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RAMSEY ULTRAFILTERS AND THE REAPING NUMBER—Con($\mathfrak{r} < \mathfrak{u}$)

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We show that it is consistent that the reaping number $\mathfrak{r}$ is less than $\mathfrak{u}$, the size of the smallest base for an ultrafilter. To show that our forcing preserves certain ultrafilters, we prove a general partition theorem involving Ramsey ideals.

1. Introduction and definitions

1.1. Definition. A set x (*weakly*) *reaps* a family $\mathfrak{A} \subseteq [\omega]^\omega$ iff

$$\forall a \in \mathfrak{A}\ [a \cap x \text{ and } a \cap (\omega - x) \text{ are infinite (nonempty)}].$$

Let

$$\mathfrak{r} = \min\{|\mathfrak{A}|: \mathfrak{A} \subseteq [\omega]^\omega, \text{ and no set reaps } \mathfrak{A}\}.$$

(This definition is from [2]. An equivalent definition is in [7]. It is clear that the analogously defined cardinal $\mathfrak{r}_{\text{weakly}}$ is equal to $\mathfrak{r}$.)

$\mathfrak{i}$ is the size of the smallest maximal independent family of subsets of ω, $\mathfrak{u}$ is the size of the smallest base for an ultrafilter on ω, $\mathfrak{b}$ is the size of the smallest unbounded family in ω^ω, $\mathfrak{d}$ is the size of the smallest dominating family in ω^ω, and $\mathfrak{s}$ is the size of the smallest splitting family. $\mathfrak{c} = 2^{\aleph_0}$ is the size of the continuum. (These cardinals are defined and discussed in [6]. See also [12] for open problems relating to these cardinals, and topological questions in which these cardinals play a role.)

1.2. Fact. (1) $\omega_1 \leqslant \mathfrak{r} \leqslant \mathfrak{c}$;

(2) $\mathrm{MA} \to \mathfrak{r} = \mathfrak{c}$;

(3) $\mathfrak{b} \leqslant \mathfrak{r}$;

(4) $\mathfrak{r} \leqslant \mathfrak{i}$;

(5) $\mathfrak{r} \leqslant \mathfrak{u}$;

(6) $\mathfrak{s} \leqslant \mathfrak{d}$, $\mathfrak{b} \leqslant \mathfrak{d}$;

(7) $\mathfrak{d} \leqslant \mathfrak{i}$.

Proof. (1) is trivial, (2) follows from (3). (6) is well known. For (7), see [12].

(3) (This proof is similar to the one for $\mathfrak{b} \leqslant \mathfrak{u}$ in [11].) Let $\mathfrak{A} \subseteq [\omega]^{\omega}$, $|\mathfrak{A}| < \mathfrak{b}$. Let M satisfy some large enough fragment of ZFC, $\mathfrak{A} \subseteq M$, $|M| < \mathfrak{b}$. Let f be a strictly increasing function satisfying $f(n) > n$ for all n, such that every function in $\omega^{\omega} \cap M$ is eventually bounded by f. Let $a_0 = 0$, $a_{n+1} = f(a_n)$. It can easily be checked that

$$(\forall a \in M \cap [\omega]^{\omega})\, \forall^{\infty} n\ (a \cap [a_n, a_{n+1}) \neq \emptyset),$$

so

$$X = \bigcup_{n \in \omega} [a_{2n}, a_{2n+1})$$

reaps every set in M.

For (4), let $\mathfrak{B} \subseteq [\omega]^{\omega}$ be a maximal independent family. Let $\mathfrak{A}$ be the set of all (nontrivial) Boolean combinations of $\mathfrak{B}$. Then $|\mathfrak{A}| = |\mathfrak{B}|$, and any set reaping $\mathfrak{A}$ would contradict the maximality of $\mathfrak{B}$.

For (5), note that a base for an ultrafilter cannot be reaped. $\square$

1.3. Remarks. Martin's axiom implies that all these cardinals are equal to $\mathfrak{c}$, in fact many consequences of MA are already implied by certain relations between these cardinals. (See [6, 12].)

Given the above relations (3)–(6) (that are proved in ZFC), the question arises whether this list is complete. The most general question in this direction is:

(A) Given a set $\{b, r, i, \ldots\}$ of cardinals satisfying the appropriate relations (i.e., $b \leqslant r \leqslant i$, etc.), is there a model in which $\mathfrak{b} = b$, $\mathfrak{r} = r$, $\mathfrak{i} = i, \ldots$?

A less ambitious list of questions is given by the following scheme:

(B) For which $x \neq y$ in $\{\mathfrak{b}, \mathfrak{r}, \ldots\}$ does ZFC prove $x \leqslant y$?

Adding $\aleph_2$ many random reals (with countable support, or with the measure algebra) to a model of GCH will generate a model in which $\mathfrak{r} = \mathfrak{i} = \mathfrak{u} = \aleph_2$ and $\mathfrak{s} = \mathfrak{d} = \mathfrak{b} = \aleph_1$, and adding $\aleph_2$ many Cohen reals will generate a model in which $\mathfrak{d} = \mathfrak{i} = \mathfrak{u} = \mathfrak{r} = \aleph_2$ and $\mathfrak{s} = \mathfrak{b} = \aleph_1$. In [3] is shown that $\mathfrak{s} = \mathfrak{d} = \mathfrak{i} = \aleph_2 \wedge \mathfrak{b} = \mathfrak{u} = \aleph_1$ is consistent. Adding $\aleph_1$ many random reals to a model of $\mathfrak{b} = \mathfrak{c}$ will make $\mathfrak{s} = \aleph_1$ and leave $\mathfrak{b} = \mathfrak{c}$.

These models show that 1.2(3)–(6) (plus the trivial consequences, like $\mathfrak{b} \leqslant \mathfrak{u}$) is a complete list of relations that can occur in (B), except for possibly $\mathfrak{u} = \mathfrak{r}$ and $\mathfrak{u} \leqslant \mathfrak{i}$.[1]

In this paper we will give positive answers to several instances of question (A), and we will show in particular that $\mathfrak{u} = \mathfrak{r}$ does not follow from ZFC. All the

[1] *Note added in proof.* Recently the second author has established Con($\mathfrak{i} < \mathfrak{u}$).

models we produce will satisfy $\mathfrak{c} = \aleph_2$, so that the cardinals considered will all be in $\{\aleph_1, \aleph_2\}$.

The strategy for making $\mathfrak{u} = \aleph_2$ is as follows: Start with a model V_0 of ZFC + GCH, and construct an increasing chain of models using a countable support iteration $\bar{Q} = \langle P_\alpha, Q_\alpha\colon \alpha < \omega_2\rangle$ of proper forcing notions Q_α. Letting $V_\alpha = V^{P_\alpha}$, we have to ensure that all ultrafilters D in V_α (for $\alpha < \omega_2$) are 'killed' in some model V_β ($\alpha < \beta < \omega_2$), i.e., we have to add a new set $x \in [\omega]^\omega \cap (V_\beta - V_\alpha)$ such that neither x nor $\omega - x$ will contain a set in D.

To get $V_{\omega_2} \models \mathfrak{r} = \aleph_1$, we will try to ensure

$$V_\alpha \models \text{“}[\omega]^\omega \cap V_0 \text{ cannot be reaped”}$$

for all $\alpha < \omega_2$. It will be sufficient to have, for all $\alpha < \omega_2$,

$$V_\alpha \models \text{“there is a } D \in V_0, D \subseteq [\omega]^\omega \cap V_0, \text{ that generates an ultrafilter in } [\omega]^\omega \cap V_\alpha\text{,”}$$

because a base for an ultrafilter cannot be reaped.

So we are looking for a class of forcing notions $\{Q_D\colon D \text{ an ultrafilter}\}$, such that

(1) Q_D kills D, but

(2) Q_D kills only a 'small' set of ultrafilters, where 'small' can refer to any $<\aleph_2$-closed ideal of ultrafilters.

Many well-known forcing notions (such as Cohen forcing) do not satisfy the second demand, as they kill all ultrafilters of the ground model. Other forcing notions (such as Sacks forcing) do not kill any ultrafilter at all, or can only kill ultrafilters of a certain type (e.g., the forcing in [3] cannot kill P-points).

For technical reasons, we will concentrate on the set of Ramsey ultrafilters; a set of ultrafilters will be 'small' if it contains at most $\aleph_1$ many Ramsey ultrafilters, i.e., we will work with forcing notions Q_D that kill at most $\aleph_1$ many Ramsey ultrafilters. (Note that there are $\aleph_2$ many Ramsey ultrafilters in the ground model.)

1.4. Theorem. $\mathrm{ZFC} \vdash \mathrm{Con}(\mathrm{ZFC}) \rightarrow \mathrm{Con}(\mathrm{ZFC} + \mathfrak{r} = \aleph_1 < \aleph_2 = \mathfrak{u} = \mathfrak{i} = \mathfrak{s} = \mathfrak{d} = \mathfrak{c}) \wedge \mathrm{Con}(\mathrm{ZFC} + \mathfrak{s} = \mathfrak{d} = \mathfrak{r} = \aleph_1 < \aleph_2 = \mathfrak{u} = \mathfrak{i} = \mathfrak{c})$.

The proof is due to the second author.

1.5. Contents of the paper. In Section 2 we will postulate properties for a class of forcing notions $\{Q_I\colon I \text{ an ideal on } \omega\}$ and show that a certain iteration of these forcing notions yields a model for $\mathfrak{r} = \aleph_1$ and $\mathfrak{u} = \aleph_2$. In Section 3 we describe Q_I and prove some basic properties of it. In Section 4 we prove a general partition lemma in the spirit of the classical theorems of Hindman [8], Baumgartner [1] and Carlson and Simpson [4, 5]. In Section 5 we apply the result of Section 4 to show that Q_I kills only few Ramsey ultrafilters. In Section 6 we describe how the iteration in 2.4 can be modified to ensure either $\mathfrak{s} = \mathfrak{d} = \mathfrak{i} = \aleph_2$ or $\mathfrak{s} = \mathfrak{d} = \aleph_1$ and $\mathfrak{i} = \aleph_2$.

2. The model

2.1. Notation. (a) In this paper, 'ideal' will mean "an ideal (on ω) that contains all finite sets".

(b) If $I \subseteq \mathfrak{P}(\omega)$ (such that ω cannot be covered by a finite union of elements from I), then let $\bar{I}$ denote the ideal generated by I, similarly for filters.

(c) If I, I' are ideals, then I' is called a 'quotient' of I, if there is a function $f \in \omega^\omega$ such that $I' = f^*(I) =^{\mathrm{df}} \{A \subseteq \omega : f^{-1}(A) \in I\}$ (or $I' \leqslant I$ in the Rudin–Keisler order). (Note that by (a), $f^{-1}(n) \in I$ for all n.)

(d) I^* is the filter dual to the ideal I, $I^+ = \mathfrak{P}(\omega) - I$.

(e) $\mathfrak{P}(\omega)/I = \mathfrak{P}(\omega)/I^*$ is the quotient of $\mathfrak{P}(\omega)$ modulo the equivalence relation $A \sim B \leftrightarrow (A - B) \cup (B - A) \in I$.

(f) For $A \subseteq \omega$, $\omega - A \in I^+$, $I + A$ is the ideal generated by $I \cup \{A\}$.

2.2. Lemma. *Assume that $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$. For every ideal I (on ω) there exists a forcing notion Q_I with the following properties:*

(1) *Q_I adds a real $x_I \subseteq \omega$ such that for every $y \in I^+ \cap V$*

$$\vdash_{Q_I} \text{“}|x_I \cap y| = |(\omega - x_I) \cap y| = \aleph_0\text{”}.$$

(2) *For every maximal ideal J that is dual to a Ramsey ultrafilter: if $\vdash_{Q_I}$ “$\bar{J}$ is not maximal”, then there is a quotient I' of I such that $I' \subseteq J$.*

(3) *Q_I is proper, $|Q_I| = \aleph_1$.*

(4) *Q_I is ω^ω-bounding.*

This lemma will be proved in Sections 3 and 5.

2.3. Corollary. *Under the assumptions of Lemma 2.2, we have:*

(1) $\Vdash_{Q_I}$ *“$\bar{I}$ is not maximal”.*

(2) *Fix a sequence $\langle D_i : i < \omega_2 \rangle$ of distinct Ramsey ultrafilters. If*

$$\textit{For every } I' \leqslant I \textit{ there are } \leqslant\aleph_1 \textit{ many maximal ideals extending } I', \tag{$*$}$$

then there is a $j < \omega_2$ such that

$$(\forall i \in (j, \omega_2)) \Vdash_{Q_I} \text{“}\bar{D}_i \textit{ is an ultrafilter”}.$$

(In particular this is true if I is maximal.) To show that 2.2(2) *implies* 2.3(2), *note that I has only $\leqslant 2^{\aleph_0} = \aleph_1$ many quotients.*

(3) $\Vdash_{Q_I} 2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$.

2.4. Proof of Con($\mathfrak{r} < \mathfrak{u}$) from the lemma. Start with a model V_0 satisfying $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. Construct a countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$, where for each $\alpha < \omega_2$

$$\Vdash_\alpha \text{“}Q_\alpha = Q_{I_\alpha},\ I_\alpha \text{ a maximal ideal on } \omega\text{”}.$$

(We will write $\Vdash_\alpha$ for $\Vdash_{P_\alpha}$. $V_\alpha = V^{P_\alpha}$ will be the intermediate model constructed in stage α, and $V_{\omega_2} = V^{P_{\omega_2}}$ will be the final model.)

It is easy to see that for each $\alpha < \omega_2$ we will have $\Vdash_\alpha 2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ (so in the final model we will have $\mathfrak{c} \leqslant \aleph_2$).

Hence in each intermediate model there are only $\aleph_2$ ultrafilters, and using a bookkeeping argument we can (by Corollary 2.3(1)) ensure that for each $\alpha < \omega_2$:

$$\text{If } U \in V_\alpha, \text{ then } V_{\omega_2} \vDash \bar{U} \text{ is not an ultrafilter.}$$

By properness, every real of V_{ω_2} appears in some intermediate model V_α, so no ultrafilter in V_{ω_2} can be generated by less than $\aleph_2$ sets, hence

$$V_{\omega_2} \vDash \mathfrak{u} = \mathfrak{c} = \aleph_2.$$

To show $V_{\omega_2} \vDash \mathfrak{r} = \aleph_1$, it is enough to prove that

$$V_{\omega_2} \vDash [\omega]^\omega \cap V_0 \text{ cannot be reaped.}$$

In V there are $\aleph_2$ many distinct Ramsey ultrafilters $\{D_i: i < \aleph_2\}$. We will show that $\forall \alpha < \omega_2\, \exists i_\alpha < \omega_2$ such that

$$V_\alpha \vDash \text{“}\forall j \in (i_\alpha, \omega_2)\text{: } \bar{D}_j \text{ is a (Ramsey) ultrafilter”.}$$

(It is enough to show that $\bar{D}_j$ is an ultrafilter, then by 2.2(3) and 2.2(4) and [10] it will follow that $\bar{D}_j$ is Ramsey.)

We can take $i_0 = 0$, and for limit ordinals δ it follows from [3] that we can let $i_\delta = \sup\{i_\alpha: \alpha < \delta\}$. Applying Corollary 2.3(3) to $\langle \bar{D}_j: i_\alpha < j < \omega_2 \rangle$ in V_α we get $i_{\alpha+1}$. (Using the $\aleph_2$-cc of P_α (see [10]), we may assume $i_\alpha \in V_0$.)

So for each $\alpha < \omega_2$ there is a $D \in L$ such that

$$V_\alpha \vDash \bar{D} \text{ is an ultrafilter.}$$

But any real $r \in V_{\omega_2}$ is in some V_α. Since in V_α there is an ultrafilter generated by sets of V_0, there is an x in V_0 such that $r \subseteq x$ or $r \cap x = \emptyset$, so r does not reap $[\omega]^\omega \cap V_0$.

2.5. Remark. If CH does not hold in the ground model V_0, there may be no Ramsey ultrafilters in V_0. However, we can force Ramsey ultrafilters with a σ-complete forcing R. Then the above argument shows that in $V^{R * P_{\omega_2}}$ the family $V_0^R \cap [\omega]^\omega = V_0 \cap [\omega]^\omega$ cannot be reaped, so again $V_0 \cap [\omega]^\omega$ cannot be reaped in $V^{P_{\omega_2}}$.

3. Construction of Q_I

3.1. Definition. If I is an ideal (on ω), an *I-partition* is a partition (of ω) all of whose classes are in I.

3.2. Definition. For any ideal I on ω, we will define a *forcing* Q_I:

(1) Q_I is the forcing with conditions of the form $p = \langle h, E; E_0, E_1, E_2, \ldots \rangle$

where

h is a function with domain ω and range $\subseteq\{+1,-1\}$,

$\langle E, E_0, E_1, \ldots\rangle$ is an I-partition of ω.

(2) If $p=\langle h, E; E_0, E_1, E_2, \ldots\rangle$, we write h^p, E^p, E_i^p for h, E, E_i.

(3) If $p=\langle h, E; E_0, E_1, E_2, \ldots\rangle$ and $p'=\langle h', E'; E_0', E_0', E_2', \ldots\rangle$, then we let $p'\geqslant p$ (p' is stronger than p), if:

$E\subseteq E'$,

$\langle E, E_0, E_1, E_2, \ldots\rangle$ is a finer partition than $\langle E', E_0', E_1', \ldots\rangle$
(i.e. each E_i is contained in some E_j' or in E'),

$h \mid E = h' \mid E$,

for all i: $h' \mid E_i \in \{h \mid E_i, -h \mid E_i\}$
(where $-h$ is defined by $-h(n)=-(h(n))$ for all n),

(i.e., to make p stronger, you can merge classes in the partition, and/or flip the value of h on some of the classes E_i, but not on E).

A generic set G will define a generic function

$$g=\bigcup\{h^p \mid E^p : p\in G\}.$$

So a condition $p=\langle h, E; E_0, E_1, E_2, \ldots\rangle$ forces that $h \mid E\subseteq g$, and for all i, $g \mid E_i\in\{h \mid E_i, -h \mid E_i\}$.

3.3. Fact. $x =^{\mathrm{df}} \{n\in\omega : g(n)=1\}$ *satisfies* 2.2(1).

3.4. Definition. For $p, q\in Q_I$, define

$$p\leqslant_0 q \;\leftrightarrow\; p\leqslant q \text{ and } E^p=E^q,$$

$$p\leqslant_n q \;\leftrightarrow\; p\leqslant_0 q \text{ and } (\forall j<n)\, E_j^p=E_j^q \text{ and } (\forall k<n)\, k\in E^q\cup\bigcup_{j\leqslant n} E_j^q.$$

(Note that it says "$\forall j<n$", but in the last clause the union is taken over all E_j^q for all $j\leqslant n$. Hence for any condition $p\in Q_I$ and for any n there is $q\in Q_I$ such that $p\leqslant_n q$, which can be obtained by merging all classes E_m^p $(m\geqslant n)$ that contain numbers $<n$, to obtain E_n^q.)

3.5. Fact. *If* $p\leqslant_0 p_0\leqslant_1 p_1\leqslant_2 p_2\leqslant_3\cdots$, *then there exists a condition* q *such that* $\forall n\, p_n\leqslant q$.

(*Proof.* Let $E^q=E^p$, $E_n^q=E_n^{p_n}=E_n^{p_j}$ for $j\geqslant n$, $h^q \mid E_n^q=h^{p_n} \mid E_n^{p_n}$.)

3.6. Notation. If $s:[0,n)\to\{+1,-1\}$, let $q=p^s$ be the condition defined by

$$E^q=E^p\cup\bigcup_{i<n}E_i^p,\quad h^q\mid E_i^p=s(i)\times h^p\mid E_i^p,\qquad \text{for } i\in[0,n),$$

$$E_j^q=E_{j+n}^p,\qquad h^q\mid E_j^q=h^p\mid E_{j+n}^p.$$

3.7. Lemma. *Let $p\in Q_I$, $\boldsymbol{\alpha}$ a Q_I-name of an ordinal, $n<\omega$. Then there exists a condition $q, p\leqslant_n q$, and a set F of ordinals of size 2^{n+1} such that*

$$q\Vdash \boldsymbol{\alpha}\in F.$$

Proof. Assume that $n=0$. Let $r_1\geqslant p$, $r_1\Vdash\boldsymbol{\alpha}=\alpha_1$. Let $r_1'=\langle f,E^{r_1};E_0^{r_1},E_1^{r_1},\ldots\rangle$, where f is defined by

$$f(n)=\begin{cases}-h^{r_1}(n) & \text{for } n\in E^{r_1}-E^p,\\ h^{r_1}(n), & \text{otherwise.}\end{cases}$$

Let $r_2\geqslant r_1'$, $r_2\Vdash\boldsymbol{\alpha}=\alpha_2$, and define

$$q=\langle h^{r_2},E^p;E^{r_2}-E^p,E_0^{r_2},E_1^{r_2},\ldots\rangle.$$

Then clearly $q\Vdash\boldsymbol{\alpha}\in\{\alpha_1,\alpha_2\}$: Let $q'\geqslant q$ force a value to $\boldsymbol{\alpha}$. Then without loss of generality $E^{q'}\supseteq E^{r_2}\supseteq E^{r_1'}=E^{r_1}$. Then either $h^{q'}\supseteq h^{r_2}\mid(E^{r_2}-E^p)$ or $h^{q'}\supseteq -h^{r_2}\mid(E^{r_2}-E^p)$. In the first case $q'\geqslant r_2$ and in the second case $q'\geqslant r_1$, hence $q'\Vdash\boldsymbol{\alpha}\in\{\alpha_1,\alpha_2\}$.

Now we can consider the case $n>0$. Let $s_1,\ldots,s_{2^n}$ enumerate ${}^n2$. Let $p_0\geqslant_n p$ and define a sequence $p_0\leqslant_n p_1\leqslant_n\cdots\leqslant_n p_{2^n}$ as follows. Given p_{j-1}, let r_j be a $\leqslant_0$-extension of $p_{j-1}^{s_j}$ (see 3.6) forcing $\boldsymbol{\alpha}\in\{\alpha_{j,1},\alpha_{j,2}\}$, and let

$$p_j=\langle h^{r_j},E;E_0,\ldots,E_{n-1},E_0^{r_j},E_0^{r_j},\ldots\rangle;$$

then $p_j\geqslant_n p_{j-1}$. Finally, let $F=\{\alpha_{1,1},\alpha_{1,2},\ldots,\alpha_{2^n,1},\alpha_{2^n,2}\}$, and let $q=p_{2^n}$. $\square$

This lemma easily implies the next theorem.

3.8. Theorem. *Q_I is proper and ω^ω-bounding.*

Proof (*for ω^ω-bounding. Properness is proved similarly*). Let $p\Vdash \boldsymbol{f}:\omega\to\omega$. Using Lemma 3.7, construct a sequence $p\leqslant_0 p_0\leqslant_1 p_1\leqslant_2 p_2\leqslant_3\cdots$ and a sequence $\langle F_n: n<\omega\rangle$ of finite sets such that $p_n\Vdash$ "$\boldsymbol{f}(n)\in F_n$". By Fact 3.5, there is $q, q\geqslant p_n$ for all n. Finally, let $g(n)=\max F_n$, then $q\Vdash\boldsymbol{f}\leqslant g$. $\square$

Note that the condition q given by Lemma 3.7 and Fact 3.5 satisfies:

if $s:n\to\{+1,-1\}$, then q^s decides $\boldsymbol{f}\mid n$.

In the next section we will prove a partition theorem that will be used to prove the remaining claim.

3.9. Main Lemma. *If J is a maximal Ramsey ideal in V, and no quotient of I is included in J, then J generates a maximal ideal in the extension V^{Q_I}.*

(For a direct proof of Lemma 3.9, see [9].)

4. A partition theorem

4.1. Definition. Let J be an ideal on ω. J is a *Ramsey ideal*, if in the following game G_J player ZERO does not have a winning strategy:
start with $A_{-1} = \omega$;
in the jth move:
player ZERO chooses a set $A_j \subseteq A_{j-1}$, $A_j \in J^+$;
then player PLUS chooses an element $k_j \in A_j$;
in the end player PLUS wins if $\{k_j : j < \omega\} \in J^+$.

4.2. Remarks. (1) The ideal of finite sets is trivially Ramsey.
(2) A maximal ideal I is Ramsey iff for every $f : \omega \to \omega$ there exists a set $A \in I^+$ such that: $f \restriction A$ is constant or 1-1 (iff the dual filter is a Ramsey ultrafilter).

4.3. Convention. Fix a Ramsey ideal J for this section. (We will show that without loss of generality we may assume J is a maximal ideal.)

Also, fix an ideal I, and assume that for no finite-to-one function $f \in \omega^\omega$, $f^*(J) \supseteq f^*(I)$. (See Notation 2.1(c).)

4.4. Definition. (1) A sequence $\langle w_i : i < \omega \rangle$ of finite pairwise disjoint subsets of ω is called *almost convex* iff for all $i \in \omega$:

$$\max w_i < \max w_{i+1}, \qquad \min w_i < \min w_{i+1}, \qquad \max w_i + 1 < \min w_{i+2},$$

$$\bigcup_{i<\omega} w_i \text{ is an 'interval' } [\min w_0, \omega).$$

(2) A finite sequence $\langle w_i : i < n \rangle$ of disjoint sets is called *almost convex,* if the first three conditions above are satisfied (for all indices i for which they make sense), and in addition $\bigcup_{i<n} w_i$ is an interval $[\min w_0, \max w_{n-1})$.
(3) A (finite or infinite) sequence $\langle w_i : i < \alpha \rangle$ $(\alpha \leqslant \omega)$ of finite intervals $\subseteq \omega$ is called *neat,* if they are adjacent intervals in the proper order, i.e., $\max w_i + 1 = \min w_{i+1}$ for all $i < \alpha$.

4.5. Definition. We say that $\mathfrak{W} = (W, \Sigma)$ is a *word system,* iff:
(a) W is a set of finite partial functions on ω with range $\subseteq \{+1, -1\}$ or some other fixed finite set. For $f \in W$ we let $a_f = \min \operatorname{dom}(f)$;
(b) Σ is a partial function from $W^{<\omega}$ to $[W]^{<\omega}$;
(c) $\Sigma(f_0, \ldots, f_{n-1})$ is well-defined whenever $\langle \operatorname{dom}(f_i) : i < n \rangle$ is almost convex;
(d) if $f \in \Sigma(f_0, \ldots, f_{n-1})$, then $\operatorname{dom}(f) = \bigcup_i \operatorname{dom}(f_i)$;
(e) if $0 = k_0 < k_1 < \cdots < k_p$, $f_0, \ldots, f_{k_p-1} \in W$, $\Sigma(f_0, \ldots, f_{k_p-1})$ well-defined,

and for each $l<p$

$$g_l \in \Sigma(f_{k_l}, \ldots, f_{k_{l+1}-1}),$$

then

$$\Sigma(g_0, \ldots, g_{p-1}) \subseteq \Sigma(f_0, \ldots, f_{k_p-1});$$

(f) if $\Sigma(f_0, \ldots, f_{n-1})$ is well-defined, then $f_0 \cup \cdots \cup f_{n-1} \in \Sigma(f_0, \ldots, f_{n-1})$;
(g) we let $\Sigma_-(f_0, \ldots, f_{n-1}) = \{f \in \Sigma(f_0, \ldots, f_{n-1}): f \supseteq f_0 \cup f_{n-1}\}$.

4.6. Remarks. (1) When we apply this theorem, we will use the word system $\mathfrak{W}^* = (W^*, \Sigma^*)$ given by W^* = all partial finite functions from ω into $\{-1, +1\}$, and

$$\Sigma^*(f_0, \ldots, f_{n-1}) = \{f: \mathrm{dom}(f) = \bigcup \mathrm{dom}(f_i), \forall i<n\, f \mid \mathrm{dom}(f_i) \in \{f_i, -f_i\}\}.$$

(2) We will fix a word system $\mathfrak{W}$ for this section.

4.7. Definition. For $q \in \omega$, let W_J^q be the set of all sequences $\bar{f} = \langle f_i: i \in \omega \rangle$ satisfying

(1) $\langle \mathrm{dom}(f_i): i < \omega \rangle$ is an almost convex partition of $[q, \omega)$;

(2) $E(\bar{f}) =^{\mathrm{df}} \left\{a_{f_i}: \bigcup_{j<i} \mathrm{dom}(f_j) = [q, a_{f_i})\right\} \in J^+$.

(Remember that $a_f =^{\mathrm{df}} \min \mathrm{dom}(f)$.) For $\bar{f} = \langle f_i: i \in \omega \rangle$, $r \in \omega$, let $\bar{f} \mid r = \langle f_i: i < r \rangle$ and $\bar{f} \mid [r, \omega) = \langle f_i: r \leqslant i < \omega \rangle = \langle f_{i+r}: i \in \omega \rangle$.

4.8. Definition. For $\bar{f}, \bar{g} \in W_J^q$, let $\bar{f} \leqslant \bar{g}$ iff for some almost convex partition $\langle w_i: i < \omega \rangle$ of ω, for all i, $g_i \in \Sigma(f_j: j \in w_i)$.

(Similarly for finite sequences $\bar{f} = \langle f_0, \ldots, f_{n-1} \rangle$, $\bar{g} = \langle g_0, \ldots, g_{m-1} \rangle$, we write $\bar{f} \leqslant \bar{g}$ if $\langle \mathrm{dom}(f_0), \ldots, \mathrm{dom}(f_{n-1}) \rangle$ is almost convex and for some almost convex partition $\langle w_i: i < m \rangle$ of $[0, n)$, for all $i < m$, $g_i \in \Sigma(f_j: j \in w_i)$.)

4.9. Fact. (1) *$\leqslant$ is reflexive and transitive.*
(2) *Note that if $\langle w_i: i < \omega \rangle$ and $\langle d_j: j < \omega \rangle$ are almost convex partitions of ω, then so is $\langle \bigcup_{j \in w_i} d_j: i < \omega \rangle$.*

4.10. Definition. $\bar{f}$ is *neat*, if each $\mathrm{dom}(f_i)$ is an interval, i.e., if $\langle \mathrm{dom}(f_i): i < \omega \rangle$ is neat.

4.11. Remark. The set of neat sequences is dense.

4.12. Definition. (1) For $\bar{f} \in W_J^q$, define $\mathrm{wd}(\bar{f})$ (the words below $\bar{f}$) by

$$\mathrm{wd}(\bar{f}) = \{g\colon \text{for some } l,\ g \in \Sigma(\bar{f} \mid l), \text{ and } \mathrm{dom}(g) \text{ is an interval } [q, a_{f_l})\}.$$

(Note that by 4.7(2), this is, in some sense, a 'large' set.)

(2) Let $IW_J^q = \{g\colon \exists \bar{f} \in W_J^q, r \in \omega, g \in \Sigma(\bar{f} \mid r), \mathrm{dom}(g) \text{ an interval}\} = \{\mathrm{wd}(\bar{f})\colon \bar{f} \in W_J^q\}$.

4.13. Definition. If $\bar{f} \in W_J^q$, $X \subseteq IW_J^q$, we say that X is *J-large above $\bar{f}$*, if:

$$\text{for every } \bar{f}' \in W_J^q, \text{ if } \bar{f}' \geqslant \bar{f}, \text{ then } \mathrm{wd}(\bar{f}') \cap X \neq \emptyset.$$

4.14. Fact. (1) *If $\bar{f} \leqslant \bar{f}'$, then* $\mathrm{wd}(\bar{f}) \supseteq \mathrm{wd}(\bar{f}')$.

(2) *If X is J-large above $\bar{f}$, $\bar{f}' \geqslant \bar{f}$, then X is J-large above $\bar{f}'$.*

(3) *X is J-large above $\bar{f}$ iff $X \cap \mathrm{wd}(\bar{f})$ is.*

(4) *If X is J-large above $\bar{f}$, $J \subseteq J'$, $\bar{f} \in W_{J'}^q$, then X is J'-large above f.*

4.15. Fact. *If $X_1 \cup X_2 \cup \cdots X_d$ is J-large above $\bar{f}$, then there exists $\bar{f}' \geqslant \bar{f}$, such that for some $l \leqslant d$, X_l is J-large above $\bar{f}'$.*

Proof. It is enough to show it for $d = 2$. If X_1 is not J-large above $\bar{f}$, then there exists a sequence $\bar{f}' \geqslant \bar{f}$ such that $X_1 \cap \mathrm{wd}(\bar{f}') = \emptyset$. If X_2 is not J-large above $\bar{f}'$, then there exists $\bar{f}'' \geqslant \bar{f}'$ such that $X_2 \cap \mathrm{wd}(\bar{f}'') = \emptyset$. But $\mathrm{wd}(\bar{f}'') \subseteq \mathrm{wd}(\bar{f}')$, so $\mathrm{wd}(\bar{f}'') \cap X = \emptyset$, contradiction. $\square$

4.16. Definition. We say that $(\mathfrak{W}, J)$ *satisfies the finite partition hypothesis* (FPH), if, whenever X is large above $\bar{f}, \bar{f} \in W_J^q$, then we can find $r_2 > r_1 > 0$ and $\langle g^0, g^1, g^2 \rangle \geqslant \bar{f} \mid r_1$, $h \in \Sigma(\bar{f} \mid [r_1, r_2))$ satisfying

(a) $\mathrm{dom}(g^0 \cup g^1)$, $\mathrm{dom}(g^2)$ and $\mathrm{dom}(h)$ are adjacent intervals,

(b) $\forall g \in \Sigma_-(g^0, g^1, g^2)$ ($g \in X$ and $g \cup h \in X$).

(We will show in the next section that this is true for the word system $\mathfrak{W}^*$.)

In the next section we will see that the following lemma is sufficient to prove the Main Lemma 3.9.

4.17. Theorem. *Assume J is a Ramsey ideal, I an ideal, for every finite-to-one function $f\colon \omega \to \omega$ and for every $A \neq \omega \bmod J$, $f^*(J + A) \not\supseteq f^*(I)$ (see 2.1), and $(\mathfrak{W}, J)$ satisfies FPH. Then: if $\bar{f} \in W_J^q$, and X is J-large above $\bar{f}$, then there exists $\bar{g}$ such that*

(i) $\bar{f} \leqslant \bar{g}$;

(ii) $E(\bar{g}) \supseteq \{a_{g_{3i}}\colon i < \omega\} \in J^+$ (*so* $\mathrm{dom}(g_{3i} \cup g_{3i+1} \cup g_{3i+2}) = [a_{g_{3i}}, a_{g_{3i+3}})$);

(iii) $\bigcup_{i<\omega} \mathrm{dom}(g_{3i+1}) = \omega \pmod{I}$;

(iv) *if $g_i' \in \Sigma_-(g_{3i}, g_{3i+1}, g_{3i+2})$ for all $i > 0$, then $\forall j > 0 \bigcup_{i<j} g_i' \in X$.*

(*Note.* Without loss of generality $q = 0$, and $\bar{f}$ is neat.)

4.18. Remarks. (1) We may assume that J is maximal: if J is not maximal, we can do the following: without loss of generality CH holds. (Otherwise force CH with a σ-complete forcing, then no reals are added.) Then we can inductively find a $\subseteq^*$-decreasing sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ such that:

(I) $A_\alpha \notin J$;

(II) $\forall f : \omega \to \omega\ \exists \alpha < \omega_1 : f \restriction A_\alpha$ is 1-1 or constant;

(III) for every $A \subseteq \omega$ there is α such that $A_\alpha \subseteq^* A$ or $A_\alpha \subseteq^* (\omega - A)$;

(IV) for all finite-to-one $f : \omega \to \omega$ there is an α such that $f^{-1}(A_\alpha) \in I$.

Then $J' = \{A \subseteq \omega : \exists \alpha\, A \cap A_\alpha \in I\}$ is a maximal Ramsey ideal extending J such that for no finite-to-one f, $f^*(J') \supseteq f^*(I)$. Then X is still J'-large. Now apply the theorem for J', then the resulting sequence $\bar{g}$ will be in $W^q_{J'} \subseteq W^q_J$, so it satisfies the conclusion of the original theorem.

(2) Why do we use 'almost convex' at all?

(a) If we use only convex domains (i.e., intervals), we do not know how to prove FPH in the case that we need.

(b) If we waive the almost convexity, then we still have to define the connection to the ideal, a connection we have not checked. However, if we omit all references to the ideal and replace "$\in J^+$" by "infinite", then we could prove a similar theorem with essentially the same proof. We could have omitted "almost convexity", and instead of W^q have $\bar{f} \in W^\tau$ (for $\tau \subseteq \omega$ finite) meaning $\bigcup \operatorname{dom}(f_i) = \omega - \tau$.

(3) We could relax the condition on the domain of Σ, but we have to at least assume that $\langle g^0, g^1, g^2 \rangle \in \operatorname{dom}(\Sigma)$ in 4.16(b), and we would also like to have that the neat sequences are still dense (see Remark 4.11 and also Remark 4.24).

(4) We could change FPH, and accordingly the theorem. E.g., we can have a function Σ^f defined on some convex $\bar{f} \in IW$ (and omit $f_1 \cup \cdots \cup f_n \in \Sigma(f_1, \ldots, f_n)$). In FPH instead of (b) we would have

$$\Sigma^f(f_1, \Sigma^f(f_2, \ldots, \Sigma^f(f_{n-1}, f_n) \cdots)) \subseteq X.$$

4.19. Proof of Theorem 4.17. To prove Theorem 4.17, we will do the following.

(A) Define a game $G(I, J)$ with two players, ZERO and PLUS.

(B) Prove that ZERO does not have a winning strategy.

(C1) Provide ZERO with a strategy, which on the side builds a sequence $\bar{g}$.

(C2) Prove that this strategy is well-defined.

(D) Prove that any play in which ZERO followed the strategy from C but PLUS won, produces a $\bar{g} \in W^0_J$ satisfying the requirements.

(A) For any ideals I, J on ω, we will define a game $G(I, J)$.

4.20. Definition. $G(I, J)$ is the following game.

Let $k_{-1} = 0$.

In the jth move ($j \geq 0$), player ZERO chooses an integer $a_j > k_{j-1}$, an infinite set $B_j \subseteq [a_j, \omega)$, and for each $b \in B_j$ a set $K_{j,b} \in J^+$.

Then player PLUS chooses $b_j \in B_j$, $k_j \in K_{j,b_j}$.

In the end (after ω moves), player PLUS wins if

$$\{k_j : j \in \omega\} \in J^+ \quad \text{and} \quad \bigcup_{j\in\omega} [a_j, b_j) = \omega \pmod I.$$

(*Note.* It cannot hurt ZERO to make $K_{j,b}$ smaller, so without loss of generality $K_{j,b} \subseteq (b, \omega)$.)

(B) Now assume that the ideals I and J are as in the hypothesis of Theorem 4.17. Then we can state the following fact.

4.21. Fact. *ZERO does not have a winning strategy in this game.*

Proof. Let σ be a strategy for ZERO. Choose a sequence $\langle N_i : i < \omega\rangle$ of finite submodels of $H((2^{\aleph_0})^+)$ and sequences $\langle \kappa_i : i < \omega\rangle$, $\langle m_i : i < \omega\rangle$ of natural numbers such that:

(a) N_0 contains I, J, σ, and all the (finitely many) functions we may need later on (see the following remark);

(b) $N_i \subseteq N_{i+1}$;

(c) $N_i \cap \omega = m_i$;

(d) if $h \in N_i$, h a function, $x_1, \ldots, x_n \in N_i$, $\langle x_1, \ldots, x_n\rangle \in \operatorname{dom}(h)$, all x_i distinct, then $h(x_1, \ldots, x_n) \in N_{i+1}$;

(e) $m_i < \kappa_i < m_{i+1}$;

(f) $\{\kappa_i : i \in \omega\} \in J^+$;

(g) $\kappa_i \in \bigcap \{A : A \in J^+, A \in N_i\}$.

This can easily be done playing the 'Ramsey game' for J (see Definition 4.1). $\square$ (Fact 4.21)

Remark. We may assume

(h) if $A \in N_i \cap [\omega]^\omega$, then $A \cap [m_i, m_{i+1}) \neq \emptyset$;

(we only have to put the function $F(A, k) = \min\{i \in A : i > k\}$ into N_0.)

Without loss of generality $\{\kappa_{5i+3} : i \in \omega\} \in J^+$. Consider the function $f \in \omega^\omega$ defined by

$$f(n) = i \leftrightarrow m_{5i} \leqslant n < m_{5i+5}.$$

By assumption on J and I, we can find a set $u \subseteq \omega$, such that

$$\bigcup_{i\in u} [m_{5i}, m_{5i+5}) \in J^+, \qquad \bigcup_{i\notin u} [m_{5i}, m_{5i+5}) = \omega \pmod I.$$

Let $u = \{\alpha_l : l < \omega\}$, in increasing order. Without loss of generality $\alpha_0 > 0$. Note that $\{\kappa_{5\alpha_l+3} : l < \omega\} \in J^+$, as J^+ is a filter.

We will define a play of the game $G(I, J)$ in which ZERO uses her strategy.

By induction we will preserve the fact that everything chosen in the first $j-1$ moves ($j \geqslant 1$) belongs to $N_{5\alpha_{j-1}+4}$.

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In the jth move, player ZERO uses her strategy σ to choose a_j, B_j, $\langle K_{j,b}: b \in B_j \rangle$, all in $N_{5\alpha_{j-1}+5}$. By the remark above, PLUS can choose a b_j in $[m_{5\alpha_j}, m_{5\alpha_j+1}) \cap B_j$, so $[a_j, b_j) \supseteq [5\alpha_{j-1}+5, 5\alpha_j)$. As $K_{j,b_j} \in N_{5\alpha_j+2}$, we can let

$$k_j = \kappa_{5\alpha_j+3} \in \bigcap \{A: A \in J^+, A \in N_{5\alpha_j+2}\} \subseteq K_{j,b_j}.$$

Clearly everything chosen so far is in $N_{5\alpha_j+4}$ (as $k_j < m_{5\alpha_j+4}$), so we have completed the induction step. At the end of the game we have

$$\{k_j: j \in \omega\} = \{\kappa_{5\alpha_j+3}: j \in \omega\} \in J^+$$

and

$$\bigcup_{j\in\omega} [a_j, b_j) \supseteq \bigcup_{j\in\omega} [m_{5\alpha_j+5}, m_{5\alpha_{j+1}}) \supseteq \bigcup_{i\notin u} [m_{5i}, m_{5i+5}) = \omega \pmod I,$$

so player PLUS wins this play, and σ was not a winning strategy.

(C1) We will define a strategy for $G(I, J)$.

4.22. Definition. Assuming that X is J-large above $\bar f \in W_J^0$, $\bar f$ neat, define a strategy as follows.

Let $\bar f^{-1} = \bar f$, $X_{-1} = X$. (Remember that by the rules of the game, $k_{-1} = 0$.) In move j, player ZERO will define a_j, B_j, $\langle K_{j,b}: b \in B_j \rangle$. On the side, ZERO will also define $\bar f^j \in W_J^{k_j}$, $X_j \subseteq \mathrm{wd}(\bar f^j)$, $\langle g_{3j}, g_{3j+1}, g_{3j+2} \rangle$.

In the jth move ($j \in \omega$), player ZERO chooses an integer $a_j > k_{j-1}$, an infinite set B_j, and defines, for each $b \in B_j$, a set $K_{j,b} \in J^+$.

$K_{j,b}$ is the set of k satisfying: there are functions g_{3j}, g_{3j+1}, g_{3j+2}, an integer r_j, and a sequence $\bar f^j$ such that

(1) $\langle g_{3j}, g_{3j+1}, g_{3j+2} \rangle \geqslant \bar f^{j-1} \restriction r_j$;
(2) $\mathrm{dom}(g_{3j} \cup g_{3j+1})$ and $\mathrm{dom}(g_{3j+2})$ are intervals with union $= [k_{j-1}, k)$;
(3) $\bar f^j \geqslant \bar f^{j-1} \restriction [r_j, \omega)$;
(4) $[a_j, b) \subseteq \mathrm{dom}(g_{3j+1})$;
(5) $\Sigma_-(g_{3j}, g_{3j+1}, g_{3j+2})$ is well-defined and $\subseteq X_{j-1}$;
(6) the set

$$X_j = \{h \in \mathrm{wd}(\bar f^j): \forall g \in \Sigma_-(g_{3j}, g_{3j+1}, g_{3j+2})\ g \cup h \in X_{j-1}\}$$

is J-large above $\bar f^j$.

(Note that $K_{j,b}$ decreases as b increases. Also note that without loss of generality all $\bar f^j$ are neat.)

We will prove below that it is always possible to choose a_j such that for infinitely many b's the set $K_{j,b}$ is in J^+.

Then player PLUS chooses $b_j \in B_j$, $k_j \in K_{j,b_j}$ (and thus implicitly f^j, g_{3j}, etc.)

(*Note.* To be precise, player ZERO really chooses a_j, B_j, and a function F_j mapping each $\langle b, k \rangle$ ($b \in B_j$, $k \in K_{j,b}$) to a tuple $\langle g_{3j,b,k}, g_{3j+1,b,k}, g_{3j+2,b,k}, X_{j,b,k}$, $\bar f^{j,b,k}$, $r_{j,b,k} \rangle$. By selecting k_j and b_j player PLUS implicitly selects $X_j = X_{j,b_j,k_j}$, etc.)

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(C2) We have to prove that this strategy is always well-defined, i.e., that it is always possible to choose a_j and B_j as required (i.e., there is a_j such that $\{b: K_{j,b} \in J^+\}$ is infinite).

4.23. Fact. *Let $\bar{f} \in W_J^q$, X J-large above $\bar{f}$. Then we can find functions g_0^*, g_1^*, g_2^*, integers l^* and q^*, a sequence $\bar{f}^* \in W_J^{q^*}$ and set $X^* \subseteq \mathrm{wd}(\bar{f}^*)$ such that*

(a) $\langle g_0^*, g_1^*, g_2^* \rangle \geqslant \bar{f} \restriction l^*$;

(b) *$\mathrm{dom}(g_0^* \cup g_1^*)$ and $\mathrm{dom}(g_2^*)$ are intervals whose union is $[q, q^*)$;*

(c) $\bar{f} \restriction [l^*, \omega) \leqslant \bar{f}^* \in W_J^{q^*}$;

(d) *X^* is J-large above $\bar{f}^*$;*

(e) *if $h \in X^*$, $g \in \Sigma_-(g_0^*, g_1^*, g_2^*)$, then $g \in X$ and $g \cup h \in X^*$; $\langle g_0^*, g_1^*, g_2^* \rangle \in \mathrm{dom}(\Sigma)$, $\langle g, h \rangle \in \mathrm{dom}(\Sigma)$.*

Proof. We use the game from the definition of "J is a Ramsey ideal". We shall describe a strategy for player ZERO. Without loss of generality $X \subseteq \mathrm{wd}(\bar{f})$.

Let $k_{-1} = q$, $\bar{f}^{-1} = \bar{f}$. In move j player ZERO on the side chooses g_j, r_j, $\bar{f}^j$ such that

(i) $a_{f_0^j} = k_{j-1}$;

(ii) $\bar{f}^{j-1} \restriction [r_j, \omega) \leqslant \bar{f}^j$;

(iii) $g_j \in \Sigma(\bar{f}^{j-1} \restriction r_j)$, $g_0 = \emptyset$;

(iv) for all $\langle g^0, g^1, g^2 \rangle \geqslant \langle g_1, \ldots, g_j \rangle$ such that $\mathrm{dom}(g^0 \cup g^1)$ is an interval (hence also $\mathrm{dom}(g^2)$): if $\Sigma_-(g^0, g^1, g^2) \subseteq X$ and well-defined, then there is no h satisfying

$$h \in \mathrm{wd}(f^j), \qquad \text{if } g \in \Sigma_-(g^0, g^1, g^2), \text{ then } g \cup h \in X.$$

Then ZERO lets $A_j = E(\bar{f}^j)$. PLUS has to choose $k_j \in A_j$.

Note. It is not clear whether player ZERO can follow this strategy, i.e., she may be stuck sometimes. In this case she just gives up, and PLUS wins immediately. (We will show in fact this is the only way for PLUS to win.)

This strategy is not a winning strategy (as the additional option of giving up is no advantage for player ZERO). So there is a play in which PLUS wins. But if the play lasts ω moves, then PLUS has produced $\{k_j: j \in \omega\} \in J^+$, so $\bar{g} = \langle g_i: i < \omega \rangle \in W_J^q$, and $\bar{g} \geqslant \bar{f}$. But then X is J-large above $\bar{g}$, so by FPH we can find a j and g^0, g^1, g^2 contradicting (iv).

So for some j player ZERO is stuck in the jth step. The choice of r_j was certainly possible (in fact dictated by the value of k_{j-1}). But she found that for all $\bar{f}^j \in W_J^{k_{j-1}}$, if $\bar{f}^j \geqslant \bar{f}^{j-1} \restriction [r_j, \omega)$ and $g_j \in \Sigma(\bar{f}^{j-1} \restriction r_j)$, then there are $\langle g^0, g^1, g^2 \rangle \geqslant \langle g_0, \ldots, g_j \rangle$ in $\mathrm{dom}(\Sigma)$, $\mathrm{dom}(g^0 \cup g^1)$ and $\mathrm{dom}(g^2)$ are intervals, and $h \in \mathrm{wd}(\bar{f}^j)$ such that

$$\forall g \in \Sigma_-(g^0, g^1, g^2): g \in X \text{ and } g \cup h \in X_j,\ \langle g, h \rangle \in \mathrm{dom}(\Sigma). \qquad (*)$$

For each possible $\langle g^0, g^1, g^2 \rangle$, consider the set

$$Z_{g^0, g^1, g^2} = \{h \in \mathrm{wd}(\bar{f}^{j-1} \restriction [r_j, \omega]): (*)\}.$$

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As $\mathrm{dom}(g^0 \cup g^1 \cup g^2) = [q, k_j)$ is a fixed finite set, there are only finitely many such sets $Z_{\langle g^0, g^1, g^2 \rangle}$ and their union is J-large above $f^{j-1} \restriction [r_j, \omega)$. Hence we can find some $\langle g_0^*, g_1^*, g_2^* \rangle$ and a sequence $\bar{f}^*$, $\bar{f}^{j-1} \leq \bar{f}^*$, such that $Z_{g_0^*, g_1^*, g_2^*}$ is large above $\bar{f}^*$.

This is as required (letting $X^* = Z_{g_0^*, g_1^*, g_2^*}$), so Fact 4.23 is proved. □ (Fact 4.23)

4.24. Remark. We could change the definition of a word system 4.5 by only requiring $\Sigma(f_0, \ldots, f_{n-1})$ to be defined whenever $\langle \mathrm{dom}(f(i)) : i < n \rangle$ is neat. Then we could still prove Fact 4.23. If J is the ideal of finite sets, we could use Fact 4.23 to prove that the neat sequences are dense by inductively constructing a neat sequence above $\bar{f}$.

Now Fact 4.23 almost proved (C2), i.e., that the strategy from (C1) is well-defined. We only have to strengthen Fact 4.23 to the next fact.

4.25. Fact. *Let $\bar{f} \in W_J^q$, X J-large above $\bar{f}$. Let $E \subseteq E(\bar{f})$, $E \in J^+$, and let $G : \omega \to \omega$ be a function satisfying $\forall i$, $i < G(i)$. Then we can find functions g_0^*, g_1^*, g_2^*, integers l^* and q^*, a sequence $\bar{f}^*$ and a set X^* such that* (a)–(e) *of Fact* 4.23 *hold, and in addition*

(f) $q^* \in E$;

(g) *for some* i, $[i, G(i)] \subseteq \mathrm{dom}(g_1^*)$.

Proof. Let $B \in J^+$, $B \subseteq E$ and $i \in B \to G(i) < \min(B - \{0, \ldots, i\})$. (Such a B can be found using the property that J is Ramsey.)

Let $0 = r_0 < r_1 < \cdots$ be such that $B = \{a_{f_{r_l}} : l \in \omega\}$. Now replace $\bar{f}$ by $\bar{f}'$, where $\bar{f}'_l \in \Sigma(\bar{f} \restriction [r_l, r_{l+1}))$ and apply Fact 4.23. □ (Fact 4.25)

4.26. Conclusion. Now we can show that the strategy of (C) is well-defined: consider the situation at the jth move. For each $a \in [k_{j-1}, \omega)$ let $u_a = u_{a,j} = \{b : K_{j,b} \in J^+\}$. (Remember that $K_{j,b}$ was defined by the strategy in step (C).) We have to show that for some a, u_a is infinite. If not, define

$$G(a) = \max(\{a\} \cup u_a) + 2.$$

Apply Fact 4.25 to G. But then $G(a) \in u_a$ (by definition of u_a), a contradiction. This finishes the proof of (C2).

(D) The strategy described in (C) cannot be a winning strategy. Hence there exists a play in which ZERO follows her strategy but PLUS wins. It is easy to see that the sequence $\bar{g} = \langle g_0, g_1, \ldots \rangle$ defined by this play will satisfy the requirements of Theorem 4.17: Assume that $g_i' \in \Sigma_-(g_{3i}, g_{3i+1}, g_{3i+2})$, for all $i < j$. Then by Definition 4.22(5) we get that $g_{j-1}' \in X_{j-2}$. Using (6) and the definition of the X_i's we get for $l = j-2, \ldots, 0$ that $\bigcup_{l \leq i < j} g_i' \in X_{l-1}$, so eventually $\bigcup_{i<j} g_i' \in X$. So we finish. □

5. Proof of Lemma 3.9

In order to apply the result of the previous section to the proof of the Main Lemma 3.9 from Section 3, we first have to show that whenever J is a maximal Ramsey ideal, then $(\mathfrak{W}^*, J)$ satisfies FPH.

5.1. Lemma (Letting $\mathfrak{W} = \mathfrak{W}^*$, see Remark 4.6). *If X is J-large $\bar{f} \in W_J^q$, then for some $\bar{f}' \geqslant \bar{f}$ and some r with $a_{f_r} \in E(\bar{f}')$,*

$$\{g \in \mathrm{wd}(\bar{f}' \mid [r, \omega)) \colon \textit{for some } e_0 \neq e_1 \textit{ in } \Sigma_-(\bar{f} \mid r),\ e_0 \cup g \in X \textit{ and } e_1 \cup g \in X\}$$

is J-large above $\bar{f}' \mid [r, \omega)$.

Proof. Without loss of generality $\bar{f}$ is neat. We will try to build by induction (playing the Ramsey game on the side) a sequence $h = \langle h_i \colon i < \omega \rangle \geqslant \bar{f}$, $h_i \in \Sigma(\bar{f} \mid [r_i, r_{i+1}))$ such that for each i

(α) there is at most one $e \in \Sigma_-(\bar{h} \mid i)$ such that $e \cup h_i \in X$;

(β) there is at most one $e \in \Sigma_-(\bar{h} \mid i)$ such that $e \cup (-h_i) \in X$.

Case 1: Assume the induction gets stuck in some stage i. Then we have $\mathrm{wd}(\bar{f} \mid [r_i, \omega)) = Z_- \cup Z_+$, where

$$Z_+ = \{h \in \mathrm{wd}(\bar{f} \mid [r_i, \omega)) \colon \text{for some } e_0 \neq e_1,\ e_0 \cup h \in X,\ e_1 \cup h \in X\},$$
$$Z_- = \{h \in \mathrm{wd}(\bar{f} \mid [r_i, \omega)) \colon \text{for some } e_0 \neq e_1,\ e_0 \cup (-h) \in X,\ e_1 \cup (-h) \in X\}.$$

One of these sets is J-large above some $\bar{f}' \geqslant \bar{f}$. As $Z_- = \{-g \colon g \in Z_+\}$, we are done.

Case 2: Assume that the induction succeeds. Then we have found a sequence $\bar{h} \geqslant \bar{f}$ in W_J^q such that

$$\text{if } l \in \omega,\ g \in \{h_l, -h_l\}, \text{ then for at most one } e \in \Sigma_-(\bar{f} \mid l) \text{ is } e \cup g \in X.$$

Now we can easily build by induction (playing the Ramsey game on the side) a sequence $\bar{h}' \geqslant \bar{h}$ such that for each r, $\mathrm{wd}(\bar{h}' \mid r) \cap X = \emptyset$, so $\mathrm{wd}(\bar{h}') \cap X = \emptyset$. $\square$

5.2. Corollary ($\mathfrak{W} = \mathfrak{W}^*$). *If X is J-large above $\bar{f} \in W_J^q$, then for some $r \in \omega$, some $\bar{f}' \geqslant \bar{f}$ and some $e_0 \neq e_1$ in $\Sigma_-(\bar{f} \mid r)$ and some $\bar{f}' \geqslant \bar{f} \mid [r, \omega)$ the set*

$$\{g \colon g \in \mathrm{wd}(\bar{f}'),\ e_0 \cup g \in X,\ e_1 \cup g \in X\}$$

is large above $\bar{f}'$.

Proof. Fix r as in the conclusion of Lemma 5.1. There are only finitely many possible pairs $\langle e_0, e_1 \rangle$. So we can use the previous claim and then apply Fact 4.15. $\square$

5.3. Lemma. *If X is J-large above $\bar{f}, \bar{f}$ neat, then for some $h_1, h_2 \in X$, h_1 is a proper initial segment of h_2.*

Proof. Assume otherwise. Let $h_0 \in \mathrm{wd}(\bar{f}) \cap X$, say $h_0 \in \Sigma(\bar{f} \restriction r_0)$. Then no extension of h_0 is in X. Let $h_1 \in \Sigma(\bar{f} \restriction [r_0, r_1))$ be such that $r_1 > r_0$ and

$$\text{if some extension on } (-h_0) \text{ is in } X \cap \mathrm{wd}(\bar{f} \restriction [r_0, \omega)), \text{ then } -(h_0 \cup h_1) \in X.$$

Let $r_2 > r_1$, $h_2 \in \bar{f} \restriction [r_1, r_2)$ arbitrary. Let

$$\bar{f}' = \langle h_0 \cup h_1 \cup h_2, f_{r_2}, f_{r_2+1}, \ldots \rangle.$$

Then $\bar{f}' \geqslant \bar{f}$. Take any $g \in \mathrm{wd}(\bar{f}') \cap X$. Then either $g \supseteq h_0 \cup h_1 \cup h_2$ or $g \supseteq -(h_0 \cup h_1 \cup h_2)$. But then in the first case h_0 has an extension in X and in the second case $-h_0$ has an extension in X, hence by definition $-(h_0 \cup h_1) \in X$, and $-(h_0 \cup h_1)$ has an extension g in X. So in both cases we reach a contradiction. $\square$

Putting the last two claims together we get FPH. Let $e_0 \neq e_1$, g, h, r, be such that

$$g, g \cup h \in \mathrm{wd}(\bar{f} \restriction [r, \omega)),$$
$$e_0 \cup g, e_1 \cup g, e_0 \cup g \cup h, e_1 \cup g \cup h \in X.$$

Now define $g^1 = e_0 \restriction \{i: e_0 \neq e_1\}$, and let g^0 and g^2 be such that $g^0 \cup g^1 \cup g^2 = e_0$ and $\mathrm{dom}(g^0 \cup g^1) = [q, \max \mathrm{dom}(g^1)]$. It is easy to check that this works. (Note that e_0 and e_1 agree on $\mathrm{dom}(f_0) \cup \mathrm{dom}(f_{r-1})$. This ensures $\min \mathrm{dom}(g^0) < \min \mathrm{dom}(g^1)$, etc.)

5.4. Proof of Lemma 3.9. Let J be an ideal dual to a Ramsey ultrafilter D, and assume that for no quotient I' of I we have $J \supseteq I'$. Then also for every finite-to-one function $f, f^*(J) \not\supseteq f^*(I)$, and the same is true for every quotient of I. Let $\boldsymbol{\tau}$ be a name for a subset of ω. (We will also denote the characteristic function of this set by $\boldsymbol{\tau}$.) Let p be a condition in Q_I. It is enough to find a condition $q \geqslant p$ and a set $T \in J$ such that $q \Vdash \boldsymbol{\tau} \subseteq T$ or $q \Vdash \boldsymbol{\tau} \cap T = \emptyset$.

Without loss of generality we may assume that for all n, for all $s: [0, n) \to \{+1, -1\}$, p^s decides whether $|s|$ is in $\boldsymbol{\tau}$.

As $\{q \in Q_I: q \geqslant p\}$ is naturally isomorphic to $Q_{I'}$, where I' is the quotient of I modulo the equivalence relation induced by p, we may without loss of generality assume that p is the empty condition, i.e., $E^p = \emptyset$, $E_i = \{i\}$. (Note that I' has the same properties that were initially assumed about I, and the operation $p \to p^s$ is respected by this isomorphism). Also without loss of generality, let h^p be the function that is constantly equal to 1. (So now p^s is the condition with $E^{p^s} = [0, n)$, $h^{p^s} \supseteq s$, and we still have that p^s decides whether $|s|$ is in $\boldsymbol{\tau}$.)

Let $X = \{s: s: [0, n) \to \{+1, -1\}, n \in \omega\}$. Then $X = X_0 \cup X_1$, where $X_i = \{s \in X: p^s \Vdash \boldsymbol{\tau}(|s|) = i\}$. As X is J-large above any sequence, we can find a sequence $\bar{f}$ such that either X_0 or X_1 is J-large above $\bar{f}$. Without loss of generality, assume that X_1 is J-large above $\bar{f}$ (otherwise replace $\boldsymbol{\tau}$ by $\omega - \boldsymbol{\tau}$). Find a sequence $\bar{g}$ as in

Theorem 4.17. Now define a condition q as follows:

$$E^q = \bigcup_{i\in\omega} (\mathrm{dom}(g_{3i}) \cup \mathrm{dom}(g_{3i+2})),$$

$$E_i^q = \mathrm{dom}(g_{3i+1}), \qquad h^q = \bigcup_{j\in\omega} g_j.$$

Note that by (iii) $E^q \in I$, and each E_i^q, being a finite set, is also in I, so this definition is legitimate. Then clearly we have $q \in Q_I$, $q \geqslant p$. To finish the proof of Lemma 3.9, it is (by (ii)) enough to show that for all i

$$q \Vdash a_{g_{3i}} \in \mathfrak{r}.$$

Assume that this is not true, and take any condition $r \geqslant q$ that forces $a_{g_{3i}} \notin \mathfrak{r}$. Then without loss of generality $E^r \supseteq [0, a_{g_{3i}})$.

Let, for $l < i$, $g_l' = h^r \upharpoonright [a_{g_{3l}}, g_{g_{3l+3}})$. Then as $r \geqslant g$, $g_l' \in \Sigma_-(g_{3l}, g_{3l+1}, g_{3l+2})$. Let $s = \bigcup_{l<3i} g_l'$; then, by the lemma, $s \in X_1$, and, by construction, $r \geqslant p^s$. Hence $r \Vdash a_{g_{3i}} \in \mathfrak{r}$, a contradiction. $\square$

6. Conclusions

We will finish the proof of Theorem 1.4 by describing how the iteration in 2.4 can be modified to adjust the values of $\mathfrak{s}$ and $\mathfrak{i}$.

6.1. Proof of Con($\mathfrak{r} = \aleph_1 < \aleph_2 = \mathfrak{u} = \mathfrak{i} = \mathfrak{s} = \mathfrak{d} = \mathfrak{c}$). We will do a countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$, where for even α, $Q_\alpha = Q_{I_\alpha}$ for some maximal Ramsey ideal I_α, and for odd α, Q_α is the (first) forcing Q in [3]. This forcing is proper, and it adds a new subset of ω that is not split by any old infinite set, so we will have $V_{\omega_2} \vDash \mathfrak{s} = \mathfrak{d} = \mathfrak{i} = \aleph_2$. This forcing Q also does not destroy any P-points, so the same arguments as the ones in 2.4 show that $V_{\omega_2} \vDash \mathfrak{u} = \aleph_2 \wedge \mathfrak{r} = \aleph_1$.

To show the second part of Theorem 1.4, we first need a few simple facts about independent families.

6.2. Definition. (1) For any infinite family $\mathfrak{X} \subseteq [X]^\omega$ (X a countable set), let

$$\mathrm{FI}(\mathfrak{X}) = \{f : f \text{ a finite partial function from } \mathfrak{X} \text{ to } \{+1, -1\}\}.$$

(2) For $f \in \mathrm{FI}(\mathfrak{X})$, let

$$X_f = \bigcap_{A\in\mathrm{dom}(f)} A^{f(A)},$$

(where $A^{+1} = A$, $A^{-1} = (X - A)$).

(There should be an index indicating what the base set X is, but this will always be clear from the context.)

(3) $\mathfrak{X}$ is *independent* (on X), if for all $f \in \mathrm{FI}(\mathfrak{X})$, X_f is infinite (iff for all $f \in \mathrm{FI}(\mathfrak{X})$, $X_f \neq \emptyset$).

6.3. Definition. If $\mathfrak{X}$ is an independent family, let

$$\begin{aligned} I_{\mathfrak{X}} &= \{A \subseteq \omega : \forall f\ \exists g \supseteq f\ X_g \cap A \text{ finite}\} \\ &= \{A \subseteq \omega : \forall f\ \exists g \supseteq f\ X_g \cap A = \emptyset\}. \end{aligned}$$

Clearly, I is an ideal, containing all finite sets. For all f, $X_f \notin I_{\mathfrak{X}}$.

6.4. Definition. A maximal independent family is *dense*, if for every $A \in I_{\mathfrak{X}}^+$ there is $f \in \mathrm{FI}(\mathfrak{X})$ such that $X_f \subseteq A$.

6.5. Definition. If $\mathfrak{X}$ is a maximal independent family, $f \in \mathrm{FI}(\mathfrak{X})$, then let $\mathfrak{X} \mid X_f = \{A \cap X_f : A \in \mathfrak{X} - \mathrm{dom}(f)\}$.

6.6. Lemma. *For every maximal independent family $\mathfrak{X}$ there is $f \in \mathrm{FI}(\mathfrak{X})$ such that for every $g \supseteq f$, $\mathfrak{X} \mid X_g$ is a maximal independent family.*

Proof. Let $\langle f_n : n \in \omega \rangle$ be a maximal family in $\mathrm{FI}(\mathfrak{X})$ with the properties:
all f_n's are incompatible;
$\mathfrak{X} \mid f_n$ is not a maximal independent family.
(The Δ-system lemma implies that such a family must indeed have size $\leqslant \omega$.)

For each n let $A_n \subseteq X_{f_n}$ be independent from $\mathfrak{X} \mid X_{f_n}$. Let $A = \bigcup_n A_n$. There is a function f such that $X_f \cap A$ is finite (or $X_f \cap (\omega - A)$ is finite). f cannot be compatible with any f_n, otherwise $X_{f \cup f_n} \cap A \supseteq X_{f \cup f_n} \cap A_n$ would be finite (or similarly for $\omega - A$).

Now consider $\mathfrak{X} \mid X_f$. For any $g \supseteq f$, $\mathfrak{X} \mid X_g$ must be maximal, since otherwise g would be a contradiction to the maximality of $\langle f_n : n \in \omega \rangle$, being incompatible with each f_n. $\square$

6.7. Lemma. *If $\mathfrak{X}$ is a maximal independent family, and for every $f \in \mathrm{FI}(\mathfrak{X})$, $\mathfrak{X} \mid X_f$ is a maximal independent family, then $\mathfrak{X}$ is dense.*

Proof. Assume $A \notin I_{\mathfrak{X}}$. Then there is an $f \in \mathrm{FI}(\mathfrak{X})$ such that

$$\forall g \supseteq f\ A \cap X_g \text{ infinite.}$$

But as $\mathfrak{X} \mid X_f$ is still maximal, there must be a function $g \in \mathrm{FI}(\mathfrak{X})$, $g \supseteq f$, such that

$$(\omega - A) \cap X_g = \emptyset.$$

Hence $X_g \subseteq A$. $\square$

6.8. Lemma. *If $\mathfrak{X}$ is a dense maximal independent family, then $\mathfrak{P}(\omega)/I_{\mathfrak{X}}$ is ccc.*

Proof. Assume that we have a sequence $\langle A_i : i < \omega_1 \rangle$ of elements of $I_{\mathfrak{X}}^+$ such that for all $i \neq j$, $A_i \cap A_j \in I_{\mathfrak{X}}$. For each $i < \omega_1$, let $f_i \in \mathrm{FI}(\mathfrak{X})$ be such that $X_{f_i} \subseteq A_i$.

Then for $i \neq j$, $X_{f_i} \cap X_{f_j} \in I_{\mathfrak{X}}$, so f_i and f_j must be incompatible (see Definition 6.3). But $\mathrm{FI}(\mathfrak{X})$ is ccc, by the Δ-system lemma. $\square$

We will prove $\mathrm{Con}(\mathrm{ZFC} + \mathfrak{s} = \mathfrak{d} = \mathfrak{r} = \aleph_1 < \aleph_2 = \mathfrak{u} = \mathfrak{i} = \mathfrak{c})$ by an interation similar to the one in 2.4.

However, some of the ideals I_α will not be maximal. To show that $V_0 \cap [\omega]^\omega$ cannot be reaped in V_{ω_2}, we will select a 'separated' family of $\aleph_2$ many Ramsey ultrafilters in the ground model and show that in every intermediate model 'most' of them are still ultrafilters.

Remember that our ground model V_0 satisfies $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. The following construction takes place in V_0.

6.9. Definition. For each $\alpha < \omega_1$, let

$$\langle A_\eta : \eta \in {}^\alpha\omega_1 \rangle$$

be a family such that for all $\eta \in T =^{\mathrm{def}} \bigcup_{\alpha<\omega_1} {}^\alpha\omega_1$

(1) $A_\eta \in [\omega]^\omega$;

(2) $\eta \subseteq \nu \rightarrow A_\nu \subseteq^* A_\eta$;

(3) $i \neq j \rightarrow |A_{\eta\hat{\ }i} \cap A_{\eta\hat{\ }j}| < \aleph_0$;

(4) for all $B \in [\omega]^\omega$ there is some $\alpha < \omega_1$ such that

$$(\forall \eta \in {}^\alpha\omega_1)(B \supseteq^* A_\eta \text{ or } B \supseteq^* (\omega - A_\eta));$$

(5) for all $f : \omega \rightarrow \omega$, for some $\alpha < \omega_1$

$$(\forall \eta \in {}^\alpha\omega_1)(f \restriction A_\eta \text{ is constant or 1-1}).$$

Such families can easily be found by induction. (Clearly, (4) and (5) continue to hold if we replace "$\exists \alpha < \omega_1$" by "$\exists \alpha' < \omega_1 \, \forall \alpha \geq \alpha'$".)

6.10. Definition. For $\eta \in \lim T = {}^{\omega_1}\omega_1$, let

$$D_\eta = \{X \subseteq \omega : \exists \alpha \, A_{\eta\restriction\alpha} \subseteq^* X\}.$$

Then the collection $\bar{D} = \langle D_\eta : \eta \in \lim T \rangle$ is a collection of $\aleph_2$ many distinct Ramsey ultrafilters.

6.11. Lemma. *For any sequence $\langle \eta_i : i < \omega_1 \rangle$ of distinct elements of* $\lim T$, *there is a subsequence $\langle \nu_\xi : \xi < \omega_1 \rangle = \langle \eta_{i_\xi} : \xi < \omega_1 \rangle$ and a sequence $\langle A_\xi : \xi < \omega_1 \rangle$ of almost disjoint sets such that for all ξ, $A_\xi \in D_{\nu_\xi}$.*

Proof. *Case* 1: $\langle \eta_i : i < \omega_1 \rangle$ has a limit point η, i.e.,

$$\forall \alpha \, \exists i \; \eta_i \restriction \alpha = \eta \restriction \alpha \text{ and } \eta_i \neq \eta.$$

In this case we can find by induction a sequence $\langle i_\xi : \xi < \omega_1 \rangle$ and an increasing sequence $\langle \alpha_\xi : \xi < \omega_1 \rangle$ such that

$$\eta_{i_\xi} \restriction \alpha_\xi = \eta \restriction \alpha_\xi \quad \text{and} \quad \eta_{i_\xi} \restriction \alpha_{\xi+1} \neq \eta \restriction \alpha_{\xi+1}.$$

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Then the family

$$\langle A_{\eta_{i_\xi}|\alpha_{\xi+1}} : \xi < \omega_1 \rangle$$

will separate $\langle D_{\eta_{i_\xi}} : \xi < \omega_1 \rangle$.

Case 2: $\langle \eta_i : i < \omega_1 \rangle$ has no limit point, in particular no η_i is a limit point, so

$$\forall j \, \exists \alpha_j \, \forall i \, (i \neq j \rightarrow \eta_i \upharpoonright \alpha_j \neq \eta_j \upharpoonright \alpha_j).$$

In this case, let $A_j = A_{\alpha_j}$, $\nu_\xi = \eta_{i_\xi} = \eta_\xi$. □

6.12. Lemma. *If I is an ideal, $\mathfrak{P}(\omega)/I$ is ccc, then $\{D_\eta : \eta \in T, D_\eta \supseteq I^*\}$ is countable.*

Proof. Assume not, then by the previous lemma we can find sequences $\langle D_{\eta_i} : i < \omega_1 \rangle$, $\forall i \, D_{\eta_i} \supseteq I^*$, and $\langle A_i : i < \omega_1 \rangle$ such that $A_i \in D_{\eta_i}$, and $i \neq j \rightarrow |A_i \cap A_j| < \aleph_0$. But as $D_{\eta_i} \subseteq I^+$, $\langle A_i/I : i < \omega_1 \rangle$ is an uncountable antichain in $\mathfrak{P}(\omega)/I$, a contradiction. □

6.13. Fact. *If I is an ideal such that $\mathfrak{P}(\omega)/I$ is ccc, then for every quotient I' of I, $\mathfrak{P}(\omega)/I'$ is ccc.*

(*Proof.* Let $f : \omega \rightarrow \omega$, $I' = \{A \subseteq \omega : f^{-1}(A) \in I\}$. Let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be an uncountable antichain modulo I'. Then $\langle f^{-1}(A_\alpha) : \alpha < \omega_1 \rangle$ is an uncountable antichain modulo I.)

6.14. Conclusion. If CH holds, and $\mathfrak{P}(\omega)/I$ is ccc, then there are at most $\aleph_1$ many η such that D_η^* extends some quotient of I.

6.15. Proof of Con(ZFC + $\mathfrak{s} = \mathfrak{d} = \mathfrak{r} = \aleph_1 < \aleph_2 = \mathfrak{u} = \mathfrak{i} = \mathfrak{c}$). As before, we do a countable support iteration $\langle P_\alpha, Q_\alpha : \alpha < \omega_2 \rangle$, where each Q_α is of the form Q_{I_α}. For even α,

$$P_\alpha \Vdash I_\alpha \text{ is a maximal ideal on } \omega.$$

As before, this will ensure that $P_{\omega_2} \Vdash \mathfrak{u} = \aleph_2$.

For odd α

$$P_\alpha \Vdash I_\alpha \text{ is the ideal } I_{\mathfrak{X}_\alpha} \text{ associated with some dense maximal independent family } \mathfrak{X}_\alpha \text{ on } \omega.$$

From Lemma 2.2(1) it is easy to see that for $I = I_{\mathfrak{X}}$, $Q_I \Vdash \mathfrak{X}$ is not maximal, so we can ensure that $P_{\omega_2} \Vdash \mathfrak{i} = \aleph_2$.

Finally, why is $\mathfrak{r} = \aleph_1$? Looking at the proof in 2.4, we see that it is enough to show that $\langle D_\eta : \eta \in T \rangle$ satisfies the hypothesis $(*)$ of Corollary 2.3(2). But this follows from Lemma 6.12, Fact 6.13 and Lemma 6.8. □

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