \bar a_{\bar\eta,n} z_{\bar\eta}$ where $\langle a_{\bar\eta,n} : n < \omega\rangle \in {}^{\omega}\mathbb{Z}$ was defined above using $h(\bar\eta)$

(∗) (a) for every $t \in G_1'$ let $\mathrm{supp}(x)$ be the minimal subset X_t of $X_{\mathbf{s}} = \{x_{\rho,\bar{\eta}}\colon \rho \in \Omega_{\mathbf{x}}', \bar{\eta} \in \Omega_{\mathbf{x}}\} \cup \{y_{\rho,\bar{\eta}}\colon \rho \in \Omega_{\mathbf{x}}'$ and $\bar{\eta} \in \Lambda_{\mathbf{w}}\}$ such that: $t \in \Sigma\{\mathbb{Q}x : x \in X_*\}$; used in part (2)

(∗) for $\rho \in \Omega'$ we define an embedding h_ρ from $G_{\mathbf{w}}$ into G_1' by (see ⊞4 below):

(a) $h_\rho(z) = z_\rho$

(b) $h_\rho(x_{\bar{\eta}}) = x_{\rho,\bar{\eta}}$ for $\bar{\eta} \in \Omega_{\mathbf{w}}$

(c) $h_\rho(y_{\bar{\eta},n}) = y_{\rho,\bar{\eta},n}$.

Now

⊞1 let $G_{\mathbf{w}}'$ be the subgroup of $G_{\mathbf{w},1}'$ generated by $\{X_{\rho,\bar{\eta}}\colon \rho \in \Omega_{\mathbf{w}}'$ and $\bar{\eta} \in \Omega_{\mathbf{w}}\} \cup \{z\} \cup \{y_{\rho,\bar{\eta},n}\colon \rho \in \Omega_{\mathbf{w}}', \bar{\eta} \in \Lambda_{\mathbf{w}}$ and $n < \omega\}$

⊞2 $G_{\mathbf{w},0}' \subseteq G_{\mathbf{x}}'$ is dense in the $\mathbb{Z}$-adic topology.

[Why? Just look at each $y_{\rho,\bar{\eta},n}$.]

⊞3 for $\rho \in \Omega_{\mathbf{x}}'$

(a) h_ρ is a well defined homomorphism

(b) h_ρ is indeed an embedding

(c) $\mathrm{Rang}(h_\rho) \subseteq G_{\mathbf{x}}'$

(d) $\mathrm{Rang}(h_\rho)$ is a pure subgroup of $G_{\mathbf{x}}'$

(e) $h_{<>}$ is ?

[Why? For clause (a) note the definition of $y_{\rho,\bar{\eta},n}$, also the other clauses are obvious.]

⊞4 $\mathrm{Hom}(G_{\mathbf{w}}', \mathbb{Z}) = 0$.

[Why? Let $g \in \mathrm{Hom}(G_{\mathbf{w}}', \mathbb{Z})$. For each $\rho \in \Omega_{\mathbf{x}}'$, the function $g \circ h_\rho$ is a homomorphism from $G_{\mathbf{x}}$ into $\mathbb{Z}$ hence by the previous claim 3.14, $(G \circ h_\rho)(z) = 0$. This means that $0 = (g \circ h_\rho)(z) = g(h_\rho(z)) = g(z_\rho)$ hence $g(z) = 0$, using $\rho = \langle\rangle$ and $g(x_{\rho,\bar{\eta}}) = 0$ for $\rho \in \Omega_{\mathbf{x}}', \bar{\eta} \in \Omega_{\mathbf{x}}$ using $z_{\rho\hat{\ }\langle\bar{z}\rangle} = X_{\rho,\bar{\eta}}$. By the choice of $G_{\mathbf{w},0}'$ this implies $g \restriction G_{\mathbf{x},0}'$ is zero and by ⊞3 this implies $g \restriction G_{\mathbf{w}}'$ is zero, as promised.] □

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