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UNIFORMIZATION AND SKOLEM FUNCTIONS IN THE CLASS OF TREES

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Abstract. The monadic second-order theory of trees allows quantification over elements and over arbitrary subsets. We classify the class of trees with respect to the question: does a tree T have definable Skolem functions (by a monadic formula with parameters)? This continues [6] where the question was asked only with respect to choice functions. A natural subclass is defined and proved to be the class of trees with definable Skolem functions. Along the way we investigate the spectrum of definable well orderings of well ordered chains.

§1. Introduction. The Uniformization Problem. The *uniformization problem* for a theory $\mathscr{T}$ in a language $\mathscr{L}$ can be formulated as follows: Suppose $\mathscr{T} \vdash (\forall \bar{Y})(\exists \bar{X})$ $\psi(\bar{X}, \bar{Y})$ where ψ is an $\mathscr{L}$-formula and $\bar{X}$, $\bar{Y}$ are tuples of variables. Is there another $\mathscr{L}$-formula ψ^* that uniformizes ψ i.e., such that

$$\mathscr{T} \vdash (\forall \bar{Y})(\forall \bar{X})[\psi^*(\bar{X}, \bar{Y}) \Rightarrow \psi(\bar{X}, \bar{Y})] \quad \text{and} \quad \mathscr{T} \vdash (\forall \bar{Y})(\exists! \bar{X})\psi^*(\bar{X}, \bar{Y})\ ?$$

Here $\exists!$ means "there is a unique".

The monadic second-order logic is the fragment of the full second-order logic that allows quantification over elements and over monadic (unary) predicates only. The monadic version of a first-order language $\mathscr{L}$ can be described as the augmentation of $\mathscr{L}$ by a list of quantifiable set variables and by new atomic formulas $t \in X$ where t is a first order term and X is a set variable. The monadic theory of a structure $\mathscr{M}$ is the theory of $\mathscr{M}$ in the extended language where the set variables range over all subsets of $|\mathscr{M}|$ and $\in$ is the membership relation. (A formal definition is given in Section 3.)

In [6] we dealt with the following question: for what trees $(T, \lhd)$ it is the case that there is a finite sequence $\bar{P}$ of subsets of T and a formula $\varphi(x, X, \bar{Z})$ in the monadic language of trees such that

$$T \models \varphi(a, A, \bar{P}) \;\Rightarrow\; a \in A,\; T \models (\forall X)(\exists y)[X \neq \emptyset \Rightarrow \varphi(y, X, \bar{P})] \text{ and}$$
$$T \models \varphi(a, A, \bar{P}) \wedge \varphi(b, A, \bar{P}) \;\Rightarrow\; a = b?$$

When the answer for that is positive we will say that T has a monadically definable choice function and that φ defines a choice function from non-empty subsets of T. On the other hand, a negative answer to the choice function problem for T implies a

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negative answer to the uniformization problem for the monadic theory of T, with the formula $\phi(x, Y)$ that says "if Y is not empty then $x \in Y$" being a counter-example.

Gurevich and Shelah ([4]) gave a negative answer to the choice function problem for the monadic theory of the tree $({}^{\omega>}2, \lhd)$ and hence to the uniformization problem that was first raised by Rabin ([7]). In [6] the question was what trees do have a definable choice function (by a monadic formula with parameters). The class of trees (which includes the class of linear orders) was split into two natural subclasses, the class of wild trees and the class of tame trees and the following dichotomy result was proved:

THEOREM. *Let T be a tree. If T is wild then there is no definable choice function on T (by a monadic formula with parameters). If T is tame then there is even a definable well ordering of the elements of T by a monadic formula (with parameters) $\varphi(x, y, \bar{P})$.*

Here we are concerned with giving a complete answer to the uniformization problem for the monadic theory of trees. The class of tame trees is split into the class of strictly tame trees that have 'only' definable choice functions and the class of very tame trees for which we have definable Skolem functions for each monadic formula, hence the uniformization property (i.e., for each $\psi(\bar{X}, \bar{Y}\bar{P})$ there is $\psi^*(\bar{X}, \bar{Y}, \bar{P}, \bar{Q})$ that uniformizes ψ). Roughly speaking, a tree T is tame if it does not embed $({}^{\omega>}2, \lhd)$, has a uniformly bounded amount of splitting and has 'tame' branches, that is branches that are scattered linear chains with a uniformly bounded Hausdorff degree. T is very tame if the branches of T are very tame in uniform way. By this we mean that for some $k < \omega$, each definable well ordering of a branch of T is of order type at most ω^k.

The paper is based on the previous [6] but it can be read independently. The relevant definitions and results are presented in the next section and two proofs that are used here are given in the appendix.

In Section 3 we cite some composition theorems that enable the computing of the monadic theory of a structure from the monadic theories of its substructures. Theorems of this sort allow us to deduce the failure of the uniformization property in a tree from the failure in some sub-branch.

Section 4 investigates the spectrum of definable well orderings of definably well orderable linear chains. It turns out that the following dichotomy holds: for every definably well orderable chain $(C, <)$ either each definable well ordering of C is of order type $\geq \omega^\omega$ or, for some $k < \omega$, each definable well ordering of C is of order type in an interval $[\omega^k, \omega^{k+1})$. Chains with the second property are called very tame. We also characterize all the definable well orderings (by a monadic formula) of ordinals and show that each of them is definable by a first order formula, in a language expanded by a finite number of unary predicates (so quantification over unary predicates is obsolete).

In Section 5 we show that the chain $(\omega^\omega, <)$ does not have the uniformization property and deduce, by applying the composition theorems, that trees containing a branch of order type $\geq \omega^\omega$ or that have branches of unbounded order types below ω^ω do not have the uniformization property.

Trees without this property are called very tame and in Section 6 we complete the proof of

THEOREM. $(T, \lhd)$ *has the uniformization property if and only if* T *is very tame.*

§2. **Tame trees.** The purpose of this section is to fix the notations and present the relevant definitions and results from [6].

DEFINITION 2.1. A tree is a partially ordered set $(T, \lhd)$ such that for every $\eta \in T$, $\{\nu : \nu \lhd \eta\}$ is linearly ordered by $\lhd$. $\trianglelefteq$ means $\lhd$ or $=$.

Note, a chain $(C, <)$ and even a pure set I is a tree.

DEFINITIONS AND NOTATIONS 2.2. Let $(T, \lhd)$ be a tree

(1) $S \subseteq T$ is a *convex subset* if $\eta, \nu \in S$ and $\eta \lhd \sigma \lhd \nu \in T$ implies $\sigma \in S$. When S is a convex subset of T we say that $(S, \lhd)$ is a *subtree* of $(T, \lhd)$. If T is a chain we use the term a *convex segment* or just a *segment*.

(2) $B \subseteq T$ is a *sub-branch* of T if B is convex and $\lhd$-linearly ordered.

(3) $B \subseteq T$ is a *branch* of T if B is a maximal sub-branch of T.

(4) $A \subseteq T$ is an *initial segment* of T if A is a sub-branch that is $\lhd$-downward closed. η is *above* [*strictly above*] an initial segment A if $\nu \in A \Rightarrow \nu \trianglelefteq \eta$ [$\nu \in A \Rightarrow \nu \lhd \eta$]. In these cases we write $A \trianglelefteq \eta$ [$A \lhd \eta$].

(5) For $\eta \in T$, $T_{\geq \eta}$ is the sub-tree $(\{\nu \in T : \eta \trianglelefteq \nu\}, \lhd)$. $T_{>\eta}$ is the sub-tree $(T_{\geq \eta} \setminus \{\eta\}, \lhd)$. For $A \subseteq T$ an initial segment, $T_{\geq A}$ and $T_{>A}$ are defined naturally (and are equal if A does not have a $\lhd$-maximal element).

(6) For $\eta \in T$ we denote by $\mathrm{Suc}(\eta)$ or $\mathrm{Suc}_T(\eta)$ the set of $\lhd$-immediate successors of η (which may be empty).

(7) For $\eta, \nu \in T$ we denote the *common initial segment of* η *and* ν *in* T by $\eta \sqcap \nu$. This is defined to be the initial segment $\{\tau : \tau \trianglelefteq \eta \,\&\, \tau \trianglelefteq \nu\}$. However, when $\eta \sqcap \nu$ has a maximal element we may identify it with this element.

(8) If there is an $\eta \in T$ that satisfies $(\forall \nu \in T)[\eta \trianglelefteq \nu]$ we say that T has a root and denote η by $\mathrm{r}(T)$.

(9) $\eta, \nu \in T$ are *incomparable in* T and we write $\eta \perp \nu$, if neither $\eta \trianglelefteq \nu$ nor $\nu \trianglelefteq \eta$. $X \subseteq T$ is an *anti-chain of* T if X consists of pairwise incomparable elements of T.

(10) When $B \subseteq T$ is a sub-branch and $A \subseteq B$ is an initial segment we say that $\sigma \in T$ *cuts* B *at* A if for every $\eta \in A$ and $\nu \in B \setminus A$ we have $\eta \lhd \sigma \,\&\, \nu \perp \sigma$.

(11) A *gap* in T is a pair (A_1, A_2) where $A_1 \cap A_2 = \emptyset$, $A_1 \cup A_2$ is a sub-branch, A_1 is an initial segment, (so $\eta \in A_1, \nu \in A_2 \Rightarrow \eta \lhd \nu$) without a $\lhd$-maximal element, A_2 without a $\lhd$-minimal element, and there is some $\sigma \in T$ that cuts $A_1 \cup A_2$ at A_1.

(12) *Filling a gap* (A_1, A_2) in T is adding a node τ to T such that $\eta \in A_1 \Rightarrow \eta \lhd \tau$, $\nu \in A_2 \Rightarrow \tau \lhd \nu$ and for every σ as above we have $\tau \lhd \sigma$.

We recall now the definition of the Hausdorff degree of a chains (linearly ordered set) and define tame chains.

DEFINITION 2.3. Let $(C, <)$ be a chain. The *Hausdorff degree of* C, $\mathrm{Hdeg}(C)$ is defined as follows:

(i) if there is an embedding $f : \mathbb{Q} \hookrightarrow C$ we stipulate $\mathrm{Hdeg}(C) = \infty$.

(ii) if there is no such embedding (and in this case we say that C is a *scattered chain* we define $\mathrm{Hdeg}(C)$ recursively:

$\mathrm{Hdeg}(C) = 0$ iff C is finite.

$\mathrm{Hdeg}(C) = \alpha$ iff $\bigwedge_{\beta<\alpha} \mathrm{Hdeg}(C) \neq \beta$ and $C = \sum_{i\in I} C_i$ where I is well ordered or inversely well ordered and for every $i \in I$, $\bigvee_{\beta<\alpha} \mathrm{Hdeg}(C_i) = \beta$.

FACT 2.4. (1) C is a scattered chain if and only if $\mathrm{Hdeg}(C)$ is well defined (i.e., there is one and only one ordinal α such that $\mathrm{Hdeg}(C) = \alpha$).

(2) Let C be a scattered chain with $\mathrm{Hdeg}(C) = \alpha$, C' the completion of C and $D \subseteq C'$. Then C' and D are scattered and $\mathrm{Hdeg}(D) \leq \mathrm{Hdeg}(C') = \alpha$.

PROOF. (1) By [5].

(2) By induction on α. ⊣

DEFINITION 2.5. $(C, <)$ is a *tame chain* if and only if C is scattered and $\mathrm{Hdeg}(C) < \omega$.

LEMMA 2.6. *Let $(C, <^*)$ be a chain. Then the following are equivalent:*

(a) *C is tame,*

(b) *there is a finite sequence $\bar{P} = \langle P_0, \dots, P_{k-1}\rangle$ of subsets of C and a formula $\varphi(x, y, \bar{P})$ in the monadic language of order that defines a well ordering of the elements of C.*

Moreover, there is some α such that some $\varphi(x, y, \bar{P})$ defines a well ordering of C isomorphic to $(\alpha, <)$ and there is a finite sequence $\bar{Q} = \langle Q_0, \dots, Q_{\ell-1}\rangle$ of subsets of α and a monadic formula $\varphi^{-1}(x, y, \bar{Q})$ that defines an ordering of α that is isomorphic to $(C, <^*)$. (In fact, it is shown in Section 4 that this holds for every definable well order of C.)

PROOF. See A1 in the appendix. ⊣

The amount of splitting in a tree T is the number of cones above initial segments of T.

DEFINITION 2.7. Let $(T, \lhd)$ be a tree and $A \subseteq T$ an initial segment.

(1) The binary relation $\sim_A^0$ on $T \setminus A$ is defined by

$$x \sim_A^0 y \iff (\forall t \in A)[t \lhd x \equiv t \lhd y].$$

(It's an equivalence relation that says "x and y cut A at the same place".)

(2) The binary relation $\sim_A^1$ on $T \setminus A$ is defined by

$$x \sim_A^1 y \iff [x \sim_A^0 y] \,\&\, (\exists z \in T \setminus A)[z \unlhd x \,\&\, z \unlhd y \,\&\, z \sim_A^0 x].$$

(It's an equivalence relation that refines $\sim_A^0$ by dividing each $\sim_A^0$-equivalence class into disjoint cones.)

We are now able to define wild and tame trees.

DEFINITION 2.8. (1) A tree T is called *wild* if either

(i) $\mathrm{Sup}\{|T_{>A}/\sim_A^1| : A \subseteq T \text{ an initial segment}\} \geq \aleph_0$ or

(ii) There is a branch $B \subseteq T$ and an embedding $f : \mathbb{Q} \to B$ or

(iii) All the branches of T are scattered linear orders but $\mathrm{Sup}\{\mathrm{Hdeg}(B) : B \text{ a branch of } T\} \geq \omega$ or

(iv) There is an embedding $f : {}^{\omega>}2 \to T$.

(2) A tree T is *tame for* (n^*, k^*) if the value in (i) is $\leq n^*$, (ii) and (iv) do not hold and the value in (iii) is $\leq k^*$

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(3) A tree T is *tame* if T is tame for (n^*, k^*) for some $n^*, k^* < \omega$.

The following is the content of [6], the proof of the part (3) $\Rightarrow$ (2) which is relevant for our purposes is given in Theorem A2 in the Appendix.

THEOREM 2.9. *Let* $(T, \lhd)$ *be a tree. Then the following are equivalent:*
(1) *T has a definable choice function.*
(2) *T has a definable well ordering.*
(3) *T is tame.*

§3. **Composition theorems.** The *monadic language of trees* $\mathcal{L}$ is the monadic version of the language of partial orders $\{\lhd\}$. Formally, we let $\mathcal{L} = (\text{Sing}, \text{Empty}, \lhd, \subseteq)$ where 'Sing' and 'Empty' are unary predicates, $<$ and $\lhd$ are binary relations. ($\mathcal{L}$ is a first order language.)

Given a tree T we define the *monadic theory of* T as the first order theory of the model $\mathcal{M}_T := (\mathcal{P}(T); \text{Sing}, \text{Empty}, \lhd, \subseteq)$ where

$$\mathcal{M}_T \models \text{Empty}(X) \iff X = \emptyset,$$
$$\mathcal{M}_T \models \text{Sing}(X) \iff X = \{x\} \text{ for some } x \in T,$$
$$\mathcal{M}_T \models X \lhd Y \iff X = \{x\}, Y = \{y\} \text{ and } T \models x \lhd y,$$
$\subseteq$ is interpreted in $\mathcal{M}_T$ as the usual inclusion relation.

We will not distinguish between T and $\mathcal{M}_T$ and write for example $T \models \text{Sing}(X)$ and $T \models X \lhd Y$.

The definable relation $\trianglelefteq$ and $\in$ will be used freely thus we will write $T \models X \trianglelefteq Y$ and $T \models X \in Y$ (meaning $\mathcal{M}_T \models \text{Sing}(X) \,\&\, X \subseteq Y$).

When C is a chain (linearly ordered set) we replace $\lhd$ and $\trianglelefteq$ by $<$ and $\leq$ respectively.

Note: everything that is defined in Definitions and Notations 2.2 is definable by a monadic formula.

In fact we will never deal with the full monadic theory of a tree T, the objects that we will be interested in are partial theories—finite approximations of the monadic theory of T. $\text{Th}^n(T; \bar{P})$ is essentially the monadic theory of $(T; \bar{P}, \lhd)$ restricted to sentences of quantifier depth n.

NOTATIONS. C, D and I denote chains. S, T and Γ denote trees.

Lower case and Greek letters (x, y, a, b, η, ν) are used to denote elements, uppercase letters (X, Y, A, P, Q) denote subsets.

$\bar{a}$ and $\bar{P}$ denote finite sequences of elements and subsets, their lengths are $\lg(\bar{a})$ and $\lg(\bar{P})$. We will write $\bar{a} \in T$ and $\bar{P} \subseteq T$ instead of $\bar{a} \in {}^{\lg(\bar{a})}T$ and $\bar{P} \in {}^{\lg(\bar{P})}\mathcal{P}(T)$.

When $\bar{P}$ and $\bar{Q}$ are of the same length we will write $\bar{P} \cup \bar{Q}$ to denote $\langle P_0 \cup Q_0, \dots, P_{\ell-1} \cup Q_{\ell-1}\rangle$. Similarly we write $\bigcup_{i \in I} \bar{P}^i$ (assuming $\lg(\bar{P}^i)$ is constant). $\bar{P} \cap S$ means $\langle P_0 \cap S, \dots, P_{\ell-1} \cap S\rangle$.

$\bar{P} {}^\wedge \bar{Q}$ is the sequence $\langle P_0, \dots, Q_0 \dots\rangle$.

DEFINITION 3.1. For a tree T, $\bar{A} \subseteq T$, and a natural number n, define by induction $t = \text{Th}^n(T; \bar{A})$;

for $n = 0$:

$$t = \{\phi(\bar{X}) : \phi(\bar{X}) \in \mathscr{L},\ \phi(\bar{X}) \text{ quantifier free},\ T \models \phi(\bar{A})\},$$

for $n = m + 1$:

$$t = \{\mathrm{Th}^m(T; \bar{A}^\wedge B) : B \in \mathscr{P}(T)\}.$$

We may regard $\mathrm{Th}^n(T; \bar{A})$ as the set of $\varphi(\bar{X})$ that are boolean combinations of monadic formulas of quantifier depth $\leq n$, such that $T \models \varphi(\bar{A})$.

CONVENTION. When $x \in T$ we will usually write $\mathrm{Th}^n(T;x)$ instead of $\mathrm{Th}^n(T;\{x\})$.

DEFINITION 3.2. $\mathscr{T}_{n,\ell}$ is the set of all the possibilities for $\mathrm{Th}^n(T; \bar{P})$ where T is a tree and $\lg(\bar{P}) = \ell$.

FACT 3.3. (A) For every formula $\psi(\bar{X}) \in \mathscr{L}$ there is an n such that from $\mathrm{Th}^n(T; \bar{A})$ we can effectively decide whether $C \models \psi(\bar{X})$. If n is minimal with this property we will say that ψ is of depth n and write $\mathrm{dp}(\psi) = n$.

(B) If $m \geq n$ then $\mathrm{Th}^n(T; \bar{A})$ can be effectively computed from $\mathrm{Th}^m(T; \bar{A})$.

(C) For every $t \in \mathscr{T}_{n,\ell}$ there is a monadic formula $\psi_t(\bar{X})$ with $\mathrm{dp}(\psi) = n$ such that for every $\bar{A} \subseteq T$ with $\lg(\bar{A}) = \ell$, $T \models \psi_t(\bar{A}) \iff \mathrm{Th}^n(T; \bar{A}) = t$.

(D) The set of possibilities $T_{n,\ell}$ is effectively computable.

PROOF. Easy. ⊣

The first, and strongest composition theorem applies to chains.

DEFINITION 3.4. If C, D are chains then $C + D$ is the chain that is obtained by adding a copy of D after C.

If $\langle C_i : i \in I\rangle$ is a sequence of chains and I is a chain then $\sum_{i<\alpha} C_i$ is the chain that is obtained by the concatenation of the C_i's along I.

THEOREM 3.5 (composition theorem for linear orders). (1) *If* $\lg(\bar{A}) = \lg(\bar{B}) = \lg(\bar{A}') = \lg(\bar{B}') = \ell$, *and*

$$\mathrm{Th}^m(C; \bar{A}) = \mathrm{Th}^m(C'; \bar{A}') \text{ and } \mathrm{Th}^m(D; \bar{B}) = \mathrm{Th}^m(D'; \bar{B}')$$

then

$$\mathrm{Th}^m(C + D; \bar{A} \cup \bar{B}) = \mathrm{Th}^m(C' + D'; \bar{A}' \cup \bar{B}').$$

(2) *If* $\mathrm{Th}^m(C_i; \bar{A}^i) = \mathrm{Th}^m(D_i; \bar{B}^i)$ *and* $\lg(\bar{A}^i) = \lg(\bar{B}^i) = \ell$ *for each* $i \in I$, *then*

$$\mathrm{Th}^m\left(\sum_{i\in I} C_i; \bigcup_{i\in I} \bar{A}^i\right) = \mathrm{Th}^m\left(\sum_{i\in I} D_i; \bigcup_{i\in I} \bar{B}^i\right).$$

PROOF. By [8] Theorem 2.4 (where a more general theorem is proved), or directly by induction on m. ⊣

The above theorem allows us to define an operation of addition of partial theories of chains.

NOTATION 3.6.

(1) When, for some $m, \ell \in \mathbb{N}$, $t_1, t_2, t_3 \in T_{m,\ell}$ then $t_1 + t_2 = t_3$ means: there are chains C and D such that

$$t_1 = \mathrm{Th}^m(C; A_0, \dots, A_{\ell-1}) \ \& $$
$$t_2 = \mathrm{Th}^m(D; B_0, \dots, B_{\ell-1}) \ \& \ t_3 = \mathrm{Th}^m(C + D; \bar{A} \cup \bar{B}).$$

(By the composition theorem, the choice of C and D is immaterial.)

(2) $\sum_{i \in I} \mathrm{Th}^m(C_i; \bar{A}^i)$ is $\mathrm{Th}^m(\sum_{i \in I} C_i; \bigcup_{i \in I} A^i)$, (assuming $\lg(\bar{A}^i) = \lg(\bar{A}^j)$ for $i, j \in I$).

(3) If D is a sub-chain of C and $\bar{A} \subseteq C$ then $\mathrm{Th}^m(D; \bar{A})$ abbreviates $\mathrm{Th}^m(D; \bar{A} \cap D)$.

(4) For C a chain, $a < b \in C$ and $\bar{P} \subseteq C$ we denote by $\mathrm{Th}^n(C; \bar{P}) \restriction_{[a,b)}$ the theory $\mathrm{Th}^n([a,b); \bar{P} \cap [a,b))$.

We also have a monadic version of the Feferman-Vaught theorem.

THEOREM 3.7. *For every $n, \ell < \omega$ there is $m = m(n, \ell) < \omega$, effectively computable from n and ℓ, such that whenever I is a chain, for $i \in I$ C_i is a chain, $\bar{Q}_i \subseteq C_i$ and $\lg(\bar{Q}_i) = \ell$,*

$$\text{if } (C; \bar{Q}) = \sum_{i \in I} (C_i; \bar{Q}_i) := \left(\sum_{i \in I} C_i; \bigcup_{i \in I} \bar{Q}_i \right)$$

and if for $t \in \mathcal{T}_{n,\ell}$ $P_t := \{i \in I : \mathrm{Th}^n(C_i; \bar{Q}_i) = t\}$ and $\bar{P} := \langle P_t : t \in \mathcal{T}_{n,\ell} \rangle$ then from $\mathrm{Th}^m(I; \bar{P})$ we can effectively compute $\mathrm{Th}^n(C; \bar{Q})$,

PROOF. Again, by [8, Theorem 2.4]. ⊣

By the composition theorem, the colouring $(\alpha, \beta) \mapsto \mathrm{Th}^n(C; \bar{P}) \restriction_{[\alpha,\beta)}$ for $\alpha < \beta \in C$ and $\bar{P} \subseteq C$ is an additive (even σ-additive) colouring. For such colourings we can prove strong Ramsey theorems.

DEFINITION 3.8. (a) A *colouring* of a chain C is a function $f : [C]^2 \to I$ where $[C]^2$ is the set of unordered pairs of distinct elements of C and I is a finite set (the set of colours).

(b) The colouring f is *additive* if for every $x_1 < y_1 < z_1$ and $x_2 < y_2 < z_2$ in C

$$[f(x_1, y_1) = f(x_2, y_2) \ \& \ f(y_1, z_1) = f(y_2, z_2)] \Rightarrow f(x_1, z_1) = f(x_2, z_2).$$

In this case a partial operation $+$ is well defined on I:

$$i_1 + i_2 = i_3$$
$$\iff (\exists x, y, z \in C)\big[x < y < z \ \& \ f(x, y) = i_1 \ \& \ f(y, z) = i_2 \ \& \ f(x, z) = i_3\big],$$

(compare with Notation 3.6(1)).

(c) A sub-chain $D \subseteq C$ is *homogeneous* (for f) if there is an $i_0 \in I$ such that for every $x, y \in D$, $f(x, y) = i_0$.

(d) f is *σ-additive* if whenever $\{x_i\}_{i<\omega}$ and $\{y_i\}_{i<\omega}$ are increasing in C, $x = \mathrm{Sup}\{x_i\}$ and $y = \mathrm{Sup}\{y_i\}$ if for $i < \omega$ $f(x_i, x_{i+1}) = f(y_i, y_{i+1})$ then $f(x_0, x) = f(y_0, y)$.

(e) In the case of σ-additive colourings we can define naturally infinite sums $\sum_{n<\omega} i_n \in I$.

THEOREM 3.9 (Ramsey theorem for additive colourings). *Let $f\colon [C]^2 \to I$ be an additive colouring where C is isomorphic to a limit ordinal and I finite. Then there is $D \subseteq C$, cofinal and homogeneous for f.*

PROOF. This is Theorem 1.1 in [8]. $\dashv$

Given a tree T and a branch $B \subseteq T$ without gaps we can consider T as being a sum of sub-trees $T^*_{\geq\eta}$ for $\eta \in B$. Thus, we can formulate a Feferman-Vaught theorem.

NOTATION 3.10. (1) Let T be a tree and $B \subseteq T$ a branch. $(B^*, \lhd^*)$ is the chain that is obtained from B by filling the gaps in B (see Definition 2.2) where $\lhd^*$ is the natural extension of the linear order in B. Call B^* the completion of B in T.

(2) Let T, B and B^* be as above. For $\eta \in B^*$ let $T^*_{\geq\eta}$ be defined by $\nu \in T^*_{\geq\eta}$ iff $[\eta \unlhd \nu]\,\&\,(\forall\tau \in T)[\eta \lhd \tau \unlhd \nu \Rightarrow \tau \notin B]$. For $\eta \in B^* \setminus B$ (so η is a 'gap') and $\tau \in T$, $\eta \unlhd \nu$ has the obvious meaning. $T^*_{\geq\eta} \subseteq T$ is a sub-tree and if $\eta \in B$ then $\eta \in T^*_{\geq\eta}$. Moreover, assuming T has a root, $T = \bigcup_{\eta\in B^*} T^*_{\geq\eta}$ (and the union is disjoint).

(3) Let T, B and B^* be as above and $\bar{Q} \subseteq T$ with $\lg(\bar{Q}) = \ell$. For $t \in \mathscr{T}_{n,\ell}$ let $P^{B^*}_t(T;\bar{Q}) \subseteq B^*$ be the set $\{\eta \in B^* : \mathrm{Th}^n(T^*_{\geq\eta};\bar{Q}) = t\}$.

(4) $\bar{P}^{B^*}_{n,\ell}(T;\bar{Q})$ is the sequence (partition of B^*) $\langle P^{B^*}_t(T;\bar{Q}) : t \in \mathscr{T}_{n,\ell}\rangle$.

THEOREM 3.11 (composition theorem along a branch). *Let T be a tree, $B \subseteq T$ a branch, B^* its completion in T, and let $T^{\setminus B} := \{\eta \in T : \nu \in B \Rightarrow \eta \perp \nu\}$.*

Then for every $n, \ell \in \mathbb{N}$ there is $m = m(n,\ell) \in \mathbb{N}$ such that if $\bar{Q} \subseteq T$ is of length ℓ then

$\mathrm{Th}^n(T;\bar{Q})$ can be effectively computed from

$$\mathrm{Th}^m(B^*;\bar{P}^{B^*}_{n,\ell}(T;\bar{Q}),B) \text{ and } \mathrm{Th}^m(T^{\setminus B};\bar{Q}).$$

Moreover, if $\bar{R} \subseteq B$ then $\mathrm{Th}^n(T;\bar{Q},\bar{R})$ *can be effectively computed from* $\mathrm{Th}^m(B^*;\bar{P}^{B^*}_{n,\ell}(T;\bar{Q}),B,\bar{R})$ *and* $\mathrm{Th}^m(T^{\setminus B};\bar{Q})$.

PROOF. This is Theorem 1.13 in [6]. $\dashv$

REMARKS.

(1) We don't really need B as a parameter when $n > 2$ because $\eta \in B$ iff the unique t such that $\eta \in P^{B^*}_t(T;\bar{Q})$ is a theory of a tree with a root.

(2) B^* is interpretable in T using monadic formulas and each $T^*_{\geq\eta}$ is definable in T by a monadic formula.

§4. Well orderings of ordinals. Tame chains have definable well orderings of their elements. We will define a partial function log with range $\mathbb{N} \cup \{\infty\}$ and show that log is well defined for every tame chain. By the moreover clause in Lemma 2.6 it suffices to prove that log is well defined on ordinals.

DEFINITION 4.1. Let log: $\{\text{tame chains}\} \to \mathbb{N} \cup \{\infty\}$ be defined by:
$\log(C) = \infty$ iff each $\varphi(x,y,\bar{P})$ that defines a well ordering on the elements of C,

defines a well ordering of order type $\geq \omega^\omega$,

$\log(C) = k$ iff there each $\varphi(x, y, \bar{P})$ that defines a well ordering on the elements of C, defines a well ordering of order type in the segment $[\omega^k, \omega^{k+1})$.

If neither holds, $\log(C)$ is undefined.

log is well defined on finite and uncountable chains. All the chains and ordinals are from now on infinite.

DEFINITION 4.2. Let α, β be ordinals. $\alpha \to \beta$ means the following: "there is $\varphi(x, y, \bar{P})$ (monadic, with parameters $\bar{P} \subseteq \alpha$) that defines a well ordering on the elements of α of order type β".

FACTS 4.3. (1) $\alpha \to \beta \;\&\; \beta \to \gamma \;\Rightarrow\; \alpha \to \gamma$.
(2) $\alpha \to \gamma \;\&\; \gamma \geq \alpha \cdot \omega \;\Rightarrow\; \alpha \to \alpha \cdot \omega$.
(3) $\alpha \geq \omega \;\Rightarrow\; \alpha \to \alpha + \alpha$.

PROOF. Straightforward. $\dashv$

NOTATION. Suppose $\alpha \to \beta$ holds by $\varphi(x, y, \bar{P})$. Define a bijection $f_\varphi = f : \alpha \to \beta$ by $f(i) = j$ iff i is the j'th element in the well order defined by φ.

LEMMA 4.4. *For any ordinal* α, $\alpha \not\to \alpha \cdot \omega$.

PROOF. Assume that α is minimal such that $\alpha \to \alpha \cdot \omega$, let $\varphi(x, y, \bar{P})$ witness that and $f = f_\varphi$ be as above. It follows that:

(i) $\alpha \geq \omega$,

(ii) α is a limit ordinal (by $\alpha \to \alpha + 1$ and 4.3.),

(iii) for $\beta < \alpha$, $\{f(i) : i < \beta\}$ does not contain a final segment of $\alpha \cdot \omega$ [suppose not: by the Composition Theorem 3.5, for $i_1, i_2 < \beta$ $\varphi(i_1, i_2, \bar{P})$ depends only on $\mathrm{Th}^{\mathrm{dp}(\varphi)}(\beta; i_1, i_2, \bar{P})$ so φ can be "restricted" to β i.e., $\beta \to \alpha \cdot \omega$ hence by Facts 4.3(2) $\beta \to \beta \cdot \omega$ and this contradicts the minimality of α].

(iv) $\beta < \alpha \Rightarrow \beta \cdot \omega \leq \alpha$ (by Facts 4.3(3))

(v) $\beta < \alpha \;\&\; \gamma \geq \alpha \;\Rightarrow \beta \not\to \gamma$ (otherwise, by (iv) and Facts 4.3 $\beta \to \beta \cdot \omega$).

Let $Q \subseteq \alpha$ be the following subset: $x \in Q$ iff for some $k < \omega$, $\alpha \cdot 2k \leq f(x) < \alpha \cdot (2k+1)$. Let E an equivalence relation on α defined by xEy iff for some $\ell < \omega$, $f(x)$ and $f(y)$ belong to the segment $[\alpha \cdot \ell, \alpha \cdot (\ell+1))$. Clearly there is a monadic formula $\psi(x, y, \bar{P}, Q)$ that defines E moreover, some monadic formula $\theta(X, \bar{P}, Q)$ expresses the statement "$\bigvee_{i<\omega}(X = Q_i)$" where $\langle Q_i : i < \omega\rangle$ are the E-equivalence classes.

Let $n := \max\{\mathrm{dp}(\varphi), \mathrm{dp}(\psi), \mathrm{dp}(\theta)\} + 5$, and $m := |\mathcal{T}_{n,\lg(\bar{P})+2}|$.

Let $\delta = \mathrm{cf}(\alpha)$ and $X = \{x_i\}_{i<\delta}$ be strictly increasing and cofinal in α. By Theorem 3.9 applied to the additive colouring $h(i,j) = \mathrm{Th}^n(\alpha; \bar{P}, Q, x_i) \restriction_{[x_i, x_j)}$ we get a cofinal subsequence $\{\beta_j\}_{j<\delta}$ such that $t := \mathrm{Th}^n(\alpha; \bar{P}, Q, \beta_{j_1}) \restriction_{[\beta_{j_1}, \beta_{j_2})}$ is constant for $j_1 < j_2 < \delta$. Clearly $t + t = t$ and t determines the theory $t^- := \mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[\beta_j, \beta_\ell)}$. Now for each $j < \delta$ we have

$$
\begin{aligned}
&\mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[\beta_j, \alpha)} \\
&\quad = \mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[\beta_j, \beta_{j+1})} + \mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[\beta_{j+1}, \alpha)} \\
&\quad = t^- + \mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[\beta_{j+1}, \alpha)}
\end{aligned}
$$

$$= \mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[\beta_0,\beta_{j+1})} + \mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[\beta_{j+1},\alpha)}$$
$$= \mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[\beta_0,\alpha)} .$$
$$= \text{Hence } (\dagger)\text{: for every } j < \delta,\ \mathrm{Th}^n(\alpha; \bar{P}, Q) \restriction_{[\beta_j,\alpha)} \text{ is constant.}$$

Note that each E-equivalence class Q_i is unbounded in α since if some $\beta < \alpha$ contained some E-equivalence class Q_i it would easily follow that $\beta \to \gamma$ for some $\gamma \geq \alpha$ contradicting fact (v).

Fix some $1 < j < \delta$ let $x < \beta_j$ and let $Q_{i(x)}$ be the E-equivalence class containing x. Since $Q_{i(x)}$ is unbounded in α there is some $j < \ell < \delta$ such that $[\beta_j, \beta_\ell) \cap Q_{i(x)} \neq \emptyset$. This statement is expressible by $\mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_\ell)$ which is equal to

$$\begin{aligned} &\mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_\ell) \restriction_{[0,\beta_j)} + \mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_\ell) \restriction_{[\beta_j,\beta_\ell)} \\ &\quad + \mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_\ell) \restriction_{[\beta_\ell,\alpha)} = \mathrm{Th}^n(\alpha; \bar{P}, Q, x, \emptyset, \emptyset) \restriction_{[0,\beta_j)} \\ &\quad + \mathrm{Th}^n(\alpha; \bar{P}, Q, \emptyset, \beta_j, \emptyset) \restriction_{[\beta_j,\beta_l)} + \mathrm{Th}^n(\alpha; \bar{P}, Q, \emptyset, \emptyset, \beta_\ell) \restriction_{[\beta_\ell,\alpha)} . \end{aligned}$$

But the second theory is determined by t hence equal to $\mathrm{Th}^n(\alpha; \bar{P}, Q, \emptyset, \beta_j, \emptyset) \restriction_{[\beta_j,\beta_{j+1})}$ and the third theory is equal by ($\dagger$) to $\mathrm{Th}^n(\alpha; \bar{P}, Q, \emptyset, \emptyset, \beta_{j+1}) \restriction_{[\beta_{j+1},\alpha)}$. We conclude:

$$\mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_\ell) = \mathrm{Th}^n(\alpha; \bar{P}, Q, x, \beta_j, \beta_{j+1}).$$

Therefore for every $x < \beta_j$, $[\beta_j, \beta_{j+1}) \cap Q_{i(x)} \neq \emptyset$.

Finally, let $j < \delta$ be such that the segment $[0, \beta_j)$ intersects $m + 1$ different E-equivalence classes, say $Q_{i_0}, \dots, Q_{i_m}$. By the previous argument we have $[\beta_j, \beta_{j+1}) \cap Q_{i_k} \neq \emptyset$ for every $k \leq m$. By the choice of m there are different $a, b \in \{i_0, \dots, i_m\}$ such that

$$(*) \qquad \mathrm{Th}^n(\alpha; \bar{P}, Q, Q_a) \restriction_{[\beta_j,\beta_{j+1})} = \mathrm{Th}^n(\alpha; \bar{P}, Q, Q_b) \restriction_{[\beta_j,\beta_{j+1})} .$$

Let $R \subseteq \alpha$ be $([0, \beta_j) \cap Q_a) \cup (([\beta_j, \beta_{j+1}) \cap Q_b) \cup ([\beta_{j+1}, \alpha) \cap Q_a)$. We claim that $\mathrm{Th}^n(\alpha, \bar{P}, Q, R) = \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a)$. Indeed:

$$\begin{aligned} &\mathrm{Th}^n(\alpha, \bar{P}, Q, R) \\ &= \mathrm{Th}^n(\alpha, \bar{P}, Q, R) \restriction_{[0,\beta_j)} + \mathrm{Th}^n(\alpha, \bar{P}, Q, R) \restriction_{[\beta_j,\beta_{j+1})} + \mathrm{Th}^n(\alpha, \bar{P}, Q, R) \restriction_{[\beta_{j+1},\alpha)} \\ &= \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a) \restriction_{[0,\beta_j)} + \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_b) \restriction_{[\beta_j,\beta_{j+1})} \\ &\quad + \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a) \restriction_{[\beta_{j+1},\alpha)} = (\text{by } (*)) \\ &\mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a) \restriction_{[0,\beta_j)} + \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a) \restriction_{[\beta_j,\beta_{j+1})} + \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a) \restriction_{[\beta_{j+1},\alpha)} \\ &= \mathrm{Th}^n(\alpha, \bar{P}, Q, Q_a). \end{aligned}$$

But Q_a is an E-equivalence class while R is not. Since $\mathrm{Th}^n(\alpha, \bar{P}, Q, Z)$ determines the truth value of: "Z is an E-equivalence class" we get a contradiction. $\dashv$

LEMMA 4.5. *If $\beta < \alpha$ then $\alpha \to \beta$ if and only if there are $\gamma_1, \gamma_2 \leq \alpha$ such that $\gamma_1 + \gamma_2 = \alpha$ and $\gamma_2 + \gamma_1 = \beta$.*

PROOF. The 'if' part is trivial. For the 'only if' part let's prove first:

SUBCLAIM. $\omega + \omega \not\to \omega$.

PROOF OF THE SUBCLAIM. Assume that $\varphi(x, y, \bar{P})$ well orders $\omega + \omega$ of order type ω and that $\mathrm{dp}(\varphi) = n$, $\lg(\bar{P}) = \ell$. Let $x <^* y$ mean $(\omega + \omega, <) \models \varphi(x, y, \bar{P})$.

Find, using Ramsey theorem for additive colourings, $\{x_i\}_{i<\omega}$ increasing and unbounded in $[0, \omega)$ satisfying, for $i < j < \omega$ and for some $s_0 \in \mathscr{T}_{n,\ell+2}$ and $t_0 \in \mathscr{T}_{n,\ell+2}$

$$\mathrm{Th}^n(\omega + \omega; x_i, \emptyset, \bar{P}) \restriction_{[x_i, x_j)} = s_0, \quad \mathrm{Th}^n(\omega + \omega; \emptyset, \emptyset, \bar{P}) \restriction_{[x_i, x_j)} = t_0,$$

similarly, let $\{y_j\}_{j<\omega}$ be increasing and unbounded in $[\omega, \omega + \omega)$ satisfying for $j < k < \omega$ and for some $s_1 \in \mathscr{T}_{n,l+2}$ and $t_1 \in \mathscr{T}_{n,l+2}$

$$\mathrm{Th}^n(\omega + \omega; \emptyset, y_j, \bar{P}) \restriction_{[y_j, y_k)} = s_1, \quad \mathrm{Th}^n(\omega + \omega; \emptyset, \emptyset, \bar{P}) \restriction_{[y_i, y_k)} = t_1.$$

We may also assume that $i_1 < i_2 \Rightarrow x_{i_1} <^* x_{i_2}$ and $j_1 < j_2 \Rightarrow y_{j_1} <^* y_{j_2}$. Note also that by homogeneity $t_0 + t_0 = t_0$ and $t_1 + t_1 = t_1$.

We will show now that for $0 < i < \omega$ and $0 < j < \omega$, $\mathrm{Th}^n(\omega + \omega; x_i, y_j, \bar{P})$ is constant. Indeed:

$$\begin{aligned}
t^* := &\mathrm{Th}^n(\omega + \omega; x_i, y_j, \bar{P}) \\
= &\mathrm{Th}^n(\omega + \omega; \emptyset, \emptyset, \bar{P}) \restriction_{[0,x_0)} + \mathrm{Th}^n(\omega + \omega; \emptyset, \emptyset, \bar{P}) \restriction_{[x_0,x_i)} \\
&+ \mathrm{Th}^n(\omega + \omega; x_i, \emptyset, \bar{P}) \restriction_{[x_i,x_{i+1})} + \mathrm{Th}^n(\omega + \omega; \emptyset, \emptyset, \bar{P}) \restriction_{[x_{i+1},\omega)} \\
&+ \mathrm{Th}^n(\omega + \omega; \emptyset, \emptyset, \bar{P}) \restriction_{[\omega,y_0)} + \mathrm{Th}^n(\omega + \omega; x_i, \emptyset, \bar{P}) \restriction_{[y_i,y_j)} \\
&+ \mathrm{Th}^n(\omega + \omega; \emptyset, y_j, \bar{P}) \restriction_{[y_j,y_{j+1})} + \mathrm{Th}^n(\omega + \omega; \emptyset, \emptyset, \bar{P}) \restriction_{[y_{j+1},\omega+\omega)} .
\end{aligned}$$

Call the sum $t^* = r_1 + r_2 + \cdots + r_8$. Now r_1 is constant, $r_2 = t_0 \cdot i = t_0$ (by $t_0 + t_0 = t_0$), r_3 is s_0, $r_4 = t_0 \cdot \omega$ hence is constant, r_5 is constant, $r_6 = t_1 \cdot j = t_1$, $r_7 = s_1$ and $r_8 = t_0 \cdot \omega$ hence is constant. Therefore t^* does not depend on i and j.

Now as $\{y_j\}_{j<\omega}$ is unbounded with respect to $<^*$, there is some $j < \omega$ such that $x_1 <^* y_j$. This is expressed by $\mathrm{Th}^n(\omega + \omega; x_1, y_j, \bar{P})$ which we have just seen to be independent of i and j hence

$$(\forall 0 < i < \omega)(\forall 0 < j < \omega)\big[(\omega + \omega, <) \models \varphi(x_i, y_j, \bar{P})\big]$$

it follows that $\mathrm{otp}(\omega + \omega, <^*) \geq \omega + 1$, a contradiction. This proves $\omega + \omega \not\to \omega$.

Returning to the proof of the claim, let β be the minimal ordinal such that there exists some $\alpha > \beta$ with $\alpha \to \beta$ but there aren't any $\gamma_1, \gamma_2 \leq \alpha$ with $(\gamma_1 + \gamma_2 = \alpha)$ & $(\gamma_2 + \gamma_1 = \beta)$. Call such a β *weird* and let $\alpha > \beta$ the first ordinal witnessing the weirdness of β. By transitivity of $\to$ β is a limit ordinal. Moreover, $\gamma < \beta \Rightarrow \beta \not\to \gamma$ (otherwise, by transitivity $\alpha \to \gamma$ and γ is weird) hence if $\beta = \gamma_1 + \gamma_2$ then $\gamma_2 + \gamma_1 \geq \beta$. By looking at the first γ such that $\gamma \cdot k_0 \geq \beta$ for some $k_0 < \omega$ it follows that there are two possible cases: either $(*)$ for all γ $\gamma < \beta \Rightarrow (\gamma + \gamma < \beta)$, hence $\gamma < \beta \Rightarrow (\gamma \cdot \omega \leq \beta)$ and $\gamma < \beta \Rightarrow (\mathrm{otp}([\gamma, \beta)) = \beta)$, or $(**)$ $\beta = \gamma \cdot k_0$ for some $1 < k_0 < \omega$.

First case: $(*)$ holds. Let $\alpha = \beta + \gamma$ what can γ be? If $\gamma < \beta$ then by $(*)$ $\gamma + \beta = \beta$ and α does not witness the weirdness of β. Therefore $\alpha \geq \beta + \beta$.

Let $\varphi(x, y, \bar{P})$ demonstrate $\alpha \to \beta$ and assume $\mathrm{dp}(\varphi) = n$ and $\lg(\bar{P}) = \ell$. As above $x <^* y$ means $(\alpha, <) \models \varphi(x, y, \bar{P})$ and finally let $\delta = \mathrm{cf}(\beta)$.

Now $\text{otp}(\alpha, <^*) = \beta$ but what is $\text{otp}([0, \beta), <^*)$? (that is, of $<^*$ restricted to $\beta \subseteq \alpha$). Clearly, as for $\gamma_1, \gamma_2 < \beta$ we have $\text{Th}^n(\alpha, \bar{P}, \gamma_1, \gamma_2) = \text{Th}^n(\alpha, \bar{P}, \gamma_1, \gamma_2) \restriction_{[0,\beta)} + \text{Th}^n(\alpha, \bar{P}, \emptyset, \emptyset) \restriction_{[\beta,\alpha)}$ we can compute $<^* \restriction_{[0,\beta)}$ in the segment $[0, \beta)$ and so $\beta \to \text{otp}([0, \beta), <^*)$. Therefore $\beta = \text{otp}([0, \beta), <^*)$ (otherwise, by $(*)$, $\text{otp}([0, \beta), <^*)$ is weird and $< \beta$).

Similarly we can show that $\text{otp}([\beta, \beta + \beta), <^*) = \beta$.

Now proceed as before: choose $\{x_i\}_{i<\delta} \subseteq [0, \beta)$ and $\{y_j\}_{j<\delta} \subseteq [\beta, \beta + \beta)$ that are increasing homogeneous and unbounded with respect to $<^*$ and use them to show that $\text{otp}(\alpha, <^*) \geq \beta + 1$.

Second case: $(**)$ holds i.e., $\beta = \gamma \cdot k_0$.

Let ε be the first ordinal such that for some $k < \omega$ $\varepsilon \cdot k$ is weird. Look at γ: if $\gamma = \gamma_1 + \gamma_2$ and $\gamma_2 + \gamma_1 < \gamma$ we would have $\alpha \to \beta = \gamma \cdot (k_0 - 1) + \gamma \to \gamma \cdot (k_0 - 1) + \gamma_2 + \gamma_1 < \beta$ and by transitivity this is a contradiction to the minimality of β. Hence as before, we have either $\gamma_1 < \gamma \Rightarrow (\gamma_1 + \gamma_1 < \gamma)$ and in this case $\gamma = \varepsilon$ or $\gamma = \gamma_1 \cdot k_1$ for some $k_1 < \omega$. Repeat the same argument to get $\gamma_1 = \varepsilon$ or $\gamma_1 = \gamma_2 \cdot k_2$. After finitely many steps we are bound to get $\beta = \varepsilon \cdot k$ for some $k < \omega$ and $\varepsilon_1 < \varepsilon \Rightarrow \varepsilon_1 \cdot \omega \leq \varepsilon$ and of course $\varepsilon_1 < \varepsilon \Rightarrow \varepsilon \not\to \varepsilon_1$.

Let $\varphi(x, y, \bar{P})$ demonstrate $\alpha \to \beta$ and $<^*$ be as usual. Assume $\delta := \text{cf}(\beta) = \text{cf}(\varepsilon)$. Let $\alpha = \beta + \varepsilon^*$. If $\varepsilon^* < \varepsilon$ then $\varepsilon^* + \beta = \beta$ and α doesn't witness weirdness, therefore $\varepsilon^* \geq \varepsilon$.

Proceed as before: as $\alpha \geq \varepsilon \cdot (k+1)$ we can choose $\{x_i^0\}_{i<\delta}, \{x_i^1\}_{i<\delta}, \dots, \{x_i^k\}_{i<\delta}$ with $\{x_i^\ell\}_{i<\delta} \subseteq [\varepsilon \cdot \ell, \varepsilon(\ell + 1))$, unbounded with respect to $<^*$, increasing and homogeneous.

By the Composition Theorem it will follow that $\text{otp}([\varepsilon \cdot \ell, \varepsilon(\ell + 1)), <^*) \geq \varepsilon$ and by homogeneity we can assume that for $0 < i, j < \omega$ and $\ell \leq k$, $x_i^\ell <^* x_j^{\ell+1}$. It follows that $\text{otp}(\alpha, <^*) \geq (\varepsilon \cdot k) + 1 = \beta + 1$ and a contradiction. $\dashv$

The previous lemmas yield a characterization of the definable well orderings of ordinals: they are exactly the well orderings that are definable in expansions of the first order language of order by a finite number of unary predicates.

THEOREM 4.6. *For every ordinal α, every definable (by a monadic formula) well ordering $<^*$ of α is obtained by the following process: let $\langle P_0, P_1, \dots, P_{n-1} \rangle$ be a partition of α and*

$$i <^* j \iff \big[(\exists k < n)(i \in P_k \,\&\, j \in P_k \,\&\, i < j)\big] \\ \vee \big[i \in P_{k_1} \,\&\, j \in P_{k_2} \,\&\, k_1 < k_2\big].$$

PROPOSITION 4.7. $\text{Log}(C)$ *is well defined for every tame chain C.*

PROOF. Let $(C, <^*)$ be tame and let $(\alpha, <)$ and $(\beta, <)$ be results of a definable well orderings of $(C, <^*)$ where in addition (by Lemma 2.6) there is $\psi(x, y, \bar{Q})$ that defines C in α. So $\alpha \to \beta$ and by Lemmas 4.4 and 4.5 $\alpha < \omega^\omega \iff \beta < \omega^\omega$ and $\alpha \in [\omega^k, \omega^{k+1}) \iff \beta \in [\omega^k, \omega^{k+1})$. $\dashv$

§5. $(\omega^\omega, <)$ and longer chains. The purpose of this section is to show that the chain $(\omega^\omega, <)$ does not have the uniformization property. From this we will be able to deduce that the same is true for every tame chain C with $\log(C) = \infty$.

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The following lemma is a part of Theorem 3.5(B) in [8]:

LEMMA 5.1. *Let I be a well ordered chain of order type $\geq \omega^k$. Let $f : I^2 \to \{t_0, t_1 \dots, t_{\ell-1}\}$ be a σ-additive colouring and assume that for $\alpha < \beta \in I$, $f(\alpha, \beta)$ depends only on the order type in I of the segment $[\alpha, \beta)$.*

Then there is $i < \ell$ such that for some $p \leq \ell$, for every $r \geq p$, if $\operatorname{otp}([\alpha, \beta)) = \omega^r$ *then $f(\alpha, \beta) = t_i$. Moreover, $t_i + t_i = t_i$.*

PROOF. To avoid triviality assume $k > \ell$. For $\alpha < \beta$ in I with $\operatorname{otp}([\alpha, \beta)) = \delta$, denote $f(\alpha, \beta)$ by $t(\delta)$ (makes sense by the assumptions).

By the pigeon-hole principle there are $p < \ell$, $p < s \leq k$ and some t_i with $t(\omega^p) = t(\omega^s) = t_i$. Now $\omega^{p+2} = \sum_{i<\omega}(\omega^{p+1} + \omega^p)$ and by the σ-additivity of f:

$$
\begin{aligned}
t\left(\omega^{p+2}\right) &= t\left(\sum_{i<\omega}(\omega^{p+1} + \omega^p)\right) = \sum_{i<\omega} t\left(\omega^{p+1} + \omega^p\right) \\
&= \sum_{i<\omega}\left(t(\omega^{p+1}) + t(\omega^p)\right) = \sum_{i<\omega}\left(t(\omega^{p+1}) + t(\omega^s)\right) = \sum_{i<\omega} t\left(\omega^{p+1} + \omega^s\right) \\
&= \sum_{i<\omega} t\left(\omega^s\right) = \sum_{i<\omega} t\left(\omega^p\right) = t\left(\sum_{i<\omega}\omega^p\right) = t\left(\omega^{p+1}\right).
\end{aligned}
$$

Hence

$$t\left(\omega^{p+2}\right) = t\left(\omega^{p+1}\right).$$

(This is false only if $k = \ell + 1$ and $t(\omega^\ell) = t(\omega^{\ell+1})$ but then the lemma holds trivially.)

Assuming $p + 3 \leq s$, as $\omega^{p+3} = \sum_{i<\omega}(\omega^{p+2} + \omega^{p+1})$ we have

$$
\begin{aligned}
t\left(\omega^{p+3}\right) &= t\left(\sum_{i<\omega}(\omega^{p+2} + \omega^{p+1})\right) = \sum_{i<\omega} t\left(\omega^{p+2} + \omega^{p+1}\right) \\
&= \sum_{i<\omega}\left(t(\omega^{p+2}) + t(\omega^{p+1})\right) = \sum_{i<\omega}\left(t(\omega^{p+1}) + t(\omega^{p+1})\right) \\
&= \sum_{i<\omega} t\left(\omega^{p+1}\right) = t\left(\sum_{i<\omega}\omega^{p+1}\right) = t\left(\omega^{p+2}\right).
\end{aligned}
$$

Hence

$$t\left(\omega^{p+3}\right) = t\left(\omega^{p+2}\right).$$

So for every $j > 0$, $t(\omega^{p+1}) = t(\omega^{p+j})$ and in particular $t(\omega^{p+1}) = t(\omega^s) = t(\omega^p) = t_i$.

This proves the first part of the lemma. As for the moreover clause, since $\omega^{p+1} = \omega^p + \omega^{p+1}$ we have

$$t_i = t\left(\omega^{p+1}\right) = t\left(\omega^p + \omega^{p+1}\right) = t\left(\omega^p\right) + t\left(\omega^{p+1}\right) = t_i + t_i. \qquad \dashv$$

PROPOSITION 5.2. *The formula $\varphi(X, Y)$ saying "if Y is without a last element then $X \subseteq Y$ is an ω-sequence unbounded in Y (and if not then $X = \emptyset$)" cannot be uniformized in $(\omega^\omega, <)$.*

Moreover, if $\psi_m(X, Y, \bar{P}_m)$ uniformize φ on ω^m then one of the sets $\{\mathrm{dp}(\psi_m) : m < \omega\}$ or $\{\lg(\bar{P}_m) : m < \omega\}$ is unbounded.

PROOF. Suppose the second statement fails, then:

(†) there is a formula $\psi(X, Y, \bar{Z})$ such that for an unbounded set $I \subseteq \omega$, for every $m \in I$ there is $\bar{P}_m \subseteq \omega^m$ such that $\psi(X, Y, \bar{P}_m)$ uniformizes φ on ω^m (i.e., ψ chooses a cofinal ω-sequence X in every unbounded Y).

Let $n = \mathrm{dp}(\psi)+1$ and $M := |\mathscr{T}_{n,\lg(\bar{Z})+2}|$. Let $m \in I$ be large enough ($m > 2M+3$ will do), denote $\bar{P}_m = \bar{P}$ and let's show that $\psi(X, Y, \bar{P})$ does not work for ω^m and some subset $Y_1 \subseteq \omega^m$ that will be defined below.

When $\alpha < \omega^m$ we have $\alpha = \omega^{m-1}k_{m-1} + \omega^{m-2}k_{m-2} + \cdots + \omega k_1 + k_0$. Let $k(\alpha) := \min\{i : k_i \neq 0\}$ and let $A_k := \{\alpha < \omega^m : k(\alpha) = k\}$. Note that $\mathrm{otp}(A_k) = \omega^{m-k}$.

For $k \in \{1, 2, \ldots, m-1\}$ we will choose $Y_k \subseteq A_k$ with $\mathrm{otp}(Y_k) = \mathrm{otp}(A_k) = \omega^{m-k}$ such that for $\alpha < \beta$ in Y_k:

$$(*) \qquad \mathrm{Th}^n(\omega^m; \bar{P}, Y_k) \restriction_{[\alpha,\beta)} \text{ depends only on } \mathrm{otp}\left([\alpha, \beta) \cap Y_k\right)$$

we will start with $k = m-1$ and proceed by inverse induction:

Let $A_{m-1} = \langle \alpha_j : j < \omega \rangle$. Let for $\ell < p < \omega$, $h(\ell, p) := \mathrm{Th}^n(\omega^m; \bar{P}, \alpha_\ell) \restriction_{[\alpha_\ell, \alpha_p)}$. Let $J \subseteq \omega$ be homogeneous with respect to this colouring namely, for some fixed theory t_{m-1}, for every $\ell < p$ in J,

$$\mathrm{Th}^n(\omega^m; \bar{P}, \alpha_\ell) \restriction_{[\alpha_\ell, \alpha_p)} = t_{m-1}.$$

By the Composition Theorem, for every $\ell < p$ in J,

$$\mathrm{Th}^n(\omega^m; \bar{P}, Y_{m-1}) \restriction_{[\alpha_\ell, \alpha_p)} = t_{m-1} \cdot |Y_{m-1} \cap [\alpha_\ell, \alpha_p)|$$

and this proves $(*)$ for Y_{m-1}.

Rename Y_{m-1} by $\langle \alpha_i : i < \omega \rangle$. In each segment $[\alpha_i, \alpha_{i+1})$ choose $\langle \beta^i_\ell : 0 < \ell < \omega \rangle \subseteq A_{m-2}$ increasing and cofinal such that for every $\ell < p < \omega$ the theory $\mathrm{Th}^n(\omega^m; \bar{P}, \beta^i_\ell) \restriction_{[\beta^i_\ell, \beta^i_p)}$ is constant.

Returning to Y_{m-1}, for $i < j < \omega$ let

$$h_1(i, j) := \left\langle \mathrm{Th}^n(\omega^m; \bar{P}) \restriction_{[\alpha_i, \beta^{j-1}_1)}, \mathrm{Th}^n(\omega^m; \bar{P}, \beta^{j-1}_1) \restriction_{[\beta^{j-1}_1, \beta^{j-1}_2)} \right\rangle$$

w.l.o.g. (by thinning out and re-renaming and noting that we don't harm $(*)$) Y_{m-1} is homogeneous with respect to this colouring.

Hence, for some theories t^* and t_{m-2}, for every $i < j < \omega$ we have

$$h_1(i, j) = \langle t^*, t_{m-2} \rangle.$$

Let $Y_{m-2} := \langle \beta^i_\ell : 0 < \ell < \omega, i < \omega \rangle$, clearly $\mathrm{otp}(Y_{m-2}) = \omega^2$. Let's check $(*)$ for Y_{m-2}:

Firstly, note that for $\ell < p < \omega$,

$$\mathrm{Th}^n(\omega^m; \bar{P}, Y_{m-2}) \restriction_{[\beta^i_\ell, \beta^i_p)} = t_{m-2} \cdot (p - l).$$

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Secondly, for $i < j < \omega$ $\mathrm{Th}^n(\omega^m; \bar{P}, Y_{m-2}) \restriction_{[\beta^i_\ell, \beta^j_p)}$

$$= \mathrm{Th}^n(\omega^m; \bar{P}, Y_{m-2}) \restriction_{[\beta^i_\ell, \alpha_{i+1})} + \mathrm{Th}^n(\omega^m; \bar{P}, Y_{m-2}) \restriction_{[\alpha_{i+1}, \alpha_{i+2})}$$
$$+ \cdots + \mathrm{Th}^n(\omega^m; \bar{P}, Y_{m-2}) \restriction_{[\alpha_{j-1}, \alpha_j)} + \mathrm{Th}^n(\omega^m; \bar{P}, Y_{m-2}) \restriction_{[\alpha_j, \beta^j_p)}$$

where the first theory is equal to $t_{m-2} \cdot \omega$, the last theory is $t^* + t_{m-2} \cdot (p - l)$, and the middle theories are $t^* + t_{m-2} \cdot \omega$. These observations prove $(*)$ for Y_{m-2}.

For defining Y_{m-3} let's restrict ourselves to a segment $[\alpha_i, \alpha_{i+1})$ where $\alpha_i, \alpha_{i+1} \in Y_{m-1}$. In this segment we have defined $\langle \beta^i_\ell : 0 < \ell < \omega \rangle \subseteq Y_{m-2}$. Now choose in each $[\beta^i_\ell, \beta^i_{\ell+1})$ an increasing cofinal sequence $\langle \gamma^{i,\ell}_j : 0 < j < \omega \rangle$ such that for $j < p < \omega$, $\mathrm{Th}^n(\omega^m; \bar{P}, \gamma^{i,\ell}_j) \restriction_{[\gamma^{i,\ell}_j, \gamma^{i,\ell}_p)}$ is constant.

For $0 < \ell < p < \omega$ let

$$h^i_1(\ell, p) := \left\langle \mathrm{Th}^n(\omega^m; \bar{P}) \restriction_{[\beta^i_\ell, \gamma^{i,p-1}_1)}, \mathrm{Th}^n(\omega^m; \bar{P}, \gamma^{i,p-1}_1) \restriction_{[\gamma^{i,p-1}_1, \gamma^{i,p-1}_2)} \right\rangle$$

and again w.l.o.g. we may assume that $\langle \beta^i_\ell : 0 < \ell < \omega \rangle$ is homogeneous with respect to h^i_1.

Next, for $i < j < \omega$ define

$$h_2(i, j) := \left\langle \mathrm{Th}^n(\omega^m; \bar{P}) \restriction_{[\alpha_i, \gamma^{j-1,1}_1)}, \mathrm{Th}^n(\omega^m; \bar{P}, \gamma^{j-1,1}_1) \restriction_{[\gamma^{j-1,1}_1, \gamma^{j-1,1}_2)} \right\rangle$$

by thinning out and renaming we may assume that Y_{m-1} is homogeneous with respect to h_2, now Y_{m-2} is also thined out but each new $\langle \beta^i_\ell : 0 < \ell < \omega \rangle$ which is some old $\langle \beta^{i^*}_\ell : 0 < \ell < \omega \rangle$ is still homogeneous.

As a result we will have, for some theories t^{**}, t^{***}, t_{m-3}:

$$(\forall i < j < \omega)(\forall 0 < \ell < p < \omega)\left[h^i_1(\ell, p) = \langle t^{**}, t_{m-3} \rangle \;\&\; h_2(i, j) = \langle t^{***}, t_{m-3} \rangle\right].$$

Let $Y_{m-3} := \{\gamma^{i,\ell}_j : i < \omega, 0 < \ell < \omega, 0 < j < \omega\}$, as before $(*)$ holds by noting that if for example $i_1 < i_2 < \omega$ and $\ell_1 < \ell_2$ then

$$\mathrm{Th}^n(\omega^m; \bar{P}, \gamma^{i_1,\ell_1}_{j_1}) \restriction_{[\gamma^{i_1,\ell_1}_{j_1}, \gamma^{i_2,\ell_2}_{j_2})} = t_{m-3} \cdot \omega$$
$$+ (t^{**} + t_{m-3} \cdot \omega) \cdot \omega + [t^{***} + (t^{**} + t_{m-3} \cdot \omega) \cdot \omega] \cdot (i_2 - i_1 - 1)$$
$$+ t^{***} + t_{m-3} \cdot \omega + (t^{**} + t_{m-3} \cdot \omega)(\ell_2 - \ell_1) + t^{**} + t_{m-3} \cdot (j_2 - 1)$$

and similarly for the other possibilities.

$Y_{m-4}, Y_{m-5}, \ldots, Y_1$ are defined by using the same prescription i.e., $Y_{m-\ell}$ is defined by taking a homogeneous sequence between two successive elements of $Y_{m-\ell-1}$ then homogeneous sequences between two successive elements of $Y_{m-\ell-2}$ by using colouring of the form $h_1, h_2, \ldots$. The thinning out and w.l.o.g.'s for already defined Y_{m-k}'s are not necessary (as we are interested only in $(*)$ for Y_1) but they ease notations considerably.

We will show now that ψ doesn't choose an unbounded ω-sequence in Y_1 that is, for every ω-sequence $X \subseteq Y_1$ there is a different ω-sequence $X' \subseteq Y_1$ such that $\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X')$.

By $(*)$, for $\alpha < \beta$ in Y_1 the σ-additive colouring $f(\alpha, \beta) := \mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\alpha,\beta)}$ depends only on $\mathrm{otp}([\alpha, \beta) \cap Y_1)$ hence we can apply Lemma 5.1 and conclude that for some $p \leq m/2$, for every $r \geq p$, $\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\alpha,\beta)}$ is equal to some

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fixed theory t whenever $\mathrm{otp}([\alpha,\beta) \cap Y_1) = \omega^r$. (Remember that f has at most M possibilities and that $m > 2M$). Moreover, we know that $t + t = t$.

Assume now that for some $X \subseteq Y_1$, $\psi(X, Y_1, \bar{P})$ holds, so X is a cofinal ω-sequence. Let $X = \{\delta_i : i < \omega\}$. As $\mathrm{otp}(Y_1) = \omega^{m-1}$ for unboundedly many i's we have $\mathrm{otp}([\delta_i, \delta_{i+1}) \cap Y_1) \geq \omega^{m-2} > \omega^p$.

Let $\beta_i := \mathrm{otp}([\delta_i, \delta_{i+1}) \cap Y_1)$ and when $\mathrm{otp}([\alpha,\beta) \cap Y_1) = \varepsilon$ denote $\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\alpha,\beta)}$ by $t(\varepsilon)$ (by $(*)$ it doesn't matter which α and β we use). We are interested in $\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X)$ which is

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \emptyset) \restriction_{[0,\delta_0)} + \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_0) \restriction_{[\delta_0,\delta_1)} + \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_1) \restriction_{[\delta_1,\delta_2)} + \cdots .$$

Now $\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \restriction_{[\delta_i,\delta_{i+1})}$ is determined by $\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\delta_i,\delta_{i+1})} = t(\beta_i)$ (as δ_i is the first element in $[\delta_i, \delta_{i+1}) \cap Y_1$), and abusing notations we will say

$$(**) \qquad \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) \simeq t(\delta_0) + \sum_{i<\omega} t(\beta_i).$$

Let $i < \omega$ be such that $\beta_i \geq \omega^{m-2}$ and let $j > i$ be the first with $\beta_j \geq \omega^{m-2}$.

First case: $j = i + 1$.

Let $\beta_i = \mathrm{otp}([\delta_i, \delta_{i+1}) \cap Y_1) = \omega^{m-2} \cdot k_1 + \varepsilon_1$ and $\beta_{i+1} = \mathrm{otp}([\delta_{i+1}, \delta_{i+2}) \cap Y_1) = \omega^{m-2} \cdot k_2 + \varepsilon_2$ where $k_1, k_2 \geq 1$ and $\varepsilon_1, \varepsilon_2 < \omega^{m-2}$.

Define $\gamma :=$ the $\omega^{m-2} \cdot k_1 + \omega^{m-3} + \varepsilon_1$'th successor of δ_i in Y_1. So $\delta_{i+1} < \gamma < \delta_{i+2}$ but $\mathrm{otp}([\gamma, \delta_{i+2}) \cap Y_1) = \beta_{i+1}$ hence

$$\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\gamma,\delta_{i+2})} = \mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\delta_{i+1},\delta_{i+2})} = t(\beta_{i+1})$$

hence

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \gamma) \restriction_{[\gamma,\delta_{i+2})} = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_{i+1}) \restriction_{[\delta_{i+1},\delta_{i+2})} .$$

On the other hand,

$$\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\delta_i,\gamma)} = t(\omega^{m-2} \cdot k_1) + t(\omega^{m-3}) + t(\varepsilon_1)$$

but $m - 3 \geq p$ hence $t(\omega^{m-3}) = t(\omega^{m-2}) = t$ moreover $t + t = t$ and it follows that

$$\begin{aligned} t(\omega^{m-2} \cdot k_1) + t(\omega^{m-3}) &= t(\omega^{m-2}) \cdot k_1 + t(\omega^{m-3}) \\ &= t(\omega^{m-2}) \cdot (k_1 + 1) = t(\omega^{m-2}) \cdot k_1 = t(\omega^{m-2} \cdot k_1) \end{aligned}$$

hence

$$\mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\delta_i,\gamma)} = t(\omega^{m-2} \cdot k_1) + t(\varepsilon_1) = \mathrm{Th}^n(\omega^m; \bar{P}, Y_1) \restriction_{[\delta_i,\delta_{i+1})} = t(\beta_i)$$

hence

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \restriction_{[\delta_i,\gamma)} = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \restriction_{[\delta_i,\delta_{i+1})} .$$

Now all other relevant theories are left unchanged therefore, letting $X' := X \setminus \{\delta_{i+1}\} \cup \{\gamma\}$ we get $X \neq X'$ but

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X').$$

General case: $j = i + \ell$.

Look at $\delta_{i+1}, \delta_{i+2}, \dots, \delta_{i+\ell-1}, \delta_{i+\ell} = \delta_j$. We'll define $\gamma_1, \gamma_2, \dots, \gamma_\ell$ with $\delta_{i+k} < \gamma_k < \delta_{i+k+1}$ for $0 < k < \ell$ and $\gamma_\ell = \delta_{i+\ell} = \delta_j$. This will be done by 'shifting' the δ_{i+k}'s by ω^{m-3} (remember that $\beta_{i+k} < \omega^{m-2}$ for $0 < k < \ell$).

Assume as before that $\beta_i = \mathrm{otp}([\delta_i, \delta_{i+1}) \cap Y_1) = \omega^{m-2} \cdot k_1 + \varepsilon_1$ where $k_1 \geq 1$ and $\varepsilon_1 < \omega^{m-2}$.

Define $\gamma_1 :=$ the $\omega^{m-2} \cdot k_1 + \omega^{m-3} + \varepsilon_1$'th successor of δ_i in Y_1, $\gamma_2 :=$ the β_{i+1}'th successor of γ_1 in Y_1, $\gamma_3 :=$ the β_{i+2}'th successor of γ_2 in Y_1 and so on, γ_ℓ will clearly be equal to δ_j.

As before we have for $1 < k \leq \ell$, (by preserving the order types)

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \gamma_k) \restriction_{[\gamma_k, \gamma_{k+1})} = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_{i+k}) \restriction_{[\delta_{i+k}, \delta_{i+k+1})} .$$

and (using $t + t = t$)

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \restriction_{[\delta_i, \gamma_1)} = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, \delta_i) \restriction_{[\delta_i, \delta_{i+1})} .$$

Letting $X' := X \setminus \{\delta_{i+1}, \delta_{i+2}, \dots, \delta_{j-1}\} \cup \{\gamma_1, \gamma_2, \dots, \gamma_{\ell-1}\}$ we get $X \neq X'$ but

$$\mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X) = \mathrm{Th}^{n-1}(\omega^m; \bar{P}, Y_1, X').$$

Since $\mathrm{dp}(\psi) = n - 1$, X is not the unique ω-sequence chosen by ψ from Y_1. Therefore, ψ does not uniformize φ on ω^m, a contradiction. This proves the second part of the claim.

Now the first part follows from the second: if there was a formula $\psi(X, Y, \bar{P})$ with $\mathrm{dp}(\psi) = n$ uniformizing $\varphi(X, Y)$ in $(\omega^\omega, <)$, then "X is the chosen ω-sequence" would be expressed by $\mathrm{Th}^n(\omega^\omega; X, Y, \bar{P})$. Now for every $m < \omega$ and $X \subseteq Y \subseteq \omega^m$ we have

$$\begin{aligned}\mathrm{Th}^n(\omega^\omega; X, Y, \bar{P}) &= \mathrm{Th}^n(\omega^\omega; X, Y, \bar{P}) \restriction_{[0,\omega^m)} + \mathrm{Th}^n(\omega^\omega; X, Y, \bar{P}) \restriction_{[\omega^m,\omega^\omega)} \\ &= \mathrm{Th}^n(\omega^\omega; X, Y, \bar{P}) \restriction_{[0,\omega^m)} + \mathrm{Th}^n(\omega^\omega; \emptyset, \emptyset, \bar{P}) \restriction_{[\omega^m,\omega^\omega)} .\end{aligned}$$

The second theory does not depend on the choice of X and Y, hence for $X \subseteq Y \subseteq \omega^m$, $\psi(X, Y, \bar{P})$ depends only on $\mathrm{Th}^n(\omega^\omega; X, Y, \bar{P}) \restriction_{[0,\omega^m)}$. Therefore we can have, for each $m < \omega$ some $\psi_m(X, Y, \bar{P}_m)$ that uniformizes φ, with depth n and parameters $\bar{P} \cap [0, \omega^m)$ and this contradicts what we have just proved. ⊣

THEOREM 5.3. *If C has the uniformization property then C is tame and* $\log(C) \neq \infty$.

PROOF. By [6] chains that are not tame do not have even a definable choice function.

If $\alpha \geq \omega^\omega$ then the $\varphi(X, Y)$ saying "if Y has a cofinal ω-sequence then X is one" cannot be uniformized in $(\alpha, <)$. (Choose a suitable $Y \subseteq \omega^\omega \subseteq \alpha$ and apply the composition theorem as in the end of the previous proof.) If $(C, <_C)$ is tame, has the uniformization property and $\log(C) = \infty$ then by Lemma 2.6 $<_C$ is definable in some $\alpha \geq \omega^\omega$ and $<_\alpha$ is definable in $(C, <_C)$. To uniformize $\varphi(X, Y)$ in $(\alpha, <)$ just translate it to a formula about the definable $<_\alpha$ in $(C, <_C)$ and use the uniformization property in C. But $<_C$ is definable in α and $(\alpha, <_C) \cong (C, <_C)$ so we can uniformize φ in α. This is a contradiction. ⊣

§6. **Very tame trees.** In this section we define very tame trees and prove that these are exactly the trees that have the uniformization property. We start with proving that very tame chains (scattered with log $< \infty$) have the uniformization property. Call $\psi(\bar{X}, \bar{Y}, \bar{P})$ *uniformizable* in C if $C \models (\forall \bar{Y})(\exists \bar{X})[\psi(\bar{X}, \bar{Y}, \bar{P})]$.

PROPOSITION 6.1. *If the ordinals α and β have the uniformization property then so do $\alpha + \beta$ and $\alpha\beta$.*

PROOF. $\alpha + \beta$ is similar to $\alpha + \alpha = \alpha \cdot 2$ and we concentrate on multiplication.

Let $\varphi(X, Y, \bar{Q})$ be uniformizable in $\alpha\beta$ with $\mathrm{dp}(\varphi) = n$ and $\lg(\bar{Q}) = \ell$. $\alpha\beta$ is $\sum_{\gamma<\beta}[\alpha\gamma, \alpha(\gamma+1))$ where each segment $[\alpha\gamma, \alpha(\gamma+1))$ is isomorphic to α hence has the uniformization property. Let $K \subseteq \alpha\beta$ be $\{\alpha\gamma : \gamma < \beta\}$, K is isomorphic to β hence also has the uniformization property. To choose a unique X we first choose $X \cap [\alpha\gamma, \alpha(\gamma+1))$, then we partition K where for each $\alpha\gamma \in K$ describe the instructions for choosing $X \cap [\alpha\gamma, \alpha(\gamma+1))$. Using the uniformization property of K we choose a unique set of instruction and obey them to choose X.

Let $\langle t_0, \dots, t_{a-1}\rangle$ be an enumeration of the the theories in $\mathscr{T}_{n,\ell+2}$. For $i < a$ and $X, Y \subseteq \alpha\beta$ define $P_i(X, Y, \bar{Q}) \subseteq K := \{\alpha\gamma : \gamma < \beta\}$ by

$$P_i(X, Y, \bar{Q}) := \{\alpha\gamma : \mathrm{Th}^n(\alpha\beta; X, Y, \bar{Q}) \restriction_{[\alpha\gamma, \alpha\gamma+\alpha)} = t_i\}$$

it follows that, for every $X, Y \subseteq \alpha\beta$, $\bar{P} = \bar{P}(X, Y, \bar{Q}) = \langle P_0(X, Y, \bar{Q}), \dots, P_{a-1}(X, Y, \bar{Q})\rangle$ is a partition of K that is definable from $X, Y, \bar{Q}$ and K.

$\alpha\beta = \sum_{\gamma<\beta}[\alpha\gamma, \alpha\gamma + \alpha)$ and by the Feferman-Vaught Theorem 3.7 there is $m = m(n, \ell)$ such that $\mathrm{Th}^m(K; \bar{P}(X, Y, \bar{Q}))$ determines $\mathrm{Th}^n(\alpha\beta; X, Y, \bar{Q})$.

Let $\mathscr{R} = \{r_0, \dots, r_{c-1}\} \subseteq \mathscr{T}_{m,\lg(\bar{P})}$ be the set of theories that satisfy, for every $X, Y \subseteq \alpha\beta$:

$$\mathrm{Th}^m(K; \bar{P}(X, Y, \bar{Q})) \in \mathscr{R} \Rightarrow \alpha\beta \models \varphi(X, Y, \bar{Q}).$$

Now let $\langle s_0, \dots, s_{b-1}\rangle$ be an enumeration of the theories in $\mathscr{T}_{n+1,\ell+1}$. For $i < b$ and $Y \subseteq \alpha\beta$ define $R_i^0(Y, \bar{Q}) \subseteq K$ by

$$R_i^0(Y, \bar{Q}) := \{\alpha\gamma : \mathrm{Th}^{n+1}(\alpha\beta; Y, \bar{Q}) \restriction_{[\alpha\gamma, \alpha\gamma+\alpha)} = s_i\}$$

as before, for every $Y \subseteq \alpha\beta$, $\bar{R}^0 = \bar{R}^0(Y, \bar{Q}) = \langle R_0^0(Y, \bar{Q}), \dots, R_{b-1}^0(Y, \bar{Q})\rangle$ is a partition of K that is definable from $Y, \bar{Q}$ and K.

Now let $\bar{R}^1 = \langle R_0^1, \dots, R_{a-1}^1\rangle$ be any partition of K. We will say that $\bar{R}^0(Y, \bar{Q})$ and $\bar{R}^1$ are **coherent** if

(1) $\alpha\gamma \in (R_i^0 \cap R_j^1)$ implies that for every chain C, $B \subseteq C$ and $\bar{D} \subseteq C$ of length ℓ:

if $\mathrm{Th}^{n+1}(C; B, \bar{D}) = s_i$ then $(\exists A \subseteq C)\left[\mathrm{Th}^n(C; A, B, \bar{D}) = t_j\right]$,

(2) $\mathrm{Th}^m(K; \bar{R}^1) \in \mathscr{R}$.

Since a, b and c are finite, there is a formula $\theta_1(\bar{U}, \bar{W})$ (with $\lg(\bar{U}) = b$ and $\lg(\bar{W}) = a$) such that for any $\bar{R}^0, \bar{R}^1 \subseteq K$,

$K \models \theta_1(\bar{R}^0, \bar{R}^1)$ iff $\bar{R}^0$ and $\bar{R}^1$ are coherent partitions of K.

Moreover, as $K \cong \beta$ and β has the uniformization property, there exists $\bar{S} \subseteq K$ and a formula $\theta_2(\bar{U}, \bar{W}, \bar{S})$ such that for every $\bar{R}^0 \subseteq K$

if $(\exists \bar{W})\theta_1(\bar{R}^0, \bar{W})$ then $(\exists! \bar{W})[\theta_2(\bar{R}^0, \bar{W}, \bar{S}) \,\&\, \theta_1(\bar{R}^0, \bar{W})]$. Let $\theta(\bar{U}, \bar{W}, \bar{S}) := \theta_1 \wedge \theta_2$.

Now let $Y \subseteq \alpha\beta$, let $\bar{R}^0 = \bar{R}^0(Y, \bar{Q})$ and suppose that $\bar{R}^0$ and some $\bar{R}^1$ are coherent partitions of K. When $\alpha\gamma \in (R_i^0 \cap R_j^1)$, we know by the first clause in the definition of coherence that

$$(\exists X \subseteq \alpha\beta)\left[\mathrm{Th}^n(\alpha\beta; X, Y, \bar{Q}) \restriction_{[\alpha\gamma,\alpha\gamma+\alpha)} = t_j\right].$$

Now as $[\alpha\gamma, \alpha\gamma + \alpha) \cong \alpha$ and α has the uniformization property, there is $\bar{T}_\gamma \subseteq [\alpha\gamma, \alpha\gamma + \alpha)$ and a formula $\psi_j^\gamma(X, Y, \bar{T}_\gamma)$ (of depth $k(n, \ell)$ that depends only on n and ℓ) that uniformizes the formula that says "$\mathrm{Th}^n(\alpha\beta; X, Y, \bar{Q}) \restriction_{[\alpha\gamma,\alpha\gamma+\alpha)} = t_j$". It follows that when $\psi_j^\gamma(X, Y, \bar{T}_\gamma)$ holds, $X \cap [\alpha\gamma, \alpha\gamma + \alpha)$ is unique.

W.l.o.g. all $\bar{T}_\gamma$ have the same length (as we have a finite number of possibilities for t_j) and even, by taking prudent disjunctions, $\psi_j^\gamma(X, Y, \bar{T}_\gamma) = \psi_j(X, Y, \bar{T}_\gamma)$ (i.e., ψ does not depend on γ—only on t_j). Let $\bar{T} = \bigcup_{\gamma<\beta} \bar{T}_\gamma$ (the union is disjoint). We are ready to define $U(X, Y, \bar{Q}, \bar{T}, \bar{S})$ that uniformizes $\varphi(X, Y, \bar{Q})$:

$U(X, Y, \bar{Q}, \bar{T}, \bar{S})$ says: "for every partition $\bar{R}^0$ of K that is equal to [the definable] $\bar{R}^0(Y, \bar{Q})$ every $\bar{R}^1$ that is a [in fact the only] partition that satisfies $\theta(\bar{R}^0, \bar{R}^1, \bar{S})$, *if* $\alpha\gamma \in R_j^1$ and $D = [\alpha\gamma, \alpha\gamma + \alpha)$ [$\alpha\gamma$ and $\alpha\gamma + \alpha$ are two successive elements of K] *then* $D \models \psi_j(X \cap D, Y \cap D, \bar{Q} \cap D, \bar{T} \cap D)$".

Check that $U(X, Y, \bar{Q}, \bar{T}, \bar{S})$ does the job: clause (1) in the definition of coherence and the ψ_j's guarantee that X is unique, clause (2) guarantees that $U(X, Y, \bar{Q}, \bar{T}, \bar{S}) \Rightarrow \varphi(X, Y, \bar{Q})$. ⊣

FACT 6.2. Every finite chain has the uniformization property.

PROOF. Order the subsets of a finite chain C lexicographically. ⊣

THEOREM 6.3. $(\omega, <)$ *has the uniformization property.*

PROOF. By [1]. ⊣

COROLLARY 6.4. *An ordinal α has the uniformization property iff $\alpha < \omega^\omega$. A chain $(C, <)$ has the uniformization property iff* $\log(C) \neq \infty$.

PROOF. By Theorem 5.3 if C has the uniformization property then $\log(C) \neq \infty$. By Proposition 6.1, Fact 6.2 and Theorem 6.3 if $\alpha < \omega^\omega$ then α has uniformization property. Use the interdefinability from Lemma 2.6 to conclude for a general C. ⊣

DEFINITION 6.5. $(T, \lhd)$ is *very tame* if
(1) T is tame.
(2) $\mathrm{Sup}\{\mathrm{Log}(B) : B \subseteq T,\ B \text{ a branch}\} < \omega$.

LEMMA 6.6. *If $(T, \lhd)$ is not very tame then $(T, \lhd)$ does not have the uniformization property.*

PROOF. If T is not tame then by Theorem 2.9 it doesn't have even a definable choice function.

If T is tame but not very tame then either there is a a branch $B \subseteq T$ with $\log(B) = \infty$ or it has branches of unbounded log. Again by Theorem 2.9 there is a definable well ordering of T that, when restricted to branches, can be used to formulate the statement "X is a cofinal ω-sequence in Y". If T has the uniformization property this is uniformized but by the Composition Theorem 3.11 we can do that inside the branch without using the full structure of T. This contradicts Proposition 5.2. $\dashv$

We are now ready to prove the main theorem.

THEOREM 6.7. $(T, \lhd)$ *has the uniformization property iff* $(T, \lhd)$ *is very tame.*

PROOF. Assume T is (n^*, k^*) tame and for every branch $B \subseteq T$ we have $\log(B) < r^*$. Let $\varphi(X, Y, \bar{Q})$ be uniformizable in T with $\mathrm{dp}(\varphi) = n$ and $\lg(\bar{Q}) = \ell$.

By the proof of Theorem 2.9 (see the Appendix) T can be well ordered in the following way: using a set $\bar{K}_0 \subseteq T$ we can, in a definable way, partition T into a disjoint union of sub-branches $\{A_\eta : \eta \in \Gamma\}$ where Γ is a well founded tree. We let $K \subseteq T$ be a set of representatives $\{u_\eta : \eta \in \Gamma\}$ and denote:

(α) $\eta^+ := \{\nu \in \Gamma : \nu$ is an immediate successor of η in $\Gamma\}$

(β) $K_{\eta^+} \subseteq T$ is $\{u_\nu : \nu \in \eta^+\}$

(γ) $T_\eta := \bigcup\{A_\nu : \eta \trianglelefteq \nu$ in $\Gamma\}$.

The following holds:

(i) The tree relation on Γ, $\lhd_\Gamma$, is definable in T as a relation between the members of K.

(ii) A_η, T_η and K_{η^+} are definable from u_η, A_η is a branch of T_η.

(iii) $\{T_\nu : \nu \in \eta^+\}$ are the equivalence classes with respect to the equivalence relation $\sim^1_{A_\eta}$ on T_η. (See Definition 2.7.)

(iv) $K_{\eta^+} = \{u_\nu : \nu \in \eta^+\}$ is a set of representatives of the $\sim^1_{A_\eta}$-equivalence classes hence has a definable linear ordering which is isomorphic to the completion of A_η in T_η (up to taking $\leq n^*$ many copies of each point in this completion).

(v) By very-tameness, $\log(K_{\eta^+}) \leq r^*$.

To ease the notations we assume that $\bar{K}_0$ and K belong to the parameters $\bar{Q}$ in φ.

To uniformize $\varphi(X, Y, \bar{Q})$ (i.e., define a unique $X^* \subseteq T$) we do the following: given $Y \subseteq T$ we will use the decomposition $T = \bigcup_{\eta \in \Gamma} A_\eta$ and the fact that each A_η has the uniformization property, to define a unique $X_\eta \subseteq A_\eta$. As in the proof of Proposition 6.1, for each $u_\eta \in K$ we describe the instructions for choosing $X \cap A_\eta$ and $X \cap T_\eta$ and then, using the fact that Γ is well founded we choose a unique set of instructions that will yield a unique $X^* \subseteq T$ when they are followed.

So let $Y \subseteq T$ and we want to define some $X^* = X^*(Y, \bar{Q}) \subseteq T$. Let $T = A_{\langle\rangle} \cup \bigcup_{\eta \in \langle\rangle^+} T_\eta$. $K_{\langle\rangle^+}$ has a natural structure of a chain and $\log(K_{\langle\rangle^+})$, $\log(A_{\langle\rangle}) < r^*$. Denoting the completion of $A_{\langle\rangle}$ in T by B^* we see by (i)–(iv) above that B^* is interpretable in $K_{\langle\rangle^+}$ therefore, by Theorem 3.11, there is some $m = m(n, \ell)$ such that when $X \subseteq T$ is given, from $\mathrm{Th}^m(A_{\langle\rangle}; X, Y, \bar{Q})$ and $\langle \mathrm{Th}^m(T_\eta; X, Y, \bar{Q}) : \eta \in \langle\rangle^+ \rangle$ we can compute $\mathrm{Th}^n(T; X, Y, \bar{Q})$.

Let $\langle s_0, \dots, s_{b-1}\rangle$ be an enumeration of the theories in $\mathscr{T}_{n+1,\ell+1}$.

Define $\bar{P}^1(Y, \bar{Q})_{\langle\rangle} = \langle P^1_0(Y, \bar{Q})_{\langle\rangle}, \dots, P^1_{b-1}(Y, \bar{Q})_{\langle\rangle}\rangle$ a partition of $K_{\langle\rangle^+}$ by

$$\eta \in \mathscr{P}^1_i(Y, \bar{Q})_{\langle\rangle} \iff \mathrm{Th}^{n+1}(T_\eta; Y, \bar{Q}) = s_i.$$

By the previous remarks $\bar{P}^1(Y, \bar{Q})_{\langle\rangle}$ is definable from $u_{\langle\rangle}$, Y and $\bar{Q}$.

Define $\bar{P}^2(Y, \bar{Q})_{\langle\rangle} = \langle P^2_0(Y, \bar{Q})_{\langle\rangle}, \dots, P^2_{b-1}(Y, \bar{Q})_{\langle\rangle}\rangle$ a partition of $K_{\langle\rangle^+}$ by

$$\eta \in \mathscr{P}^1_i(Y, \bar{Q})_{\langle\rangle} \iff \mathrm{Th}^{n+1}(A_\eta; Y, \bar{Q}) = s_i.$$

Again, $\bar{P}^2(Y,\bar{Q})_{\langle\rangle}$ is definable from $u_{\langle\rangle}$, Y and $\bar{Q}$.

Let $\langle t_0, \dots, t_{a-1}\rangle$ be an enumeration of the theories in $\mathscr{T}_{n,\ell+2}$.

A partition of $K_{\langle\rangle^+}$, $\bar{R}^1 = \langle R^1_0, \dots, R^1_{a-1}\rangle$ is coherent with $\bar{P}^1(Y,\bar{Q})_{\langle\rangle}$ if $P^1_i(Y,\bar{Q})_{\langle\rangle} \cap R^1_j \neq \emptyset$ implies

"for every tree S and $B, \bar{C} \subseteq S$ with $\lg(\bar{C}) = \ell$, if $\mathrm{Th}^{n+1}(S; B, \bar{C}) = s_i$ then there is $A \subseteq S$ such that $\mathrm{Th}^n(S; A, B, \bar{C}) = t_j$".

Similarly a partition of $K_{\langle\rangle^+}$, $\bar{R}^2 = \langle R^2_0, \dots, R^2_{a-1}\rangle$ is coherent with $\bar{P}^2(Y,\bar{Q})_{\langle\rangle}$ if $P^2_i(Y,\bar{Q})_{\langle\rangle} \cap R^2_j \neq \emptyset$ implies

"for every chain S and $B, \bar{C} \subseteq S$ with $\lg(\bar{C}) = \ell$, if $\mathrm{Th}^{n+1}(S; B, \bar{C}) = s_i$ then there is $A \subseteq S$ such that $\mathrm{Th}^n(S; A, B, \bar{C}) = t_j$".

Finally, a pair of partitions of $K_{\langle\rangle^+}$, $\langle \bar{R}^1, \bar{R}^2\rangle$ is **t*-coherent** with the pair $\langle \bar{P}^1(Y,\bar{Q})_{\langle\rangle}, \bar{P}^2(Y,\bar{Q})_{\langle\rangle}\rangle$ if

(1) $\bar{R}^1$ is coherent with $\bar{P}^1(Y,\bar{Q})_{\langle\rangle}$,

(2) $\bar{R}^2$ is coherent with $\bar{P}^2(Y,\bar{Q})_{\langle\rangle}$, and

(3) For every $X \subseteq T$, if $\mathrm{Th}^n(A_{\langle\rangle}; X, Y, \bar{Q}) = t^*$ and if for every $\eta \in \langle\rangle^+$ $\left[\mathrm{Th}^n(T_\eta; X, Y, \bar{Q}) = t_i \iff u_\eta \in R^1_i\right]$, then $T \models \varphi(X, Y, \bar{Q})$.

As $T \models (\exists X)\varphi(X, Y, \bar{Q})$ there are t^* (that will be fixed from now on), $\bar{R}^1$ and $\bar{R}^2$ such that $\langle \bar{R}^1, \bar{R}^2\rangle$ is t^*-coherent with the pair $\langle \bar{P}^1(Y,\bar{Q})_{\langle\rangle}, \bar{P}^2(Y,\bar{Q})_{\langle\rangle}\rangle$.

Moreover, "$\langle \bar{R}^1, \bar{R}^2\rangle$ is t^*-coherent with the pair $\langle \bar{P}^1(Y,\bar{Q})_{\langle\rangle}, \bar{P}^2(Y,\bar{Q})_{\langle\rangle}\rangle$" is determined by $\mathrm{Th}^k(K_{\langle\rangle^+}; \bar{R}^1, \bar{R}^2, \bar{P}^1(Y,\bar{Q})_{\langle\rangle}, \bar{P}^2(Y,\bar{Q})_{\langle\rangle})$ where k depends only on n and l.

The first two clauses are clear (since a and b are finite) and for the third clause use Theorem 3.11 as above. So the statement is expressed by a uniformizable formula $\psi^1(\bar{R}^1, \bar{R}^2, \bar{P}^1(Y,\bar{Q})_{\langle\rangle}, \bar{P}^2(Y,\bar{Q})_{\langle\rangle})$ of depth k. As by a previous remark $\mathrm{Log}(K_{\langle\rangle^+}) < r^*$ there is $\bar{S}_{\langle\rangle} \subseteq K_{\langle\rangle^+}$ and a formula $\psi_{\langle\rangle}(\bar{U}_1, \bar{U}_2, \bar{W}_1, \bar{W}_2, \bar{S}_{\langle\rangle})$ that uniformizes ψ^1.

To conclude the first step use $\mathrm{Log}(A_{\langle\rangle}) < r^*$ to define, by a formula $\theta_{\langle\rangle}(X, Y \cap A_{\langle\rangle}, \bar{Q} \cap A_{\langle\rangle}, \bar{O}_{\langle\rangle})$ and a sequence of parameters $\bar{O}_{\langle\rangle} \subseteq A_{\langle\rangle}$, a unique $X_{\langle\rangle} \subseteq A_{\langle\rangle}$ that will satisfy $\mathrm{Th}^n(A_\eta; X_{\langle\rangle}, Y, \bar{Q}) = t^*$.

The result of the first step is the following:

(a) We have defined $X_{\langle\rangle} \subseteq A_{\langle\rangle}$ using $\bar{O}_{\langle\rangle} \subseteq A_{\langle\rangle}$ and $\theta_{\langle\rangle}$. $X_{\langle\rangle}$ is the intersection of the eventual X^* with $A_{\langle\rangle}$.

(b) We have chosen $\bar{R}^1_{\langle\rangle^+}, \bar{R}^2_{\langle\rangle^+} \subseteq K_{\langle\rangle^+}$ using ψ and $\bar{S}_{\langle\rangle}$.

(c) $\bar{R}^1_{\langle\rangle^+}$ and $\bar{R}^2_{\langle\rangle^+}$ tell us what are (for $\eta \in \langle\rangle^+$) the theories $\mathrm{Th}^n(T_\eta; X^*, Y, \bar{Q})$ and $\mathrm{Th}^n(A_\eta; X_\eta, Y, \bar{Q})$ respectively: if $u_\eta \in R^1_i$ then the eventual $X^* \cap T_\eta \subseteq T_\eta$ will satisfy $\mathrm{Th}^n(T_\eta; X^* \cap T_\eta, Y, \bar{Q}) = t_i$ and if $u_\eta \in R^2_j$ then the soon to be defined $X_\eta \subseteq A_\eta$ will satisfy $\mathrm{Th}^n(A_\eta; X_\eta, Y, \bar{Q}) = t_j$.

We will proceed by induction on the level of η in Γ (remember, all the levels are $< \omega$) to define $\bar{S}_\eta, \bar{O}_\eta \subseteq A_\eta$ and $\bar{R}^1_{\eta^+}, \bar{R}^2_{\eta^+} \subseteq K_{\eta^+}$ and $X_\eta \subseteq T_\eta$.

The induction step:

We are at $\nu \in \Gamma$ where $\nu \in \eta^+$ and we want to define $\bar{S}_\nu, \bar{O}_\nu \subseteq A_\nu$, $\bar{R}^1_{\nu^+}, \bar{R}^2_{\nu^+} \subseteq K_{\nu^+}$ and $X_\nu \subseteq T_\nu$. Now as $\bar{R}^1_{\eta^+}$ and $\bar{R}^2_{\eta^+}$ are defined, u_ν belongs to one member of $\bar{R}^1_{\eta^+}$

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say the i_1'th and to one member of $\bar{R}^2_{\eta^+}$ say the i_2'th. This implies that there is some $X'_\nu \subseteq T_\nu$ such that $\mathrm{Th}^n(T_\nu; X'_\nu, Y, \bar{Q}) = t_{i_1}$ and $\mathrm{Th}^n(A_\nu; X'_\nu \cap A_\nu, Y, \bar{Q}) = t_{i_2}$.

Let $\bar{P}^1(Y, \bar{Q})_\nu$ and $\bar{P}^2(Y, \bar{Q})_\nu$ be partitions of K_{ν^+} that are defined as in the first step by saying, for $\tau \in \nu^+$, what are $\mathrm{Th}^{n+1}(T_\tau; Y, \bar{Q})$ and $\mathrm{Th}^{n+1}(A_\tau; Y, \bar{Q})$. $\langle \bar{R}^1_{\nu^+}, \bar{R}^2_{\nu^+} \rangle \subseteq K_{\nu^+}$ will be a pair that is t_{i_1}, t_{i_2}-**coherent** with $\langle \bar{P}^1(Y, \bar{Q})_\nu, \bar{P}^2(Y, \bar{Q})_\nu \rangle$ that is:

(1) $\bar{R}^1_{\nu^+}$ is coherent with $\bar{P}^1(Y, \bar{Q})_\nu$,
(2) $\bar{R}^2_{\nu^+}$ is coherent with $\bar{P}^2(Y, \bar{Q})_\nu$, and
(3) For every $X \subseteq T_\nu$ if $\mathrm{Th}^n(A_\nu; X, Y, \bar{Q}) = t_{i_2}$ and for every $\tau \in \nu^+$ $\left[\mathrm{Th}^n(T_\tau; X, Y, \bar{Q}) = t_i \iff u_\tau \in \text{the } i\text{'th member of } R^1_{\nu^+}\right]$, then $\mathrm{Th}^n(T_\nu; X, Y, \bar{Q}) = t_{i_1}$.

Using $\mathrm{Log}(K_{\nu^+}) < r^*$ choose $\bar{S}_{\nu^+} \subseteq K_{\nu^+}$ and $\psi_{i_1,i_2}(\bar{R}^1, \bar{R}^2, \bar{P}^1(Y, \bar{Q})_{\nu^+}, \bar{P}^2(Y, \bar{Q})_{\nu^+}, \bar{S}_{\nu^+})$ that uniformizes the formula that says "$\langle \bar{R}^1, \bar{R}^2 \rangle$ is t_{i_1}, t_{i_2}-coherent with $\langle \bar{P}^1(Y, \bar{Q})_\nu, \bar{P}^2(Y, \bar{Q})_\nu \rangle$". We may assume that ψ_{i_1,i_2} depends only on i_1 and i_2 and that $\lg(\bar{S}_{\nu^+})$ is constant.

Use $\mathrm{Log}(A_\nu) < r^*$ to define, by a formula $\theta_{i_2}(X, Y \cap A_\nu, \bar{Q} \cap A_\nu, \bar{O}_\nu)$ and a sequence of parameters $\bar{O}_\nu \subseteq A_\nu$, a unique $X_\nu \subseteq A_\nu$ that will satisfy $\mathrm{Th}^n(A_\nu; X_\nu, Y, \bar{Q}) = t_{i_2}$. Again, we may assume that θ_{i_2} depends only on i_2 and that $\lg(\bar{O}_\nu)$ is constant.

So $\bar{S}_\nu$, $\bar{O}_\nu$, $\bar{R}^1_{\nu^+}$, $\bar{R}^2_{\nu^+}$ and X_ν are defined and we have concluded the inductive step. (Note that nothing will really go wrong if ν doesn't have any successors in Γ.)

Let $\bar{O} = \bigcup_{\eta \in \Gamma} \bar{O}_\eta$, $\bar{S} = \bigcup_{\eta \in \Gamma} \bar{S}_\eta$. The uniformizing formula $U(X, Y, \bar{Q}, \bar{O}, \bar{S})$ says:

"$X \cap A_{\langle\rangle}$ is defined as in the first step, and for every pair of partitions $\langle \bar{P}^1, \bar{P}^2 \rangle$ of K that agrees on each K_{η^+} with [the definable] $\langle \bar{P}^1_{\eta^+}(Y, \bar{Q}), \bar{P}^2_{\eta^+}(Y, \bar{Q}) \rangle$, (and agrees with $\langle \bar{P}^1_{\langle\rangle}, \bar{P}^2_{\langle\rangle} \rangle$ on $K_{\langle\rangle^+}$), and for every $\langle \bar{R}^1, \bar{R}^2 \rangle$ that is a [in fact the only] pair of partitions that satisfies for every $u_\eta \in K$: if $u_\eta \in P^1_{i_1} \cap P^2_{i_2}$ then $\psi_{i_1,i_2}(\bar{R}^1 \cap K_{\eta^+}, \bar{R}^2 \cap K_{\eta^+}, \bar{P}^1 \cap K_{\eta^+}, \bar{P}^2 \cap K_{\eta^+}, \bar{S} \cap K_{\eta^+})$ holds, (and agrees with $\langle \bar{R}^1_{\langle\rangle}, \bar{R}^2_{\langle\rangle} \rangle$ on $K_{\langle\rangle^+}$), for every $u_\eta \in K$ if $u_\eta \in R^2_i$ then $\theta_i(X \cap A_\eta, Y \cap A_\eta, \bar{Q} \cap A_\eta, \bar{O} \cap A_\eta)$ holds."

$U(X, Y, \bar{Q}, \bar{O}, \bar{S})$ does the job because it defines $X \cap A_\eta$ uniquely on each A_η and because, (by the conditions of coherence) the union of the parts, X, satisfies $\varphi(X, Y, \bar{Q})$. Note also that U does not depend on Y. ⊣

The concluding remark shows that we can't hope to generalize the results to the class of partial orders.

FACT 6.8. Every partial order P can be embedded in a partial order $(P^*, <^*)$ in which P is well orderable in a definable way.

PROOF. Given a partial order $(P, <_P)$ let $\langle p_i : i < \lambda \rangle$ an enumeration of the elements of P (λ an ordinal). Let $Q := \langle q_i : i < \lambda \rangle$ and $R := \langle r_i : i < \lambda \rangle$ be lists of copies of the elements of P. P^* will be $P \cup Q \cup R$ and the partial order $<^*$ on P^* will be the transitive closure of the following rules:

(1) the partial order $<_P$ between the elements of P,
(2) $i < j < \lambda \Rightarrow q_i <^* q_j$,
(3) $i < \lambda \Rightarrow r_i <^* q_i$,
(4) $i < \lambda \Rightarrow q_i <^* p_i$.

Now, R is definable (the elements without a predecessor), Q is definable (the immediate successors of the elements of R) and the correspondence $p_i \mapsto q_i$ is defined by "p_i is the immediate successor of q_i that does not belong to Q".

This defines a well ordering of the elements of P. ⊣

Appendix.

LEMMA A.1. *Let C be a scattered chain with* $\mathrm{Hdeg}(C) = n$. *Then there are $\bar{P} \subseteq C$, $\lg(\bar{P}) = n-1$, and a formula (depending on n only) $\varphi_n(x, y, \bar{P})$ that defines a well ordering of C.*

PROOF. By induction on n we will prove the existence of a formula $\varphi_n(x, y, \bar{Z})$ such that for every scattered C with $\mathrm{Hdeg}(C) = n$ there is $\bar{P} \subseteq C$ such that $\varphi_n(x, y, \bar{P})$ well orders the elements of C:

$n \leq 1$: $\mathrm{Hdeg}(C) \leq 1$ implies $(C, <_C)$ is well ordered or inversely well ordered. A well ordering of C is easily definable from $<_C$.

$\mathrm{Hdeg}(C) = n+1$: Suppose $C = \sum_{i \in I} C_i$ and each C_i is of Hausdorff degree n. By the induction hypothesis there are a formula $\varphi_n(x, y, \bar{Z})$ and a sequence $\langle \bar{P}^i : i \in I \rangle$ with $\bar{P}^i \subseteq C_i$, $\bar{P}^i = \langle P_1^i, \dots, P_{n-1}^i \rangle$ such that $\varphi_n(x, y, \bar{P}^i)$ defines a well ordering of C_i.

Let for $0 < k < n$, $P_k := \bigcup_{i \in I} P_k^i$ (disjoint union) and $P_n := \bigcup\{C_i : i \text{ even}\}$ (if I is well ordered then $i \in I$ is even means i is a limit ordinal or $i = \alpha + 2k$ where α is limit and $k < \omega$. When I is inversely well ordered invert it).

We will define an equivalence relation $\sim$ by $x \sim y$ iff $\bigwedge_i (x \in C_i \Leftrightarrow y \in C_i)$.

$\sim$ and $[x]$, (the equivalence class of an element x), are easily definable from P_n and $<_C$. We can also decide from P_n if I is well or inversely well ordered (by looking at subsets of C consisting of non equivalent elements) and define $<'$ to be $<_C$ if I is well ordered and the inverse of $<_C$ if not. $\varphi_{n+1}(x, y, P_1, \dots, P_n)$ will be defined by:

$$\varphi_{n+1}(x, y, \bar{P}) \Leftrightarrow \left[x \not\sim y \,\&\, x <' y\right] \vee \left[x \sim y \,\&\, \varphi_n(x, y, P_1 \cap [x], \dots, P_{n-1} \cap [x])\right].$$

$\varphi_{n+1}(x, y, \bar{P})$ well orders C.

Now if $<_{\mathrm{w}}$ is the well order of C definable by φ_n then there is a sequence of parameters $\bar{Q} \subseteq C$ and a formula $\varphi_n^{-1}(x, y, \bar{Z})$ such that

$$(C, <_{\mathrm{w}}) \models \varphi_n^{-1}(x, y, \bar{Q}) \iff x <_C y$$

the proof is similar by induction on n (noting that when $C = \sum_{i \in I} C_i$ then each C_i is a convex set with respect to $<_{\mathrm{w}}$). ⊣

THEOREM A.2. *Let T be a tame tree. Then there are $\bar{Q} \subseteq T$ and a monadic formula $\varphi(x, y, \bar{Q})$ that defines a well ordering of T.*

PROOF. Assume T is (n^*, k^*) tame. Since ${}^{\omega>}2$ is not embeddable in T for every $x \in T$, $\mathrm{rk}(x)$ is well defined.

(rk is defined by $\mathrm{rk}(\eta) \geq \alpha + 1 \iff$ there are $\nu_1, \nu_2 \in T$ with $\eta \trianglelefteq \nu_1$ and $\eta \trianglelefteq \nu_2$ such that $\nu_1 \perp \nu_2$, $\mathrm{rk}(\nu_1) \geq \alpha$ and $\mathrm{rk}(\nu_2) \geq \alpha$.)

We will partition T into a disjoint union of sub-branches, indexed by the nodes of a well founded tree Γ and reduce the problem of a well ordering of T to a problem of a well ordering of Γ.

Step 1. Define by induction on α a set $\Gamma_\alpha \subseteq {}^\alpha\mathrm{Ord}$ (this is a our set of indices), for every $\eta \in \Gamma_\alpha$ define a tree $T_\eta \subseteq T$ and a branch $A_\eta \subseteq T_\eta$.

$\alpha = 0$: Γ_0 is $\{\langle\rangle\}$, $T_{\langle\rangle}$ is T and $A_{\langle\rangle}$ is a branch (i.e., a maximal linearly ordered subset) of T.

$\alpha = 1$: Look at $(T \setminus A_{\langle\rangle})/\sim^1_{A_{\langle\rangle}}$, it's a disjoint union of trees and name it $\langle T_{\langle i\rangle} : i < i^*\rangle$, let $\Gamma_1 := \{\langle i\rangle : i < i^*\}$ and for every $\langle i\rangle \in \Gamma_1$ let $A_{\langle i\rangle}$ be a branch of $T_{\langle i\rangle}$.

$\alpha = \beta + 1$: For $\eta \in \Gamma_\beta$ denote $(T_\eta \setminus A_\eta)/\sim^1_{A_\eta}$ by $\{T_{\eta^\wedge\langle i\rangle} : i < i_\eta\}$, let $\Gamma_\alpha = \{\eta^\wedge\langle i\rangle : \eta \in \Gamma_\beta,\ i < i_\eta\}$ and choose $A_{\eta^\wedge\langle i\rangle}$ to be a branch of $T_{\eta^\wedge\langle i\rangle}$.

α limit: Let $\Gamma_\alpha = \{\eta \in {}^\alpha\mathrm{Ord} : \bigwedge_{\beta<\alpha} \eta\restriction_\beta \in \Gamma_\beta,\ \bigwedge_{\beta<\alpha} T_{\eta\restriction_\beta} \neq \emptyset\}$, let for $\eta \in \Gamma_\alpha$ $T_\eta = \bigcap_{\beta<\alpha} T_{\eta\restriction_\beta}$ and A_η a branch of T_η. (T_η may be empty.)

Now, at some stage $\alpha \leq |T|^+$ we have $\Gamma_\alpha = \emptyset$ and let $\Gamma = \bigcup_{\beta<\alpha} \Gamma_\beta$. Clearly $\{A_\eta : \eta \in \Gamma\}$ is a partition of T into disjoint sub-branches.

Notation: having two trees T and Γ, to avoid confusion, we use x, y, s, t for nodes of T and η, ν, σ for nodes of Γ.

Step 2. We want to show that $\Gamma_\omega = \emptyset$ hence Γ is a well founded tree. Note that we made no restrictions on the choice of the A_η's and we add one now in order to make the above statement true. Let $\eta^\wedge\langle i\rangle \in \Gamma$ define $B_{\eta,i} = \{t \in A_\eta : (\forall s \in T_{\eta^\wedge\langle i\rangle})[t \trianglelefteq s]\}$, let $\gamma_{\eta,i}$ be $\min\{\mathrm{rk}(t) : t \in B_{\eta,i}\}$ (if $B_{\eta,i} = \emptyset$ then $\gamma_{\eta,i} = \infty$) and finally let $A_{\eta,i} := \{t \in T_{\eta^\wedge\langle i\rangle} : \mathrm{rk}(t) = \gamma_{\eta,i}\}$.

Proviso: For every $\eta \in \Gamma$ and $i < i_\eta$, $A_{\eta,i} \subseteq A_{\eta^\wedge\langle i\rangle}$.

Following this we claim: "Γ does not contain an infinite, strictly increasing sequence". Otherwise let $\{\eta_i\}_{i<\omega}$ be one, and choose $s_n \in A_{\eta_n,\eta_{n+1}(n)}$ (so $s_n \in A_{\eta_n}$). Clearly $\mathrm{rk}(s_n) \geq \mathrm{rk}(s_{n+1})$ and by the proviso we get

$$\mathrm{rk}(s_n) = \mathrm{rk}(s_{n+1}) \Rightarrow \mathrm{rk}(s_{n+1}) > \mathrm{rk}(s_{n+2})$$

therefore $\{\mathrm{rk}(s_n)\}_{n<\omega}$ contains an infinite, strictly decreasing sequence of ordinals which is absurd.

Step 3. Next we want to make "x and y belong to the same A_η" definable.

For each $\eta \in \Gamma$ choose $s_\eta \in A_\eta$, and let $Q \subseteq T$ be the set of representatives. Let $h\colon T \to \{d_0, \ldots, d_{n^*-1}\}$ be a colouring that satisfies: $h\restriction_{A_{\langle\rangle}} = d_0$ and for every $\eta^\wedge\langle i\rangle \in \Gamma$, $h\restriction_{A_{\eta^\wedge\langle i\rangle}}$ is constant and, when $j < i$ and $s_{\eta^\wedge\langle j\rangle} \sim^0_{A_\eta} s_{\eta^\wedge\langle j\rangle}$ we have $h\restriction_{A_{\eta^\wedge\langle i\rangle}} \neq h\restriction_{A_{\eta^\wedge\langle j\rangle}}$. This can be done as T is (n^*, d^*) tame.

Using the parameters $D_0, \ldots, D_{n^*-1}$ ($x \in D_i$ iff $h(x) = d_i$), we can define $\bigvee_\eta[x, y \in A_\eta]$ by "x, y are comparable and the sub-branch $[x, y]$ (or $[y, x]$) has a constant colour".

Step 4. As every A_η has Hausdorff degree at most k^*, we can define a well ordering of it using parameters $P^\eta_1, \ldots, P^\eta_{k^*}$ and by taking $\bar{P}$ to be the (disjoint) union of the $\bar{P}^\eta$'s we can define a partial ordering on T which well orders every A_η.

By our construction $\eta \vartriangleleft \nu$ if and only if there is an element in A_ν that cuts A_η i.e., is above a proper initial segment of A_η. (Caution, if T does not have a root this may not be the case for $\langle\rangle$ and a $< n^*$ number of $\langle i\rangle$'s and we may need parameters for expressing that). Therefore, as by step 3 "being in the same A_η" is definable, we can define a partial order on the sub-branches A_η (or the representatives s_η) by $\eta \vartriangleleft \nu \Rightarrow A_\eta \leq A_\nu$.

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Next, note that "ν is an immediate successor of η in Γ" is definable as a relation between s_ν and s_η hence the set $A_\eta^+ := A_\eta \cup \{s_{\eta \wedge \langle i \rangle}\}$ is definable from s_η. Now the order on A_η induces an order on $\{s_{\eta \wedge \langle i \rangle} / \sim_{A_\eta}^0\}$ which is can be embedded in the completion of A_η hence has Hdeg$\leq k^*$. Using additional parameters $Q_1^\eta, \dots, Q_{k^*}^\eta$, we have a definable well ordering on $\{s_{\eta \wedge \langle i \rangle} / \sim_{A_\eta}^0\}$. As for the ordering on each $\sim_{A_\eta}^1$ equivalence class (finite with $\leq n^*$ elements), define it by their colours (i.e., the element with the smaller colour is the smaller according to the order).

Using $\bar{D}$, $\bar{P}$, Q and $\bar{Q} = \cup_\eta \bar{Q}^\eta$ we can define a partial ordering which well orders each A_η^+ in such a way that every $x \in A_\eta$ is smaller then every $s_{\eta \wedge \langle i \rangle}$.

Summing up we can define (using the above parameters) a partial order on subsets of T that well orders each A_η, orders sub-branches A_η, A_ν when the indices are comparable in Γ and well orders all the "immediate successors" sub-branches of a sub-branch A_η.

Step 5. The well ordering of T will be defined by $x < y \iff$

(a) x and y belong to the same A_η and $x < y$ by the well order on A_η; or

(b) $x \in A_\eta$, $y \in A_\nu$ and $\eta \lhd \nu$; or

(c) $x \in A_\eta$, $y \in A_\nu$, $\sigma = \eta \wedge \nu$ in Γ (defined as a relation between sub-branches), $\sigma^\wedge\langle i \rangle \lhd \eta$, $\sigma^\wedge\langle j \rangle \lhd \nu$ and $s_{\sigma^\wedge\langle i \rangle} < s_{\sigma^\wedge\langle j \rangle}$ in the order of A_σ^+.

Note that $<$ is a linear order on T and every A_η is a convex and well ordered sub-chain. Moreover $<$ is a linear order on Γ and the order on the s_η's is isomorphic to a lexicographic order on Γ.

Why is the above (which is clearly definable with our parameters) a well order? Because of the above note and because a lexicographic ordering of a well founded tree is a well order, provided that immediate successors are well ordered. $\dashv$

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