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Antichains in Products of Linear Orders

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Abstract. We show that:

(1) For many regular cardinals λ (in particular, for all successors of singular strong limit cardinals, and for all successors of singular ω-limits), for all $n \in \{2, 3, 4, \ldots\}$: There is a linear order L such that L^n has no (incomparability-)antichain of cardinality λ, while L^{n+1} has an antichain of cardinality λ.

(2) For any nondecreasing sequence $\langle \lambda_n : n \in \{2, 3, 4, \ldots\} \rangle$ of infinite cardinals it is consistent that there is a linear order L such that, for all n: L^n has an antichain of cardinality λ_n, but no antichain of cardinality λ_n^+.

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1. Introduction

For any nontrivial linear ordering L and any natural number $n > 1$, the set L^n (ordered by the 'pointwise' or 'product' order) is partial order which is not linear any more. A natural measure for its nonlinearity is obtained by considering the possible sizes (cardinalities) of antichains in L^n (that is, sets of pairwise incomparable elements).

Haviar and Ploščica in [2] asked: Can there be a linear ordering L and an infinite cardinal number λ such that for some natural number $n > 1$, the partial order L^n does not have antichains of size λ, while L^{n+1} has such antichains?

Farley [1] has constructed, for any *singular* cardinal κ, a linear order L of size κ such that L^3 has an antichain of cardinality κ while L^2 does not.

We will be mainly interested in this question for *regular* cardinals.

First we show in ZFC that there are many successor cardinals λ (including $\aleph_{\omega+1}$) with the following property:

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For every $n \geq 2$ there is a linear order J of size λ such that J^n has no antichain of size λ, while J^{n+1} does.

This proof is given in Section 2. It uses a basic fact from pcf theory.

In Section 3 we show that partial orders L^n can all be very different as far as the possible sizes of antichains in these orders are concerned. More precisely, we show that for any nondecreasing sequence of infinite cardinals $\langle \lambda_n : 2 \leq n < \omega \rangle$ there is a cardinal-preserving forcing extension of the universe in which we can find a linear order L such that for all $n \in \{2, 3, \ldots\}$: in L^n there are antichains of cardinality λ_n, but no larger ones.

For example, it is consistent that there is a linear order L such that L^2 has no uncountable antichain, while L^3 does.

Here we use forcing. The heart of this second proof is the well-known Δ-system lemma.

2. A ZFC Proof

Let μ be a regular cardinal. We will write D_μ for the filter of cobounded sets, i.e.,

$$D_\mu = \{A \subseteq \mu : \exists i < \mu \; \mu \setminus i \subseteq A\}.$$

For any sequence $\langle \lambda_i : i < \mu \rangle$ of cardinals, we write $\prod_{i<\mu} \lambda_i$ for the set of all functions f with domain μ satisfying $f(i) < \lambda_i$ for all i. The relation $f \sim_{D_\mu} g \Leftrightarrow \{i < \mu : f(i) = g(i)\} \in D_\mu$ is an equivalence relation. We call the quotient structure $\prod_i \lambda_i / D_\mu$ (and we often do not distinguish between a function f and its equivalence class). $\prod_i \lambda_i / D_\mu$ is partially ordered by the relation

$$f <_{D_\mu} g \quad \text{iff} \quad \{i < \mu : f(i) < g(i)\} \in D_\mu.$$

DEFINITION 2.1. For any partial order $(P, \leq)$ and any regular cardinal λ we say $\lambda = \operatorname{tcf}(P)$ ("λ is the true cofinality of P") iff there is an increasing sequence $\langle p_i : i < \lambda \rangle$ such that $\forall p \in P \, \exists i < \lambda : p \leq p_i$.

Remark 2.2. (1) For any partial order P there is at most one regular cardinal λ which can be the true cofinality of P, that is: if $\langle p_i : i < \lambda \rangle$ and $\langle p'_j : j < \lambda' \rangle$ are both as above, then $\lambda = \lambda'$.

(2) Every linear order has a true cofinality.

(3) If P has true cofinality, then P is upward directed.

(4) There are partial orders which are upward directed but have no true cofinality, for example $\omega \times \omega_1$.

THEOREM 2.3. *Assume that*

(1) $\langle \lambda_i : i < \mu \rangle$ *is an increasing sequence of regular cardinals,*
(2) *For each* $j < \mu$, $|\prod_{i<j} \lambda_i| < \lambda_j$,

(3) *λ is regular and* $\operatorname{tcf}(\prod_{i<\mu} \lambda_i / D_\mu) = \lambda$,
(4) $n \geq 2$.

Then there is a linear order J of size λ such that

- *J^{n+1} has an antichain of size λ,*
- *J^n has no antichain of size λ.*

To show that there are indeed many cardinals λ of the form $\operatorname{tcf} \prod_{i<\mu} \lambda_i / D_\mu$ as in our theorem, we use Theorem II.1.5 from [3, p. 50]:

THEOREM 2.4. *If κ is singular and $\operatorname{cf} \kappa = \mu$, then there is a strictly increasing sequence $\langle \lambda_i : i < \mu \rangle$ of regular cardinals such that $\kappa = \sum_{i<\mu} \lambda_i$ and $(\prod_{i<\mu} \lambda_i, <_{D_\mu})$ has true cofinality κ^+.*

It is clear that in the above theorem we can replace $\langle \lambda_i : i < \mu \rangle$ by any increasing cofinal subsequence $\langle \lambda_{f(j)} : j < \mu \rangle$.

CONCLUSION 2.5. *Whenever $\lambda = \kappa^+$, where κ is a singular strong limit cardinal, or even only the following holds:*

$$\kappa \text{ is singular}, \text{ and } \forall \kappa' < \kappa : (\kappa')^{<\operatorname{cf}(\kappa)} < \kappa,$$

then we can find a sequence $\langle \lambda_i : i < \operatorname{cf}(\kappa) \rangle$ as in the assumption of Theorem 2.3.

For example, if $\lambda = \aleph_{\omega+1}$, then there is an increasing sequence $\langle n_k : k \in \omega \rangle$ of natural numbers such that $\operatorname{tcf}(\prod_{k\in\omega} \aleph_{n_k} / D_\omega) = \aleph_{\omega+1}$. So Theorem 2.3 implies, for any $n > 1$ there is a linear order J such that J^n has no antichain of size $\aleph_{\omega+1}$, whereas J^{n+1} has one.

Proof. Let $\mu = \operatorname{cf}(\kappa)$. We can start with a sequence $\langle \lambda_i : i < \mu \rangle$ such that $\prod_{i<\mu} \lambda_i / D_\mu$ has true cofinality $\lambda = \kappa^+$. Choose $f \colon \mu \to \mu$ increasing cofinal such that $\prod_{j<i} \lambda_{f(j)} < \lambda_{f(i)}$ for all $i < \mu$, then $\langle \lambda_{f(j)} : j < \mu \rangle$ is as required. □

The proof of Theorem 2.3 will occupy the rest of this section. The assumption of Theorem 2.3 says $\operatorname{tcf}(\prod_{i<\mu} \lambda_i / D_\mu) = \lambda$, so we may fix a sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in $\prod_{i<\mu} \lambda_i$ such that for all $\alpha < \beta < \lambda$ we have $f_\alpha <_{D_\mu} f_\beta$.

We start by writing $\mu = \bigcup_{\ell=0}^{n} A_\ell$ as a disjoint union of $n+1$ many D_μ-positive (i.e., unbounded) sets. For $\ell = 0, \ldots, n$ we define a linear order $<_\ell$ on λ as follows:

DEFINITION 2.6. For any two distinct functions $f, g \in \prod_{i<\mu} \lambda_i$ we define

$$d(f, g) = \sup\{i < \mu : f \restriction i = g \restriction i\} = \max\{i < \mu : f \restriction i = g \restriction i\}. \tag{1}$$

That is: if $f \neq g$, then $d(f, g) = \min\{j : f(j) \neq g(j)\}$ is the first point where f and g differ.

For $\alpha, \beta \in \lambda$ we define $\alpha <_\ell \beta$ iff:

$$\begin{aligned} &\text{letting } i := d(f_\alpha, f_\beta), \\ &\qquad\qquad \text{either } i \in A_\ell \text{ and } f_\alpha(i) < f_\beta(i) \\ &\qquad\qquad\quad \text{or } i \notin A_\ell \text{ and } f_\alpha(i) > f_\beta(i). \end{aligned} \tag{2}$$

We leave it to the reader to check that $<_\ell$ is indeed a linear order on λ.

We now define J to be the 'ordinal sum' of all the orders $<_\ell$:

DEFINITION 2.7. Let

$$J = \bigcup_{\ell=0}^{n} \{\ell\} \times (\lambda, <_\ell)$$

with the 'lexicographic' order, i.e., $(\ell_1, \alpha_1) < (\ell_2, \alpha_2)$ iff $\ell_1 < \ell_2$, or $\ell_1 = \ell_2$ and $\alpha_1 <_{\ell_1} \alpha_2$.

CLAIM 2.8. *J^{n+1} has an antichain of size λ.*

Proof. Let $\vec{t}_\alpha = \langle (0, \alpha), \ldots, (n, \alpha) \rangle \in J^{n+1}$.

For any $\alpha \neq \beta$ we have to check that $\vec{t}_\alpha$ and $\vec{t}_\beta$ are incomparable. Let $i^* = d(f_\alpha, f_\beta)$, and find ℓ^* such that $i^* \in A_{\ell^*}$. Wlog assume $f_\alpha(i^*) < f_\beta(i^*)$. Then $\alpha <_{\ell^*} \beta$, but $\alpha >_\ell \beta$ for all $\ell \neq \ell^*$, i.e., $(\ell^*, \alpha) <_J (\ell^*, \beta)$, but $(\ell, \alpha) >_J (\ell, \beta)$ for all $\ell \neq \ell^*$. □

Finishing the proof of 2.3. It remains to show that J^n does not have an antichain of size λ. Towards a contradiction, assume that $\langle \vec{t}_\beta : \beta < \lambda \rangle$ is an antichain in J^m, $m \leq n$, and m as small as possible. Let $\vec{t}_\beta = (t_\beta(1), \ldots, t_\beta(m)) \in J^m$. For $k = 1, \ldots, m$ we can find functions ℓ_k, ξ_k such that

$$\forall \beta < \lambda \; \forall k\colon \; t_\beta(k) = (\ell_k(\beta), \xi_k(\beta)).$$

Thinning out we may assume that the functions $\ell_1, \ldots, \ell_m$ are constant. We will again write $\ell_1, \ldots, \ell_m$ for those constant values.

We may also assume that for each k the function $\beta \mapsto \xi_k(\beta)$ is either constant or strictly increasing. If any of the functions ξ_k is constant we get a contradiction to the minimality of m, so all the ξ_k are strictly increasing. So we may moreover assume that $\beta < \gamma$ implies $\xi_k(\beta) < \xi_{k'}(\gamma)$ for all k, k', and in particular $\beta \leq \xi_k(\beta)$ for all β, k.

Now define $g_\beta^+, g_\beta^- \in \prod_{i<\mu} \lambda_i$ for every $\beta < \lambda$ as follows:

$$\begin{aligned} g_\beta^+(i) &= \max(f_{\xi_1(\beta)}(i), \ldots, f_{\xi_n(\beta)}(i)), \\ g_\beta^-(i) &= \min(f_{\xi_1(\beta)}(i), \ldots, f_{\xi_n(\beta)}(i)). \end{aligned} \tag{3}$$

CLAIM. *The set*

$$C := \{i < \mu : \; \forall \beta \; \{g_\gamma^-(i) : \gamma > \beta\} \textit{ is unbounded in } \lambda_i\} \tag{4}$$

is in the filter D_μ.

Proof. Let $S = \mu \setminus C$, or more explicitly:

$$S := \{i < \mu : \exists \beta < \lambda \, \exists s < \lambda_i \;\; \{g_\gamma^-(i) : \gamma > \beta\} \subseteq s\}.$$

We have to show that S is a bounded set.

For each $i \in S$ let $\beta_i < \lambda$ and $h(i) < \lambda_i$ be such that $\{g^-_\gamma(i) : \gamma > \beta_i\} \subseteq h(i)$. Let $\beta^* = \sup\{\beta_i : i \in S\} < \lambda$, and extend h arbitrarily to a total function on μ. Since the sequence $\langle f_\alpha : \alpha < \lambda\rangle$ is cofinal in $\prod_{i<\mu} \lambda_i/D_\mu$, we can find $\gamma > \beta^*$ such that $h <_{D_\mu} f_\gamma$.

We have $\gamma \leq \xi_k(\gamma)$ for all $k \in \{1, \dots, m\}$, so the sets

$$X_k := \{i < \mu : h(i) < f_{\xi_k(\gamma)}(i)\}$$

are all in D_μ. Now if S were positive mod D_μ (i.e., unbounded), then we could find $i_* \in S \cap X_1 \cap \cdots \cap X_m$. But then $i_* \in X_1 \cap \cdots \cap X_m$ implies

$$h(i_*) < g^-_\gamma(i_*),$$

and $i_* \in S$ implies

$$g^-_\gamma(i_*) < h(i_*),$$

a contradiction.

This shows that C is indeed a set in the filter D_μ. □

We will now use the fact that $m < n + 1$. Let

$$\ell^* \in \{0, \dots, n\} \setminus \{\ell_1, \dots, \ell_m\}.$$

Since A_{ℓ^*} is positive mod D_μ, we can pick

$$i^* \in A_{\ell^*} \cap C. \tag{5}$$

Using the fact that $i^* \in C$ and definition (4) we can find a sequence $\langle\beta_\sigma : \sigma < \lambda_{i^*}\rangle$ such that

$$\forall \sigma < \sigma' < \lambda_{i^*}\colon\ g^+_{\beta_\sigma}(i^*) < g^-_{\beta_{\sigma'}}(i^*), \tag{6}$$

We now restrict our attention from $\langle\vec{t}_\beta : \beta < \lambda\rangle$ to the subsequence $\langle\vec{t}_{\beta_\sigma} : \sigma < \lambda_{i^*}\rangle$; we will show that this sequence cannot be an antichain. For notational simplicity only we will assume $\beta_\sigma = \sigma$ for all $\sigma < \lambda_{i^*}$.

Recall that $\vec{t}_\sigma = \langle(\ell_1, \xi_1(\sigma)), \dots, (\ell_m, \xi_m(\sigma))\rangle$. For each $\sigma < \lambda_{i^*}$ define

$$\vec{x}_\sigma := \langle f_{\xi_1(\sigma)} \restriction i^*, \dots, f_{\xi_n(\sigma)} \restriction i^*\rangle \in \left(\prod_{j<i^*} \lambda_j\right)^m.$$

Since $|\prod_{j<i^*} \lambda_j| < \lambda_{i^*}$, there are only $< \lambda_{i^*}$ many possible values for $\vec{x}_\sigma$, so we can find $\sigma_1 < \sigma_2 < \lambda_{i^*}$ such that $\vec{x}_{\sigma_1} = \vec{x}_{\sigma_2}$.

Now note that by (3) and (6) we have

$$f_{\xi_k(\sigma_1)}(i^*) \leq g^+_{\sigma_1}(i^*) < g^-_{\sigma_2}(i^*) \leq f_{\xi_k(\sigma_2)}(i^*). \tag{7}$$

Hence $d(f_{\xi_k(\sigma_1)}, f_{\xi_k(\sigma_2)}) = i^*$ for $k = 1, \dots, m$.

Since $i^* \in A_{\ell^*}$ we have for all k: $i^* \notin A_{\ell_k}$. From (1), (2), (7) we get

$$\xi_k(\sigma_1) <_{\ell_k} \xi_k(\sigma_2) \quad \text{for } k = 1, \ldots, m.$$

Hence $(\ell_k, \xi_k(\sigma_1)) < (\ell_k, \xi_k(\sigma_2))$ for all k, which means $\vec{t}_{\sigma_1} < \vec{t}_{\sigma_2}$. □

3. Consistency

THEOREM 3.1. *Assume $\aleph_0 \leq \lambda_2 \leq \lambda_3 \leq \cdots$, where $\lambda_n^{<\kappa} = \lambda_n$ for all n, and $\kappa^{<\kappa} = \kappa$.*

Then there is a forcing notion $\mathbb{P}$ which satisfies the κ^+-cc and is κ-complete, and a $\mathbb{P}$-name $\underset{\sim}{I}$ of a subset of 2^κ (where 2^κ is endowed with the lexicographic order, which is inherited by $\underset{\sim}{I}$) such that

$$\Vdash_{\mathbb{P}} \forall n > 1\colon \underset{\sim}{I}^n \text{ has antichains of size } \lambda_n\text{, but no larger ones.}$$

Remark 3.2. At first reading, the reader may want to consider the special case $\kappa = \omega$, $\lambda_{n+2} = \aleph_n$. Note that 2^ω is order isomorphic to the Cantor set, a subset of the real line $\mathbb{R}$, so we obtain as a special case of Theorem 3.1:

> Consistently, there is a set $I \subseteq \mathbb{R}$ such that for each n, I^{n+1} admits much larger antichains than I^n.

NOTATION 3.3. (1) We let $\lambda_1 = 0$, $\lambda_\omega = \sup\{\lambda_n : n < \omega\}$.

(2) It is understood that 2^α is linearly ordered lexicographically, and $(2^\alpha)^m$ is partially ordered by the pointwise order.

(3) For $\alpha \leq \beta \leq \kappa$, $\eta \in 2^\alpha$, $\nu \in 2^\beta$, we define

$$\eta \trianglelefteq \nu \quad \text{iff} \quad \nu \text{ extends } \eta\text{, i.e., } \eta \subseteq \nu.$$

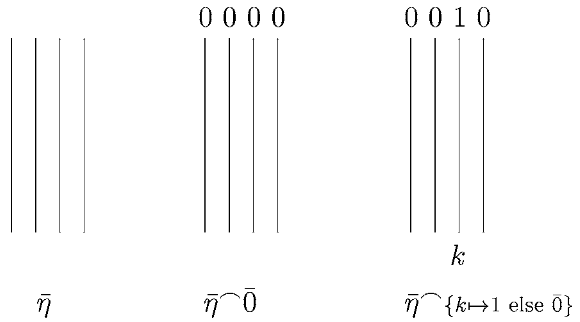
(4) For $\bar{\eta} = \langle \eta(0), \ldots, \eta(n-1)\rangle \in (2^\alpha)^n$, $\bar{\nu} = \langle \nu(0), \ldots, \nu(n-1)\rangle \in (2^\beta)^n$, we let

$$\bar{\eta} \trianglelefteq \bar{\nu} \quad \text{iff} \quad \eta(0) \trianglelefteq \nu(0), \ldots, \eta(n-1) \trianglelefteq \nu(n-1).$$

(5) For $\eta \in 2^\alpha$, $i \in \{0, 1\}$ we write $\eta^\frown i$ for the element $\nu \in 2^{\alpha+1}$ satisfying $\eta \trianglelefteq \nu$, $\nu(\alpha) = i$.

DEFINITION 3.4. Let $\bar{\eta} \in (2^\alpha)^m$, $k \in \{0, \ldots, m-1\}$, $m \geq 2$. We define $\bar{\eta}^\frown \bar{1}$, $\bar{\eta}^\frown \bar{0}$, $\bar{\eta}^\frown\{k \mapsto 1 \text{ else } \bar{0}\}$, $\bar{\eta}^\frown\{k \mapsto 0 \text{ else } \bar{1}\}$ in $(2^{\alpha+1})^m$ as follows: All four are $\trianglelefteq$-extensions of $\bar{\eta}$, and:

- $\bar{\eta}^\frown \bar{0}(n) = \eta(n)^\frown 0$ for all $n < m$.
- $\bar{\eta}^\frown \bar{1}(n) = \eta(n)^\frown 1$ for all $n < m$.
- $\bar{\eta}^\frown\{k \mapsto 0 \text{ else } \bar{1}\}(n) = \eta(n)^\frown 1$ for all $n \neq k$, $\bar{\eta}^\frown\{k \mapsto 0 \text{ else } \bar{1}\}(k) = \eta(n)^\frown 0$.
- $\bar{\eta}^\frown\{k \mapsto 1 \text{ else } \bar{0}\}(n) = \eta(n)^\frown 0$ for all $n \neq k$, $\bar{\eta}^\frown\{k \mapsto 1 \text{ else } \bar{0}\}(k) = \eta(n)^\frown 1$.



FACT 3.5. (1) If $\alpha \le \beta \le \kappa$, $\bar\eta, \bar\eta' \in (2^\alpha)^n$ are incomparable, $\bar\nu, \bar\nu' \in (2^\beta)^n$, $\bar\eta \trianglelefteq \bar\nu$, $\bar\eta' \trianglelefteq \bar\nu'$, then also $\bar\nu$ and $\bar\nu'$ are incomparable.

(2) $\bar\eta^\frown\bar 0 < \bar\eta^\frown\bar 1$.

(3) $\bar\eta^\frown\{k \mapsto 0 \text{ else } \bar 1\}0$ and $\bar\eta^\frown\{k \mapsto 1 \text{ else } \bar 0\}$ are incomparable.

DEFINITION 3.6. We let $\mathbb{P}$ be the set of all "conditions"

$$p = (u^p, \alpha^p, \langle \bar\eta^p_\xi : \xi \in u^p\rangle)$$

satisfying the following requirements for all m:

- $u^p \in [\lambda_\omega]^{<\kappa}$.
- $\alpha^p < \kappa$.
- For all $\xi \in u^p \cap (\lambda_m \setminus \lambda_{m-1})$: $\bar\eta^p_\xi = \langle \eta^p_\xi(0), \dots, \eta^p_\xi(m-1)\rangle \in (2^{\alpha^p})^m$.
- For all $\xi \ne \xi'$ in $u^p \cap (\lambda_m \setminus \lambda_{m-1})$, $\bar\eta^p_\xi$ and $\bar\eta^p_{\xi'}$ are incomparable in $(2^{\alpha^p})^m$.

We define $p \le q$ ("q is stronger than p") iff

- $u^p \subseteq u^q$,
- $\alpha^p \le \alpha^q$,
- for all $\xi \in u^p$, $\bar\eta^p_\xi \trianglelefteq \bar\eta^q_\xi$.

FACT 3.7. (1) For all $\alpha < \kappa$: The set $\{p \in \mathbb{P} : \alpha^p \ge \alpha\}$ is dense in $\mathbb{P}$.

(2) For all $\xi < \lambda_\omega$: The set $\{p \in \mathbb{P} : \xi \in u^p\}$ is dense in $\mathbb{P}$.

FACT AND DEFINITION 3.8. We let $\langle \underset{\sim}{\bar\nu}_\xi : \xi < \lambda_\omega\rangle$ be the "generic object", i.e., a name satisfying

$$\forall m \in \omega\ \forall p \in \mathbb{P}\ \forall \xi \in u^p \cap (\lambda_m \setminus \lambda_{m-1}) : p \Vdash_{\mathbb{P}} \underset{\sim}{\bar\nu}_\xi \in (2^\kappa)^m,$$
$$\forall p \in \mathbb{P}\ \forall \xi \in u^p : p \Vdash \bar\eta^p_\xi \trianglelefteq \underset{\sim}{\bar\nu}_\xi.$$

(This definition makes sense, by Fact 3.7.)

Clearly, $\Vdash \xi, \xi' \in \lambda_m \setminus \lambda_{m-1} \Rightarrow \underset{\sim}{\bar\nu}_\xi, \underset{\sim}{\bar\nu}_{\xi'}$ incompatible.

We let $\Vdash \underset{\sim}{I} = \bigcup_{m=2}^\infty \{\nu_\xi(\ell) : \xi \in \lambda_m \setminus \lambda_{m-1}, \ell < m\}$.

LEMMA 3.9. *Let* $\mathbb{P}$, $\underset{\sim}{I}$ *be as in* 3.6 *and* 3.8.

Then $\Vdash_{\mathbb{P}} \underset{\sim}{I}^m$ *has antichains of size* λ_m*, but no larger ones.*

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It is clear that $\mathbb{P}$ is κ-complete, and κ^+-cc is proved by an argument similar to the Δ-system argument below. So all the λ_m stay cardinals.

We can show by induction that $\Vdash \alpha(\underset{\sim}{I}^m) > \lambda_m$, i.e., $\underset{\sim}{I}^m$ has an antichain of size λ_m: This is clear if $\lambda_m = \lambda_{m-1}$ (and void if $m = 0$); if $\lambda_m > \lambda_{m-1}$ then $\langle \bar{\nu}_\xi : \xi \in \lambda_m \setminus \lambda_{m-1} \rangle$ will be forced to be antichain.

It remains to show that (for any m) there is no antichain of size λ_m^+ in $\underset{\sim}{I}^m$.

Fix $m^* \in \omega$, and assume wlog that $\lambda_{m^*+1} > \lambda_{m^*}$.

[Why is this no loss of generality? If $\lambda_{m^*} = \lambda_\omega$, then the cardinality of $\underset{\sim}{I}$ is at most λ_{m^*}, and there is nothing to prove. If $\lambda_{m^*} = \lambda_{m^*+1} < \lambda_\omega$, then replace m^* by $\min\{m \geq m^* : \lambda_m < \lambda_{m+1}\}$].

Towards a contradiction, assume that there is a condition p and a sequence of names $\langle \bar{\underset{\sim}{\rho}}_\beta : \beta < \lambda_{m^*}^+ \rangle$ such that

$$p \Vdash \langle \bar{\underset{\sim}{\rho}}_\beta : \beta < \lambda_{m^*}^+ \rangle \text{ is an antichain in } \underset{\sim}{I}^{m^*}.$$

Let $\bar{\underset{\sim}{\rho}}_\beta = (\bar{\underset{\sim}{\rho}}_\beta(n) : n < m^*)$. For each $\beta < \lambda_{m^*}^+$ and each $n < m^*$ we can find a condition $p_\beta \geq p$ and

$$m_n(\beta) \in \omega, \qquad \ell_n(\beta) < m_n(\beta), \qquad \xi_n(\beta) \in \lambda_{m_n(\beta)} \setminus \lambda_{m_n(\beta)-1}$$

such that

$$p_\beta \Vdash \rho_\beta(n) = \nu_{\xi_n(\beta)}(\ell_n(\beta)).$$

We will now employ a Δ-system argument.

We define a family $\langle \zeta^\beta : \beta < \lambda_{m^*}^+ \rangle$ of functions as follows: Let i_β be the order type of u^{p_β}, and let

$$u^\beta = u^{p_\beta} = \{\zeta^\beta(i) : i < i_\beta\} \quad \text{in increasing enumeration.}$$

By 3.7(2) may assume $\xi_n(\beta) \in u^\beta$, say $\xi_n(\beta) = \zeta^\beta(i_n(\beta))$.

By thinning out our alleged antichain $\langle \bar{\underset{\sim}{\rho}}_\beta : \beta < \lambda_{m^*}^+ \rangle$ we may assume

- For some $i^* < \kappa$, for all β: $i_\beta = i^*$.
- For some $\alpha^* < \kappa$, for all β: $\alpha^{p_\beta} = \alpha^*$.
- For each $i < i^*$ there is some $m_{\langle i \rangle}$ such that for all β: $\zeta^\beta(i) \in \lambda_{m_{\langle i \rangle}} \setminus \lambda_{m_{\langle i \rangle}-1}$.
- For each $i < i^*$ there is some $\bar{\eta}_{\langle i \rangle} \in (2^{\alpha^*})^{m_{\langle i \rangle}}$ such that for all β: $\bar{\eta}^{p_\beta}_{\zeta^\beta(i)} = \bar{\eta}_{\langle i \rangle}$. (Here we use $\lambda_m^{<\kappa} = \lambda_m$.)
- The family $\langle u^\beta : \beta < \lambda_{m^*}^+ \rangle$ is a Δ-system, i.e., there is some set $u^* \in [\lambda_\omega]^{<\kappa}$ such that for all $\beta \neq \gamma$: $u^\beta \cap u^\gamma = u^*$.
- Moreover: there is a set $\Delta \subseteq i^*$ such that for all β: $u^* = \{\zeta^\beta(i) : i \in \Delta\}$. Since ζ^β is increasing, this also implies $\zeta^\beta(i) = \zeta^\gamma(i)$ for $i \in \Delta$.
- For each $n < m^*$, the functions m_n, ℓ_n and i_n are constant. (Recall that these functions map $\lambda_{m^*}^+$ into ω.) We will again write m_n, ℓ_n, i_n for these constant values.

Note that for $i \in i^* \setminus \Delta$ all the $\zeta^\beta(i)$ are distinct elements of $\lambda_{m_{\langle i \rangle}}$, hence:

$$i \notin \Delta \text{ implies } \lambda^+_{m^*} \leq \lambda_{m_{\langle i \rangle}}, \text{ hence } m_{\langle i \rangle} > m^*.$$

Now pick $k^* \leq m^*$ such that $k^* \notin \{\ell_n : n < m^*\}$. Pick any distinct $\beta, \gamma < \lambda^+_{m^*}$. We will find a condition q extending p_β and p_γ, such that $q \Vdash \bar{\varrho}_\beta \leq \bar{\varrho}_\gamma$.

We define q as follows:

- $u^q := u_\beta \cup u_\gamma = u^* \dot{\cup} \{\zeta^\beta(i) : i \in i^* \setminus \Delta\} \dot{\cup} \{\zeta^\gamma(i) : i \in i^* \setminus \Delta\}$.
- $\alpha^q = \alpha^* + 1$.
- For $\xi \in u^*$, say $\xi = \zeta^\beta(i) = \zeta^\gamma(i)$, recall that $\bar{\eta}^{p_\beta}_\xi = \bar{\eta}_{\langle i \rangle} = \bar{\eta}^{p_\gamma}_\xi$. We let $\bar{\eta}^q_\xi = \bar{\eta}_{\langle i \rangle}{}^\frown \bar{0}$ (see 3.3).
- For $\xi = \zeta^\beta(i)$, $i \in i^* \setminus \Delta$, we have $\bar{\eta}^{p_\beta}_\xi = \bar{\eta}_{\langle i \rangle} \in (2^{\alpha^*})^{m_{\langle i \rangle}}$, where $m_{\langle i \rangle} > m^*$. So $k^* \leq m^* < m_{\langle i \rangle}$, hence $\bar{\eta}_{\langle i \rangle}{}^\frown\{k^* \mapsto 1 \text{ else } \bar{0}\}$ is well-defined. We let

$$\bar{\eta}^q_\xi = \bar{\eta}_{\langle i \rangle}{}^\frown\{k^* \mapsto 1 \text{ else } \bar{0}\}.$$

- For $\xi = \zeta^\gamma(i)$, $i \in i^* \setminus \Delta$, we let

$$\bar{\eta}^q_\xi = \bar{\eta}_{\langle i \rangle}{}^\frown\{k^* \mapsto 0 \text{ else } \bar{1}\}.$$

We claim that q is a condition. The only nontrivial requirement is the incompatibility of all $\bar{\eta}^q_\xi$: Let $\xi, \xi' \in u^q$, $\xi \neq \xi'$, and assume that $\xi, \xi' \in \lambda_m \setminus \lambda_{m-1}$ for some m.

If $\xi, \xi' \in u^\beta$, then the incompatibility of $\bar{\eta}^q_\xi$ and $\bar{\eta}^q_{\xi'}$ follows from the incompatibility of $\bar{\eta}^{p_\beta}_\xi$ and $\bar{\eta}^{p_\beta}_{\xi'}$. The same argument works for $\xi, \xi' \in u_\gamma$.

So let $\xi \in u_\beta \setminus u^*$, $\xi' \in u_\gamma \setminus u^*$. Say $\xi = \zeta^\beta(i)$, $\xi' = \zeta^\gamma(i')$.

If $i \neq i'$, then $\bar{\eta}_{\langle i \rangle} = \bar{\eta}^{p_\beta}_{\zeta^\beta(i)} = \bar{\eta}^{p_\gamma}_{\zeta^\gamma(i)}$ and $\bar{\eta}_{\langle i' \rangle} = \bar{\eta}^{p_\gamma}_{\zeta^\gamma(i')}$ are incompatible (because p_γ is a condition). From $\bar{\eta}_{\langle i \rangle} \trianglelefteq \bar{\eta}^q_\xi$ and $\bar{\eta}_{\langle i' \rangle} \trianglelefteq \bar{\eta}^q_{\xi'}$ we conclude that also $\bar{\eta}^q_\xi$ and $\bar{\eta}^q_{\xi'}$ are incompatible.

Finally, we consider the case $i = i'$.

We have

$$\bar{\eta}^q_\xi = \bar{\eta}_{\langle i \rangle}{}^\frown\{k^* \mapsto 0 \text{ else } \bar{1}\}, \qquad \bar{\eta}^q_{\xi'} = \bar{\eta}_{\langle i \rangle}{}^\frown\{k^* \mapsto 1 \text{ else } \bar{0}\}$$

so by 3.7(3), $\bar{\eta}^q_\xi$ and $\bar{\eta}^q_{\xi'}$ are incompatible.

This concludes the construction of q. We now check that $q \Vdash \bar{\varrho}_\beta \leq \bar{\varrho}_\gamma$, i.e., $q \Vdash \bar{\varrho}_\beta(n) \leq \bar{\varrho}_\gamma(n)$ for all n. Clearly, $q \Vdash \bar{\varrho}_\beta(n) = \bar{\nu}_{\zeta^\beta(i_n)}(\ell_n) \trianglerighteq \bar{\eta}^q_{\zeta^\beta(i_n)} = \eta\langle i_n \rangle{}^\frown\bar{0}$. Here we use the fact that $k^* \neq \ell_n$. Similarly, $q \Vdash \bar{\varrho}_\gamma(n) = \bar{\nu}_{\zeta^\beta(i_n)}(\ell_n) \trianglerighteq \eta_{\langle i_n \rangle}{}^\frown\bar{1}$.

Hence $q \Vdash \bar{\varrho}_\beta \leq \bar{\varrho}_\gamma$.

This concludes the proof of Theorem 3.1

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References

1. Farley, J.: Cardinalities of infinite antichains in products of chains, *Algebra Universalis* **42** (1999), 235–238.
2. Ploščica, M. and Haviar, M.: On order-polynomial completeness of lattices, *Algebra Universalis* **39** (1998), 217–219.
3. Shelah, S.: *Cardinal Arithmetic*, Oxford Logic Guides 29, Oxford University Press, 1994.