IMPLAUSIBLE SUBGROUPS OF INFINITE SYMMETRIC GROUPS

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1. *Introduction*

Throughout this paper, S will denote the symmetric group on the set ω of natural numbers. We shall be studying the possibilities for the induced action of subgroups $G \leqslant S$ on the power set $\mathscr{P}(\omega)$. Our work was motivated by the following:

QUESTION. Suppose that $\aleph_0 < \kappa < 2^{\aleph_0}$. Does there exist a subgroup $G \leqslant S$ which has exactly κ orbits on $\mathscr{P}(\omega)$?

Recently Läuchli and Neumann [3] have given a partial answer to this question. Using only ZFC, they have shown that the group of piecewise order preserving permutations of $\mathbb{Q}$ has exactly $\aleph_1$ orbits on $\mathscr{P}(\mathbb{Q})$. However, it seems unlikely that their approach can be modified to yield examples with exactly κ orbits for $\aleph_1 < \kappa < 2^{\aleph_0}$. In this paper, we shall give a complete answer to the question, under the assumption of Martin's Axiom (MA).

Since there are only countably many finite and cofinite subsets of ω, the question really concerns the action of subgroups $G \leqslant S$ on the set of moieties of ω, that is, those subsets $X \subseteq \omega$ such that $|X| = |\omega \backslash X| = \aleph_0$. Let $\mathscr{P}(\omega)/\mathrm{Fin}(\omega)$ be the quotient of the boolean algebra $\mathscr{P}(\omega)$ by the ideal $\mathrm{Fin}(\omega)$ of finite subsets. Then it is easily checked that if $G \leqslant S$ has $\kappa \geqslant \aleph_0$ orbits in its induced action on $\mathscr{P}(\omega)/\mathrm{Fin}(\omega)$, then G also has κ orbits on $\mathscr{P}(\omega)$. More generally, let $\mathscr{I}$ be an arbitrary ideal of $\mathscr{P}(\omega)$ and let $\mathscr{F}$ be the dual filter. Define

$$S_{(\mathscr{I})} = \{\pi \in S \mid \mathrm{fix}(\pi) \in \mathscr{F}\},$$

$$S_{\{\mathscr{I}\}} = \{\pi \in S \mid \Delta \in \mathscr{I} \Leftrightarrow \pi(\Delta) \in \mathscr{I}\}.$$

Then each $\pi \in S_{\{\mathscr{I}\}}$ induces an automorphism of the boolean algebra $\mathscr{B} = \mathscr{P}(\omega)/\mathscr{I}$, and $S_{(\mathscr{I})}$ is the kernel of the associated homomorphism from $S_{(\mathscr{I})}$ into $\mathrm{Aut}(\mathscr{B})$. Suppose that $S_{(\mathscr{I})} \leqslant G \leqslant S_{\{\mathscr{I}\}}$ and that G has $\kappa \geqslant \aleph_0$ orbits on $\mathscr{P}(\omega)/\mathscr{I}$. Since $S_{(\mathscr{I})} \leqslant G$, each orbit on $\mathscr{P}(\omega)/\mathscr{I}$ corresponds to at most countably many orbits on $\mathscr{P}(\omega)$. Hence G also has κ orbits on $\mathscr{P}(\omega)$.

If $\mathscr{I}$ is an ideal and $X \in \mathscr{P}(\omega)$, then $[X] = \{Y \in \mathscr{P}(\omega) \mid X \equiv Y \bmod \mathscr{I}\}$. (While it would be more accurate to write $[X]_{\mathscr{I}}$, the relevant ideal $\mathscr{I}$ will always be clear from the context.)

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THEOREM 1 (MA). *Suppose that* $0 < \kappa < 2^{\aleph_0}$. *Let* $\{X_i \mid i < \kappa\}$ *be a set of moieties such that for all* j *with* $i < j < \kappa$, $X_i \not\equiv X_j \bmod \mathrm{Fin}(\omega)$. *Then there exists a subgroup* $G \leqslant S$ *such that* $\{[X_i] \mid i < \kappa\} \cup \{[\phi], [\omega]\}$ *is a complete set of orbit representatives for the action of* G *on* $\mathscr{P}(\omega)/\mathrm{Fin}(\omega)$.

It is perhaps worth pointing out that we do not need the full force of Martin's Axiom in the proof of Theorem 1. All we require is that the set $\mathbb{R}$ of real numbers cannot be covered by $2^{\aleph_0}$ nowhere dense subsets. Our second result gives examples of subgroups of S which have rather surprising actions on $\mathscr{P}(\omega)$.

THEOREM 2 (MA). *Suppose that* $\aleph_0 \leqslant \kappa < 2^{\aleph_0}$. *Then there exists an ideal* $\mathscr{I}$ *such that;*

(i) $\mathscr{P}(\omega)/\mathscr{I} \simeq \mathscr{P}(\kappa)$, *and*

(ii) $S_{\{\mathscr{I}\}}$ *induces* $\mathrm{Sym}(\kappa)$ *on* $\mathscr{P}(\omega)/\mathscr{I}$.

These groups are also relevant to the above question. For suppose that $\kappa = \aleph_\theta < 2^{\aleph_0}$, where θ is a cardinal such that $\aleph_0 < \theta < 2^{\aleph_0}$. Then $S_{\{\mathscr{I}\}}$ has θ orbits on $\mathscr{P}(\omega)/\mathscr{I} \simeq \mathscr{P}(\kappa)$, and hence also has θ orbits on $P(\omega)$.

It is a little disturbing to find $\mathrm{Sym}(\kappa)$ involved in $\mathrm{Sym}(\omega)$ for $\kappa > \aleph_0$. However, this part of Theorem 2 does not require Martin's Axiom.

THEOREM 3 (ZFC). *Suppose that* $2^\kappa = 2^{\aleph_0}$. *Then there exist subgroups* $N \leqslant G \leqslant S$ *such that* $N \lhd G$ *and* $G/N \simeq \mathrm{Sym}(\kappa)$.

Proof. By Theorem 5.1 [**4**], there is an embedding of the free group on $2^{\aleph_0}$ generators into S. The result follows.

At this point, some readers may be wondering whether we can perhaps even embed $\mathrm{Sym}(\kappa)$ into S. Fortunately there are limits to the amount of pathological behaviour that S can exhibit.

THEOREM 4 (ZFC). *If* $\kappa > \theta$, *then* $\mathrm{Sym}(\kappa)$ *does not embed in* $\mathrm{Sym}(\theta)$.

Proof. Clearly we can assume that κ and θ are infinite cardinals. Suppose that $G \leqslant \mathrm{Sym}(\theta) = P$ with $G \simeq \mathrm{Sym}(\kappa)$. Let $\alpha \in \theta$ and let P_α be the stabiliser of α in P. Then $[P:P_\alpha] = \theta$ and so $[G:G \cap P_\alpha] \leqslant \theta$. By [**1**], every proper subgroup of G has index at least κ. Hence $G \leqslant P_\alpha$. But then $G \leqslant \bigcap_{\alpha \in \theta} P_\alpha = 1$, which is a contradiction.

We shall be assuming a familiarity with Martin's Axiom. A clear account can be found in Kunen [**2**], whose notation we will follow. In particular, if $\mathbb{P}$ is a partial ordering and $p, q \in \mathbb{P}$ are conditions, then $p \leqslant q$ will mean that p extends q.

2. *The proof of Theorem* 1

Let $\{Y_\alpha \mid \alpha < 2^{\aleph_0}\}$ be the set of moieties of ω. We shall define inductively a chain of subgroups $G_\alpha, \alpha < 2^{\aleph_0}$, so that the following conditions are satisfied.

2.1. $|G_\alpha| \leqslant |\alpha| + \aleph_0$.

2.2. For all $\gamma < \alpha$, there exists $\pi \in G_\alpha$ and $i < \kappa$ such that $\pi(Y_\gamma) \equiv X_i \bmod \mathrm{Fin}(\omega)$.

2.3. For all $i < j < \kappa$ and $\pi \in G_\alpha$, $\pi(X_i) \not\equiv X_j \bmod \mathrm{Fin}(\omega)$.

Then $G = \bigcup_{\alpha < 2^{\aleph_0}} G_\alpha$ will satisfy the conclusion of Theorem 1. Let $G_0 = 1$. Suppose that G_γ has been defined for all $\gamma < \alpha$. If α is a limit ordinal, we define $G_\alpha = \bigcup_{\gamma < \alpha} G_\gamma$.

Suppose that $\alpha = \beta + 1$. If there exists $\pi \in G_\beta$ and $i < \kappa$ such that $\pi(Y_\beta) \equiv X_i \bmod \operatorname{Fin}(\omega)$, then we set $G_{\beta+1} = G_\beta$. Suppose that this does not occur.

Let $\mathbb{P}$ be the set of finite partial injective functions p from ω to ω which satisfy

2.4. if $n \in \operatorname{dom} p \cap Y_\beta$, then $p(n) \in X_0$, and if $n \in \operatorname{dom} p \cap (\omega \setminus Y_\beta)$, then $p(n) \in \omega \setminus X_0$.

We partially order $\mathbb{P}$ by letting $p \leqslant q$ if and only if $q \subseteq p$. Clearly $\mathbb{P}$ is c.c.c.

Let $w(\bar{g}, x)$ be a word from the free product $G_\beta * \langle x \rangle$. If $p \in \mathbb{P}$, then we regard $w(\bar{g}, p)$ as a partial function from ω to ω. Fix $i \neq j < \kappa$, $n \in \omega$ and a word $w(\bar{g}, x)$. Let $\mathscr{D}$ consist of those $p \in \mathbb{P}$ such that either

2.5. there exists $m > n$ such that $m \in X_i$ and $w(\bar{g}, p)(m) \notin X_j$; or

2.6. there exists $m > n$ such that $m \notin X_i$ and $w(\bar{g}, p)(m) \in X_j$.

We shall prove that $\mathscr{D}$ is a dense subset of $\mathbb{P}$. Martin's Axiom then yields a permutation $\pi \in S$ such that $\pi(Y_\beta) = X_0$, and $G_{\beta+1} = \langle G_\beta, \pi \rangle$ satisfies conditions 2.1 to 2.3. If x does not occur in $w(\bar{g}, x)$, then $\mathscr{D} = \mathbb{P}$ (since G_β satisfies 2.3). Therefore we can suppose that

$$w(\bar{g}, x) = g_1 x^{n_1} g_2 \dots g_r x^{n_r} g_{r+1}$$

where $n_k \in \mathbb{Z} \setminus \{0\}$, $g_k \in G_\beta$, and $g_k X 1$ for $1 < k < r+1$. Let $p \in \mathbb{P}$ be an arbitrary condition. First suppose that

$$|g_{r+1}[X_i] \cap Y_\beta| = |g_{r+1}[\omega \setminus X_i] \cap Y_\beta| = \aleph_0.$$

Then there exist integers $m_1, m_2 > n$ such that

2.7. $m_1 \in X_i$ and $m_2 \in \omega \setminus X_i$;

2.8. $g_{r+1}(m_1)$, $g_{r+1}(m_2) \in Y_\beta \setminus (\operatorname{dom} p \cup \operatorname{ran} p)$.

So we can find $q_1, q_2 \leqslant p$ such that

$$g_r q_1^{n_r} g_{r+1}(m_1) = g_r q_2^{n_r} g_{r+1}(m_2) \notin \operatorname{dom} q_1 \cup \operatorname{dom} q_2 \cup \operatorname{ran} q_1 \cup \operatorname{ran} q_2.$$

Continuing in this manner, we obtain $q_3 \leqslant q_1$ and $q_4 \leqslant q_2$ such that

$$w(\bar{g}, q_3)(m_1) = w(\bar{g}, q_4)(m_2).$$

If $w(\bar{g}, q_3)(m_1) \notin X_j$, then $q_3 \in \mathscr{D}$. Otherwise, $q_4 \in \mathscr{D}$.

The other cases are handled in a similar fashion. For example, suppose that $|g_{r+1}[\omega \setminus X_i] \cap Y_\beta| < \aleph_0$. Since $g_{r+1}[X_i] \not\equiv Y_\beta \bmod \operatorname{Fin}(\omega)$, this implies that

$$|g_{r+1}[X_i] \cap (\omega \setminus Y_\beta)| = |g_{r+1}[\omega \setminus X_i] \cap (\omega \setminus Y_\beta)| = \aleph_0$$

and an obvious modification of the previous argument deals with this case.

3. *The proof of Theorem* 2

Let T be the tree of finite binary sequences, ordered by $s \leqslant t$ if and only if s is an initial segment of t. Let $T_n = {}^n 2$ be the nth level of T. We shall begin by choosing a set $\mathscr{B} = \{\eta_i \mid i < \kappa\}$ of branches of T. Then we will use Martin's Axiom to construct for each $\pi \in \operatorname{Sym}(\kappa)$ a level-preserving permutation $\Pi \in \operatorname{Sym}(T)$ such that

$$\Pi(\eta_i) \equiv \eta_{\pi(i)} \bmod \operatorname{Fin}(T)$$

for each $i < \kappa$. Once this is accomplished, Theorem 2 will follow quite easily. If we start with an arbitrary set $\mathscr{B}$ of branches, then the partial ordering $\mathbb{P}_\pi$ of Lemma 2 may not be c.c.c. The set $\mathscr{B}$ which is constructed in Lemma 1 is specifically designed to avoid this difficulty.

If $A = \{a_i \mid i < \lambda\}$ is a set with fixed enumeration, then $[A]^n$ denotes the set of n-tuples $\langle a_{i_1}, \dots, a_{i_n} \rangle$ with $i_1 < \dots < i_n$. If η is a branch of T, then $\eta(n) = \eta \cap T_n$ and $\eta \restriction k = \eta \cap [\bigcup_{n<k} T_n]$.

LEMMA 1 (MA). *There exists a set* $\mathscr{B} = \{\eta_i \mid i < \kappa\}$ *of branches of* T *together with a set of functions* $\{d_n \mid d_n : [\mathscr{B}]^n \to \omega\}$, *such that the following condition is satisfied.*

3.1. *If* $\langle \theta_1, \dots, \theta_n \rangle, \langle \nu_1, \dots, \nu_n \rangle \in [\mathscr{B}]^n$ *and* $d_n(\theta_1, \dots, \theta_n) = d_n(\nu_1, \dots, \nu_n)$, *then there exists* $k \in \omega$ *such that*

(i) $\theta_i \restriction k = \nu_i \restriction k$ *for* $1 \leqslant i \leqslant n$;

(ii) $\theta_i \restriction k \neq \theta_j \restriction k$ *for* $1 \leqslant i < j \leqslant n$;

(iii) *if* $\theta_i \neq \nu_i$, *then* $\theta \restriction k+1 \neq \nu_i \restriction k+1$ *for* $1 \leqslant i \leqslant n$.

Proof. Let $\mathbb{P}$ be the partial ordering whose typical element has the form $p = \langle f; \langle d_n \mid n \in \omega \rangle \rangle$ where

(a) f is a function such that $\operatorname{dom} f = X \times m$ for some finite $X \subseteq \kappa$ and some $m \in \omega$;

(b) for each $\alpha \in X$ and $n \in m$, $f_\alpha(n) = f(\alpha, n) \in T_n$ and f_α is a branch of $\bigcup_{n<m} T_n$;

(c) if $\alpha \neq \beta \in X$, then $f_\alpha \neq f_\beta$;

(d) for each $n \in \omega$, d_n is a function from $[\{f_\alpha \mid \alpha \in X\}]^n$ to ω such that if $d_n(f_{\alpha_1}, \dots, f_{\alpha_n}) = d_n(f_{\beta_1}, \dots, f_{\beta_n})$, then there exists $k \in m$ satisfying conditions (i) to (iii) of 3.1. (In particular, $d_n = \phi$ if $n > |X|$.)

The ordering on $\mathbb{P}$ is the obvious one. A straightforward Δ-system argument shows that $\mathbb{P}$ is c.c.c., and the result follows easily.

From now on, η_i always denotes the ith element of $\mathscr{B}$. If $\theta \in \mathscr{B}$ and $\pi \in \operatorname{Sym}(\kappa)$, then $\pi(\theta) = \eta_{\pi(i)}$, where $\theta = \eta_i$.

LEMMA 2 (MA). *For each* $\pi \in \operatorname{Sym}(\kappa)$, there exists a function $G : \mathscr{B} \to \omega$ *and a level-preserving permutation* $\Pi \in \operatorname{Sym}(T)$ *such that* $\Pi(\eta_i(n)) = \eta_{\pi(i)}(n)$ *for all* $i < \kappa$ *and* $n \geqslant G(\eta_i)$.

Proof. First we construct the function G. Let $\mathbb{P}_\pi$ be the partial ordering whose elements are the finite functions g satisfying

(a) $\operatorname{dom} g \subseteq \mathscr{B}$ and $\operatorname{ran} g \subseteq \omega$;

(b) for $\eta_\alpha, \eta_\beta \in \operatorname{dom} g$ and $n \in \omega$, if $g(\eta_\alpha), g(\eta_\beta) \leqslant n$ then $\eta_\alpha(n) = \eta_\beta(n)$ if and only if $\eta_{\pi(\alpha)}(n) = \eta_{\pi(\beta)}(n)$.

The set $\mathbb{P}_\pi$ is ordered by reverse inclusion.

We shall prove that $\mathbb{P}_\pi$ is c.c.c. Let $\{g_i \mid i < \aleph_1\}$ be a subset of $\mathbb{P}_\pi$, and let $a_i = \operatorname{dom} g_i$. We may suppose that $|a_i| = |a_j| = n$ for all $i < j < \aleph_1$. By the Δ-system lemma, we may also suppose that

(i) there exists a fixed set r such that $a_i \cap a_j = r$ for all $i < j < \aleph_1$.

Let $a_i = \{\theta^i_l \mid 1 \leqslant l \leqslant n\}$ and let $b_i = \{\theta^i_l, \pi(\theta^i_l) \mid 1 \leqslant l \leqslant n\}$. We can assume that there is an integer m such that $|b_i| = m$ for all $i < \aleph_1$. Let $\sigma_i = \langle \nu^i_1, \dots, \nu^i_m \rangle \in [\mathscr{B}]^m$, where $b_i = \{\nu^i_t \mid 1 \leqslant t \leqslant m\}$. Then we can suppose that for all $i, j < \aleph_1$

(ii) for all $\theta \in r$, there exists a fixed $t \leqslant m$ such that $\theta = \nu^i_t$;

(iii) for all $l \leqslant n$, there exist fixed $s, t \leqslant m$ such that $\theta^i_l = \nu^i_s$ and $\pi(\theta^i_l) = \nu^i_t$;

(iv) $d_m(\sigma_i) = d_m(\sigma_j)$, where d_m is the function given by Lemma 1;

(v) for all t with $1 \leqslant t \leqslant n$, $g_i(\theta^i_t) = g_j(\theta^j_t) = N_t$. (In particular, $g_i \restriction r = g_j \restriction r$.)

Choose $N > N_1, \dots, N_n$. We can also suppose that

(vi) for all j, t with $i < j < \aleph_1$ and $1 \leqslant t \leqslant m$, $\nu^i_t \restriction N = \nu^j_t \restriction N$.

We claim that for any $i < j < \aleph_1$, $g = g_i \cup g_j \in \mathbb{P}_\pi$. It is only necessary to verify that g satisfies (b). Suppose that $\eta_\alpha, \eta_\beta \in \operatorname{dom} g$. If both are included in either $\operatorname{dom} g_i$ or

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$\operatorname{dom} g_j$, then (b) holds. So we can assume that $\eta_\alpha \in \operatorname{dom} g_i$ and $\eta_\beta \in \operatorname{dom} g_j$, say $\eta_\alpha = v_s^i$ and $\eta_\beta = v_t^j$.

First suppose that $s = t$. Since $d_m(\sigma_i) = d_m(\sigma_j)$, conditions (i) to (iii) of 3.1 hold with respect to some $k \geqslant N$. In particular, we have that

$$v_s^i \restriction k = v_s^j \restriction k, \quad \pi(v_s^i) \restriction k = \pi(v_s^j) \restriction k$$

and

$$v_s^i \restriction k+1 \neq v_s^j \restriction k+1, \quad \pi(v_s^i) \restriction k+1 \neq \pi(v_s^j) \restriction k+1.$$

Hence (b) holds in this case.

Now suppose that $s \neq t$. Since $v_s^i \restriction k = v_s^j \restriction k$ and $\pi(v_s^i) \restriction k = \pi(v_s^j) \restriction k$, no problems occur for $n < k$. For $n \geqslant k$, $v_s^i(n) \neq v_t^j(n)$ and $\pi(v_s^i)(n) \neq \pi(v_t^j)(n)$. Hence (b) also holds in this case. This completes the proof that $\mathbb{P}_\pi$ is c.c.c.

By Martin's Axiom, there exists a function $G: \mathscr{B} \to \omega$ such that for all $\eta_\alpha, \eta_\beta \in \mathscr{B}$ and $n \in \omega$, if $G(\eta_\alpha), G(\eta_\beta) \leqslant n$ then $\eta_\alpha(n) = \eta_\beta(n)$ if and only if $\eta_{\pi(\alpha)}(n) = \eta_{\pi(\beta)}(n)$.

Finally we define the level-preserving permutation Π of T. Let $n \in \omega$ and $s \in T_n$. If there exists $\alpha < \kappa$ such that $\eta_\alpha(n) = s$ and $G(\eta_\alpha) \leqslant n$, then define $\Pi(s) = \eta_{\pi(\alpha)}(n)$. Clearly $\Pi(s)$ is well-defined in this case. Suppose that there exists $s' \in T_n$ and $\beta < \kappa$ such that $G(\eta_\beta) \leqslant n$, $\eta_\beta(n) = s'$ and

$$\Pi(s') = \eta_{\pi(\beta)}(n) = \eta_{\pi(\alpha)}(n) = \Pi(s).$$

Then $s' = \eta_\beta(n) = \eta_\alpha(n) = s$. Hence Π can be extended to a level-preserving permutation of T. Clearly G and Π satisfy the requirements of the lemma.

We now construct an ideal $\mathscr{I}$ such that $\mathscr{P}(\omega)/\mathscr{I} \simeq \mathscr{P}(\kappa)$ and $S_{\{\mathscr{I}\}}/S_{(\mathscr{I})} \simeq \operatorname{Sym}(\kappa)$. The next lemma allows us to construct inductively the filter on η_α which will be dual to $\mathscr{I} \cap \mathscr{P}(\eta_\alpha)$.

LEMMA 3 (MA). *Suppose that* $\{\mathscr{F}_\alpha \subseteq \mathscr{P}(\eta_\alpha) \mid \alpha < \kappa\}$ *satisfies the following conditions:*

(i) *for all* $\alpha < \kappa$, $|\mathscr{F}_\alpha| < 2^{\aleph_0}$;

(ii) *for all* $\alpha < \kappa$ *and finite subsets* $Z \subseteq \mathscr{F}_\alpha$, $|\bigcap Z| = \aleph_0$;

(iii) *if* $S \subseteq \omega$ *and* $\alpha < \beta < \kappa$, *then* $\{\eta_\alpha(n) \mid n \in S\} \in \mathscr{F}_\alpha$ *if and only if* $\{\eta_\beta(n) \mid n \in S\} \in \mathscr{F}_\beta$.

Then for each $X \subseteq \kappa$, *there exists* $X^* \subseteq T$ *such that*

(a) *if* $\alpha < \kappa$ *and* $A \in \mathscr{F}_\alpha$, *then* $|A \cap X^*| = \aleph_0$ *if and only if* $\alpha \in X$;

(b) *for each* $\alpha < \kappa$ *and* $\beta \in X$, *let* $T_\beta^\alpha = \{\eta_\alpha(n) \mid \eta_\beta(n) \in \eta_\beta \cap X^*\}$, *and let* $\mathscr{F}_\alpha^+ = \mathscr{F}_\alpha \cup \{T_\beta^\alpha \mid \beta \in X\}$. *Then* $\{\mathscr{F}_\alpha^+ \mid \alpha < \kappa\}$ *also satisfies conditions* (i) *to* (iii).

Proof. Let $\mathscr{A} = \{\eta_\beta \mid \beta \notin X\}$. Let $\mathbb{P}_\mathscr{A}$ be the partial ordering whose elements have the form $p = \langle s, F \rangle$, where s is a finite subset of T and F is a finite subset of $\mathscr{A}$, ordered by $\langle s', F' \rangle \leqslant \langle s, F \rangle$ if and only if

(1) $s \subseteq s'$ and $F \subseteq F'$;

(2) for all $\theta \in F$, $\theta \cap s' \subseteq s$.

By [2, II.2.14], $\mathbb{P}_\mathscr{A}$ is c.c.c.

Fix finite subsets $Y \subseteq X$, $Z \subseteq \mathscr{F}_0$ and an integer $n \in \omega$. Let

$$\mathscr{D} = \{\langle s, F \rangle \in \mathbb{P}_\mathscr{A} \mid \text{there exists } m > n \text{ such that } \eta_0(m) \in \textstyle\bigcap Z \text{ and } \eta_\alpha(m) \in s \text{ for all } \alpha \in Y\}.$$

We shall show that $\mathscr{D}$ is dense in $\mathbb{P}_\mathscr{A}$. The lemma then follows easily. Let $\langle s_0, F_0 \rangle \in \mathbb{P}_\mathscr{A}$. There exists $k \in \omega$ such that if $\eta_\alpha \neq \eta_\beta \in F_0 \cup \{\eta_\gamma \mid \gamma \in Y\}$, then $\eta_\alpha(k) \neq \eta_\beta(k)$. Choose

$m > \max\{k, n\}$ such that $\eta_0(m) \in \bigcap Z$, and let $s = s_0 \cup \{\eta_\beta(m) \mid \beta \in Y\}$. Then $\langle s, F_0\rangle \in \mathscr{D}$ and $\langle s, F_0\rangle \leqslant \langle s_0, F_0\rangle$.

Proof of Theorem 2 (MA). Since Martin's Axiom holds, $2^\kappa = 2^{\aleph_0}$. Hence we can use Lemma 3 to construct sets $\{U_\alpha \subseteq \mathscr{P}(\eta_\alpha) \mid \alpha < \kappa\}$ which satisfy the following conditions.

(i) If $Z \subseteq U_\alpha$ is a finite subset, then $|\bigcap Z| = \aleph_0$.

(ii) If $S \subseteq \omega$ and $\alpha < \beta < \kappa$, then $\{\eta_\alpha(n) \mid n \in S\} \in U_\alpha$ if and only if

$$\{\eta_\beta(n) \mid n \in S\} \in U_\beta.$$

(iii) For each $X \in \mathscr{P}(\kappa)$, there exists $X^* \in \mathscr{P}(T)$ such that

(a) if $\alpha \in X$, then $X^* \cap \eta_\alpha \in U_\alpha$;

(b) if $\alpha \notin X$, then there exists $A \in U_\alpha$ such that $|X^* \cap A| < \aleph_0$.

Let $\mathscr{F}_0$ be a nonprincipal ultrafilter on η_0 such that $U_0 \subseteq \mathscr{F}_0$. For each $\alpha < \kappa$, define the nonprincipal ultrafilter $\mathscr{F}_\alpha$ on η_α by

$$\{\eta_\alpha(n) \mid n \in S\} \in \mathscr{F}_\alpha \Leftrightarrow \{\eta_0(n) \mid n \in S\} \in \mathscr{F}_0$$

for each $S \subseteq \omega$. Then $\{\mathscr{F}_\alpha \mid \alpha < \kappa\}$ also satisfies conditions (i) to (iii). For each $\alpha < k$, let $\mathscr{I}_\alpha$ be the maximal ideal on η_α which is dual to $\mathscr{F}_\alpha$. Define

$$\mathscr{I} = \{Y \subseteq T \mid Y \cap \eta_\alpha \in \mathscr{I}_\alpha \text{ for all } \alpha < \kappa\}.$$

Then $\mathscr{I}$ is an ideal on T, and it is easily verified that the mapping

$$[S] \mapsto \{\alpha \in \kappa \mid S \cap \eta_\alpha \in \mathscr{F}_\alpha\}$$

provides an isomorphism from $\mathscr{P}(\omega)/\mathscr{I}$ onto $\mathscr{P}(\kappa)$. In particular, $[\eta_\alpha] \mapsto \{\alpha\}$.

Let $\pi \in \mathrm{Sym}(\kappa)$ and let $\Pi \in \mathrm{Sym}(T)$ be the corresponding level-preserving permutation, given by Lemma 2. For each $S \subseteq \omega$ and $\alpha < \kappa$,

$$\Pi(\{\eta_\alpha(n) \mid n \in S\}) \equiv \{\eta_{\pi(\alpha)}(n) \mid n \in S\} \bmod \mathrm{Fin}(T).$$

Since $\{\mathscr{I}_\alpha \mid \alpha < \kappa\}$ satisfies (ii), $\Pi \in S_{\{\mathscr{I}\}}$. Also Π induces π on $\mathscr{P}(\omega)/\mathscr{I}$. Hence $S_{\{\mathscr{I}\}}$ induces $\mathrm{Sym}(\kappa)$ on $\mathscr{P}(\omega)/\mathscr{I}$.

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