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Two Problems on $\aleph_0$-Indecomposable Abelian Groups

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1. INTRODUCTION

Following the program outlined on the Problem Session of the Honolulu Conference on Abelian Group Theory (Dec. 1983–Jan. 1984), the second author of this paper formulated the following conjectures and problems in his Udine talk [5]:

Conjecture 1. Let T be a countable reduced torsion group. There exists an $\aleph_0$-indecomposable (i.e., not decomposable into a direct sum of $\aleph_0$ nonzero summands) extension G of T by a completely decomposable torsion-free group of rank n if and only if T can be presented as a direct sum $T = F \oplus H$, where F is a finite group and for any prime p and nonnegative integer k

$$\mathbf{f}_k^p(H) + \mathbf{f}_{k+1}^p(H) \leqslant n$$

(where symbols $\mathbf{f}_k^p(H)$ denote the Ulm–Kaplansky invariants of H).

This criterion would not change if we were to omit the words "completely decomposable."

Problem 2. Given a countable reduced torsion group T and a torsion-free group R. Find the necessary and sufficient condition for the existence of an $\aleph_0$-indecomposable extension of T by R.

Problem 3. Is it true that for any countable reduced torsion group T

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without infinite bounded summands, there exists an $\aleph_0$-indecomposable. extension of T by a countable torsion-free group H?

The two problems considered in this paper are Conjecture 1 and Problem 3.

Theorem 3 proves Conjecture 1, and thus completely describes torsion parts of countable $\aleph_0$-indecomposable abelian groups of finite torsion-free rank in terms of their Ulm–Kaplansky invariants. This result generalizes the main result of [3].

The negative answer on Problem 3 was quite surprising (at least to the second author). It is obtained in Theorem 5, which shows that allowing torsion-free rank to grow from finite to countable does not contribute anything toward $\aleph_0$-indecomposability of a group, namely, if G is a countable $\aleph_0$-indecomposable abelian group, then already some subgroup of finite torsion-free rank H of G with the same torsion part as G is $\aleph_0$-indecomposable.

A new tool used in this paper is the Test lemma, which is a combinatorial description of the property of a countable group to be $\aleph_0$-indecomposable. An immediate application of it (Theorem 2) shows that the class of countable $\aleph_0$-indecomposable groups of finite torsion-free rank is closed under finite direct sums.

And finally, Problem 2 is not addressed here. It will be the sole subject of another joint paper of the authors, which is forthcoming.

2. Preliminaries

All groups in this paper are abelian. We will mostly use the notations and terminology of [1]. In addition, N will stand for the set of natural numbers; $I_n = \{1, 2,..., n\}$; $N^* = \{0\} \cup N$; P: the set of prime numbers; (m, n): the greatest common divisor of $m, n \in \mathbf{Z}$; $\mathbf{h}_p(g)_G$: the p-height of g in G $(g \in G, p \in P)$; $\mathbf{f}_k^p(G)$: the Ulm–Kaplansky invariants $(p \in P, k \in N^*)$; $B[p^n]$: the direct sum of a set of cyclic groups $\mathbf{Z}(p^n)$ of order p^n $(p \in P, n \in N)$.

Let $\{G_i\}_{i \in \Delta}$ be a set of groups. The subgroup $\mathbf{S}_{i \in \Delta} G_i$ of the direct product $\prod_{i \in \Delta} G_i$ consisting of all sequences $\{g_i\}_{i \in \Delta}$ $(g_i \in G_i)$, such that for any natural n almost all components of $\{g_i\}_{i \in \Delta}$ satisfy $g_i \in nG_i$, is called the regular direct sum of the groups G_i, $i \in \Delta$ (it is also called $\mathbf{Z}$-adic completion).

The group G is said to be fully reduced if $\bigcap_{n \in N} nG = 0$.

Lemma 1. [4, Theorem 4.3]. *Let G be a fully reduced group and*

$$B = \bigoplus_{p \in P} \bigoplus_{n \in N} B[p^n]$$

an arbitrarily fixed basic subgroup of the torsion part tG of G (δ: $B \to G$ is the corresponding embedding).

Then there exist a group $\hat{G}$ and a monomorphism σ: $G \to \hat{G}$ such that

$$\hat{G} = {}_{p\in P}\mathbf{S}_{n\in N}\, B[p^n] \oplus G',$$

where the group G' is the regular direct sum of some set of additive groups of rings of p-adic integers (with distinct or identical p), the subgroup σG is pure in $\hat{G}$, and the composition $\sigma\delta$ is the canonical embedding of the direct sum into the regular direct sum.

THEOREM 0 [3, Theorem 1]. *If a basic subgroup B_0 of the maximal torsion subgroup $T(G)$ of a group G contains a direct summand of G of the same cardinality as G, then any basic subgroup B of $T(G)$ has the same property.*

Let us take this opportunity to thank Adolf Mader for pointing out that Lemma 4 in [3] is incorrect. All other results in [3] remain unchanged and correct. This lemma should be replaced by the following lemma:

LEMMA 2. *Let B be a basic subgroup of a reduced countable torsion group T. There exists an $\aleph_0$-indecomposable extension of T by a finite rank torison-free group R if and only if there exists an $\aleph_0$-indecomposable extension of B by R.*

Proof. Lemma 3 in [3] proves exactly the sufficient condition.

Let G be a $\aleph_0$-indecomposable extension of T by R, $G_1 = \bigcap_{n\in N} nT$. We get the commutative diagram (D) with exact rows and columns.

$$\begin{array}{ccccc}
 & & G_1 & = & G_1 \\
 & & \downarrow 1 & & \downarrow \\
B & \xrightarrow{1} & G & \longrightarrow & G/B \\
\delta_1 \downarrow & & \downarrow \delta & & \downarrow \\
B' & \xrightarrow{\beta} & G' & \longrightarrow & G'/B'
\end{array} \qquad \text{(D)}$$

It can be checked straightforwardly that $B' = \delta(B)$ is a basic subgroup of the torsion part $T(G')$ of G', and δ_1 is an isomorphism.

Assume G' is $\aleph_0$-decomposable. Since $T(G) = T$ is reduced and $\mathbf{r}_0(G') < \aleph_0$, G' has a countable direct summand decomposable into a direct sum of finite cyclic groups. Due to Theorem 0, there exists a countable subgroup B'_1 of B' such that $\beta B'_1$ is a direct summand of G', i.e., $G' = \beta B'_1 \oplus L'$.

Denote $B_1 = \delta_1^{-1}(B_1')$ and $L = \delta^{-1}(L')$. Obviously, $G = \langle B_1, L \rangle$. Also, $B_1 \cap L \subseteq T_1$, i.e., $B_1 \cap L \subseteq B \cap T_1 = 0$. Therefore we get a direct decomposition $G = B_1 \oplus L$, which shows that G is $\aleph_0$-decomposable. Contradiction, therefore G' is $\aleph_0$-indecomposable.

Note that $T(G')$ being countable without elements of infinite height is a direct sum of cyclics, therefore $T(G') \cong B$. G' is an $\aleph_0$-indecomposable extension of B by R. The necessary condition is proven.

THEOREM 1 [3, Theorem 2]. *Let T be a countable reduced torsion group. There exists an $\aleph_0$-indecomposable extension of T by a torsion-free group of the torsion-free rank 1 if and only if T can be presented as a direct sum $T = F \oplus H$, where F is a finite group and for every prime p and non-negative integer k*

$$\mathbf{f}_k^p(H) + \mathbf{f}_{n+1}^p(H) \leqslant 1. \tag{0}$$

3. Countable Countably Indecomposable Groups of Finite Torsion-Free Rank, n-Decomposable for Any Finite n

DEFINITION. We would say that a group G passes the test if there exists a finite subset A in G such that there is no direct decomposition

$$G = F \oplus G_1 \tag{1}$$

with a finite non-zero subgroup F, and $A \subseteq G_1$.

We would also say in this case that G passes the test with its corresponding subset A.

LEMMA 3. *Let $G = F' \oplus G'$ be a direct decomposition of a group G with a finite subgroup F'. Then G passes the test if and only if G' passes the test.*

Proof. Let σ be the projection of G onto G'. If G passes the test with its corresponding subset A, then G' passes the test with $\sigma(A)$.

If G' passes the test with its corresponding subset A', then G passes the test with $F' \cup A'$.

TEST LEMMA. *Let G be a mixed countable reduced group of finite torsion-free rank. Then G is $\aleph_0$-indecomposable if and only if it passes the test.*

Proof. Let A be a finite subset of G and $G = \oplus_{n \in N} G_n$ a decomposition of G into a direct sum of non-zero summands. There exists $m \in N$ such that $A \subseteq \oplus_{n<m} G_n$ and G_m is torsion. Therefore G_m has a finite non-zero direct summand F, and a prohibited decomposition (1) exists. Thus G does not pass the test. The sufficient condition is proven.

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Necessary Condition. Assume that G does not pass the test. Let $\{g_i\}_{i\in N}$ enumerate G. We will define by induction on i finite non-zero subgroups F_i, subgroups H_i and finite subsets A_i satisfying all the inclusions of the following diagram

$$\begin{array}{ccccccc} A_1 \subseteq & A_2 \subseteq & \cdots & \subseteq A_i \subseteq & \cdots \\ \cap\!| & \cap\!| & \cdots & \cap\!| & \\ H_1 \supseteq & H_2 \supseteq & \cdots & \supseteq H_i \supseteq & \cdots \end{array} \tag{2}$$

and the following direct decompositions:

$$H_i = F_{i+1} \oplus H_{i+1} \tag{3}$$

$$G = \left(\bigoplus_{j \leqslant i} F_j \right) \oplus H_i. \tag{4}$$

Let $A_1 = \emptyset$.

Let $H_1 = G$, $A_2 = \{g_1\}$. There exists a direct decomposition $H_1 = F_2 \oplus H_2$, where $1 < |F_2| < \aleph_0$ and $A_2 \subseteq H_2$.

Let $i \in N$. Due to Lemma 3, H_i does not pass the test. In particular, for the subset $A_i \subseteq H_i$ there exists a direct decomposition $H_i = F_{i+1} \oplus H_{i+1}$ with $1 < |F_{i+1}| < \aleph_0$ and $A_i \subseteq H_{i+1}$. We get

$$G = \left(\bigoplus_{j \leqslant i+1} F_j \right) \oplus H_{i+1}. \tag{5}$$

In accordance with decomposition (5), we can present $g_i \in G$ as

$$g_i = \beta_i + \alpha_i, \tag{6}$$

where $\beta_i \in \bigoplus_{j \leqslant i+1} F_j$, and $\alpha_i \in H_{i+1}$. Define $A_{i+1} = A_i \cup \{\alpha_i\}$.

In the end, the following decomposition takes place:

$$G = \left(\bigoplus_{i \in N} F_i \right) \oplus \left(\bigcap_{i \in N} H_i \right). \tag{7}$$

Indeed due to enumeration, $g \in G$ implies $g = g_i$ for some $i \in N$. Due to (5), (6), $g_i = \beta_i + \alpha_i$, where $\beta_i \in \bigoplus_{j \leqslant i+1} F_j \subseteq \bigoplus_{j \in N} F_j$, and $\alpha_i \in A_{i+1}$. Due to inclusions of (2), $\alpha_i \in A_j$ for every $j \geqslant i+1$, therefore $\alpha_i \in H_j$ for every $j \geqslant i+1$, so $\alpha_i \in \bigcap_{j \in N} H_j$, and finally $g_i \in (\bigoplus_{j \in N} F_j) \oplus (\bigcap_{j \in N} H_j)$.

Thus, G is $\aleph_0$-decomposable.

Remark 1. The statement and the proof of the Test Lemma as well as the following Theorem 2 would not change if we were to replace the requirement of finite rank of $G/T(G)$ by a weaker condition of $G/T(G)$ to be $\aleph_0$-indecomposable.

Remark 2. The necessary condition of the Test Lemma holds for arbitrary countable groups.

THEOREM 2. *The class K of countable $\aleph_0$-indecomposable groups of finite torsion-free rank is closed under finite direct sums.*

Proof. Let $G_1, G_2 \in K$. It means that G_1, G_2 pass the test with their finite subsets A_1, A_2, respectively.

Let

$$G = G_1 \oplus G_2, \tag{8}$$

$A = A_1 \cup A_2$. It suffices to show that a decomposition

$$G = F \oplus G' \tag{9}$$

is impossible, where $1 < |F| < \aleph_0$, and $A \subseteq G'$.

Assume there exists a decomposition (9). Without loss of generality we can assume that F is a cyclic p-group of the order, say p^n. Let T be a projection of G onto F, and $F = \langle x \rangle$. In accordance with decomposition (8), $x = x_1 + x_2$, where $x_i \in G_i$ $(i = 1, 2)$. Since $Tx = x$, $o(Tx_i) = p^n$ for at least one $i = 1, 2$. Without loss of generality we can assume that $o(Tx_1) = p^n$, therefore $Tx_1 = kx$, where $(k, p) = 1$. Let S be the isomorphism from $\langle x \rangle$ onto $\langle x_1 \rangle$ determined by $x \mapsto^S k^{-1} x_1$ (k^{-1} is the inverse of k in $\mathbf{Z}/p^n\mathbf{Z}$). Denote $T^1 = ST$. The restriction of T^1 to G_1 is a projection of G_1 onto $\langle x_1 \rangle$, such that $T^1 x_1 = x_1$ and $T^1(A_1) = 0$. Therefore there is a direct decomposition $G_1 = \langle x_1 \rangle \oplus \operatorname{Ker} T^1$, which shows that G_1 does not pass the test.

This contradiction shows that decomposition (9) is impossible, and therefore the group G is $\aleph_0$-indecomposable.

The following theorem is a generalization of the main result from [3].

THEOREM 3. *Let T be a countable reduced torsion group. The following conditions are equivalent:*

(1) *There exists an $\aleph_0$-indecomposable extension G of T by a completely decomposable torsion-free group of rank n.*

(2) *There exists an $\aleph_0$-indecomposable extension G of T by a torsion-free group of rank n.*

(3) *T can be presented as a direct sum $T = F \oplus H$, where F is a finite group and*

$$(\forall p \in P)(\forall k \in N^*)\colon \mathbf{f}_k^p(H) + \mathbf{f}_{k+1}^p(H) \leqslant n. \tag{10}$$

In the proof we will need the following lemma:

LEMMA 4. (Follows from Theorem 3.2 of [2].) *Let the group D be a direct sum of cyclics:*

$$D = \oplus_{i \in I_n} \langle \alpha_i \rangle, \qquad \text{where } o(\alpha_i) = p^m \text{ or } o(\alpha_i) = p^{m+1} \text{ for } i \in I_n, \text{ and } d \in D.$$

Then there exists a direct decomposition

$$D = \langle \beta \rangle \oplus D'$$

such that $d \in \langle \beta \rangle$, $\beta \in D$.

Proof of Theorem 3. (1) → (2) Obvious.

(2) → (3) Let G be an $\aleph_0$-indecomposable group with $T(G) = T$ and $G/T(G) \cong \bar{G}$ ($\tau\colon G \to \bar{G}$ corresponding epimorphism), where $\bar{G}$ is a torsion-free group of rank n.

Due to Lemma 2, we can assume without loss of generality that T is a direct sum of cyclics.

If for some $p \in P$, $k \in N^*$, $\mathbf{f}_k^p(T) = \aleph_0$, then G has a bounded countable summand, i.e., G is an $\aleph_0$-decomposable group. That contradicts the condition (2), and we can conclude, therefore, that

$$(\forall p \in P)(\forall k \in N^*)\colon \mathbf{f}_k^p(T) < \aleph_0. \tag{11}$$

Assume that T does not satisfy the condition (3). Due to (11), only two situations are possible.

Case 1. T has a direct summand $T_1 = \oplus_{p \in P_1} D_p$, where P_1 is an infinite set of primes and

$$(\forall p \in P_1)(\exists m(p) \in N^*)\colon \mathbf{f}_{m(p)}^p(D_p) + \mathbf{f}_{m(p)+1}^p(D_p) \geqslant n$$

and

$$\mathbf{f}_k^p(D_p) = 0 \qquad \text{for } k < m(p) \text{ and } k > m(p) + 1.$$

By appropriate selection of inverse images under epimorphism τ of a system of generators of $\bar{G}$ and numerous applications of Lemma 4, it is not difficult to show that in this case the group G is $\aleph_0$-decomposable.

Case 2. T has a direct summand T_1 which is a p-group and

$$T_1 = \bigoplus_{i \in N} D_i, \tag{12}$$

where $(\forall i \in N)(\exists m_i \in N^*)$: $\mathbf{f}^p_{m_i}(D_i) + \mathbf{f}^p_{m_i+1}(D_i) \geqslant n$ and

$$\mathbf{f}^p_k(D_i) = 0 \qquad \text{for } k < m_i \text{ and } k > m_i + 1,$$

and $m_{i+1} > m_i + 1$ $(i \in N)$.

Due to Lemma 2, we can assume without loss of generality that $\bigcap_{n \in N} nG = 0$. According to Lemma 1, we can present G as a pure subgroup of

$$\hat{G} = \prod_{i \in N} D_i \oplus G' \tag{13}$$

with T_1 (please see (12)) canonically embedded into G.

Let $\bar{G}_1$ be a pure subgroup of $\bar{G}$ of rank 1, and G_1 the complete inverse image of $\bar{G}_1$ under epinorphism τ. Assume that $\bar{G}_1$ is a p-divisible group. Let $\bar{G}_1$ be generated by $\bar{S} = \{\bar{r}_i\}_{i \in N} \cup \{\bar{s}_j\}_{j \in N}$ with relations $0 \neq \bar{r}_1 = p^{i-1}\bar{r}_i$ and $\bar{r}_1 = k_j \bar{s}_j$, $(p, k_j) = 1$.

We will use below the following notations: for $g \in \hat{G}$, $[g]_i$ will stand for the component of g in the factor D_i of the decomposition (13); $[g]_1^n = \sum_{i \in I_n} [g]_i$.

Elements $\{r_i\}_{i \in N}$ of G and positive integers $\{v_i\}_{i \in N}$ are defined as follows: r_1 is an inverse image of $\bar{r}_1$ under epimorphism τ, $v_1 = 1$. Assume $r_i \in G$ and $v_i \in N$ are chosen. There exist $r_{i+1} \in G$ and $v_{i+1} > v_i$ such that

$$r_i - [r_i]_1^{v_i} = pr_{i+1} + t_i,$$

where $[r_{i+1}]_1^{v_i} = 0$ and $t_i \in \bigoplus_{v_i < j < v_{i+1}} D_j$.

Let $i \in N$. Due to Lemma 4 applied to $[r_i]_{v_i} \in D_{v_i}$, there exists a direct decomposition

$$D_{v_i} = \langle \beta_i \rangle \oplus D'_{v_i} \tag{14}$$

such that $[r_i]_{v_i} \in \langle \beta_i \rangle$.

The following decomposition is now obvious:

$$\langle T, \{r_i\}_{i \in N} \rangle = \bigoplus_{i \in N} D'_{v_i} \oplus M,$$

where $M = \langle \{r_i\}_{i \in N}; \{\beta_i\}_{i \in N}; \bigoplus_{j \in N \setminus \{v_i\}_{i \in N}} D_j \rangle$.

Let $j \in N$. Choose an inverse image s_j of $\bar{s}_j$ under epimorphism τ such that for every $i \in N$, $[s_j]_{v_i} \in \langle \beta_i \rangle$, we have a direct decomposition

$$G_1 = \bigoplus_{i \in N} D'_{v_i} \oplus L, \tag{15}$$

where $L = \langle \{s_j\}_{j \in N}, M \rangle$.

Note that $T(G_1) = T$; $G_1/T \cong \bar{G}_1$; and

$$(\forall i \in N)(\exists m_{v_i} \in N^*)\colon \mathbf{f}^p_{m_{v_i}}(D'_{v_i}) + \mathbf{f}^p_{m_{v_i+1}}(D'_{v_i}) \geqslant n-1 \tag{16}$$

and

$$\mathbf{f}^p_k(D'_{v_i}) = 0 \qquad \text{for } k < m_{v_i} \text{ and } k > m_{v_i} + 1, \tag{17}$$

and $m_{v_{i+1}} > m_{v_i} + 1$ $(i \in N)$.

If $\bar{G}_1$ were not a p-divisible group, the proof of the existence of decomposition (15), satisfying (16) and (17) would be much simpler.

Group G, being the extension of T by the rank n torsion-free group $\bar{G}$, can be obtained as the composition of n consecutive extensions by torsion-free groups of rank 1. This process can be formalized by a straightforward induction. But the idea of construction would not be sacrificed if we stopped here after the first step. The outcome of the n-step construction is the decomposition of G into a direct sum of $\aleph_0$ non-zero summands.

Thus in both cases G is an $\aleph_0$-decomposable group. Contradiction. Group T satisfies the condition (3).

(3) $\to$ (1) The group T obviously can be decomposed into a direct sum

$$T = \bigoplus_{i=1}^{n} T_i$$

such that for every $i = 1, 2, \ldots, n$ the group T_i can be presented as a direct sum $T_i = F_i \oplus H_i$, where F_i is a finite group, and

$$(\forall p \in P)(\forall k \in N^*)\colon \mathbf{f}^p_k(H_i) + \mathbf{f}^p_{k+1}(H_i) \leqslant 1. \tag{18}$$

Due to Theorem 1, there exists an $\aleph_0$-indecomposable extension G_i of T_i by a rank 1 torsion-free group R_i.

Let $G = \bigoplus_{i=1}^n G_i$. Obviously $T(G) = T$, $G/T(G) \cong \bigoplus_{i=1}^n R_i$ is a completely decomposable group. Due to Theorem 2, G is an $\aleph_0$-indecomposable group.

Theorem 3 is proven.

COROLLARY 1. *Let T be a countable reduced torsion group. There exists an $\aleph_0$-indecomposable extension of T by a (completely decomposable) finite rank torsion-free group if and only if the set of the Ulm–Kaplansky invariants*

$$\{\mathbf{f}_n^p(T) \mid p \in P, k \in N^*\}$$

is bounded (by an integer).

In the necessary condition of Theorem 3, the following statement in fact was proven.

THEOREM 4. *Let G be a mixed group of a finite torsion-free rank n with a countable torsion part T.*

If there is an integer $m > n$ and an infinite subset $Q \subseteq P \times N^*$, such that

$$[\forall (p, k) \in Q]: \mathbf{f}_k^p(T) + \mathbf{f}_{k+1}^p(T) \geqslant m,$$

then there exists a direct decomposition

$$G = H \oplus G_1,$$

where H is a direct sum of finite cyclic groups and

$$[\forall (p, k) \in Q]: \mathbf{f}_k^p(H) + \mathbf{f}_{k+1}^p(H) \geqslant m - n.$$

4. COUNTABLE TORSION-FREE RANK DOES NOT CONTRIBUTE ANYTHING TOWARD $\aleph_0$-INDECOMPOSABILITY

THEOREM 5. (1) *Let G be a countable group, and the set of the Ulm–Kaplansky invariants $\{\mathbf{f}_k^p(T(G)) \mid p \in P, k \in N^*\}$ not bounded (by an integer). Then G is $\aleph_0$-decomposable.*

(2) *In fact, if G is a countable $\aleph_0$-indecomposable group, then some subgroup H of G of finite torsion-free rank and such that $T(G) \leqslant H \leqslant G$, is $\aleph_0$-indecomposable.*

Proof. Let G be a countable $\aleph_0$-indecomposable group with unbounded Ulm–Kaplansky invariants set. According to Remark 2, G passes the test with, say, a corresponding finite subset A. There exists a finite torsion-free rank pure subgroup H of G, such that $T(G) \leqslant H \leqslant G$ and $A \subseteq H$.

Due to Theorem 3, H is $\aleph_0$-decomposable, i.e.,

$$H = \bigoplus_{i \in N} H_i; \qquad H_i \neq 0.$$

There exists $n \in N$ such that $A \subseteq \bigoplus_{i<n} H_i$ and H_n is torsion. Therefore we can get a direct decomposition $H = F \oplus H'$, where F is a finite non-zero cyclic group, and $A \subseteq H'$. It is easy to show that the projection T of H onto F can be extended to a projection T^1 of G onto F, such that $T^1(A) = 0$. This contradicts the fact that G passes the test with the corresponding finite subset A.

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