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HOW LARGE CAN A HEREDITARILY SEPARABLE OR HEREDITARILY LINDELÖF SPACE BE?

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ABSTRACT

The main result of this paper is that if V satisfies GCH and $\omega_1 < \lambda < \mu$ are arbitrary regular cardinals, then in some cardinal preserving forcing extension W of V we have $\lambda^{<\lambda} = \lambda = 2^{\aleph_0}$, $\mu = 2^{\lambda}$ and there are a hereditarily separable $X \subset 2^{\lambda}$ with $|X| = 2^{2^{\aleph_0}} = \mu$ and a hereditarily Lindelöf $Y \subset 2^{\mu}$ with $w(Y) = 2^{2^{\aleph_0}} = \mu$. So far similar results have only been obtained under the assumption of CH.

In [1] Hajnal and Juhász proved the following inequality: if X is a Hausdorff space then $|X| \leqq 2^{2^{s(X)}}$. Considering the most important particular case $s(X) = \aleph_0$ this says that a Hausdorff space X of countable spread has cardinality $\leqq 2^{2^{\aleph_0}}$. Somewhat later, in [3], they showed that this inequality is — at least consistently — sharp by constructing a generic extension in which CH holds and there is an even hereditarily separable (in what follows HS) subspace of 2^{ω_1} that has cardinality $2^{\aleph_1} = 2^{2^{\aleph_0}}$. On the other hand S. Todorčević has recently shown in [9] that it is consistent that every Hausdorff space of countable spread be of cardinality $\leqq 2^{\aleph_0}$; in his model $2^{\aleph_0} = 2^{\aleph_1}$. Thus the natural question, originally raised by J. Gerlits, arose whether $2^{\aleph_1}$ is the real upper bound for the size of a Hausdorff space that has countable spread (or is HS)? These questions were explained in detail in [4].

There is an analogous (or dual) question concerning Hausdorff spaces that are hereditarily Lindelöf (HL, in short). Here there is the classical inequality of de Groot: $|X| \leqq 2^{h(X)}$ for X Hausdorff, in particular $|X| \leqq 2^{\aleph_0}$ if X is also HL. Here the question is whether the weight of X, for which $2^{|X|}$ is a trivial upper bound, could actually be as large as $2^{2^{h(X)}}$. Here we had the result of Hajnal and Juhász

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from [2] showing again that the answer is consistently "yes", at least in a model satisfying CH, raising again the problem whether $2^{\aleph_1}$ is the correct upper bound? Let us note here parenthetically that we do not have an analogue of Todorčević's result here; this is probably because this question is closely related to the celebrated, and still unsolved, L space problem.

The aim of this paper is to answer both of the above questions by constructing (consistent) examples of Hausdorff and 0-dimensional HS (HL) spaces X such that $|X| = 2^c$ ($w(X) = 2^c$), while $c = c^{<c}$ and 2^c are both as large as you wish. This shows that in the above inequalities the upper bounds $2^{2^{s(X)}}$ (or even $2^{2^{z(X)}}$) and $2^{2^{h(X)}}$ respectively cannot be replaced with anything better, at least in ZFC.

Actually the example we construct yields a little stronger result in that the spaces obtained will be *strongly* HS (HL), i.e. all their finite powers will also be HS (HL). This approach to the construction has the additional advantage that, via passing to the dual, one case immediately yields the other.

THEOREM 1. *Suppose that, in V, $\omega < \lambda < \mu$ are cardinals and there is a map $F : \mu \times \lambda \to \lambda$ satisfying the following two conditions* (*) *and* (**):

(*): *If $n, k \in \omega$, $\{\boldsymbol{\alpha}_\xi : \xi \in \omega_1\} \subset \mu^n$ and $\{b \cup b_\xi : \xi \in \omega_1\} \subset [\lambda]^{<\omega}$ is a Δ-system with root b and $|b_\xi| = k$ for each $\xi \in \omega_1$, then there exist $\xi < \eta < \omega_1$ such that*

(*1) *if $\nu \in b_\eta$ and $F(\alpha^i_\xi, \nu) \in b \cup b_\eta$ then*

$$F(\alpha^i_\xi, \nu) = F(\alpha^i_\eta, \nu);$$

(*2) *if $i, j < n$ and $r < k$ then*

$$F(\alpha^i_\xi, \nu^r_\eta) = F(\alpha^j_\xi, \nu^r_\eta) \quad \textit{implies} \quad F(\alpha^i_\xi, \nu^r_\xi) = F(\alpha^j_\xi, \nu^r_\xi).$$

(Of course, here $\boldsymbol{\alpha}_\xi = \langle \alpha^i_\xi : i < n \rangle$ and $b_\xi = \{\nu^r_\xi : r < k\}$, where ν^r_ξ is the rth element of b_ξ in its natural increasing order.)

(**): *For every $a \in [\mu]^{<\omega}$ we have*

$$|\{\nu \in \lambda : |\{F(\alpha, \nu) : \alpha \in a\}| = |a|\}| \geqq \omega.$$

If W is the generic extension of V obtained by adding λ Cohen reals to V then in W there is a strongly HS *space X with $|X| = \mu$ and a strongly* HL *space Y with $w(Y) = \mu$.*

PROOF. We consider W as the extension determined by a Cohen generic map

$$h : \lambda \times \lambda \to 2,$$

i.e. the notion of forcing used is the set $H(\lambda \times \lambda)$ of all finite partial functions from $\lambda \times \lambda$ to 2 with the extension of functions as the partial order.

For any $\alpha \in \mu$ we define, in W, an element $x_\alpha \in 2^\lambda$ as follows:

$$x_\alpha(\nu) = h(\nu, F(\alpha, \nu))$$

for all $\nu \in \lambda$. We first show that

$$X = \{x_\alpha : \alpha \in \mu\}$$

is as required.

To see that $|X| = \mu$ consider $\{\alpha, \beta\} \in [\mu]^2$ and any condition $p \in H(\lambda \times \lambda)$. Applying condition (**) to $a = \{\alpha, \beta\}$ we can choose $\nu \in \lambda$ not mentioned in p such that $F(\alpha, \nu) \neq F(\beta, \nu)$. Then

$$q = p \cup \{\langle\langle \nu, F(\alpha, \nu)\rangle, 0\rangle, \langle\langle \nu, F(\beta, \nu)\rangle, 1\rangle\}$$

is an extension of p that forces $x_\alpha(\nu) \neq x_\beta(\nu)$, hence clearly $x_\alpha \neq x_\beta$ in W.

Now, to see that X is strongly HS we show that for every $n \in \omega$ there is no left separated subspace in X^n of type ω_1. Since a basic neighborhood of any point $f \in 2^\lambda$ is specified by fixing a finite subset d of λ, more precisely by $f \restriction d$, we consider, in W, maps

$$s : \omega_1 \to \mu^n \quad \text{and} \quad e : \omega_1 \to [\lambda]^{<\omega}.$$

We want to show that $\{\boldsymbol{x}_{s(\xi)} : \xi \in \omega_1\}$ is not left separated by the neighborhoods determined by $e(\xi)$ for $s(\xi)$, i.e. that there are $\xi < \eta < \omega_1$ such that

$$x_{s(\xi),i} \supset x_{s(\eta),i} \restriction e(\eta)$$

for each $i < n$.

A standard Δ-system and counting argument allows us to assume that $e(\xi) \cap e(\eta) = \emptyset$ if $\xi \neq \eta$.

Let $p \in H(\lambda \times \lambda)$ be a condition that forces all this for suitable names $\dot{s}$ and $\dot{e}$ of s and e, respectively.

For each $\xi \in \omega_1$ we may choose, in V, an extension p_ξ of p as well as $\boldsymbol{\alpha}_\xi \in \mu^n$ and $d_\xi \in [\lambda]^{<\omega}$ such that

$$p_\xi \Vdash \dot{s}(\xi) = \boldsymbol{\alpha}_\xi \wedge \dot{e}(\xi) = d_\xi.$$

We may assume that for each ξ

$$D(p_\xi) = (b \cup b_\xi)^2,$$

where $\{b \cup b_\xi : \xi \in \omega_1\} \subset [\lambda]^{<\omega}$ is a Δ-system with root b, $|b_\xi| = k$ for all ξ and

$$p_\xi \restriction b^2 = p_\eta \restriction b^2$$

for all $\xi, \eta \in \omega_1$. In particular then p_ξ and p_η are compatible, hence we must have

$$d_\xi \cap d_\eta = \emptyset$$

if $\xi \neq \eta$. Thus we may also assume that $d_\xi \subset b_\xi$ for all $\xi \in \omega_1$.

Let us write $\boldsymbol{\alpha}_\xi = \langle \alpha^i_\xi : i < n \rangle$ and $b_\xi = \{\nu^r_\xi : r < k\}$. For $\xi \in \omega_1$ and $r < k$ we put

$$E^r_\xi = \{\langle i, j \rangle \in n^2 : F(\alpha^i_\xi, \nu^r_\xi) = F(\alpha^j_\xi, \nu^r_\xi)\}.$$

W.l.o.g. we can assume that

$$E^r_\xi = E^r_\eta$$

holds whenever $\xi, \eta \in \omega_1$ and $r < k$.

Let $\xi < \eta < \omega_1$ be chosen satisfying (*) with respect to this $\{\boldsymbol{\alpha}_\xi : \xi \in \omega_1\}$ and $\{b \cup b_\xi : \xi \in \omega_1\}$. We may then define an extension q of $p_\xi \cup p_\eta$ such that if $F(\alpha^i_\xi, \nu^r_\eta) \notin b \cup b_\eta$ then

$$q(\nu^r_\eta, F(\alpha^i_\xi, \nu^r_\eta)) = p_\eta(\nu^r_\eta, F(\alpha^i_\eta, \nu^r_\eta))[= q(\nu^r_\eta, F(\alpha^i_\eta, \nu^r_\eta))].$$

Since $E^r_\xi = E^r_\eta$ and (*2) hold, this can be done without any conflict for different i and j. Moreover, by (*1) this equality is automatically valid for $F(\alpha^i_\xi, \nu^r_\eta) \in b \cup b_\eta$ as well. Consequently we have

$$q \Vdash x_{\alpha^i_\xi} \supset x_{\alpha^i_\eta} \restriction d_\eta$$

for every $i < n$. It is standard to show that this implies

$$p \Vdash (\exists \xi < \eta < \omega_1)(\forall i < n)(x_{\dot{s}(\xi),i} \supset x_{\dot{s}(\eta),i} \restriction \dot{e}(\eta)),$$

hence $\{\boldsymbol{x}_{s(\xi)} : \xi \in \omega_1\}$ cannot be left separated.

Let us now turn to the dual Y of X, i.e. for $\nu \in \lambda$ we let $y_\nu \in 2^\mu$ be specified by

$$y_\nu(\alpha) = x_\alpha(\nu) = h(\nu, F(\alpha, \nu))$$

for each $\alpha \in \mu$, and

$$Y = \{y_\nu : \nu \in \lambda\}.$$

It is known that then Y is strongly HL (see e.g. [8]), and $w(Y) = \mu$ will follow if we show that Y is dense in 2^μ.

To see this let $\varepsilon \in H(\mu)$, $p \in H(\lambda \times \lambda)$ and applying (**) to $a = D(\varepsilon)$ pick $\nu \in \lambda$ such that ν does not occur in p and

$$|\{F(\alpha, \nu) : \alpha \in a\}| = |a|.$$

We may obviously find an extension q of p which satisfies

$$q(\nu, F(\alpha, \nu)) = \varepsilon(\alpha)$$

for every $\alpha \in a = D(\varepsilon)$, and then

$$q \Vdash \varepsilon \subset y_\nu.$$

This clearly shows that Y must be dense in 2^μ.

Of course, Theorem 1 is only useful if a map F with properties (*) and (**) can indeed be constructed for large λ and μ.

The next result serves exactly this purpose.

THEOREM 2. *Suppose that $\omega_1 < \lambda < \mu$ are regular cardinals in V such that $\lambda^{<\lambda} = \lambda$ and $\mu^\lambda = \mu$. Then there is a notion of forcing $\langle P, \leqq \rangle$ in V such that V^P has the same cardinals as V, $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \mu$ hold in V^P, moreover there is in V^P a map $F : \mu \times \lambda \to \lambda$ satisfying (*) and (**).*

PROOF. We intend to define P as the set of all fragments of F of size $< \lambda$, hence the elements of P will be all triples of the form

$$p = \langle A, B, f \rangle,$$

where $A \in [\mu]^{<\lambda}$, $B \in [\lambda]^{<\lambda}$ and

$$f : A \times B \to B,$$

which satisfy the appropriately restricted version of (*):

$(*)_p$: If $n, k \in \omega$, $\{\boldsymbol{\alpha}_\xi : \xi \in \omega_1\} \subset A^n$ and $\{b \cup b_\xi : \xi \in \omega_1\} \subset [B]^{<\omega}$ is a Δ-system with root b and $|b_\xi| = k$ for all $\xi \in \omega_1$, then there are $\xi < \eta < \omega_1$ such that

$(*1)_p$: if $\nu \in b_\eta$ and $f(\alpha^i_\xi, \nu) \in b \cup b_\eta$ then

$$f(\alpha^i_\xi, \nu) = f(\alpha^i_\eta, \nu);$$

$(*2)_p$: if $i, j < n$ and $r < k$ then

$$f(\alpha^i_\xi, \nu^r_\eta) = f(\alpha^j_\xi, \nu^r_\eta) \quad \text{implies} \quad f(\alpha^i_\xi, \nu^r_\xi) = f(\alpha^j_\xi, \nu^r_\xi).$$

The partial order $\leqq$ of extension on P is defined as follows: If $p = \langle A^p, B^p, f^p \rangle$ and $q = \langle A^q, B^q, f^q \rangle$ are in P then $p \leqq q$ if and only if

(i) $A^p \supset A^q$, $B^p \supset B^q$, $f^p \supset f^q$;

(ii) if $\alpha \in A^p \setminus A^q$ and $A^q \neq \emptyset$ then there is a $\beta \in A^q$ such that $f^p(\alpha, \nu) = f^p(\beta, \nu)$ for all $\nu \in B^q$;

(iii) if $\nu \in B^p \setminus B^q$ and $\{\alpha, \beta\} \in [A^q]^2$ then

$$f^p(\alpha, \nu) \neq f^p(\beta, \nu).$$

It is straightforward to check that $\leqq$ is indeed a partial order.

Next we show that $\langle P, \leqq \rangle$ is λ-closed, more precisely that if η is a limit ordinal less than λ and $p_\xi = \langle A_\xi, B_\xi, f_\xi \rangle \in P$ for $\xi \in \eta$ are such that $\xi < \zeta < \eta$ implies

$p_\xi \geqq p_\zeta$, then with $A = \bigcup\{A_\xi : \xi \in \eta\}$, $B = \bigcup\{B_\xi : \xi \in \eta\}$ and $f = \bigcup\{f_\xi : \xi \in \eta\}$ we have

$$p = \langle A, B, f\rangle \in P$$

and $p \leqq p_\xi$ for all $\xi \in \eta$. The second part of the conclusion being obvious, we concentrate on showing that $p \in P$, i.e. that $(*)_p$ is valid. If $\mathrm{cf}(\eta) \neq \omega_1$ this is obvious again, hence we may assume that actually $\eta = \omega_1$.

Now consider $\{\boldsymbol{\alpha}_\xi : \xi \in \omega_1\} \subset A^n$ and $\{b \cup b_\xi : \xi \in \omega_1\} \subset [B]^{|b|+k'}$ as in $(*)_p$. With suitable reindexing if necessary, we may assume that $A_0 \neq \emptyset$, $b \subset B_0$ and that $\boldsymbol{\alpha}_\xi \in A^n_\xi$, $b_\xi \subset B_\xi$ for each $\xi \in \omega_1$.

We may also assume that, for every $i < n$, the sequence $\langle \alpha^i_\xi : \xi \in \omega_1\rangle$ is either constant or one-to-one.

With an application of Fodor's pressing down lemma we may pass to the case in which, using the notation $\tilde{B}_\xi = \bigcup\{B_\eta : \eta < \xi\}$, we have

$$\tilde{b}_\xi = b_\xi \cap \tilde{B}_\xi \subset B_0$$

for every $\xi \in \omega_1$. Of course we may assume that the set $\{r \in k : \nu^r_\xi \in \tilde{b}_\xi\}$ is the same for all $\xi \in \omega_1$. Let us put $b'_\xi = b_\xi \setminus \tilde{B}_\xi$.

For any limit ordinal $\eta \in \omega_1$ and $\nu \in b'_\eta$ consider the sequence of values $\langle f(\alpha^i_\xi, \nu) : \xi \in \eta\rangle$ for some fixed $i < n$. By the above and by (iii) this sequence is either constant or one-to-one. Since $b \cup b_\eta$ is finite it follows that in the second case there is some ordinal $\varphi(\eta) < \eta$ such that $f(\alpha^i_\xi, \nu) \notin b \cup b_\eta$ if $\varphi(\eta) \leqq \xi < \eta$. Hence another pressing down argument allows us to assume that whenever $\xi < \eta$, $i < n$ and $\nu \in b'_\eta$ then $f(\alpha^i_\xi, \nu) \in b \cup b_\eta$ implies $f(\alpha^i_\xi, \nu) = f(\alpha^i_\eta, \nu)$, i.e. the conclusion of $(*1)_p$ holds.

Also, by (iii) again, if $\xi < \eta$ and $\nu \in b'_\eta$ then $f(\alpha^i_\xi, \nu) = f(\alpha^j_\xi, \nu)$ implies $\alpha^i_\xi = \alpha^j_\xi$, hence the conclusion of $(*2)_p$ holds trivially. Thus we see that it remains to check $(*)_p$ with $\tilde{b}_\xi$ replacing b_ξ.

Next, applying (ii), for every $\xi \in \omega_1$ and $i < n$ we may choose an ordinal $\beta^i_\xi \in A_0$ such that ($\beta^i_\xi = \alpha^i_\xi$ if $\alpha^i_\xi \in A_0$ and) for every $\nu \in B_0$

$$f(\alpha^i_\xi, \nu) = f(\beta^i_\xi, \nu).$$

We may then apply $(*)_{p_0}$ to $\{\boldsymbol{\beta}_\xi : \xi \in \omega_1\} \subset A^n_0$ and $\{b \cup \tilde{b}_\xi : \xi \in \omega_1\} \subset [B_0]^{<\omega}$ to obtain $\xi < \eta$ satisfying $(*1)_{p_0}$ and $(*2)_{p_0}$. It is obvious, however, that these ξ and η will also satisfy $(*1)_p$ and $(*)_p$ as well.

The next step in our proof is to show that $\langle P, \leqq\rangle$ satisfies the λ^+-antichain condition. Since $\lambda^{<\lambda} = \lambda$, it is straightforward to show that among any λ^+ members of P there are two with the same B and isomorphic over B.

Consequently it will suffice to show that if $p = \langle A, B, f \rangle$ and $p' = \langle A', B, f' \rangle$ are two conditions with the same B and isomorphic over B then $q = \langle A \cup A', B, f \cup f' = g \rangle$ is in P and $q \leqq p, p'$. (By p and p' being isomorphic over B we mean that $f \upharpoonright (A \cap A') \times B = f' \upharpoonright (A \cap A') \times B$, $\operatorname{tp} A = \operatorname{tp} A'$ and if $\alpha \mapsto \alpha'$ is the unique order isomorphism of A onto A' then $f(\alpha, \nu) = f'(\alpha', \nu)$ holds for every $\alpha \in A$ and $\nu \in B$.)

Let us first show that $q \in P$, i.e. that $(*)_q$ is valid. Thus we consider $\{\boldsymbol{\alpha}_\xi : \xi \in \omega_1\} \subset (A \cup A')^n$ and a Δ-system with root b, $\{b \cup b_\xi : \xi \in \omega_1\} \subset [B]^{|b|+k}$. W.l.o.g. we may assume that there is an $l \leqq k$ such that for all $\xi \in \omega_1$

$$\alpha^i_\xi \in A \leftrightarrow i < l.$$

For every $\xi \in \omega_1$ and $j \in k \setminus l$ let $\beta^i_\xi \in A$ be such that $\alpha^i_\xi = (\beta^i_\xi)'$ and for $i < l$ we simply put $\beta^i_\xi = \alpha^i_\xi$. Now $(*)_p$ can be applied to $\{\boldsymbol{\beta}_\xi : \xi \in \omega_1\}$ and $\{b \cup b_\xi : \xi \in \omega_1\}$, and it is obvious that if $\xi < \eta$ satisfy $(*1)_p$ and $(*2)_p$ then they also satisfy $(*1)_q$ and $(*2)_q$ as well.

To see, e.g., that $q \leqq p$ note first that (i) and (iii) are trivially satisfied. As for (ii), let $\alpha' \in A' \setminus A$, then for $\alpha \in A$ we have

$$g(\alpha', \nu) = f'(\alpha', \nu) = f(\alpha, \nu) = g(\alpha, \nu)$$

for all $\nu \in B$, i.e. (ii) is also valid.

Now, for any $\alpha \in \mu$ and $\nu \in \lambda$ let us put

$$D_\alpha = \{p : \alpha \in A^p\} \quad \text{and} \quad E_\nu = \{p : \nu \in B^p\}.$$

We claim that both D_α and E_ν are dense in $\langle P, \leqq \rangle$. Let us first consider $p = \langle A, B, f \rangle \in P$ and $\alpha \in \mu \setminus A$. We define an extension

$$q = \langle A \cup \{\alpha\}, B, g \rangle \in D_\alpha$$

of p as follows. If $A = \emptyset$ then $g : \{\alpha\} \times B \to B$ can be chosen arbitrarily. If there is some $\beta \in A$ then we define $g \supset f$ for all $\langle \alpha, \nu \rangle \in \{\alpha\} \times B$ by

$$g(\alpha, \nu) = f(\beta, \nu).$$

To see that $q \in P$ let us check first that $(*)_q$ holds. To this end consider $\boldsymbol{\alpha}_\xi \in (A \cup \{\alpha\})^n$ and $b \cup b_\xi \subset B$ for $\xi \in \omega_1$. If $A = \emptyset$ then $\alpha^i_\xi = \alpha$ for all ξ and i hence for any $\xi < \eta < \omega_1$ both $(*1)_q$ and $(*2)_q$ are trivially valid. If, on the other hand, $A \neq \emptyset$ then replacing every occurrence of α as α^i_ξ with β we are back in p and it is obvious that if $\xi < \eta$ satisfy $(*1)_p$ and $(*2)_p$ then they satisfy $(*1)_q$ and $(*2)_q$ as well. It is also obvious that $q \leqq p$.

Now assume that p is as above and $\nu \in \lambda \setminus B$. We then first pick a set $H \subset \lambda \setminus B$ with $\nu \in H$ such that

$$|H| = |A| + \omega,$$

and then define a function $g: A \times H \to H$ such that if $\langle \alpha, \nu \rangle$, $\langle \alpha', \nu' \rangle \in A \times H$ then $g(\alpha', \nu) = g(\alpha', \nu')$ implies $\alpha = \alpha'$ and $\nu = \nu'$. It is clear that such a g exists. We claim that

$$g = \langle A, B \cup H, f \cup g \rangle$$

is in P, hence in E_ν, and that $q \leqq p$.

To see this let $\{\boldsymbol{\alpha}_\xi : \xi \in \omega_1\} \subset A^n$ and $\{b \cup b_\xi : \xi \in \omega_1\} \subset [B \cup H]^{|b|+k}$ be as in $(*)_q$. We may assume that for each $i < n$ the sequence $\langle \alpha^i_\xi : \xi \in \omega_1 \rangle$ is either constant or one-to-one.

If $i < n$ is of the latter kind and $\nu \in b_\eta \cap H$ then the sequence $\langle g(\alpha^i_\xi, \nu) : \xi \in \eta \rangle$ is also one-to-one, hence we may assume that $g(\alpha^i_\xi, \nu) \notin b \cup b_\eta$ whenever $\xi < \eta$. Thus in this case for any two $\xi < \eta$ and $\nu \in b_\eta \cap H$ $(*1)_q$ is valid. Also, if $g(\alpha^i_\xi, \nu) = g(\alpha^j_\xi, \nu)$ and for $\xi < \eta$ and $\nu \in b_\eta \cap H$ then actually $\alpha^i_\xi = \alpha^j_\xi$, and thus, for such a ν, $(*2)_q$ is also valid.

We may assume that the set $\{r \in k : \nu^r_\xi \in b_\xi \cap B\}$ is the same for all $\xi \in \omega_1$. Thus if $\xi < \eta$ are chosen in such a way that $(*1)_p$ and $(*2)_p$ be satisfied for $\{\boldsymbol{\alpha}_\xi : \xi \in \omega_1\}$ and $\{(b \cup b_\xi) \cap B : \xi \in \omega_1\}$, then $(*2)_q$ is also satisfied Finally if $\nu \in b_\eta \cap B$ then for any $i < n$ we have $f(\alpha^i_\xi, \nu) \in B$, hence $f(\alpha^i_\xi, \nu) \in b \cup b_\eta$ implies $f(\alpha^i_\xi, \nu) \in (b \cup b_\eta) \cap B$ and thus by $(*1)_p$ $f(\alpha^i_\xi, \nu) = f(\alpha^i_\eta, \nu)$, showing that $(*1)_q$ holds. To show that $q \leqq p$, only (iii) requires checking, but it follows immediately from the fact that g is one-to-one.

Now, if $G \subset P$ is P-generic over V then

$$F = \cup \{f^p : p \in G\}$$

is a map in $V[G]$ with $D(F) = \mu \times \lambda$ and $R(F) \subset \lambda$, because both the D_α and E_ν are dense. Moreover, since $\omega_1 < \lambda$ and P is λ-closed it is easy to see that whenever $\{\boldsymbol{\alpha}_\xi : \xi \in \omega_1\} \subset \mu^n$ and $\{b \cup b_\xi : \xi \in \omega_1\} \subset [\lambda]^{<\omega}$ there is a $p \in G$ with $\{\boldsymbol{\alpha}_\xi : \xi \in \omega_1\} \subset (A^p)^n$ and $\{b \cup b_\xi : \xi \in \omega_1\} \subset [B^p]^{<\omega}$, consequently F satisfies $(*)$.

To see that F satisfies $(**)$ take any $a \in [\mu]^{<\omega}$. Again there is a condition $p \in G$ with $a \subset A^p$. Now it follows from (iii) that whenever $\nu \in \lambda \setminus B^p$ then

$$|\{F(\alpha, \nu) : \alpha \in a\}| = |a|.$$

In particular we get that the maps $F(\alpha, -) : \lambda \to \lambda$ obtained by fixing $\alpha \in \mu$ in F are pairwise distinct, hence $|\mu| \leqq 2^\lambda$ in $V[G]$.

The λ-closedness of P implies that cardinals $\leqq \lambda$ are preserved and that $\lambda^{<\lambda} = \lambda$ remains valid in $V[G]$. The λ^+-antichain condition implies that the cardinals above λ stay alive. From $\mu^\lambda = \mu$ it follows that $|P| = \mu$ as well, hence by counting names one easily gets that $2^\lambda \leqq \mu$ in $V[G]$. Thus, in fact, $2^\lambda = \mu$. This completes the proof of Theorem 2.

From Theorems 1 and 2 we immediately obtain the following result, which corresponds to our claims in the introduction.

COROLLARY 1. *If V satisfies* GCH *and $\omega_1 < \lambda < \mu$ are arbitrary regular cardinals in V then there is a cardinal preserving generic extension W of V in which $2^{\aleph_0} = \lambda = \lambda^{<\lambda}$, $2^\lambda = 2^{2^{\aleph_0}} = \mu$ and there exist a strongly* HS *0-dimensional Hausdorff space X with $|X| = 2^{2^{\aleph_0}}$ and a strongly* HL *0-dimensional Hausdorff space Y with $w(Y) = 2^{2^{\aleph_0}}$.*

PROOF. Indeed, the notion of forcing to be used is $P * H(\lambda \times \lambda)$, where P is as in Theorem 2. It is well-known that $2^{\aleph_0} = \lambda$ and $2^{2^{\aleph_0}} = \mu$ will then be valid in $W = V^{P * H(\lambda \times \lambda)}$.

Before we describe another application of the above results we first recall a result of Kunen and Roitman in [6], where they proved that if X is Hausdorff and $\mathrm{cf}(\hat{s}(X)) = \aleph_0$ then actually $\hat{s}(X) < 2^{\aleph_0}$.

S. Shelah has recently shown that the following is true in general: if X is Hausdorff then

$$\hat{s}(X) < 2^{2^{\mathrm{cf}(\hat{s}(X))}}$$

i.e. if $\mathrm{cf}(\hat{s}(X))$ is uncountable then another exponent appears in the upper bound. Now, we show that this result is best possible.

COROLLARY 2. *It is consistent with* ZFC *to have, e.g., $2^{\aleph_0} = \aleph_2 = 2^{\aleph_1}$, $2^{2^{\aleph_1}} = 2^{\aleph_2} = \aleph_{\omega_1+1}$ and there is a Hausdorff space X with $\hat{s}(X) = \aleph_{\omega_1}$.*

PROOF. Let us apply Corollary 1 to the case in which $\lambda = \aleph_2$ and $\mu = \aleph_{\omega_1+1}$. We then have in W the required cardinal arithmetic and a HS subspace S of 2^λ with $|S| = \aleph_{\omega_1}$. Now, we may apply the methods of [7] (see also [5], 7.4) to S to obtain a finer topology on S such that the resulting (obviously Hausdorff) space X satisfies $\hat{z}(X) = \hat{s}(X) = \aleph_{\omega_1}$.

In the above model, when we start with a ground model V satisfying GCH, we shall also have $2^{\aleph_\alpha} = \aleph_{\omega_1+1}$ whenever $2 \leqq \alpha \leqq \omega_1$. However, if we start with V in which $2^{\aleph_1} = \aleph_2$, $2^{\aleph_2} = \aleph_{\omega_1+1}$ and $2 \leqq \alpha < \beta < \omega_1$ implies

$$(+) \qquad 2^{\aleph_\alpha} < 2^{\aleph_\beta},$$

then we can still apply Theorems 1 and 2 for $\lambda = \aleph_2$ and $\mu = \aleph_{\omega_1+1}$ to obtain the conclusion of Corollary 2. Similarly as in [5], 7.4 it can be shown then that $o(X)$, the number of open subsets of X, satisfies

$$o(X) = 2^{<2\aleph_{\omega_1}},$$

hence, because $(+)$ remains valid in the extension, we get $\mathrm{cf}(o(X)) = \omega_1$ and at the same time $2^{\aleph_1} = \aleph_2 < o(X)$.

To conclude, let us remark that all the results of this paper may be lifted in a straightforward manner to obtain, e.g., hereditarily κ-separable spaces of cardinality 2^{2^κ} and hereditarily κ-Lindelöf spaces of weight 2^{2^κ} where $2^\kappa = \lambda = \lambda^{<\lambda}$ and $2^\lambda = \mu$ are arbitrarily prescribed regular cardinals such that $\kappa^+ < \lambda < \mu = \mu^\lambda$.

Another result that we mention here without proof is that if CH holds in V then a function $F : \mu \times \omega_1 \to \omega_1$ with $\mu = 2^{\aleph_1}$ can be constructed that satisfies $(*)$ and $(**)$, hence after adding $\aleph_1$ Cohen reals to V strong HS (resp. HL) spaces of size (weight) $2^{\aleph_1} = 2^{2^{\aleph_0}}$ will exist. This conclusion however is not new, because it has been known (cf. [4], 2.9) that strong HFD (HFC) spaces of this size (weight) exist in such a model.

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