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ON UNIVERSAL LOCALLY FINITE GROUPS

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ABSTRACT
We deal with the question of existence of a universal object in the category of universal locally finite groups; the answer is negative for many uncountable cardinalities; for example, for $2^{\aleph_0}$, and assuming G.C.H. for every cardinal whose cofinality is $>\aleph_0$. However, if $\lambda > \kappa$ when κ is strongly compact and $\mathrm{cf}\,\lambda = \aleph_0$, then there exists a universal locally finite group of cardinality λ. The idea is to use the failure of the amalgamation property in a strong sense. We shall also prove the failure of the amalgamation property for universal locally finite groups by transferring the kind of failure of the amalgamation property from LF into ULF.

1. Introduction

Universal locally finite groups were introduced by P. Hall [3], but we shall refer to the presentation of this subject in the book of O. Kegel and B. Wehrfritz [4]. The three basic facts on universal locally finite groups (u.l.f.) are:

Fact A (Theorem 6.4 in [4]). Up to isomorphism there is a unique countable u.l.f. group.

Fact B (Theorem 6.1 in [4]). Every u.l.f. group is universal with respect to countable locally finite groups (i.e., given G which is u.l.f., then for every locally finite countable group H there exists a group monomorphism $f: H \to G$).

Fact C (Theorem 6.5 in [4]). Every locally finite group can be extended to a u.l.f. group.

In this context the following questions were asked in [4]:

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QUESTION VI.1. Are any two u.l.f. groups of the same uncountable cardinality isomorphic?

QUESTION VI.3.. Does every u.l.f uncountable group G contain an isomorphic copy of every locally finite group H such that $|H| \leqq |G|$?

Or, in other words: Are the generalizations of Facts A and B true for every uncountable cardinality?

A. Macintyre and S. Shelah [5] answered Question VI.1 by proving that for every uncountable cardinal λ there are 2^λ pairwise non-isomorphic u.l.f. groups each of cardinality λ.

They also answered Question VI.3 by constructing a locally finite group H of cardinality $\aleph_1$ such that for every $\kappa \geqq \aleph_1$ there exists a u.l.f. group G of power κ such that H is not embeddable into G. Let us introduce here some definitions and notation.

DEFINITION 1. (1) $\mathrm{LF} = \{G : G$ is a group and every subgroup generated by a finite subset is finite$\}$.

(2) $\mathrm{ULF} = \{G \in \mathrm{LF} : G$ satisfies (a) and (b) below$\}$.

(a) For every finite group H, there exists a monomorphism $h : H \to G$.

(b) For every H_1, H_2 finite subgroups of G, *if* $f : H_1 \cong H_2$ *then* there exists an element $g \in G$ such that $x \to x^g$ induces f.

DEFINITION 2. Let $\mathfrak{C}$ be a category whose objects are sets (i.e. we can speak about the cardinality of an object).

(1) Let μ be a cardinal number, then $\mathfrak{C}_\mu = \{G \in \mathfrak{C} : |G| = \mu\}$ and $\mathfrak{C}_{\leq\mu} = \{G \in \mathfrak{C} : |G| \leqq \mu\}$.

(2) $G \in \mathfrak{C}$ is *μ-universal* if for every $H \in \mathfrak{C}_\mu$ there exists a monomorphism (of $\mathfrak{C}$) $f : H \to G$.

(3) Let λ be an infinite cardinal number. $\mathfrak{C}$ has the λ *amalgamation property* (λ-A.P.) if for every $G_l \in \mathfrak{C}_\lambda$ for $l = 0, 1, 2$ such that there are $\mathfrak{C}$-monomorphisms $f_l : G_0 \to G_l$ for $l = 1, 2$, there exists $G \in \mathfrak{C}_\lambda$ and monomorphisms $g_l : G_l \to G$ for $l = 1, 2$ such that $(f_1 \circ g_1) \restriction G_0 = (f_2 \circ g_2) \restriction G_0$, or, in other words, the diagram

$$\begin{array}{ccc} G_1 & & \\ {\scriptstyle f_1}\uparrow & & \\ G_0 & \xrightarrow[f_2]{} & G_2 \end{array}$$

can be completed to a commutative diagram in $\mathfrak{C}$.

By Fact B every countable u.l.f. group is $\aleph_0$-universal, so in the category $\mathrm{LF}_{\aleph_0}$ a universal object exists, and moreover every u.l.f. group is universal. This can be understood as a generalization of the fact that S_n (= the general symmetric group of n-elements) is universal for the category of finite groups of cardinality $\leqq n$.

So you may ask: "Is the parallel generalization for uncountable cardinalities true"? For example, does there exist a universal object in $LF_{\aleph_1}$? Remember that Macintyre and Shelah proved a weaker result: they proved that contrary to the situation for countable groups, not every uncountable u.l.f. group is a universal object for LF, but still, by this result, it is possible for example that there exists $G \in ULF_{\aleph_1}$ and all the $2^{\aleph_1}$ non-isomorphic u.l.f. groups isomorphic to subgroups of G. We shall prove here that this situation is impossible.

THEOREM 3. *For every uncountable cardinal* λ *which satisfies*

(1) $\lambda = \lambda^{\aleph_0}$

or

(2) *there exists a cardinal* μ *such that* $\lambda < \mu \leqq \lambda^{\aleph_0}$ *and* $2^\mu < 2^\lambda$, *there is no universal object in* ULF_λ.

COROLLARY 4. *There is no universal object in* $ULF_{2^{\aleph_0}}$, *and if* $2^{\aleph_0} < 2^{\aleph_1}$ *then also there is no universal object in* $ULF_{\aleph_1}$.

PROOF OF THE COROLLARY. The first half follows from Theorem 3 using (1), since $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\aleph_0} = 2^{\aleph_0}$.

The second half follows by taking $\mu = \aleph_0$ and $\lambda = \aleph_1$. Now since $2^{\aleph_0} < 2^{\aleph_1}$ and $\aleph_1 < 2^{\aleph_0}$ (otherwise we can apply part (1)), $\aleph_0 < 2^{\aleph_0} = \aleph_0^{\aleph_0}$, so we can apply Theorem 3.

COROLLARY. *Assuming* G.C.H., *for every uncountable cardinal* λ *with uncountable cofinality, there is no universal object in* ULF_λ.

PROOF. Use Theorem 3(1). $\lambda = \lambda^{\aleph_0}$ follows from G.C.H. and the hypothesis on λ.

So under G.C.H. we have

OPEN PROBLEM A. Let λ be uncountable cardinal of cofinality ω. Does there exist a universal object in ULF_λ? In particular, what is the answer for $\lambda = \beth_\omega$?

Under the impression of the last corollary the reader may guess that the answer to Problem A may be positive.

THEOREM. *If* κ *is a compact cardinal* (= *the logic* $L_{\kappa,\kappa}$ is compact), and $\lambda > \kappa$, *strong limit of cofinality* ω, then *there exists a universal object in* ULF_λ.

The last theorem will be proved in section 5 where a more general theorem is presented. In the context of Corollary 4 it is interesting to mention

OPEN PROBLEM B. Is the following statement consistent with ZFC: "there exists a universal object in $ULF_{\aleph_1}$"?

Clearly if such a model exists, then by Corollary 4, $2^{\aleph_0} = 2^{\aleph_1}$ must hold. In section 4 we shall answer a weaker question.

In section 2 the proof of Theorem 3 is divided into a sequence of lemmas and claims; we shall get more information during the proof. Also we shall reprove the Macintyre–Shelah result: Assuming $2^{\aleph_0} < 2^{\aleph_1}$ there are $2^{\aleph_1}$ non-isomorphic groups in $ULF_{\aleph_1}$.

The last theorem in section 2 is complementary to Corollary 4 in the sense that it presents a model of ZFC where $2^{\aleph_0} = 2^{\aleph_1}$, but there is no universal object in $ULF_{\aleph_1}$; so $2^{\aleph_0} < 2^{\aleph_1}$ is not a necessary assumption to conclude the non-existence of a universal object in $ULF_{\aleph_1}$.

The "reason" for the non-existence of a universal object in ULF_λ is the failure of a strong form of the κ-A.P. in LF_κ for every $\kappa < \lambda$. Section 3 is dedicated to the construction of a natural mapping from LF into ULF which shows that all counterexamples to the amalgamation properties in LF can be transferred into ULF.

In section 4 we prove a general theorem for categories of sets (not necessarily l.f. groups or groups at all) assuming $2^{\aleph_0} < 2^{\aleph_1}$ which together with the result of section 3 gives a new proof to the fact that there is no universal object in $ULF_{\aleph_1}$ and that there are $2^{\aleph_1}$ non-isomorphic u.l.f. groups of cardinality $\aleph_1$. The advantage over the earlier proofs in this paper is that now we can answer a version of Problem B for the general theorem from section 4.

For a general classification theory of classes of models via their amalgamation properties, the reader is referred to [7], [9], [10] (start by reading [10]). Really, the general theorems in section 4 follow directly from some theorems in [10]. We included the work here because we could present it in a way accessible to the algebraist without any knowledge of logic. In this paper we shall not use that kind of classification theory. The ideas of section 2 are motivated by some of the methods of [8] chapter VIII, but we don't assume the reader is familiar with it.

NOTATION. (1) Notice that $\langle a, b, c, \cdots \rangle$ will be used in two different ways, once as a sequence, and again as a subgroup generated by $\langle a, b, c, \cdots \rangle$.

(2) A will stand for an uncountable set.

(3) Let $S(A) = \{f : A \to A \mid f \text{ is a bijection}\}$, i.e. $S(A)$ is the set of permutations of the set A.

(4) λ, μ, κ, χ will stand for infinite cardinal numbers. Remember that a cardinal is an ordinal. α, β, γ, i, j stand for ordinals, and please recall that the ordinal α is also the set of smaller ordinals. n, k, l stand for natural numbers. ω is the first infinite ordinal and, according to our notation, also the set of natural numbers. η, ν stand for sequences and $\eta[i]$ is the i-th element of η; $l(\eta)$ is the length of η, i.e. the domain of η when η is viewed as a function, so $\eta = \{\eta[i] : i < l(\eta)\}$. ${}^{\omega}\lambda$ is the set of ω-sequences (sequences of length ω) of elements from λ (i.e. ordinals smaller than λ). $\eta \lhd \nu$ means: the sequence ω is an initial segment of the sequence ν. All groups shall be infinite unless stated otherwise.

2. Non-existence of universal groups

Definition 5. (1) A permutation $f \in \mathbf{S}(A)$ is a *finite permutation* if f moves at most finitely many elements of A.

(2) $f \in \mathbf{S}(A)$ is *almost finite* if $\{x \in A : f(x) \neq x\}$ is at most countable; we denote this set by $\mathbf{Dom}\, f$. In addition we require that there exists a family $\{W_f^n : n < \omega\}$ of finite subsets of $\mathbf{Dom}\, f$ such that $\mathbf{Dom}\, f = \bigcup_{n<\omega} W_f^n$, and $(\forall n < \omega)\ [\,|W_f^n| = |W_f^0| \wedge f \restriction W_f^n \in \mathbf{S}(W_f^n)] \wedge [n \neq k \Rightarrow W_f^n \cap W_f^k = \emptyset]$.

(3) Let $f, g \subseteq \mathbf{S}(A)$, f and g are *almost disjoint* if $|\mathbf{Dom}\, f \cap \mathbf{Dom}\, g| < \aleph_0$.

(4) Let $F \subseteq \mathbf{S}(A)$ be a family of almost finite permutations which is also pairwise almost disjoint (i.e., if $f, g \in F$ then $f \neq g \Rightarrow f$ and g are almost disjoint). $\mathrm{Sg}(A, F)$ is the subgroup of $\mathbf{S}(A)$ generated by: (a) the elements of F, and (b) the finite permutations of A.

(5) Let α be a finite natural number or ω; and let $\eta = \{\eta[i] : i < \alpha\}$ be a sequence without repetitions of elements of A (i.e. η is an injection of α into A). Denote by $f_\eta \in \mathbf{S}(A)$ the permutation which satisfies: (i) $\mathbf{Dom}\, f_\eta = \{\eta[i] : i < \alpha\}$, and (ii) for every $i < \alpha$, f_η interchanges $\eta[2i]$ by $\eta[2i+1]$.

(6) Given $\eta = \langle \eta[i] : i < \omega \rangle$, a sequence of elements without repetitions from A, denote $\eta^- = \langle \eta[1+i] : i < \omega \rangle$.

Remarks. (1) Please be careful: notice the difference between $\mathbf{Dom}\, f$ and the usual notation $\mathrm{Dom}\, f$ (= domain of the function f).

(2) $\mathrm{Sg}(A)$ is just $\mathrm{Sg}(A, \emptyset)$, i.e., all the finite permutations of the set A.

Theorem 6. *Let A and F be as in Definition* 5(4). *Then we have*:

(1) $\mathrm{Sg}(A) \in \mathrm{LF}$.

(2) $\mathrm{Sg}(A, F) \in \mathrm{LF}$.

(3) *Let η be an ω-sequence of distinct elements, and denote $\lambda = |A|$, then the triple* $\mathrm{Sg}(A)$, $\mathrm{Sg}(A, \{f_\eta\})$, $\mathrm{Sg}(A, \{f_{\eta^-}\})$ *exemplify the failure of the* λ-A.P. *in* LF.

PROOF. (1) Is well known.

(2) By (1), it suffices to prove that given $h_1, \cdots, h_n \in \mathrm{Sg}(A)$, $f_1, \cdots, f_m \in F$ then $K = \langle h_1, \cdots, h_n, f_1, \cdots, f_m \rangle$ is a finite subgroup of $\mathrm{Sg}(A, F)$. By the choice of the h's and f's there exists a finite set $W \subseteq A$ such that for all k satisfying $1 \leqq k \leqq n$, $\mathbf{Dom}\, h_k \subseteq W$, and for every $1 \leqq k$, $l \leqq m$, $k \neq l \Rightarrow \mathbf{Dom}\, f_k \cap \mathbf{Dom}\, f_l \subseteq W$. Notice that any finite $W' \subseteq A$ which includes W has the properties of W, so we may increase W. Denote $A_0 = W$, and for $1 \leqq l \leqq m$, $A_l = \mathbf{Dom}\, f_l - W$, let $A_{m+1} = A - \bigcup_{1 \leq l \leq m} \mathbf{Dom}\, f_l$.

Clearly $\{A_k : k \leqq m + 1\}$ form a partition of A. Now we shall define an equivalence relation E on K with finitely many equivalence classes and we shall prove that if $g_1, g_2 \in K$ then $g_1 E g_2 \Rightarrow g_1 = g_2$.

Given $g_1, g_2 \in K$ for $l \leqq m + 1$ define $g_1 E_l g_2$ iff $g_1 \restriction A_l = g_2 \restriction A_l$. Finally let $g_1 E g_2 \overset{df}{\Leftrightarrow} (\forall l \leqq m + 1)[g_1 E_l g_2]$. Clearly E is an equivalence relation on K. It suffices to prove that for every $l \leqq m + 1$ there are finitely many equivalence classes. It is easy, since in A_l (for $l \geqq 1$) there is only one generator $f_l \restriction A_l$ (the restriction of the others to A_l is the identity); the order of $f_l \restriction A_l$ is the divisor of $(|W_f^n|)!$ So we are done since A_0 is finite. Now $g_1 E g_2 \Rightarrow g_1 = g_2$, since every $g \in K$ satisfies $(\forall l \leqq m + 1)$ $[g \restriction A_l \in \mathbf{S}(A_l)]$ (remember that we may increase the set W).

(3) We leave the reader to check that f_η is almost finite; combining f_η, f_η , and $f_{\{\eta[0], \eta[1]\}}$ we get $\aleph_0$ distinct elements, hence $\{f_\eta, f_\eta , f_{\{\eta[0], \eta[1]\}}\}$ generates an infinite group. Hence the groups cannot be amalgamated by locally finite groups.

CLAIM 7. *For every infinite cardinal number* λ, LF_λ *has a universal object iff* ULF_λ *has a universal object.*

PROOF. Let G be a universal object of LF_λ. By Hall's theorem (Theorem 6.5 in [4]; readers who are not familiar with it and don't want to check the reference can read a generalization of it in the next section — Theorem 18) there exists $\bar{G} \in \mathrm{ULF}_\lambda$ containing G as a subgroup. It is easy to verify that $\bar{G}$ is a universal object of ULF_λ. Now assume that ULF_λ has a universal object G; clearly this object belongs to LF_λ (since $\mathrm{ULF} \subseteq \mathrm{LF}$). Again it is easy to check that the same object G is universal for LF_λ. Let $H \in \mathrm{LF}_\lambda$ be given. By the above-mentioned theorem of Hall there exists $\bar{H} \in \mathrm{ULF}_\lambda$ containing H as a subgroup. Since G is universal for ULF_λ $\bar{H}$ is embeddable into G and this embedding induces an embedding of H into G (the restriction of the embedding to H).

The next theorem proves the second half of Theorem 3:

THEOREM 8. *If λ satisfies: there exists a cardinal μ such that $\lambda < \mu \leqq \lambda^{\aleph_0}$ and $2^\lambda < 2^\mu$, then there is no universal object in* $\mathrm{LF}_{\leq\mu}$.

PROOF. Choose $\{\eta_i : i < \mu\} \subseteq {}^{\omega}\lambda$ such that $i < j < \mu \Rightarrow$ Range$(\eta_i) \cap$ Range$(\eta_j)| < \aleph_0$ and each η_i is a sequence without repetitions. Now for $S \subseteq \mu$ and $i < \mu$ define

$$g_i^S = \begin{cases} f_{\eta_i} & \text{if } i \in S, \\ f_{\bar{\eta}_i} & \text{if } i \notin S. \end{cases}$$

Clearly $\{g_i^S : i < \mu\}$ satisfies the demands from F in Definition 5(4), so we can define $G_S = \mathrm{Sg}(\lambda, \{g_i^S : i < \mu\})$. Assume by contradiction that $G^* \in \mathrm{LF}_{\leq\mu}$ is a universal object. For every $S \subseteq \mu$ there should be an embedding $h_S : G_S \to G^*$. There are 2^μ distinct subsets S of μ. The number of different embeddings of $\mathrm{Sg}(\lambda)$ into G^* is at most $|G^*|^{|\mathrm{Sg}(\lambda)|} = \mu^\lambda \leqq (2^\lambda)^\lambda = 2^\lambda$. Combining the last two sentences we have $S_1, S_2 \subseteq \mu$ such that $S_1 \neq S_2$ and $h_{S_1} \restriction \mathrm{Sg}(\lambda) = h_{S_2} \restriction \mathrm{Sg}(\lambda)$. This is a contradiction: We may assume that there exists $i \in S_1 - S_2$ so

$$\begin{array}{ccc} \mathrm{Sg}(\lambda, \{f_{\eta_i}\}) & \xrightarrow[h_{S_1}]{} & G^* \\ {\scriptstyle \mathrm{inc.}}\uparrow & & \uparrow {\scriptstyle h_{S_2}} \\ \mathrm{Sg}(\lambda) & \xrightarrow[\mathrm{inc.}]{} & \mathrm{Sg}(\lambda, \{f_{\bar{\eta}_i}\}) \end{array}$$

contradicts Theorem 5(3).

CONCLUSION 9. Assuming $2^{\aleph_0} < 2^{\aleph_1}$ neither $\mathrm{LF}_{\aleph_1}$ nor $\mathrm{ULF}_{\aleph_1}$ has universal members.

PROOF. Substitute $\lambda = \aleph_0$ and $\mu = \aleph_1$ in Theorem 8 and use Claim 6.

Now using the method of proof from Theorem 8 we can reprove a result of Macintyre and Shelah [5]. The advantage of our proof is its simplicity, but it has a disadvantage of relying on a weak continuum hypothesis; the result in [5] is proved in ZFC alone.

THEOREM 10. *Assume λ satisfies: there exists a cardinal number μ such that $\lambda < \mu \leqq \lambda^{\aleph_0}$ and $2^\lambda < 2^\mu$; then in ULF_μ there are 2^μ non-isomorphic groups.*

PROOF. For every $S \subseteq \mu$ construct $G_S \in \mathrm{LF}_\mu$ exactly as in the proof of Theorem 8. Assume by contradiction that the number of isomorphism types of elements of ULF_μ is $\chi < 2^\mu$, since $2^\lambda < 2^\mu$; choose $\kappa = \mathrm{Max}\{\chi, 2^\lambda\}$, we know that $\kappa < 2^\mu$. For every $S \subseteq \mu$ let $\bar{G}_S \in \mathrm{ULF}_\mu$ containing G_S (exists by Hall's theorem — Theorem 6.4 [4] or next section, Theorem 18). Since $\kappa \geqq \chi$ there exists $\{G_i : i < \kappa\} \subseteq \mathrm{ULF}_\mu$ such that for every $S \subseteq \mu$ there exists $i(S) < \kappa$ such that $\bar{G}_S \cong G_{i(S)}$. By the pigeonhole principle for regular cardinals (κ^+ is regular and $\kappa^+ \leqq 2^\mu$) there exists $\{S_i \subseteq \mu : i < \kappa^+\}$ of cardinality κ^+ such that

$$i < \kappa^+ \Rightarrow f_i : \bar{G}_{S_i} \cong \bar{G}_{S_0} \quad \text{and} \quad S_i \neq S_j \quad \text{for } i \neq j < \kappa^+.$$

Now the number of mappings of $\mathrm{Sg}(\lambda)$ into $\bar{G}_{S_0}$ is $\mu^\lambda \leqq (2^\lambda)^\lambda = 2^\lambda$; since $2^\lambda < \kappa^+$ there must be $i \neq j < \kappa^+$ such that $f_i \restriction \mathrm{Sg}(\lambda) = f_j \restriction \mathrm{Sg}(\lambda)$ and $f_i : \bar{G}_{S_i} \to \bar{G}_{S_0}$, $f_j : \bar{G}_{S_j} \to \bar{G}_{S_0}$ and we obtain a contradiction exactly as at the end of the proof of Theorem 8.

COROLLARY 11. *If $2^{\aleph_0} < 2^{\aleph_1}$ then there are $2^{\aleph_1}$ isomorphism types of groups from* $\mathrm{ULF}_{\aleph_1}$.

PROOF. Easy using Theorem 10.

THEOREM 12. *If λ satisfies $\lambda = \lambda^{\aleph_0}$ then there is no universal object in* $\mathrm{LF}_{\leq\lambda}$.

PROOF. Let $G \in \mathrm{LF}_\lambda$ be an arbitrary group and we shall construct a group in LF_λ which cannot be embeddable into G.

CLAIM 13. *Let $A = {}^{\omega>}\lambda$. For every $\eta \in {}^{\omega}\lambda$ there exists a function h_η whose domain is $K_\eta = \{g \in \mathrm{Sg}(A) : \mathbf{Dom}\, g \subseteq \{\eta \restriction n : n < \omega\}\}$ such that*

(+) *For every $h : \mathrm{Sg}(A) \to G$ there exists $\eta \in {}^{\omega}\lambda$ such that $h_\eta \subseteq h$.*

PROOF OF CLAIM 13. Define for every $\eta \in {}^{\omega>}\lambda$ a function $\bar{h}_\eta$ such that:

(a) $\bar{h}_\eta$ is an embedding of the permutation group $\{\eta \restriction k : k < l(\eta)\}$ into G.

(b) $\nu \lessdot \eta \Rightarrow \bar{h}_\nu \subseteq \bar{h}_\eta$.

(c) *If* h is an embedding of $\mathrm{Sg}(\{\eta \restriction n : n \leqq l(\eta)\})$ which extends $\bar{h}_\eta$ *then* there exists an $\alpha < \lambda$ such that $h = \bar{h}_{\eta\hat{}\langle\alpha\rangle}$.

The definition of $\{\bar{h}_\eta : \eta \in {}^{\omega>}\lambda\}$ is easy by induction on the length of η. Now for every branch $\eta \in {}^{\omega}\lambda$ define $h_\eta = \bigcup_{n<\omega} \bar{h}_{\eta|n}$. This is an embedding of $\mathrm{Sg}(\{\eta \restriction n : n < \omega\})$ into G.

Given $h : \mathrm{Sg}(A) \to G$ we shall construct now an η as required in (+). This is achieved by defining η by approximations $\{\eta_n \in {}^{n}\lambda : n < \omega\}$:

For $n = 0$ take the empty sequence.

For $n = k+1$ by property (c) in the assignments of $\bar{h}_\eta$ there exists α such that $\bar{h}_{\eta_k\hat{}\langle\alpha\rangle} \subseteq h$ so take $\eta_n = \eta_{k+1} = \eta_k\hat{}\langle\alpha\rangle$. By (b) $\eta = \bigcup_{k<\omega} \eta_k$ is as we wanted.

REMARK. Notice the similarity between the proof of Claim 13 and [8] VIII Lemma 2.5 (exactly as the game GI from Definition 2.1).

BACK TO THE PROOF OF CLAIM 13. For every increasing $\eta \in {}^{\omega}\lambda$ define two almost disjoint permutations of $A : g^0_\eta$, g^1_η by taking $g^0_\eta = f_{\langle \eta \restriction l : l < \omega\rangle}$ and $g^1_\eta = f_{\langle \eta^- \restriction l : l < \omega\rangle}$; note that $\langle \eta \restriction l : l < \omega\rangle$ is a sequence of elements of $A\,(= {}^{\omega>}\lambda)$. By Theorem 5(3), $\mathrm{Sg}(A, \{g^0_\eta\})$ and $\mathrm{Sg}(A, \{g^1_\eta\})$ are extensions of $\mathrm{Sg}(A)$ which cannot be amalgamated in LF. Moreover, it is impossible to extend h_η (from Claim 13) to two embeddings of $\mathrm{Sg}(K_\eta, \{g^l_\eta\})$ (for $l = 1, 0$) into G. Hence there exists an

$l_\eta \in \{0,1\}$ (depending on η) such that h_η cannot be extended to an embedding of $\mathrm{Sg}(K_\eta, \{g_\eta^{l_\eta}\})$ into G. Define, for every $\eta \in {}^\omega\lambda$, $g_\eta = g_\eta^{l_\eta}$. Now let $G^* = \mathrm{Sg}({}^{\omega>}\lambda, \{g_\eta : \eta \in {}^\omega\lambda\})$; by Theorem 5(2) $G^* \in \mathrm{LF}$. Clearly $|G^*| = \lambda^{\aleph_0} = \lambda$. Now prove that there is no embedding of G^* into G. If $h : G^* \to G$ is a monomorphism, by Claim 13 there exists $\eta \in {}^\omega\lambda$ such that $h \supseteq h_\eta$. Now $h \restriction \mathrm{Sg}(K_\eta, \{g_\eta\})$ contradicts the choice of g_η as $g_\eta^{l_\eta}$.

Now we shall prove a theorem which implies that the non-existence of a universal object in $\mathrm{ULF}_{\aleph_1}$ is consistent with $2^{\aleph_0} = 2^{\aleph_1}$. The proof follows exactly [8] VIII Theorem 1.9.

THEOREM 14. *Suppose there is a universal object in* $\mathrm{LF}_{\leq\lambda}$ *and* $\aleph_0 < \lambda \leqq 2^{\aleph_0}$. *Then there is a family F of subsets of λ, each of cardinality λ, $A \neq B \in F \Rightarrow A \cap B$ is finite, and the family has cardinality* $2^{\aleph_0}$.

PROOF. By Giorgetta and Shelah [2] Proposition 5.1 there is a locally finite group G, and $\{x_\eta : \eta \in {}^\omega 2\} \subseteq G$ such that:

⊛ The power of the subgroup generated by $\{x_\eta, x_\nu\}$ ($\eta \neq \nu \in {}^\omega 2$) is $g[h(\eta,\nu)]$, where h is the largest common initial segment, and the sets $A_n = \{g(\rho) : \rho \in {}^n 2\}$ are pairwise disjoint.

Now for any infinite $B \subseteq \omega$ denote by $C[B]$ a subset of $\{\eta \in {}^\omega 2 : (\forall n < \omega)[n \notin B \to \eta[n] = 0]\}$ of cardinality λ, and let $G[B] = \langle x_\eta : \eta \in C[B]\rangle$. Fix a family $\{B_i : i < 2^{\aleph_0}\}$ of infinite subsets of ω which are almost disjoint. Let, for each $i < 2^{\aleph_0}$, $G_i = G[B_i]$.

To prove our theorem assume that there exists a universal member H of $\mathrm{LF}_{\leq\lambda}$. Without loss of generality we may assume that the set of elements of H is λ. By universality of H, for every $i < 2^{\aleph_0}$ there exists a monomorphism $h_i : G_i \to H$. Let $D_i = \{h_i(x_\eta) : \eta \in C[B_i]\}$. Clearly $\{D_i : i < 2^{\aleph_0}\}$ is a family of subsets of λ; we want to prove that it is as F in the statement of this theorem. Let $i \neq j < 2^{\aleph_0}$ and assume, for the sake of contradiction, $|D_i \cap D_j| \geqq \aleph_0$, i.e. there are $\{\eta_l : l < \omega\} \subseteq C[B_i]$, $\{\nu_l : l < \omega\} \subseteq C[B_j]$ such that $h_i(x_{\eta_l}) = h_j(x_{\nu_l})$ for all $l < \omega$. Let $\{a_l < \lambda : l < \omega\}$ be such that $a_l = h_i(x_{\eta_l}) = h_j(x_{\nu_l})$ for $l < \omega$. By the choice of the family of B_i's there exists an $n = n(i,j)$ so that $B_i \cap B_j \subseteq n$, and so w.l.o.g. $\eta_l \restriction n = \eta^*$, $\nu_l \restriction n = \nu^*$ for every l. Now evaluate the power of $\langle a_1, a_2\rangle$ (= the subgroup of H generated by a_1, a_2):

$$|\langle a_1, a_2\rangle_H| = |\langle h_i(x_{\eta_1}), h_i(x_{\eta_2})\rangle_H| = |\langle x_{\eta_1}, x_{\eta_2}\rangle_{G_i}|$$

$$= |\langle x_{\eta_1}, x_{\eta_2}\rangle_G| = g(h(\eta_1,\eta_2)) \in \bigcup\{A_k : k \in B_i, k \geqq n\}.$$

Similarly $|\langle a_2, a_1\rangle_H| \in \bigcup\{A_k : k \in B_j,\ k \geqq n\}$. But this contradicts the assumption that the A_k's are pairwise disjoint and $B_i \cap B_j \subseteq n$.

CONCLUSION 15. There exists a model of $\mathrm{ZFC} + 2^{\aleph_0} = 2^{\aleph_1}$ in which there is no universal object in $\mathrm{LF}_{\aleph_1}$.

PROOF. Baumgartner [1] constructed a model of $\mathrm{ZFC} + 2^{\aleph_0} = 2^{\aleph_1}$ where there is no family of subsets of ω_1 as F in the statement of Theorem 14.

3. **The amalgamation property fails in** ULF

We shall prove the statement in the title in two different ways. The first is intended for logicians who don't want to learn any group theory but are ready to believe in Fact C. The first way has another advantage (from our point of view), namely, to convince the algebraist that forcing is an important technique not only to obtain independence results. The second proof is for the algebraist who refuses to study any logic and, philosophically, is less complicated (i.e., we do not change the universe).

By Theorem 12 the following two lemmas will imply what we want:

LEMMA 16. *Assume* $2^\lambda = \lambda^+$. *If* ULF *satisfies the amalgamation property then there exists a universal object in* ULF_{λ^+}.

LEMMA 17. *If* $V \models \mathrm{ZFC} + 2^\lambda > \lambda^+$ *then there exists a forcing notion* P *such that* $V^P \models 2^\lambda = \lambda^+$ *and does not add new subsets of* λ.

PROOF OF LEMMA 16. This was originally proved by B. Jonsson. Using $2^\lambda = \lambda^+$ and the λ-A.P. construct a model homogeneous group (which is universal) and use the Joint Mapping Property (J.M.P.) — possible by taking a direct product and applying Fact C. For more details see §2 in [10].

PROOF OF LEMMA 17. Let $P = \{f : (\exists \alpha < \lambda^+)\,[f : \alpha \to {}^\lambda 2]\}$. Clearly P collapses 2^λ to λ^+ and is λ^+-complete, so does not add new subsets of λ.

Now to the algebraic argument. First quote from [4]:

DEFINITION 18. (1) Let G be a group.

$$S(G) = \{\sigma \in \mathrm{S}(G) \colon \exists H_\sigma \text{ a finite subgroup of } G \text{ such that } |(\forall x \in G)\ [(xH_\sigma)^\sigma = xH_\sigma]\}.$$

(2) The mapping $\rho_{G} : G \to S(G)$ stands for the regular representation of G in $S(G)$ by right multiplication.

FACT D (Lemma 6.3 in [4]). (1) If $G \in \mathrm{LF}$ then $S(G) \in \mathrm{LF}_{|G|}$.

(2) If K, K^* are finite isomorphic subgroups of G then K^ρ and $K^{*\rho}$ are conjugate in $S(G)$ (we take $\rho = \rho_G$).

It is easy to verify that it suffices to prove

THEOREM 19. *Given G, $H \in \mathrm{LF}$ and a monomorphism $f : G \to H$ then there exists $F(G)$, $F(H) \in \mathrm{ULF}$ and a monomorphism $F(f) : F(G) \to G(H)$ such that the following diagram commutes:*

$$\begin{array}{ccc} F(G) & \xrightarrow{F(f)} & F(H) \\ {\scriptstyle \text{inc.}}\uparrow & & \uparrow{\scriptstyle \text{inc.}} \\ G & \xrightarrow[f]{} & H \end{array}$$

and we have that $|F(G)| = |G|$.

PROOF. We shall construct the three required objects as a direct limit of ω-sequences of triples as follows:

For every $n < \omega$ define V_n, U_n, $f_n : V_n \to U_n$ and $\rho_{V_n} : V_n \to V_{n+1}$, $\rho_{U_n} : U_n \to U_{n+1}$ such that the following diagram commutes:

$$\begin{array}{ccc} V_{n+1} & \xrightarrow{f_{n+1}} & U_{n+1} \\ {\scriptstyle \rho_{V_n}}\uparrow & & \uparrow{\scriptstyle \rho_{U_n}} \\ V_n & \xrightarrow[f_n]{} & U_n \end{array}$$

When all mappings are monomorphisms and K, K^* are finite isomorphic subgroups of U_n (V_n) then their images under regular right translation are conjugate in U_{n+1} (V_{n+1}).

If we take $V_0 = G$, $U_0 = H$, $f_0 = f$ and succeed in defining the above, then $F(G) = \varinjlim_{n<\omega} V_n$, $F(H) = \varinjlim_{n<\omega} U_n$ and $F(f) = \varinjlim_{n<\omega} f_n$ will be as required in the statement of the theorem. So we want to define the above triples by induction on $n < \omega$. For a successor stage we use the following lemma which was proved by Simon Thomas and we thank him for letting us use it here.

LEMMA 20. *Let V, U and f be as above, then there exists a monomorphism $h : S(V) \to S(U)$ such that the following diagram commutes:*

$$\begin{array}{ccc} S(V) & \xrightarrow{h} & S(U) \\ {\scriptstyle \rho_V}\uparrow & & \uparrow{\scriptstyle \rho_U} \\ V & \xrightarrow[f]{} & U \end{array}$$

PROOF. Let I be a fixed left coset representative of $f[V]$ in U ($f[V]$ is a subgroup of U since f is an embedding). So $U = \bigcup\{xf(u) : x \in I, v \in V\}$. Now in order to define $g = h(\sigma)$ for each permutation $\sigma \in S(V)$ we want to define a permutation g of $S(U)$ by $xf(v) \mapsto xf(v^{\sigma})$. If is easy to check that h is as required.

Now back to the proof of Theorem 19: Let $\{K_\alpha : \alpha < |V_n|\}$ be an enumeration of all finite subgroups of V_n and define a function $g : |V_n| \times |V_n| \to S(V_n)$ by

$$g(\alpha,\beta) = \begin{cases} \text{if } K_\alpha \cong K_\beta \text{ then } g(\alpha,\beta) \text{ conjugates } K_\alpha,\ K_\beta \text{ in } S(V_n), \\ \text{identity of } S(V_n), \text{ otherwise.} \end{cases}$$

Let $V_{n+1} = \{V_n^{\rho_{V_n}} \cup g[|V_n| \times |V_n|]\}$. Similarly define U_{n+1} and f_{n+1} to be a restriction of the function $h : S(V_n) \to S(U_n)$ from Lemma 20. By repeating the proofs it is easy to prove

THEOREM 21. LF *and* ULF *have the same non-amalgamation numbers.*

In the context of Theorem 19 we can ask

OPEN PROBLEM C. Construct a functor $F : \mathrm{LF} \to \mathrm{ULF}$.

4. Nice categories

The category of countable universal locally finite groups has the following properties:

(1) It has a unique (up to isomorphism) countable object. This is Fact A in section 1.

(2) Every object has a proper extension. Multiply it by $\mathrm{Sg}(\omega)$ and apply Hall's theorem, or Theorem 18.

(3) The amalgamation property fails (section 3).

(4) Closed under direct limits of length $\leqq \omega_1$.

The last four properties are the motivation for the following:

DEFINITION 22. Let $\mathfrak{C}$ be a category of sets. $\mathfrak{C}$ is a λ-*nice category* provided:

(1) $\mathfrak{C}_\lambda$ has a unique object up to isomorphism.

(2) $(\forall M \in \mathfrak{C}_\lambda)$ $(\exists N \in \mathfrak{C}_\lambda)$ and a monomorphism $f : M \to N$ such that $f(M) \subsetneqq N$.

(3) The λ-A.P. fails in $\mathfrak{C}$.

(4) $\mathfrak{C}$ is closed under direct limits of length $\leqq \lambda^+$.

So the above observations imply that $\mathrm{ULF}_{\aleph_0}$ is an $\aleph_0$-nice category.

Now we have two nice properties of nice categories which follow from [10] section 3 (based on the method of [9]).

THEOREM 23. *Assume $2^\lambda < 2^{\lambda^+}$. If $\mathfrak{C}$ is a λ-nice category then $\mathfrak{C}$ does not have a universal object of cardinality λ^+.*

THEOREM 24. *Assume $2^\lambda < 2^{\lambda^+}$. If $\mathfrak{C}$ is a λ-nice category then $\mathfrak{C}$ has 2^{λ^+} non-isomorphic objects of cardinality λ^+.*

So for $\lambda = \aleph_0$ the last two theorems provide an alternative proof to Corollary 4 (second part) and Corollary 11, respectively.

This is interesting in view of the open problem from the first section, since Saharon Shelah proved

THEOREM 25 (Shelah [10]). *Assume $2^{\aleph_0} > \aleph_1$ and* MA. *There exists an $\aleph_0$-nice category whose objects are models of a finite similarity type so that there exists a unique object up to isomorphism in $\mathfrak{C}_{\aleph_1}$.*

So Theorem 25 shows that both the statements of Theorems 23 and 24 are independent of ZFC.

5. A universal object may exist

We shall prove a more general theorem than that stated in the Introduction (not only for locally finite groups).

Let us assume in this section that κ is a compact cardinal and $\lambda > \kappa$ strong limit of cofinality $\aleph_0$. Let L be a fixed similarity type of cardinality $\leqq \kappa$.

THEOREM 26. *Let $\mathfrak{C}$ be the class of all L-models of a fixed $L_{\kappa,\kappa}$ sentence. There exists $\{M_i : i < 2^\kappa\} \subseteq \mathfrak{C}_\lambda$ such that for every $M \in \mathfrak{C}_\lambda$ there exists $i < 2^\kappa$ and an $L_{\kappa,\omega}$ embedding $f_i : M \to M_i$.*

PROOF. Let $\{T_i : i < 2^\kappa\}$ be the enumeration of all $L_{\kappa,\kappa}$ theories. It is enough to construct M_i ($i < 2^\kappa$) such that every $M \in \{N \in \mathfrak{C}_\lambda : N \models T_i\}$ is $L_{\kappa,\kappa}$ embeddable into M_i, so we are given a fixed T ($L_{\kappa,\kappa}$ theory) and want to construct M_T as above.

Fix an increasing sequence of cardinals $\{\mu_n : n < \omega\}$ satisfying $\mu_0 = \kappa$, $\mu_{n+1} = \mu_{n+1}^{\mu_n}$ converging to λ (exists by the choice of λ). Now to construct M_T, define $\{M_n : n < \omega\}$ $(\forall n < \omega)\|M_n\| = \mu_{n+1}$, $M_n <_{L_{\kappa,\kappa}} M_{n+1}$ and finally take $M_T = \bigcup_{n<\omega} M_n$. By induction on $n < \omega$, $n = 0$, let $\mathbf{D}$ be the union of the $L_{\kappa,\kappa}$ complete diagrams of all models of cardinality μ_0 by taking for distinct models disjoint sets of constants. $\mathbf{D}$ has cardinality 2^κ, but since distinct diagrams are disjoint the assumption that κ is compact is sufficient to conclude existence of a model M_0 of cardinality μ_1 so that all models from $\{M \in \mathfrak{C}_\kappa : M \models T\}$ are embeddable into M_1, $n > 0$. Take M_{n+1} so that it realizes all types in μ_{n+1} variables and parameters from M_n which are κ-satisfiable in M_n (every subset of cardinality $\chi < \kappa$ is realized); M_{n+1} exists by compactness of the cardinal κ. Now we want to show that M_T is universal for all models from $\mathfrak{C}_\lambda$ whose $L_{\kappa,\kappa}$ theory is T. Let $N \in \{N \in \mathfrak{C}_\lambda : N \models T\}$; by the Skolem Lowenheim theorem choose $\{N_n \in \mathfrak{C}_{\mu_n} : n < \omega\}$ such that $N_n <_{L_{\kappa,\kappa}} N$ and $N = \bigcup_{n<\omega} N_n$. Using the definition of $\{M_n : n < \omega\}$ it is easy to construct by induction on $n < \omega$ $L_{\kappa,\kappa}$-elementary embeddings $h_n : N_n \to M_n$ such that $h_n \subseteq h_{n+1}$. Now $h = \bigcup_{n<\omega} h_n$ is the required $L_{\kappa,\omega}$-elementary embedding.

Corollary 27. *Let $\mathfrak{C}$ be as in the previous theorem, and assume in addition that $\mathfrak{C}_\lambda$ has the* J.M.P. *for 2^κ (i.e. given $\{M_i : i < 2^\kappa\} \subseteq \mathfrak{C}_\lambda$ there exists $M \in \mathfrak{C}_\lambda$ such that $(\forall i < 2^\kappa)$ M_i is $L_{\kappa,\kappa}$ embeddable into M). Then $\mathfrak{C}_\lambda$ has a universal object.*

Proof. Apply the J.M.P. for 2^κ on the result of Theorem 26.

Corollary 28. ULF$_\lambda$ *has a universal object.*

Proof. ULF clearly satisfies the assumption of Theorem 26 since it is definable by an $L_{\omega_1,\omega}$ sentence. It has the J.M.P. for 2^κ by taking the direct product (with finite support) and applying Hall's theorem (Fact C).

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