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On long increasing chains modulo flat ideals

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We prove that, e.g., in ${}^{(\omega_3)}(\omega_3)$ there is no sequence of length ω_4 increasing modulo the ideal of countable sets.

This note is concerned with the depth of the partial order of the functions in ${}^{\kappa}\gamma$ modulo the ideal of the form $\mathcal{I} = [\kappa]^{<\mu}$. Let us recall the following definitions.

Definition 1 For a partial order $(P, \sqsubset)$ we define

- $\mathrm{Depth}(P, \sqsubset) = \sup\{|\mathcal{F}| : \mathcal{F} \subseteq P$ is well–ordered by $\sqsubset\}$ (the *depth*),
- $\mathrm{cf}(P, \sqsubset) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq P$ is $\sqsubset$–cofinal, that is for every $p \in P$ there is $q \in \mathcal{F}$ such that $p \sqsubseteq q$ $\}$ (the *cofinality*).

Our result (Theorem 5) states that under suitable assumptions the depth of the partial order $({}^{\kappa}\gamma, <_{[\kappa]^{<\mu}})$ is at most $|\gamma|$. In particular, letting $\mu = \aleph_1$, $\kappa = |\gamma| = \aleph_3$, we obtain that in ${}^{(\omega_3)}(\omega_3)$ there is no sequence of length ω_4 increasing modulo the ideal of countable sets.

Let $\kappa = \mathrm{cf}(\kappa) > \aleph_0$. If $\mu = \kappa$, then $\mathrm{Depth}({}^{\kappa}\kappa, <_{J^{\mathrm{bd}}_{\kappa}})$ can be (forced to be) large. But for μ bigger than $\mathrm{Depth}({}^{\kappa}\kappa, <_{J^{\mathrm{d}}_{y}k})$, having $\mu < \mathrm{Depth}({}^{\kappa}\kappa, <_{J^{\mathrm{d}}_{\mu}})$ implies pcf results (see [2], [3]).

Now, e.g., for the ideal $\mathcal{I} = [\omega_3]^{\leq\aleph_0}$ it is harder to get long increasing sequence, as above for "high μ", this leads to pcf results, e.g., the assumption that $\bar{\lambda} = \langle \lambda_i : i < \omega_3 \rangle \in {}^{\omega_3}\mathrm{Reg}$, and in $(\prod \bar{\lambda}, <_{\mathcal{I}})$ there is an increasing sequence modulo $\mathcal{I}$ of length say $> 2^{\aleph_3} + \sup\{\lambda_i : i < \omega_3\}$, is much stronger than known consistency results. Even for $I = [\omega_1]^{\leq\aleph_0}$ we do not know, for $I = [\beth_\omega]^{\leq\aleph_0}$ we know that not ([4]), so even $[\aleph_\omega]^{\leq\aleph_0}$ would be interesting good news.

We hope sometime to prove, e.g.,

Conjecture 2 *For every $\mu > \theta$, in ${}^{(\theta^{+3})}\mu$ there is no increasing sequence of length μ^+ modulo $[\theta^{+3}]^{\leq\theta}$.*

Problem 3 Is it consistent that ${}^{\theta}\theta$ contains $<_{\mathcal{I}}$-increasing sequence of length θ^+ when $\theta = \kappa^+$ and $\mathcal{I} = [\theta]^{<\kappa}$?

Notation 4 Our notation is rather standard and compatible with that of classical textbooks (like Jech [1]).

1. Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta, \dots$ (with possible subscripts). Cardinal numbers will be called $\kappa, \lambda, \mu, \theta$.

2. For a set X and a cardinal θ, $[X]^{\theta}$ (or $[X]^{<\theta}$, respectively) stands for the family of subsets of X of size θ ($< \theta$, respectively).

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Theorem 5 *Assume $\mu^+ < \kappa \le \theta$ and $\mathcal{J} = [\kappa]^{<\mu}$ and $\mathrm{cf}([\theta]^\mu, \subseteq) \le \theta$. If $\gamma < \theta^+$, then $\mathrm{Depth}({}^\kappa\gamma, <_{\mathcal{J}}) \le \theta$, i.e., there is no $<_{\mathcal{J}}$–increasing sequence $\langle f_\alpha : \alpha < \theta^+\rangle$ of functions from ${}^\kappa\gamma$.*

P r o o f. Assume towards contradiction that there is a $<_{\mathcal{J}}$–increasing sequence $\langle f_\zeta : \zeta < \theta^+\rangle \subseteq {}^\kappa\gamma$.

Let $\mathcal{S} \subseteq [\gamma]^\mu$ be cofinal of cardinality $\le \theta$ (exists as $|\gamma| \le \theta$ and $\mathrm{cf}([\theta]^\mu, \subseteq) \le \theta$). For every $s \in \mathcal{S}$ and $\beta < \kappa$, $\zeta < \theta^+$ we let

- $I(\beta) := [\beta, \beta + \mu)$,
- $f^s_\zeta \in {}^\kappa(\gamma + 1)$ be defined by $f^s_\zeta(i) = \min(s \cup \{\gamma\} \setminus f_\zeta(i))$,
- $f^{s,\beta}_\zeta \in {}^{I(\beta)}(\gamma + 1)$ be defined as $f^s_\zeta \restriction I(\beta)$.

Now, for each $s \in \mathcal{S}$ we have

$(*)_1$ (a) for every $\zeta < \theta^+$, $f^{s,\beta}_\zeta : I(\beta) \longrightarrow s \cup \{\gamma\}$,
(b) if $\zeta < \xi < \theta^+$, then $f^{s,\beta}_\zeta \le f^{s,\beta}_\xi \bmod [I(\beta)]^{<\mu}$.

For $s \in \mathcal{S}$ we define

$(*)_2$ $B_s = \{\beta < \kappa : (\forall \zeta < \theta^+)(\exists \xi > \zeta)\neg(f^{s,\beta}_\zeta = f^{s,\beta}_\xi \bmod [I(\beta)]^{<\mu})\}$.

Plainly, we may choose a sequence $\langle C^s_\beta : \beta < \kappa,\ s \in \mathcal{S}\rangle$ such that

$(*)_3$ (a) C^s_β is a club of θ^+,
(b) if $\beta \in B_s$ and $\xi, \zeta \in C^s_\beta$ are such that $\zeta < \xi$, then $\neg(f^{s,\beta}_\zeta = f^{s,\beta}_\xi \bmod [I(\beta)]^{<\mu})$,
(c) if $\beta \in \kappa \setminus B_s$, then $f^{s,\beta}_\zeta = f^{s,\beta}_\xi \bmod [I(\beta)]^{<\mu}$ whenever $\min(C^s_\beta) \le \zeta \le \xi < \theta^+$.

Then, as $|\mathcal{S}| \le \theta$ and $\kappa \le \theta$, we have

$(*)_4$ the set $C := \bigcap\{C^s_\beta : s \in \mathcal{S}$ and $\beta < \kappa\}$ is a club of θ^+ .

Choose a sequence $\langle \alpha_\varepsilon : \varepsilon < \mu^+\rangle \subseteq C$ increasing with ε. Then, for all $\varepsilon < \zeta < \mu^+$,

$(*)_5$ $u_{\varepsilon,\zeta} := \{i < \kappa : f_{\alpha_\varepsilon}(i) \ge f_{\alpha_\zeta}(i)\} \in \mathcal{J}$.

We have assumed that $\mu^+ < \kappa$, so we can find $\delta < \kappa$ such that

$(*)_6$ (a) $I(\delta) = [\delta, \delta + \mu)$ is disjoint from $\bigcup\{u_{\varepsilon,\zeta} : \varepsilon < \zeta < \mu^+\}$, and hence
(b) the sequence $\langle f_{\alpha_\varepsilon}(i) : \varepsilon < \mu^+\rangle$ is increasing for each $i \in I(\delta)$.

As $|I(\delta)| = \mu$ and $\mathcal{S} \subseteq [\gamma]^{\le\mu}$ is cofinal (for the partial order $\subseteq$), we can find $s \in \mathcal{S}$ such that

$(*)_7$ $\{f_{\alpha_0}(i), f_{\alpha_1}(i) : i \in I(\delta)\} \subseteq s$.

It follows from $(*)_6 + (*)_7$ that for every $i \in I(\delta)$

$(*)_8$ $f^s_{\alpha_0}(i) = f_{\alpha_0}(i) < f_{\alpha_1}(i) = f^s_{\alpha_1}(i)$.

As $\alpha_0 < \alpha_1$ are from C and $I(\delta) \notin \mathcal{J}$, recalling $(*)_2 + (*)_3 + (*)_4$, clearly

$(*)_9$ $\delta \in B_s$.

Therefore, as $\alpha_\varepsilon \in C \subseteq C^s_\delta$ for $\varepsilon < \mu^+$ and α_ε is increasing with ε, we have for every $\varepsilon < \mu^+$ there is $i_\varepsilon \in I(\delta)$ such that

$(*)_{10}$ (α) $f^s_{\alpha_\varepsilon}(i_\varepsilon) < f^s_{\alpha_{\varepsilon+1}}(i_\varepsilon)$, and hence there is $j_\varepsilon \in s$ such that
(β) $f^s_{\alpha_\varepsilon}(i_\varepsilon) \le j_\varepsilon < f^s_{\alpha_{\varepsilon+1}}(i_\varepsilon)$ and therefore
(γ) $f_{\alpha_\varepsilon}(i_\varepsilon) \le j_\varepsilon < f_{\alpha_{\varepsilon+1}}(i_\varepsilon)$.

But $|I(\delta)| + |s| = \mu < \mu^+$, so for some pair $(j_*, i_*) \in s \times I(\delta)$ we may choose $\varepsilon_1 < \varepsilon_2 < \mu^+$ such that

$(*)_{11}$ $j_{\varepsilon_1} = j_{\varepsilon_2} = j_*$ and $i_{\varepsilon_1} = i_{\varepsilon_2} = i_*$.

But the sequence $\langle f_{\alpha_\varepsilon}(i_*) : \varepsilon < \theta^+\rangle$ is increasing by $(*)_6$(b) (see the choice of δ), so

$$f_{\alpha_{\varepsilon_1}}(i_*) < f_{\alpha_{\varepsilon_1+1}}(i_*) \le f_{\alpha_{\varepsilon_2}}(i_*) < f_{\alpha_{\varepsilon_2+1}}(i_*).$$

It follows from $(*)_{10}(\gamma)$+$(*)_{11}$ that the ordinal j_* belongs to $[f_{\alpha_{\varepsilon_1}}(i_*), f_{\alpha_{\varepsilon_1+1}}(i_*))$ and to $[f_{\alpha_{\varepsilon_2}}(i_*), f_{\alpha_{\varepsilon_2+1}}(i_*))$, which are disjoint intervals, a contradiction. □

Similarly,

Theorem 6 *Assume that*

(a) $\mathcal{J}$ *is an ideal on* κ*,*

(b) $I_\beta \in [\kappa]^\mu$*,* $I_\beta \notin \mathcal{J}$ *for* $\beta < \kappa$*,*

(c) $\theta = |\gamma| + \kappa$ *and* $\mathrm{cf}([\theta]^\mu, \subseteq) < \lambda$*,*

(d) *if* $u_\varepsilon \in \mathcal{J}$ *for* $\varepsilon < \mu^+$*, then for some* $\beta < \kappa$ *the set* I_β *is disjoint from* $\bigcup_{\varepsilon<\mu^+} u_\varepsilon$*.*

Then there is no $<_{\mathcal{J}}$*–increasing sequence of functions from* κ *to* γ *of length* λ*.*

Proof. Without loss of generality λ is the successor of $\mathrm{cf}([\theta]^\mu, \subseteq)$ hence is regular. The proof is similar to the proof of Theorem 5. □

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