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FORCINGS WITH IDEALS AND SIMPLE FORCING NOTIONS

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ABSTRACT

Using generic ultrapower techniques we prove the following statements:

(1) for every sequence $\langle \mu_n \mid n < \omega \rangle$ of 0–1 σ-additive measures over the set of reals, there exists a set which is nonmeasurable in each μ_n,

(2) there is no nowhere prime σ-complete $\aleph_0$-dense ideal,

(3) if I is a nowhere prime ideal over a set X then add $(I) \leq d(I)$,

(4) suppose that μ is a σ-additive total nowhere prime probability measure over a set X, then add $(\mu) < d(\mu)$, in particular, if μ is a real valued measure on the continuum, then the measure algebra cannot have countable density,

(5) there is no σ-complete ideal I over a set X such that the forcing with I is isomorphic to the Cohen real forcing or to the random real forcing or to the Hechler real forcing or to the Sacks real forcing.

Some general conditions on forcing preventing it for being isomorphic to the forcing with an ideal are formulated.

The following is a classical theorem of S. Ulam:

There is no *total* σ-additive 0–1 valued measure over $\aleph_1$.

Alaoglu–Erdös [E] extended this result by showing that for every $\aleph_0$ σ-additive 0–1 valued measures over $\aleph_1$ there exists a set nonmeasurable for all these measures.

If we replace $\aleph_0$ by $\aleph_1$ above (i.e., allow $\aleph_1$ measures on $\aleph_1$), then S. Shelah [Sh2], using supercompacts, was able to prove that it is consistent to have $\aleph_1$ σ-additive 0–1 valued measures over $\aleph_1$ so that every set is measurable in one of them. Previously M. Magidor [M], using a huge cardinal, showed the same with only $\aleph_1$, as the domain of the ideal, replaced by $\aleph_3$. H. Woodin [W], using

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an almost huge cardinal, was able to produce a model of ZFC with a stronger property, namely: $\aleph_1$ carries an $\aleph_1$-dense ideal. A. Taylor [T] showed that it is possible to generalize the Alaoglu–Erdös theorem by replacing simultaneously $\aleph_0$ and $\aleph_1$ there by λ and λ^+ (in absence of a measurable cardinal).

The following natural question was raised by L. Grinblat:†

Are there $\aleph_0$ σ-additive 0–1 measures over the set of reals so that every set of reals is measurable in one of them?

This question also appears in A. Taylor [T] but only with restriction to the uniform measures (i.e., every set of the cardinality less than the cardinality of the set carrying the measure is a zero set). Clearly, if $2^{\aleph_0} = \aleph_1$, then by Alaoglu–Erdös [E] the answer is negative.

The following equivalence was proved by A. Taylor [T] (see also [Sh3]).

THEOREM. *The following two statements are equivalent for a set X:*

(a) *there are $\aleph_0$ σ-additive* 0–1 *measures over X so that every subset of X is measurable in one of them*;

(b) *there exists a σ-complete ideal I over X having a countable dense set* (*i.e., there are subsets $\{Y_n \mid n < \omega\}$ of X so that each $Y_n \notin I$ and for every $Z \notin I$ there is $n < \omega$ such that $Y_n \setminus Z \in I$*).

If $|X|$ is a measurable cardinal or there exists a measurable cardinal below $|X|$ then it is easy to construct $\aleph_0$-dense ideal (i.e., σ-complete ideal having a countable dense set). If $|X| \geqq 2^{\aleph_0}$ less than the first measurable and an ideal I on X is uniform (i.e., every $Y \subseteq X$ of cardinality less than $|X|$ belongs to I), then, as it was shown by A. Tarski [Ta] for $|X| > 2^{\aleph_0}$ and for $|X| = 2^{\aleph_0}$ by R. Frankiewicz, A. Gutek [F–Gu]†† and, independently, by J. Baumgartner, K. Kunen, A. Taylor, T. Jech and K. Prikry [J–P] and probably others: I cannot be $\aleph_0$-dense. In order to eliminate the appeal to the nonmeasurability, let us call an ideal I over X nowhere prime if for every $Y \subseteq X$, $Y \notin I$ there are disjoint $Y_0, Y_1 \subseteq Y$, $Y_0, Y_1 \notin I$. Equivalently, $I \mid Y$ is not prime for every $Y \notin I$. The classical result says that every σ-complete ideal over a set of cardinality $\leqq 2^{\aleph_0}$ is nowhere prime, since $2^{\aleph_0}$ is less than the first measurable.

In Section 1 we give the negative answer to the question above by showing that there is no σ-complete nowhere prime $\aleph_0$-dense ideal (see 1.1). The

† A. Krawczyk pointed out to us that this question was raised previously by him and A. Pelc in [Kr–P].

†† Previously it was done by A. Krawczyk and A. Pelc [Kr–P].

methods of the proof apply to show that forcing with a κ-complete ideal† cannot be isomorphic to forcing of less than κ^+ Cohen reals (see 1.4). Also the following statements are proved in Section 1:

Let I be a nowhere prime ideal over a set X. Then $\mathrm{add}(I) \leqq d(I)$†† (see 1.5).

Let $\langle \mu_\alpha \mid \alpha < \lambda \rangle$ be a sequence of κ-additive 0–1 nowhere prime measures over a set X. If $\kappa > \lambda$, then there exists a set which is nonmeasurable in each μ_α (see 1.7).

L. Grinblat was able to extend the last statement for $\lambda_0 = \aleph_0$ to nontotal real valued measures and even σ-algebras.

R. Solovay [So1] showed that the continuum can be real valued measurable cardinal. In his model the measure algebra has density $> 2^{\aleph_0}$. D. Fremlin [Fr1] asked whether it can have a countable density. We show in Section 2 (see 2.3, 2.6) that it is impossible.

In Section 2, a general condition on forcing notions for preventing it from being isomorphic to the forcing with a κ-complete ideal over κ is formulated. It is satisfied by the Cohen real forcing, as well as by the random real forcing. Also the following is shown:

Let μ be a σ-additive total nowhere prime probability measure over set X. Then $d(\mu) > \mathrm{add}(\mu)$‡ (see 2.6), where the totality means that μ is defined on all $P(X)$ and the probability means that $\mu(X) = 1$.

In particular the forcing with a κ-complete ideal over cannot be isomorphic to less than κ^+ random reals. By R. Solovay [So1] κ^+ random reals are enough.

In Section 3, we deal with forcing notions which for different reasons fail to be isomorphic to the forcing with an ideal. Namely, the reasons are the first failure of the Fubini theorem in a strong form and an example is the Hechler forcing, and the second, a strong violation of the countable chain condition, an example of which is the Sacks real forcing.

Our notations are quite standard. The main notions used in the paper can be found in books by K. Kunen [K1] and T. Jech [J] and in A. Kanamori, M. Magidor's paper [Ka–M]. We assume that the reader is familiar with forcing and generic ultrapowers. Let us review only the definition of a generic ultrapower.

† (Forcing with an ideal I over a set X means the forcing with Boolean algebra $\mathscr{P}(X)/I$.)

†† $\mathrm{add}(I)$ is the maximal λ such that I is λ-complete, $d(I)$ is the density of $\mathscr{P}(X)/I$ as a Boolean algebra.

‡ $\mathrm{add}(\mu) = \mathrm{add}\{A \mid \mu(A) = 0\}$, $d(\mu)$ is the density of $\mathscr{P}(X)/I$ as a metric space with $d(A, B) = \mu(A \triangle B)$.

Let I be a σ-complete ideal over a set X. We would like to extend I in a generic fashion to a prime ideal over X in V, i.e., for the sets in $P(X) \cap V$. The set of conditions will be the $\{Y \subseteq X \mid Y \notin I\}$ and for Y_1, Y_2 in this set let $Y_1 \geqq Y_2$ (Y_1 is stronger than Y_2) iff $Y_1 \setminus Y_2 \in I$. We further refer to this forcing as to forcing with I and to the subsets of X not in I as to I-positive sets. Let G be a generic set. Define

$$I^* = \{Y \subseteq X \mid Y \in V \text{ and } X \setminus Y \in G\}.$$

Then I^* is a prime extension of I with respect to $P(X) \cap V$. Form now ultrapower using functions in $V \cap {}^X V$ and comparing them modulo I^*. Ideal I is called precipitous if this ultrapower is well founded. In this case let M be its transitive collapse. We can define an elementary embedding

$$j: V \to M \simeq (V \cap {}^X V)/I^*$$

by setting $j(a) = [c_a]_{I^*}$ where c_a is the constant function with the value a.

The results of Section 1 are due to the first author and the results of Sections 2 and 3 are due to the second author with exceptions of forcing construction in Section 2 and extensions from κ-complete ideals over κ to σ-complete ideals over arbitrary sets.

R. Frankiewicz told us that he also found a proof of "the forcing with an ideal cannot be isomorphic to the Cohen and/or to the random reals forcings". Unfortunately we were unable to follow his proof.

1. Cohen real case: Nonexistence of $\aleph_0$-dense ideal, a nonmeasurable set for $\aleph_0$ measures on the reals, etc.

Let I be an ideal over a set X. I is called $\aleph_0$-dense, if there exists a sequence $\langle A_n \mid n < \omega \rangle$ of I-positive sets so that for every I-positive $Z \subseteq X$ there is $n < \omega$ such that $A_n \setminus Z \in I$. An ideal I is nowhere prime if for every I-positive set $Z \subset X$, $I \mid Z = \{Y \subseteq X \mid Y \cap Z \in I\}$ is not prime ideal over X.

THEOREM 1.1. *There is no σ-complete nowhere prime $\aleph_0$-dense ideal.*

REMARKS. (a) Only the part of the statement related to the cardinalities $< 2^{\aleph_0}$ is new, but the proof covers also the $\geqq 2^{\aleph_0}$ case.

(b) A more general theorem for κ-complete ideals over κ will be proved in the next section. Its proof will not appeal directly to Theorem 1.1 but will use ideas of the present proof. It seems to us that it is much easier to see the ideas here in a simpler case than to dig them up from the general one.

(c) The main tool we are going to use is generic ultrapowers. We do not know if it is possible to replace this by some elementary methods, like for example, those of Jech and Prikry [J–P] for the $\geqq 2^{\aleph_0}$ case.†

PROOF. Suppose otherwise. Let I be a σ-complete nowhere prime, $\aleph_0$-dense ideal over a set X. Then I is a precipitous ideal, we refer to T. Jech [J] for main properties of such ideals and why our I is necessary precipitous. Pick an I-positive set X^*, a cardinal κ and a function $f: X \to \kappa$ so that X^* forces in the forcing with I that κ is the critical point of the generic elementary embedding and f represents κ in the generic ultrapower. Set

$$I^* = I \mid X^* = \{Y \subseteq X \mid Y \cap X^* \in I\}.$$

Then I^* is nowhere prime, $\aleph_0$-dense ideal over X which is κ-complete. Let us use f in order to define a projection of I^* to an ideal J over κ. For $a \subseteq \kappa$ let $a \in J$ iff $f^{-1}[a] \in I^*$.

CLAIM 1. J is a κ-complete nowhere prime $\aleph_0$-dense ideal over κ.

PROOF. The definition of J implies its κ-completeness. J is nowhere prime since $\kappa \leqq 2^{\aleph_0}$. Otherwise $\kappa > 2^{\aleph_0}$ and this means that the forcing with I^* does not add reals, since κ is the critical point and I^* is $\aleph_1$-saturated, see T. Jech's book [J] for such arguments. On the other hand, the forcing I^* is isomorphic to the Cohen real forcing and so adds a Cohen real. Contradiction.

It remains to be shown that J has a countable dense set. Consider the Boolean algebra $B(I^*) = \mathcal{P}(X)/I^*$. It satisfies c.c.c. and hence is complete Boolean algebra. Then the same holds for $B(J) = \mathcal{P}(\kappa)/J$. f^{-1} generates a complete embedding e from $B(J)$ into $B(I)$, where $e([a]_J) = [f^{-1}[a]]_{I^*}$, for $a \subseteq \kappa$. Let $\{b_n \mid n < \omega\}$ be a dense countable set in $B(I^*)$. Define

$$c_n = \Pi\{a \in B(J) \mid e(a) \geqq b_n\}, \qquad n < \omega.$$

Then $\{c_n \mid n < \omega\}$ is a dense countable set in $B(J^*)$. So J is $\aleph_0$-dense.

□ of the claim.

† D. Fremlin and, later, A. Kamburelis have found such proofs.

For the rest of the proof we shall work with J which, as was shown above, is a κ-complete nowhere prime, $\aleph_0$-dense ideal over κ. Actually J is a normal ideal over κ since the function f used in its definition was forced to be the least nonconstant function.

The forcing with J is isomorphic to the Cohen real forcing. Let r be a Cohen generic real. It generates the generic elementary embedding $j: V \to M$ with M transitive and the critical point κ. By $\aleph_1$-saturateness of J, M and $V[r]$ have the same reals. In particular, $r \in M$. Pick a sequence of reals $\langle r_\alpha \mid \alpha < \kappa \rangle \in V$ representing r in M, i.e., $j(\langle r_\alpha \mid \alpha < \kappa \rangle)(\kappa) = r$. There is a condition in the Cohen forcing this. Let us assume for simplification of the notations that already the empty condition forces "$j(\langle r_\alpha \mid \alpha < \kappa \rangle)(\kappa) = \underset{\sim}{r}$". Clearly $j(\langle r_\alpha \mid \alpha < \kappa \rangle) = \langle r_\alpha \mid \alpha < j(\kappa) \rangle$ in M, $j(r_\alpha) = \tilde{r}_\alpha$ for $\alpha < \kappa$ and $r_\kappa = r$.

Working in V, we pick a sequence $\langle T_\alpha \mid \kappa < \alpha < \kappa^+ \rangle$ of canonical names of the reals $\langle r_\alpha \mid \kappa < \alpha < \kappa^+ \rangle$ so that

$$\varnothing \Vdash T_\alpha(\underset{\sim}{r}) = \underset{\sim}{r}_\alpha$$

for every α, $\kappa < \alpha < \kappa^+$. Let us view each T_α as a subset of ${}^{\omega>}2 \times {}^{\omega>}2$.

For every α, $\kappa < \alpha < \kappa^+$ let us consider the following set

$$A_\alpha = \{ r_\beta \mid \beta < \kappa, \text{ there exists } \gamma < \kappa \text{ so that } T_\alpha(r_\beta) = r_\gamma \}.$$

Claim 2. For every α, $\kappa < \alpha < \kappa^+$, A_α is a set of reals of the second category.

Proof. Otherwise, there exists a Borel meager set $B \supseteq A_\alpha$. But in a generic ultrapower $T_\alpha(r_\kappa) = r_\alpha$. Hence $r_\kappa \in j(A_\alpha) \subseteq j(B)$. But it is impossible since r_κ is a Cohen real over V and $j(B)$ is a meager Borel set with a cod in V. Contradiction. $\square$ of the claim.

Claim 3. For every α, $\kappa < \alpha < \kappa^+$, $\varnothing \Vdash \check{A}_\alpha$ is of the second category.

This holds since a countable forcing cannot turn a set of the second category into a meager set; see Kunen [K3].

Claim 4. $\varnothing \Vdash \underset{\sim}{j}''(\kappa^+) \notin \underset{\sim}{M}$, where $j: V \to M$ is a generic elementary embedding.

Proof. Otherwise $j(\kappa)^+$ will be singular in M, since $j(\kappa^+) = \bigcup_{\delta < \kappa^+} j(\delta)$. $\square$ of the claim.

Let us consider the image $j(A_\alpha)$ of A_α in a generic ultrapower M ($\kappa < \alpha < \kappa^+$). Then

$$j(A_\alpha) = \{r_\beta \mid \beta < j(\kappa), \text{ there is } \gamma < j(\kappa) \text{ so that } T_{j(\alpha)}(r_\beta) = r_\gamma\}.$$

Since the critical point κ of j is above $\aleph_0$, $T_{j(\alpha)} = T_\alpha$. Hence A_α is an initial segment of $j(A_\alpha)$. So, by Claim 3, $j(A_\alpha)$ has (in M) an initial segment of second category. Then, in V, the same holds for A_α. So there exists $\xi(\alpha) < \kappa$ such that $\{r_\beta \in A_\alpha \mid \beta < \xi(\alpha)\}$ is of the second category and for $\beta < \xi(\alpha)$, with $r_\beta \in A_\alpha$, there is $\gamma < \xi(\alpha)$ satisfying $T_\alpha(r_\beta) = r_\gamma$. Denote $\{r_\beta \in A_\alpha \mid \beta < \xi(\alpha)\}$ by $A_\alpha \mid \xi(\alpha)$.

Back in M let us consider the following set:

$$E = \{\alpha \mid j(\kappa) < \alpha < j(\kappa^+),\ r_\kappa \in A_\alpha, \text{ for some } \xi < \kappa,\ A_\alpha \mid \xi \text{ is of the second category and for } \beta < \xi \text{ with } r_\beta \in A_\alpha \text{ there is } \gamma < \xi \text{ satisfying } T_\alpha(r_\beta) = r_\gamma\}.$$

Then $E \supseteq j''((\kappa, \kappa^+))$ since for every α, $\kappa < \alpha < \kappa^+$, $r_\kappa \in j(A_\alpha) = A_{j(\alpha)}$ and $A_{j(\alpha)} \mid \xi(\alpha) = A_\alpha \mid \xi(\alpha)$. Using Claim 4, find some $\alpha^* \in E \setminus j''((\kappa, \kappa^+))$. Suppose for simplification of the notations that the empty condition already decides the values of $\underset{\sim}{\alpha}^*$ and $\xi(\underset{\sim}{\alpha}^*)$. For every condition p in the Cohen real forcing consider the set

$$S_p = \{r_\beta \mid \beta < \xi(\alpha^*) \text{ and for some } \gamma_\beta < \xi(\alpha^*)\ p \Vdash \underset{\sim}{T}_{\alpha^*}(r_\beta) = r_{\gamma_\beta}\}.$$

Claim 5. For some p, S_p is of the second category.

Proof. Suppose otherwise. Then the set $S = \bigcup \{S_p \mid p \text{ is a condition in the Cohen real forcing}\}$ is meager. But $\emptyset \Vdash \check{S} \supseteq \underset{\sim}{A}_{\alpha^*} \mid \xi(\alpha^*)$. So, in a generic extension $V[r]$, $A_{\alpha^*} \mid \xi(\alpha^*)$ is a meager set. Since $V[r]$ and the generic ultrapower M have the same reals, $A_{\alpha^*} \mid \xi(\alpha^*)$ is meager also in M. It contradicts the choice of α^*, $\xi(\alpha^*)$. □ of the claim.

Pick now p such that S_p is of the second category. We force a Cohen real r extending p. Let $j: V \to M$ be the corresponding elementary embedding. Then $S_p \subseteq A_{\alpha^*} \mid \xi(\alpha^*)$ and for $r_\beta \in S_p$ there exists $\gamma_\beta < \xi(\alpha^*)$ such that $T_{\alpha^*}(r_\beta) = r_{\gamma_\beta}$. Note that S_p and the function $\beta \mapsto \gamma_\beta$ are both in V and are bounded below κ. Using Claim 4, we can find $\alpha_0 \neq \alpha_1$, $\kappa < \alpha_0$, $\alpha_1 < \kappa^+$, having the same properties as α^*. Namely, $\xi(\alpha_i) = \xi(\alpha^*)$, $S_p \subseteq A_{\alpha_i} \mid \xi(\alpha_i)$ and for every $r_\beta \in S_p$, $T_{\alpha_i}(r_\beta) = r_{\gamma_\beta}$, where $i < 2$. It means that T_{α_0} and T_{α_1} agree on a set of the second category. The next claim shows that this is impossible.

Claim 6. Let $i_0 \neq i_1$ be ordinals between κ and κ^+, let S be a set of reals of the second category. Then there exists $s \in S$ so that $T_{i_0}(s) \neq T_{i_1}(s)$.

PROOF. Pick a countable elementary submodel N of V_χ, with χ big enough, so that $T_{i_0}, T_{i_1}, S \in N$. Since in V the empty condition in the Cohen real forcing forces

$$\text{“}T_{i_0}(\underset{\sim}{r}) = \underset{\sim}{r}_{i_0} \neq \underset{\sim}{r}_{i_1} = T_{i_1}(\underset{\sim}{r})\text{”}$$

the same holds in N. Let E be the union of all Borel meager sets with cods in N. Since N is countable, E is a meager set in V. So there is $s \in S \setminus E$. Then s is a Cohen generic real over N. Hence $N[s]$ satisfies "$T_{i_0}(s) \neq T_{i_1}(s)$". By absoluteness, then $T_{i_0}(s) \neq T_{i_1}(s)$ also in V. □ of the claim.

□ of Theorem 1.1.

The above theorem stated in the forcing terms claims that a forcing with ideal over a set cannot be isomorphic to a countable nontrivial forcing notion. The next theorem extends this.

THEOREM 1.2. *Suppose that I is a κ-complete ideal over a set X and P is a nontrivial κ-c.c. forcing notion so that for some $\langle P_\alpha \mid \alpha < \kappa \rangle$ satisfying*

(a) $P_\alpha \lessdot P$ *(i.e. P_α is a suborder of P and every maximal antichain of P_α is still maximal in P)*,

(b) $|P_\alpha| < \kappa$,

$P = \bigcup_{\alpha<\kappa} P_\alpha$. *Then the forcing I cannot be isomorphic to the forcing with P.*

REMARK. (a) K. Kunen [K2] starting with a measurable had constructed a κ-complete ideal I over κ such that the forcing with I is isomorphic to a forcing notion of cardinality κ satisfying κ-c.c.

(b) It is impossible to remove the κ-c.c. assumption by H. Woodin [W].

PROOF. Suppose otherwise. Let I be a κ-complete ideal over a set X, $P = \bigcup_{\alpha<\kappa} P_\alpha$ a forcing notion satisfying (a) and (b) so that the forcing with I is isomorphic to P. For simplification of the notations, let us assume that I is nowhere more than κ-complete. As $|P| \leqq \kappa$, I is κ^+-saturated. So generic ultrapowers are well founded and κ is the critical point of generic embeddings. As in the proof of Theorem 1.1, define the ideal J over κ. The argument of Claim 1 shows that J is κ-complete and the forcing with J is isomorphic to a forcing notion $Q \lessdot P$ of the form $\bigcup_{\alpha<\kappa} Q_\alpha$, $|Q_\alpha| < \kappa$ and $Q_\alpha \lessdot Q$. In order to show the nontriviality of Q note that for some $\alpha < \kappa$, P_α is nontrivial. So the forcing with P adds new subsets to $\delta = |P_\alpha|$. But then 2^δ cannot be less than κ, since I is κ-saturated. So J is nowhere prime. Hence Q is nontrivial. From now we shall deal only with J, Q and $\langle Q_\alpha \mid \alpha < \kappa \rangle$. Let $\delta < \kappa$ be the least cardinal so that $2^\delta \geqq \kappa$. Fix a sequence $\langle x_\alpha \mid \alpha < \kappa \rangle$ of κ distinct subsets of δ. In a generic ultrapower it moves to a longer sequence $\langle x_\alpha \mid \alpha < j(\kappa)\rangle$ and the elements $\langle x_\alpha \mid \kappa \leqq \alpha < j(\kappa)\rangle$ are new subsets of δ. Let $\langle \underset{\sim}{x}_\alpha \mid \kappa \leqq \alpha < \kappa^+\rangle$ be a sequence of canonical names of $\langle x_\alpha \mid \kappa \leqq \alpha < \kappa^+\rangle$. For every $\alpha, \kappa \leqq \alpha < \kappa^+$ there is $\beta(\alpha) < \kappa$ such that $\underset{\sim}{x}_\alpha$ is a $Q_{\beta(\alpha)}$-name. Let $C \subseteq \kappa^+$, $\delta \leqq \beta_0 < \kappa$ be so that $|C| = \kappa^+$ and $\beta(\alpha) = \beta_0$ for every $\alpha \in C$.

Collapse now β_0 to $\aleph_0$ using the usual Levy collapse. $\mathrm{Col}(\omega, \{\beta_0\}) = \{f \mid f$ is a finite partial function from ω to $\beta_0\}$. An easy application of κ-completeness of J gives that each J-positive set in $V[G]$ contains a J-positive set belonging to V, where G is a generic subset of $\mathrm{Col}(\omega, \{\beta_0\})$. So the forcings with J are the same in V and in $V[G]$. Note also that $Q_{\beta_0} < Q$ in $V[G]$. Since otherwise, in $V[G]$, $B^V(Q_{\beta_0}) \not< B^V(Q)$, where $B^V(Q_0)$, $B^V(Q)$ are the complete Boolean algebras of regular open sets generated by Q_0 and Q in V. Let then π be the projection of $B^V(Q)$ onto $B^V(Q_0)$ defined in V as follows:

$$\pi(x) = \Pi\{y \mid y \in B^V(Q_{\beta_0}), y \geqq x\}.$$

If now, in $V[G]$, $A \subseteq B^V(Q_{\beta_0})$ is a maximal antichain of $B^V(Q_{\beta_0})$ and $x \in B^V(Q)$ is incompatible with every member of A, then for some $a \in A$, $a \cdot \pi(x) \neq 0$ and $a \cdot x = 0$. Hence $x \leqq -a$. So $-a \geqq \pi(x)$ which implies $a \cdot \pi(x) = 0$. Contradiction. The same holds for all Q_α's.

In $V[G]$, Q_{β_0} is a countable forcing notion. So it is isomorphic to the Cohen real forcing. Also $\langle x_\alpha \mid \alpha < \kappa\rangle$ are now subsets of the countable ordinal δ and $\langle \underset{\sim}{x}_\alpha \mid \alpha \in C\rangle$ are names in the countable forcing Q_{β_0} of subsets of the countable ordinal δ. Let us assume, for simplification of the notations, that already in V, Q_{β_0} is the Cohen real forcing, $\langle x_\alpha \mid \alpha < \kappa\rangle$ are reals and $\langle \underset{\sim}{x}_\alpha \mid \alpha \in C\rangle$ are Cohen names of reals $\langle x_\alpha \mid \alpha \in C\rangle$ in the generic ultrapowers. From now we proceed as in the proof of Theorem 1.1. Namely, let r be a Cohen generic real, extend it to a Q-generic set and form the generic ultrapower $j: V \to M$. Let $\langle \mathrm{r}_\alpha \mid \alpha < \kappa\rangle \in V$ represent r in M. Pick in V a sequence $\langle T_\alpha \mid \alpha \in C\rangle$, $T_\alpha \subseteq {}^{\omega>}2 \times {}^{\omega>}2$ of Cohen canonical names of $\langle x_\alpha \mid \alpha \in C\rangle$ so that

$$\varnothing \Vdash T_\alpha(\underset{\sim}{r}) = \underset{\sim}{x}_\alpha$$

for every $\alpha \in C$.

The sets A_α's are defined now as follows:

$$A_\alpha = \{r_\beta \mid \beta < \kappa, \text{ there exists } \gamma < \kappa \text{ so that } T_\alpha(r_\beta) = x_\gamma\},$$

where $\alpha \in C$.

As in Theorem 1.1, each A_α is of the second category.

In order to show that A_α is still of the second category in V^Q we shall use a different argument.

CLAIM 1. For every $\alpha \in C$, $\emptyset \Vdash_Q \check{A}_\alpha$ is of the second category.

PROOF. Suppose otherwise.

Let $\underset{\sim}{B}$ be a name of a Borel meager set so that $\emptyset \Vdash_Q \underset{\sim}{B} \supseteq \check{A}_\alpha$. Since a cod of B in V^Q is a real, we can find $\tau < \kappa$ so that B is already in V^{Q_τ}. Let, in V^{Q_τ}, $B = \bigcup_{n<\omega} B_n$, where each B_n is nowhere dense. Suppose that the empty condition forces the above. Let us consider in V the following sets,

$$S_{n,p} = \{r_\beta \mid \beta < \kappa,\ p \Vdash_{Q_\tau} r_\beta \in \underset{\sim}{B}_n\}, \qquad \text{where } p \in Q_\tau,\quad n < \omega.$$

Then each $S_{n,p}$ is nowhere dense and $\bigcup \{S_{n,p} \mid n < \omega,\ p \in Q_\tau\} \supseteq A_\alpha$. Force now with Q and take the generic ultrapower. Then

$$\bigcup \{j(S_{n,p}) \mid n < \omega,\ p \in Q_\tau\} \supseteq j(A_\alpha).$$

Note that $|Q_\tau| < \kappa$. The Cohen real r introduced by the forcing which belongs to $j(A_\alpha)$ and does not belong to any of $j(S_{n,p})$ since these sets are meager with cods in V. □ of the claim.

The same proof gives a little stronger result:

CLAIM 2. For every $\alpha \in C$, $\emptyset \Vdash_Q \check{A}_\alpha$ is not a union of less than κ meager sets.

Then in a generic ultrapower M, A_α is an initial segment of $j(A_\alpha)$ which is of the second category and, moreover, cannot be a union of less than κ meager sets. Hence in V, the same holds for A_α. Namely, there are unboundedly many $\xi < \kappa$ such that $A_\alpha \mid \xi = \{r_\beta \in A_\alpha \mid \beta < \xi\}$ is not a union of less than ξ meager sets and for $\beta < \xi$, with $r_\beta \in A_\alpha$, there is $\gamma < \xi$ satisfying $T_\alpha(r_\beta) = x_\gamma$.

Now, as in Theorem 1.1, we define in M the set

$$\begin{aligned} E = \{\alpha \mid\ & \alpha \in j(C),\ r_\kappa \in A_\alpha, \text{ for unboundedly many } \xi\text{'s below } \kappa \\ & A_\alpha \mid \xi \text{ is not a union of fewer than } \xi \text{ meager sets and for } \beta < \xi, \\ & \text{with } r_\beta \in A_\alpha \text{ there is } \gamma < \xi \text{ satisfying } T_\alpha(r_\beta) = x_\gamma\}. \end{aligned}$$

Pick $\alpha^* \in E \setminus j''(C)$. $T_{\alpha^*} \subseteq {}^{\omega>}2 \times {}^{\omega>}2$, so for some $\tau < \kappa$ it is added by the forcing Q_τ. Let $\underset{\sim}{T}_{\alpha^*}$ be a Q_τ-name of T_{α^*}.

Pick $\xi(\alpha^*)$ to be an ordinal between τ and κ so that $A_{\alpha^*} \mid \xi(\alpha^*)$ is not a union of fewer than $\xi(\alpha^*)$ meager sets and for $\beta < \xi(\alpha^*)$ with $r_\beta \in A_{\alpha^*}$ there is $\gamma < \xi(\alpha^*)$ satisfying $T_{\alpha^*}(r_\beta) = x_\gamma$.

Back in V, suppose for simplicity that the empty condition decides the values of $\underset{\sim}{\tau}$, $\underset{\sim}{\alpha}^*$ and $\xi(\underset{\sim}{\alpha}^*)$. For every $p \in Q_\tau$ consider the set

$$S_p = \{r_\beta \mid \beta < \xi(\alpha^*) \text{ and for some } \gamma_\beta < \xi(\alpha^*),\ p \Vdash_{Q_\tau} \underset{\sim}{T}_{\alpha^*}(r_\beta) = x_{\gamma_\beta}\}.$$

Using the fact that $A_{\alpha^*} \mid \xi(\alpha^*)$ cannot be covered by τ meager sets, as in Claim 5 it is not hard to see that for some $p \in Q_\tau$, S_p is of the second category. The rest of the proof is as in Theorem 1.1. □

The following is immediate now.

Corollary 1.3. *Suppose that I is a κ-complete ideal over a set X and P is a nontrivial forcing notion of cardinality less than κ. Then the forcing with I cannot be isomorphic to P.*

Corollary 1.4. *Suppose that I is a κ-complete ideal over a set X. Then the forcing with I cannot be isomorphic to the Cohen forcing for adding less than κ^+-Cohen reals.*

It follows from Theorem 1.2 that at least κ^+ Cohen reals are needed.

Note that starting with a model of GCH with a measurable κ and adding κ-Cohen reals by the Cohen forcing one obtains a model where the forcing with the ideal (dual to the measurable ultrafilter) is isomorphic to the Cohen forcing for adding κ^+-Cohen reals.

It is impossible to replace κ^+ by 2^κ. Thus, start with a measurable κ and force with the Cohen forcing for adding $\kappa^{+\omega}$-Cohen reals. Then the generic extension satisfies $2^\kappa \geqq (\kappa^{+\omega})^+$ but the forcing with the ideal dual the measurable ultrafilter over κ is isomorphic to the Cohen forcing for adding $\kappa^{+\omega}$-Cohen reals.

For an ideal I over a set X let us denote by $d(I)$ the density of I, i.e., the least cardinal λ so that there exists a set Y of cardinality λ consisting of I-positive sets and satisfying the following: for every I-positive set Z there is $S \in Y$ such that $S \setminus Z \in I$. By add(I), the additivity of I, we mean the least cardinal λ so that there are λ sets in I with I-positive union.

Corollary 1.5. *Suppose that I is a nowhere prime ideal over a set X. Then* $\text{add}(I) \leqq d(I)$.

Proof. Otherwise the forcing with I will be isomorphic to a forcing notion of cardinality $d(I)$ less than the completeness of I, which is impossible by Corollary 1.3. □

By K. Kunen [K] and H. Woodin [W] it is possible to have $\mathrm{add}(I) = d(I)$.

An ideal I over a set X is called non-principal if $\{a\} \in I$ for every $a \in X$. The following follows from 1.1.

COROLLARY 1.6. *There is no σ-complete nonprincipal $\aleph_0$-dense ideal over the set of reals.*

The next results follow from the theorem of A. Taylor stated in the introduction and Corollary 1.4.

COROLLARY 1.7. *Suppose $\langle \mu_\alpha \mid \alpha < \lambda \rangle$ is a sequence of κ-additive 0–1 nowhere prime measures over a set X. If $\kappa > \lambda$, then there exists a set which is nonmeasurable in each μ_α.*

In particular for $X = 2^{\aleph_0}$, $\kappa = \aleph_1$ and $\lambda = \aleph_0$ the following holds:

COROLLARY 1.8. *Suppose that $\langle \mu_n \mid n < \omega \rangle$ is a sequence of σ-additive 0–1 measures over the set of reals. Then there exists a set of reals which is nonmeasurable in each of the measures.*

By K. Kunen [K], S. Shelah [Sh2] or H. Woodin [W] it is impossible to weaken the assumption $\kappa > \lambda$ to $\kappa = \lambda$.

L. Grinblat [Gr] was able to replace 0–1 measures in 1.8 by real valued nowhere total measures and even by nontrivial σ-algebras.

2. General conditions and the random real case

Let $\mathbb{B}$ be a complete Boolean algebra. Suppose that for some ordinal ν a generic object $G(\mathbb{B})$ of $\mathbb{B}$ can be reconstructed from a subset $r(\mathbb{B})$ of ν, i.e., for a name $\underset{\sim}{r}(\mathbb{B})$ of a subset of ν the complete subalgebra of $\mathbb{B}$ generated by $\{ \| \xi \in \underset{\sim}{r}(\mathbb{B}) \| \mid \xi \in \nu \}$ is equal to $\mathbb{B}$.

Let λ be a cardinal above ν. Consider sentences of the propositional logic L_λ where as a set of the propositional variables we shall use the set $\{ \text{“}\xi \in y\text{”} \mid \xi < \nu \}$. Recall that L_λ is defined as usual propositional logic but only conjunctions and disjunctions of any length below λ are allowed.

We attach to every $r \subseteq \nu$ the natural truth function f_r defined as follows:

$$f_r(\text{“}\xi \in y\text{”}) = \begin{cases} 1, & \xi \in r; \\ 0, & \xi \notin r. \end{cases}$$

Further we shall write $\varphi(r)$ instead of $f_r(\varphi)$ for sentence φ of L_λ. Denote the set $\{ r \subseteq \nu \mid V \vDash \varphi(r) \}$ by E_φ.

Let us define now an ideal $J_\lambda(\mathbb{B})$ over $\mathscr{P}(\nu)$. A set $H \in \mathscr{P}(\mathscr{P}(\nu))$ belongs to $J_\lambda(\mathbb{B})$ if there exists a sentence φ of L_λ so that $H \subseteq E_\varphi$ and

$$\| V[\underset{\sim}{G}(\mathbb{B})] \models \varphi(\underset{\sim}{r}(\mathbb{B})) \| = 0.$$

Then $J_\lambda(\mathbb{B})$ is a λ-complete ideal over $\mathscr{P}(\nu)$.

If $\mathbb{B}$ is the random real (or Cohen real) algebra, then $J_{\aleph_1}(\mathbb{B})$ is the ideal of measure zero sets (or meager sets). The ideal $J_\lambda(\mathbb{B})$ may be the trivial ideal. For example, if $\lambda = (2^\nu)^+$, then there exists φ in L_λ which describes exactly all the subsets of ν in V, i.e. $r \subseteq \nu$ satisfies φ iff $r \in V$. So $\mathscr{P}(\nu) \in J_\lambda(\mathbb{B})$.

Note also that if $\mathbb{B}$ satisfies λ-c.c., then there is a natural correspondence between elements of $\mathbb{B}$ and the sentences of L_λ. Since, then, $\mathbb{B} = \bigcup \{A_\alpha \mid \alpha < \lambda\}$, where $A_0 = \{ \| \check{\xi} \in \underset{\sim}{r}(\mathbb{B}) \| \mid \xi < \nu\}$, $A_\alpha = \{ -b \mid b \in A_\beta$ for some $\beta < \lambda\}$ if α is odd and $A_\alpha = \{\Sigma X \mid X \subseteq \bigcup_{\beta<\alpha} A_\beta\}$ if α is even. Replacing $\| \check{\xi} \in \underset{\sim}{r}(\mathbb{B}) \|$ by "$\xi \in y$" we obtain a sentence of L_λ from an element of $\mathbb{B}$, and the converse.

Theorem 2.1. *Suppose that $\mathbb{B}$ is a complete Boolean algebra satisfying the following conditions*:

(1) *$\mathbb{B}$ has an absolute definition*,

(2) *the forcing with $\mathbb{B}$ preserves a regular cardinal κ*,

(3) *$\mathbb{B}$ is generated by a name of a subset of some $\nu < \kappa$*,

(4) *for some $\mu < \kappa$ there are automorphisms $\langle \pi_\alpha \mid \alpha < \mu \rangle$ of $\mathbb{B}$ so that*

$$\left\| \check{\mathbb{B}} \setminus \{0\} = \bigcup_{\alpha<\mu} \check{\mu}''_\alpha(\underset{\sim}{G}(\mathbb{B})) \right\| = 1,$$

(5) *if for some $\xi < \kappa$, $\mathbb{B}$ satisfies ξ-c.c., then there exists a regular δ so that $\mu < \delta < \kappa$, for some $\delta' < \delta$, $\mathbb{B}$ satisfies $(\delta')^+$-c.c. and $\mathbb{B}$ preserves $J_\delta(\mathbb{B})$-positivity, i.e., for every $A \subseteq \mathscr{P}(\mathscr{P}(\nu))$, if $A \notin \mathbf{J}_\delta(\mathbb{B})$ in V, then $A \notin J_\delta(\mathbb{B}^{V[G(\mathbb{B})]})$ in a generic extension.*

Then *there is no κ-complete ideal I over κ so that the forcings with $\mathbb{B}$ and with I-positive subsets of κ are isomorphic.*

Proof. Suppose otherwise. Let I be a κ complete ideal over κ so that the forcings with $\mathbb{B}$ and with I-positive sets are isomorphic. The conditions (2) and (4) imply that $\mathbb{B}$ satisfies κ-c.c. Since otherwise there would be an antichain $\langle b_\alpha \mid \alpha < \lambda \rangle$ in $\mathbb{B}$ and in a generic extension $V[G(\mathbb{B})]$. $\mathbb{B}^V$ is a union of μ sets consisting of pairwise compatible elements. But $\kappa > \mu$ and it remains cardinal, which is impossible.

Then the ideal I is κ-saturated. So, by R. Solovay [So1], see also T. Jech [J], κ

is a weakly inaccessible cardinal, generic ultrapowers of V by I are well founded and contain all bounded subsets of κ which are in $V[G(I)]$. Hence, no cardinal is collapsed below κ. The weak inaccessibility together with the conditions (2) and (4) imply, as above, that $\mathbb{B}$ satisfies ξ-c.c. for some $\xi<\kappa$. Then there are $\delta, \mu<\delta<\kappa$ and $\delta'<\delta$ satisfying the condition (5).

Since the forcings with $\mathbb{B}$ and with I are isomorphic, we can define in $V[G(\mathbb{B})]$ the generic elementary embedding

$$j: V \rightarrow M \simeq V^*/G(I).$$

$r(\mathbb{B})$ will be in M, since it is a subset of $\nu<\kappa$. Then $2^\nu \geqq \kappa$ in V, since otherwise M and V would have some subsets of ν but $r(\mathbb{B})$ is in M and it is new. Let $\langle r_\alpha \mid \alpha<2^\nu\rangle$ be an enumeration of all subsets of ν in V. Pick a function $f: \kappa \rightarrow 2^\nu$, $f \in V$ so that $\langle r_{f(\alpha)} \mid \alpha<\kappa\rangle$ represents $r(\mathbb{B})$ in M, i.e., $j(\langle r_{j(\alpha)} \mid \alpha<\kappa\rangle)([\mathrm{id}])=r(\mathbb{B})$ or $r(\mathbb{B})=r_{j(f)([\mathrm{id}])}$, where [id] is the equivalence class of the identity function. W.l.o.g. let us assume that the empty condition forces this as well as the value of [id].

Claim 1. If $A \in V[G(\mathbb{B})]$ is a set of ordinals of the order type κ^+ then $j''(A) \notin M$.

Proof. Suppose otherwise. Let $A=\{\alpha_i \mid i<\kappa^+\}$ where $\langle\alpha_i \mid i<\kappa^+\rangle$ is an increasing sequence of ordinals. Set $\alpha=\bigcup_{i<\kappa^+} \alpha_i$. Then $\operatorname{cf} \alpha=\kappa^+$ both in $V[G(\mathbb{B})]$ and in V. So, in M, $\operatorname{cf} j(\alpha)=j(\kappa^+)>\kappa^+$. On the other hand, every function $f: \kappa \rightarrow \alpha$, $f \in V$ is bounded in α by some α_i $(i<\kappa^+)$. So $j''(A)$ is unbounded in $j(\alpha)$. And hence $\operatorname{cf} j(\alpha) \leqq \kappa^+$ in M. Contradiction. $\square$ of Claim 1.

Claim 2. $|\{r_{j(f)(\alpha)} \mid \alpha<j(\kappa)\} \backslash V| \geqq \kappa^+$ in $V[G(\mathbb{B})]$.

Proof. Suppose otherwise. First note that for every $\alpha<j(\kappa)$ there is $\alpha<\beta<j(\alpha)$ so that $r_{j(f)(\beta)} \notin\{r_{j(f)(\gamma)} \mid \gamma<\alpha\}$. Since otherwise, the same holds for $\langle r_{f(\gamma)} \mid \gamma<\kappa\rangle$ in V, i.e., for some $\alpha<\kappa$, $\{r_{f(\gamma)} \mid \gamma<\kappa\} \subseteq\{r_{f(\gamma)} \mid \gamma<\alpha\}$. But then, back in M, $r(\mathbb{B})=r_{j(f)([\mathrm{id}])} \in\{r_{j(f)(\gamma)} \mid \gamma<\alpha\}$ which is impossible since $r_{j(f)(\gamma)}= r_{f(\gamma)}$ for $\gamma<\kappa$.

Find $\xi<j(\kappa)$ so that $\{r_{j(f)(\eta)} \mid \xi \leqq \eta<j(\kappa)\} \subseteq V$. Define in M a sequence $\langle s_\alpha \mid \alpha<\kappa^+\rangle$ as follows. Set $s_0=r_{j(f)(\xi)}$, $s_\alpha=r_{j(f)(\eta_\alpha)}$, where η_α is the least ordinal η so that $\xi \leqq \eta<j(\kappa)$, $r_{j(f)(\eta)} \notin\{s_\beta \mid \beta<\alpha\}$. Still in M compare $\langle s_\alpha \mid \alpha<\kappa^+\rangle$ with $j(\langle r_\alpha \mid \alpha<2^\nu\rangle)=\langle r_\alpha \mid \alpha<j(2^\nu)\rangle$, i.e., the image of the enumeration of the subsets of ν in V. Define the sequence $\langle\alpha_i \mid i<\kappa^+\rangle$ as follows: α_i is the $\alpha<j(2^\nu)$ so that $s_i=r_\alpha$. Since each s_i belongs to V, the same would hold for r_{α_i}. Then r_{α_i} $(i<\kappa^+)$ appears in the enumeration $\langle r_\alpha \mid \alpha<2^\nu\rangle$ of V. So for some

$\beta_i < (2^\nu)^V$, $r_{\alpha_i} = r_{\beta_i}$. Then $\alpha_i = j(\beta_i)$. But $\{\beta_i \mid i < \kappa^+\} \in V[G(\mathbb{B})]$ forms a set of ordinals of the order type κ^+ and its image $\{\alpha_i \mid i < \kappa^+\}$ belongs to M, which contradicts Claim 1. Contradiction. $\square$ of Claim 2.

Let $\langle r_{j(f)(\alpha_i)} \mid i < \kappa^+ \rangle$ be κ^+ elements of $\{r_{j(f)(\alpha)} \mid \alpha < j(\kappa)\}$ which are not in V and are different from $r(\mathbb{B})$. Return now to V. W.l.o.g. we assume that the empty condition forces the above. Pick a $\langle T_i \mid i < \kappa^+ \rangle$ of canonical names of these subsets of ν. Each T_i can be viewed as a Borel function from $\mathscr{P}(\nu)$ to $\mathscr{P}(\nu)$ and $\varnothing \Vdash_{\mathbb{B}} \check{T}_i(\underset{\sim}{r}(\mathbb{B})) = r_{j(f)(\alpha_i)}$.

For every $i < \kappa^+$, we consider the following set:

$$A_i = \{r_{f(\beta)} \mid \beta < \kappa, \text{ there exists } \gamma < \kappa \text{ so that } T_i(r_{f(\beta)}) = r_{f(\gamma)}\}.$$

Claim 3. For every $i < \kappa^+$,

$$A_i \notin J_\delta(\mathbb{B}).$$

Proof. Suppose otherwise. Then there exists a sentence φ of L_δ so that $E_\varphi \supseteq A_i$ and $E_\varphi \in J_\delta(\mathbb{B})$. Force with $\mathbb{B}$ and form the generic ultrapower M. Then, in M still $j(E_\varphi) \supseteq j(A_i)$ but $j(\varphi) = \varphi$. But $j(T_i) = T_i$ and $T_i(r(\mathbb{B})) = r_{j(f)(\alpha_i)}$ so $r(\beta) \in j(A_i)$. Then $M \models \varphi(r(\mathbb{B}))$. Since φ is a sentence, also $V[G(\mathbb{B})] \models \varphi(r(\mathbb{B}))$, which contradicts $E_\varphi \in J_\delta(\mathbb{B})$. Contradiction. $\square$ of the claim.

Claim 4. For every $i < \kappa^+$, $\varnothing \Vdash_{\mathbb{B}}$ "$\check{A}_i \notin J_\delta(\mathbb{B})$".

This holds since A_i is $J_\delta(\mathbb{B})$ positive in V by Claim 3 and the condition (5) on $\mathbb{B}$ guarantees that the forcing with $\mathbb{B}$ preserves $J_\delta(\mathbb{B})$-positivity.

Let us consider now in a generic ultrapower M the image $j(A_i)$ of A_i $(i < \kappa^+)$.

$$j(A_i) = \{r_{j(f)(\beta)} \mid \beta < j(\kappa), \text{ there is } \gamma < j(\kappa) \text{ so that } T_{j(i)}(r_{j(f)(\beta)}) = r_{j(f)(\gamma)}\}.$$

But κ is the critical point of j. So $T_{j(i)} = T_i$ and $r_{j(f)(\beta)} = r_{f(\beta)}$ for every $\beta < \kappa$. Hence A_i is an initial segment of $j(A_i)$ and it is not in $J_\delta(\mathbb{B}^M)$, since, by Claim 4, $A_i \notin J_\delta(\mathbb{B}^{V[G(\mathbb{B})]})$. Now, back in V, A_i has $J_\delta(\mathbb{B})$-positive initial segment $\{r_{f(\beta)} \in A_i \mid \beta < \xi(i)\}$ for some $\xi(i) < \kappa$ such that for $\beta < \xi(i)$, with $r_{f(\beta)} \in A_i$, there is $\gamma < \xi(i)$ satisfying $T_i(r_{f(\beta)}) = r_{f(\gamma)}$. Denote this initial segment by $A_i \mid \xi(i)$. Let us consider in a generic ultrapower M the following set:

$$E = \{\eta < j(\kappa^+) \mid r(\mathbb{B}) \in A_\eta \text{ and, for some } \xi(\eta) < \kappa,\ A_\eta \mid \xi(\eta) \notin J_\delta(\mathbb{B}^M)\}.$$

Then $E \supseteq j''(\kappa^+)$, since for every $i < \kappa^+$, $r(\mathbb{B}) \in j(A_i) = A_{j(i)}$ and $A_{j(i)} \mid \xi(i) = A_i \mid \xi(i)$. By Claim 1, there is $i^* \in E \setminus j''(\kappa^+)$. Suppose for simplification of the notations that the empty condition forces the values of $\underset{\sim}{i}^*$ and $\underset{\sim}{\xi}(i^*)$. Let, in V,

$$F = \{\beta < \xi(i^*) \mid \| \check{r}_{f(\beta)} \in \underset{\sim}{A}_{i^*} \mid \check{\xi}(i^*) \| \neq 0\}.$$

For every $\beta \in F$ there exists $\gamma < \xi(i^*)$ so that $r_{f(\gamma)}$ is a candidate for being $T_{i^*}(r_{f(\beta)})$, i.e., $\| \check{r}_{f(\gamma)} = \underset{\sim}{T}_{i^*}(r_{f(\beta)}) \| \neq 0$. Since $\mathbb{B}$ satisfies $(\delta')^+$-c.c. and $\delta > \delta'$ the number of possible candidates is $\leqq \delta'$. Let $\langle \gamma_{\beta,\eta} \mid \eta < \delta' \rangle$ be the indexes of all such $r_{f(\gamma)}$'s.

Claim 5. There exists $\bar{\eta} < \delta'$ so that the set

$$\{r_{f(\beta)} \mid \beta < \xi(i^*) \text{ and } \| r_{f(\beta)} \in \underset{\sim}{A}_{i^*} \mid \xi(i^*) \wedge \underset{\sim}{T}_{i^*}(r_{f(\beta)}) = r_{f(\gamma_{\beta,\bar{\eta}})} \| > 0\} \notin J_\delta(\mathbb{B}).$$

Proof. Force with $\mathbb{B}$. Then for every $\beta < \xi(i^*)$ so that $r_{f(\beta)} \in A_{i^*} \mid \xi(i^*)$, there is $\eta(\beta) < \delta'$ satisfying $T_{i^*}(r_{f(\beta)}) = r_{f(\gamma_{\beta,\eta(\beta)})}$. This defines a splitting of $A_{i^*} \mid \xi(i^*)$ into $\delta' < \delta$ pieces. By δ-completeness of $J_\delta(\mathbb{B})$, there exists $\bar{\eta} < \delta'$ so that $\{r_{f(\beta)} \in A_{i^*} \mid \xi(i^*) \mid T_{i^*}(r_{f(\beta)}) = r_{f(\gamma_{\beta\bar{\eta}})}\} \notin J_\delta(\mathbb{B})$. □ of the claim.

Let $\bar{\eta} < \delta'$ be as in the claim.

For $\beta < \xi(i^*)$ denote by p_β the value $\| r_{f(\beta)} \in \underset{\sim}{A}_{i^*} \mid \xi(i^*) \wedge \underset{\sim}{T}_{i^*}(r_{f(\beta)}) = r_{f(\gamma_{\beta,\bar{\eta}})} \|$ if it is not equal to 0. Denote by Γ the set of $\beta < \xi(i^*)$ such that p_β is defined.

Claim 6. Let $\{q_\beta \mid \beta < \beta^*\} \subseteq \mathbb{B} \setminus \{0\}$ and $\{s_\beta \mid \beta < \beta^*\}$ be a $J_\delta(\mathbb{B})$-positive subset of $\mathscr{P}(\nu)$. Then some $q \in \mathbb{B}$ forces "$\{s_\beta \mid q_\beta \in \underset{\sim}{G}(\mathbb{B})\}$ is $J_\delta(\mathbb{B})$-positive".

Proof. Work in a generic extension. By the condition (4), $\mathbb{B}^V \setminus \{0\} = \bigcup_{\alpha<\mu} \pi''_\alpha(G(\mathbb{B}))$. So for every $\beta < \beta^*$ there exists $\alpha(\beta) < \mu$ such that $q_\beta \in \pi''_{\alpha(\beta)}(G(\mathbb{B}))$. This defines a partition of the set $\{s_\beta \mid \beta < \beta^*\}$ into μ pieces. Since $\{s_\beta \mid \beta < \beta^*\}$ remains $J_\delta(\mathbb{B})$-positive in $V[G(\mathbb{B})]$, there is $\bar{\alpha} < \mu$ so that $\{s_\beta \mid q_\beta \in \pi''_{\bar{\alpha}}(G(\mathbb{B}))\} \notin (J(\mathbb{B}))^{V[G(\mathbb{B})]}$. But $V[G(\mathbb{B})] = V[\pi''_{\bar{\alpha}}(G(\mathbb{B}))]$ and $\pi''_{\bar{\alpha}}(G(\mathbb{B}))$ is $\mathbb{B}$-generic over V. So some $q \in \pi''_{\bar{\alpha}}(G(\mathbb{B}))$ forces "$\{s_\beta \mid q_\beta \in \underset{\sim}{G}(\mathbb{B})\} \notin J_\delta(\mathbb{B})$".

□ of the claim.

Using Claim 6 for the sequences $\langle r_{f(\beta)} \mid \beta \in \Gamma \rangle$, $\langle p_\beta \mid \beta \in \Gamma \rangle$ find p forcing "$\{r_{f(\beta)} \mid p_\beta \in \underset{\sim}{G}(\mathbb{B})\} \in J_\delta(\mathbb{B})$". Suppose for simplicity that $p = 1_{\mathbb{B}}$ otherwise; just replace each p_β by $p \wedge p_\beta$ and work below p in $\mathbb{B}$.

Return now to a generic ultrapower M. It satisfies the following:

> "there exists a subset Γ' of Γ so that $\{r_{f(\beta)} \mid \beta \in \Gamma'\} \notin (J_\delta(\mathbb{B}))^M$,
> for every $\beta \in \Gamma'$, $r_{f(\beta)} \in A_{i^*} \mid \xi(i^*)$ and $T_{i^*}(r_{f(\beta)}) = r_{f(\gamma_{\beta,\bar{\eta}})}$".

Now, since Γ and the function $\beta \mapsto \gamma_{\beta,\bar{\eta}}$ $(\beta \in \Gamma)$ are both sets in V bounded below κ, there exists $i_0 < \kappa^+$ satisfying in V the same, i.e., for a subset Γ_0 of Γ so that $\{r_{f(\beta)} \mid \beta \in \Gamma_0\} \notin J_\delta(\mathbb{B})$, $r_{f(\beta)} \in A_{i_0} \mid \xi(i^*)$ and $T_{i_0}(r_{f(\beta)}) = r_{f(\gamma_{\beta,\bar{\eta}})}$. We shall use now Claim 6 once more and find p forcing "$\{r_{f(\beta)} \mid p_\beta \in \underset{\sim}{G}(\mathbb{B})$ and $\beta \in \check{\Gamma}_0\} \notin$

$J_\delta\mathbb{B})$". Let us form a generic ultrapower M with $p \in G(\mathbb{B})$. Denote by Γ_1 the set of all $\beta \in \Gamma_0$ s.t. $p_\beta \in G(\mathbb{B})$. Then $\{r_{f(\beta)} \mid \beta \in \Gamma_1\} \notin \mathbf{J}_\delta(\mathbb{B})$ in $V[G(\mathbb{B})]$. Consider, finally, the set $A = \{r_{f(\beta)} \mid \beta \in \Gamma_0,\ r_{f(\beta)} \in A_{i^*} \mid \xi(i^*)$ and $T_{i^*}(r_{f(\beta)}) = r_{f(\gamma_{\beta,\eta})}\}$. Since Γ_0 is in V and it is bounded below κ, $A \in M$. But $A \notin (J_\delta(\mathbb{B}))^M$, since, in $V[G(\mathbb{B})]$, $A \supseteq \{r_{f(\beta)} \mid \beta \in \Gamma_1\} \notin (J_\delta(\mathbb{B}))^{V[G(\mathbb{B})]}$. Then, $T_{j(i_0)}$ and T_{i^*} agree on a set A which is not in $J_\delta(\mathbb{B})$.

Returning to V we obtain $i_1 \neq i_2 < \kappa^+$ so that T_{i_1} and T_{i_2} agree on a $J_\delta(\mathbb{B})$-positive set. The next claim provides the desired contradiction.

Claim 7. For every $i_1 \neq i_2 < \kappa^+$, every $J_\delta(\mathbb{B})$-positive set X, $T_{i_1} \restriction X \neq T_{i_2} \restriction X$.

Proof. Suppose otherwise. Let N be an elementary submodel of V_χ for χ big enough so that $T_{i_1}, T_{i_2}, X, f, \mathbb{B}$, etc., are in N, $|N| = \delta'$, $N \supseteq \delta'$, where $\delta' < \delta$ is s.t. $\mathbb{B}$ satisfies $(\delta')^+$-c.c. Since, in V, the empty condition forces

$$\text{“}T_{i_1}(\underset{\sim}{r}(\mathbb{B})) = \underset{\sim}{r}_{j(f)(\alpha_{i_1})} \neq \underset{\sim}{r}_{j(f)(\alpha_{i_2})} = T_{i_2}(\underset{\sim}{r}(\mathbb{B}))\text{”}$$

the same holds in N.

Let $E = \bigcup \{E_\varphi \mid \varphi \in N, E_\varphi \in J_{(\delta')^+}\}$. Then $E \in J_{(\delta')^+} \subseteq J_\delta$. So $E \not\supseteq X$. Pick some $s \in X \setminus E$. Then s will be $\mathbb{B}$-generic over N so $N[s]$ satisfies "$T_{i_1}(s) \neq T_{i_2}(s)$". By absoluteness, then $T_{i_1}(s) \neq T_{i_2}(s)$ in V, which is impossible since $s \in X$. Contradiction. □ of the claim.

2.2. Remarks. (1) The absoluteness of $\mathbb{B}$ was used only between V, a generic extension and a generic ultrapower.

(2) The condition (4) can be replaced by the following weaker condition: there exists a complete Boolean algebra $\mathbb{B}^*$ which preserves $J_\delta(\mathbb{B})$-positivity and complete embeddings $\langle \pi_\alpha \mid \alpha < \mu \rangle$ $(\mu < \kappa)$, $\pi_\alpha : \mathbb{B} \to \mathbb{B}^*$ so that

$$1_{\mathbb{B}^*} = \| (\forall x \in \mathbb{B} \setminus \{0\})(\exists i < \mu)(\pi_i(x) \in G(\mathbb{B}^*)) \| .$$

(3) If $\mathbb{B}$ satisfies ξ-c.c. for some $\xi < \kappa$, then the condition (4) can be replaced by:

for every $\{q_\beta \mid \beta < \beta^*\} \subseteq \mathbb{B} \setminus \{0\}$ and every $J_\delta(\mathbb{B})$-positive set $\{s_\beta \mid \beta < \beta^*\}$ there exists $q \in \mathbb{B} \setminus \{0\}$ forcing "$\{s_\beta \mid q_\beta \in \underset{\sim}{G}(\mathbb{B})\}$ is $J_\delta(\mathbb{B})$-positive".

(4) It is possible to replace (3) by:

$$\| \text{there exists } \underset{\sim}{r} \subseteq \check{\nu} \text{ so that } V[\underset{\sim}{G}(\mathbb{B})] \cap \mathcal{P}(\nu) = V[\underset{\sim}{r}] \cap \mathcal{P}(\nu) \| > 0.$$

(5) It is impossible to remove the condition (2) since Col($\aleph_0, \kappa$), the Levy collapse of κ to $\aleph_0$, satisfies all the rest and by H. Woodin [W] it is possible to

have a normal ideal over κ (even the nonstationary ideal restricted to a stationary set over $\aleph_1$) with forcing isomorphic to $\mathrm{Col}(\aleph_0, \kappa)$.

(6) It is impossible to remove (3) and to replace (4) by "$\mathbb{B}$ is σ-centered". Just start with a measurable κ and add κ^+-Cohen reals.

Let us describe another relevant forcing construction. It will produce $\mathbb{B}$ which satisfies (1)–(3) and is σ-centered. In particular, it will be countably generated and the generic real for it is quite similar to the Cohen real. Let κ be a measurable $j: V \to N$, the corresponding elementary embedding. Add first κ Cohen reals $\langle r_\alpha \mid \alpha < \kappa \rangle$. Then j generically extends to $j_1 : V[\langle r_\alpha \mid \alpha < \kappa \rangle] \to N[\langle r_\alpha \mid \alpha < j(\kappa) \rangle]$. On the first stage we like to add codes $\langle a_\alpha \mid \kappa < \alpha < j(\kappa) \rangle$ so that each new real r_α $(\kappa < \alpha < j(\kappa))$ can be recovered using r_κ and a_α as follows: $n \in r_\alpha$ iff $|r_\kappa \cap a_\alpha \cap I_n| < \aleph_0$ where $\langle I_n \mid n < \omega \rangle$ is some fixed from the beginning partition of ω.

Then we shall add some sequence $\langle a^1_\alpha \mid \alpha < \kappa^+ \rangle$ for decoding elements of j^* $(\langle a_\alpha \mid \kappa < \alpha < j(\kappa) \rangle)$ via r_κ and so on, using finite support iteraction. The forcings for adding a_α's are just variants of the Solovay almost disjoint codding. For example, define the forcing for adding $a_{\kappa+1}$. Let

$$\mathscr{P} = \{\langle A, B \rangle \mid A \subseteq \omega, B \subset \kappa \text{ and } A, B \text{ are finite}\}.$$

For $\langle A_1, B_1 \rangle, \langle A_2, B_2 \rangle \in \mathscr{P}$ let $\langle A_1, B_1 \rangle \geqq \langle A_2, B_2 \rangle$ iff $A_1 \subseteq A_2$, $B_1 \subseteq B_2$ and for every $\alpha \in B_1$, $n \in r_{\alpha+1}$, $r_\alpha \cap I_n \cap A_1 = r_\alpha \cap I_n \cap A_2$.

Clearly, the Cohen real forcing satisfies the conditions of Theorem 2.1. Also forcings of bounded cardinalities satisfy the conditions. So the results of Section 1 for ideals over κ (except Theorem 1.2) can be derived from Theorem 2.1.

Let us turn now to the random real forcing. It was introduced by R. Solovay in his celebrated paper [So2]. We refer to this paper or to K. Kunen [K3] or to T. Jech [J] for the definition and the basic properties of this forcing.

D. Fremlin [Fr1] had asked if the forcing with a κ-complete ideal I over κ can be isomorphic to the random real forcing. The following gives the negative answer:

Theorem 2.3. *Let I be a κ-complete ideal over κ. Then the forcing with I cannot be isomorphic to the random real forcing.*

Proof. Let us check the conditions of Theorem 2.1 for the random real forcing. The only nontrivial ones are (4) and (5). Let $\mathbb{B}$ denote the random real forcing, i.e., $\mathbb{B}$ = Borel subsets of the reals/measure zero sets. The next two lemmas show that the conditions (4) and (5) are satisfied by $\mathbb{B}$.

Lemma 2.4. *There are automorphisms* $\langle \pi_n \mid n < \omega \rangle$ *of* $\mathbb{B}$ so that

$$\left\| \check{\mathbb{B}} \setminus \{0\} = \bigcup_{n<\omega} \check{\pi}''_n(G(\underset{\sim}{B})) \right\|^{\mathbb{B}} = 1.$$

Proof. For every two finite sequences of open intervals with rational endpoints $\{(a_i, a'_i) \mid i < n\}$, $\{(b_i, b'_i) \mid i < n\}$ so that $a'_i < a_{i+1}$, $a'_i - a_i = a'_{i+1} - a_{i+1}$ and $b_i < b_{i+1}$, $b'_i - b_i = b'_{i+1} - b_{i+1}$, pick the partly linear function f which moves (a_i, a'_i) onto (b_i, b'_i) and $[a'_i, a_{i+1}]$ onto $[b'_i, b_{i+1}]$ (see Fig. 1).

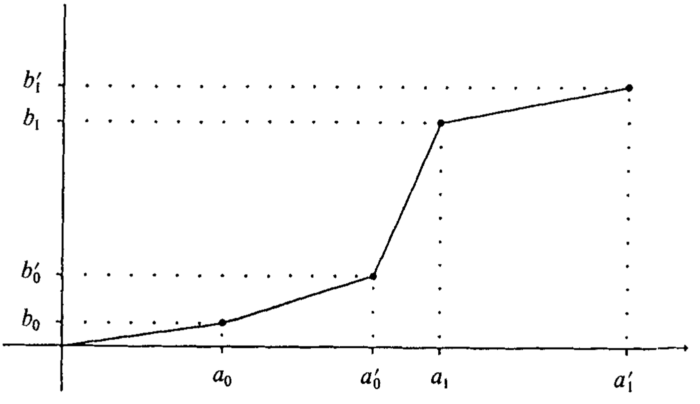

Fig. 1.

Note that for every Lebesgue measurable set $A \subseteq \bigcup_{i<n} (a_i, a'_i)$ the measure of the image of A is equal to $[(b'_0 - b_0)/(a'_0 - a_0)] \cdot \mu(A)$. Clearly f defines an automorphism of $\mathbb{B}$. Let $\{\pi_n \mid n < \omega\}$ be the set of all automorphisms which are defined in this fashion. We claim that the sequence $\langle \pi_n \mid n < \omega \rangle$ is as desired. Suppose otherwise. Then there are Borel positive sets A_0 and A_1 so that

$$[A_0] \Vdash \text{"for every } n < \omega\ [A_1] \notin \pi''_n(G(\mathbb{B}))\text{"}.$$

Shrinking A_0 or A_1 if necessary we can asume that $\mu(A_0) = \mu(A_1)$.

Let $\delta = \mu(A_0)$ and $\mathscr{E} = \delta/4$. Pick $\mathscr{U}_0$ and $\mathscr{U}_1$ to be finite unions of open intervals with rational endpoints so that $\mu(A_i \triangle \mathscr{U}_i) < \mathscr{E}$ $(i \in 2)$. Clearly, the intervals giving each of the $\mathscr{U}_i$'s can be cut into smaller intervals $\{(a_{ik}, a'_{ik}) \mid k < n\}$ so that $a'_{ik} < a_{ik+1}$ and $a'_{ik} - a_{ik} = a'_{ik+1} - a_{ik+1}$.

Let π be the automorphism generated by these intervals. Then

$$\mu(\pi''(A_0 \cap \mathscr{U}_0)) = \frac{\mu(\mathscr{U}_1)}{\mu(\mathscr{U}_0)} \cdot \mu(A_0 \cap \mathscr{U}_0) \geqq \frac{\delta - \mathscr{E}}{\delta + \mathscr{E}} \cdot (\delta - \mathscr{E}) = \frac{(\frac{3}{4}\delta)^2}{\frac{5}{4}\delta} = \frac{9\delta}{20} > \frac{\delta}{4} = \mathscr{E}.$$

Hence $\mu((\pi''(A_0 \cap \mathcal{U}_0)) \cap (A_1 \cap \mathcal{U}_1)) > 0$. Denote $(\pi''(A_0 \cap \mathcal{U}_0)) \cap (A_1 \cap \mathcal{U}_1)$ by B_1 and $\pi^{-1}{}''(B_1)$ by B_0. Then $B_0 \subseteq A_0$, $B_1 \subseteq A_1$ are both Borel of positive measure and $\pi(B_0) = B_1$, which is impossible since $[B_0] \Vdash$ there is no $n < \omega$ such that $[B_1] \in \pi''_n(G(\mathbb{B}))$. Contradiction. □

So condition (4) is satisfied with $\mu = \aleph_0$. Now in (5) we can take $\delta = \aleph_1$. Then $J_{\aleph_1}(\mathbb{B})$ became the usual ideal of the null sets. So the condition (5) claims that a nonnull set remains such after adding a random real.

This property of the random real forcing was shown by K. Kunen [K3]. For the reader's convenience we present here a proof of a little stronger property.

Lemma 2.5. *For every regular cardinal $\lambda > \aleph_0$ the random real forcing $\mathbb{B}$ preserves $J_\lambda(\mathbb{B})$-positive sets.*

Proof. First notice the following application of the Fubini Theorem: a pair (r, s) is random generic for the random forcing on the plane iff r is random over V and s is random over $V[r]$ iff s is random over V and r is random over $V[s]$.

Suppose now that $A \notin J_\lambda(\mathbb{B})$ and, in a generic extension by $\mathbb{B}$, A is in $(J_\lambda(\mathbb{B}))^{V[G(\mathbb{B})]}$. So, in $V[G(\mathbb{B})]$, there is $\xi < \lambda$ and a sentence $\varphi \in L_{\xi^+}$ such that $A \subseteq E_\varphi$. Suppose for simplicity that $1_{\mathbb{B}}$ already forces this as well as the value of ξ. Pick an elementary submodel N of V_χ for χ big enough so that $|N| = \xi$, $\xi \subseteq N$, the length of φ is contained in N and $A \in N$. Set

$$C = \bigcup \{A_c \mid c \in N \text{ is a cod of a Borel null set}\}.$$

Since $|N| = \xi$, $C \in J_\lambda(\mathbb{B})$. So $C \not\supseteq A$. Pick $r \in A \setminus C$. Then r is a random real over N. Force a random real s over V. Then s is random over $N[r]$. Hence $\langle r, s \rangle$ is generic for the plane random forcing over N. So r is random over $N[s]$. Since $N[s] \prec V_\chi[s]$, there is $\varphi \in L_{\xi^+}$ in $N[s]$ so that $E_\varphi \supseteq A$ and $E_\varphi \in (J_\lambda(\mathbb{B}))^{V[s]}$. Hence, in $V[s]$, $\| \varphi(\text{random}) \|^{\mathbb{B}} = 0$. Then the same is true in $N[s]$. But r is random over $N[s]$. So $N[s, r] \models \neg\varphi(r)$. Since φ is an L_{ξ^+}-sentence and N contains its length, it should be false also in $V[s]$. So $V[s] \models \neg \varphi(r)$. It means that $r \notin E_\varphi$, which contradicts $A \subseteq E_\varphi$. Contradiction. □

As in the Cohen real case, we shall show at least κ^+ random reals are needed in order to be isomorphic to the forcing with a κ-complete ideal. For a measure μ over a σ-algebra X, the density of μ, $d(\mu)$ is the least cardinal λ such that for some $Y \subseteq X$ of cardinality λ for every $Z \in X$ for every $\mathscr{E} > 0$ there is $S \in Y$ such that $\mu(Z \triangle S) < \mathscr{E}$.

Let $\mathrm{add}(\mu) = \min\{\lambda \mid \text{there is } Y \subseteq X,\ |Y| = \lambda,\ \mu(Y') = 0 \text{ for every } Y' \in Y$ and $\mu(\bigcup Y) \neq 0\}$.

A measure μ over a set X is called a probability measure if $\mu(X) = 1$. It is total if it is defined on all the subsets of X. μ is nowhere prime, if every set $Y \subseteq X$ s.t. $\mu(Y) \neq 0$ can be split into two nonzero sets.

Theorem 2.6. *Suppose that μ is a σ-additive, total, nowhere prime probability measure over a set X. Then $d(\mu) > \mathrm{add}(\mu)$.*

Remark. In the Solovay model with $2^{\aleph_0}$ real valued measurable [So2], $d(\mu) = (\mathrm{add}(\mu))^+$.

Proof.† Suppose otherwise. Let μ be a measure over X witnessing this. Let I be the ideal of μ-measure zero sets. Then I is $\aleph_1$-saturated nowhere prime ideal over X. Let $\kappa = \mathrm{add}(I)$. As in the proof of Theorem 1.1 define the normal ideal J over κ. The Boolean algebras $B(I) = P(X)/I$ and $B(J) = \mathcal{P}(\kappa)/J$ are complete and $B(J)$ is isomorphic to a complete subalgebra of $B(I)$. Using this, it is easy to define a total κ-complete probability measure ν over κ such that $J = \{A \subseteq \kappa \mid \nu(A) = 0\}$. Since I is $\aleph_1$-saturated and nowhere prime, $2^{\aleph_0} \geqq \kappa$. So J and ν are nowhere prime. It is not hard to see that $d(\nu) \leqq d(\mu)$. Restricting I to a positive set, if necessary, we can assume that for every $a, b \in B(J)$, the smallest cardinalities of a dense subset of $\{c \in B(J) \mid c \geqq a\}$ and $\{c \in B(J) \mid c \leqq b\}$ are the same. Now, by the D. Maharam theorem, see D. Fremlin [F2] A6Fa, then the forcing with J is isomorphic to the measure algebra $\{0, 1\}^{d(\nu)}$, i.e., to the forcing of $d(\nu)$ random reals. Let $\lambda = d(\nu)$. But the assumption $\lambda \leqq \kappa$.

Now we proceed as in the proof of Theorem 1.2. Pick a sequence $\langle x_\alpha \mid \alpha < \kappa \rangle$ consisting of κ different reals. A generic elementary embedding j (obtained by forcing with J) moves $\langle x_\alpha \mid \alpha < \kappa \rangle$ to $\langle x_\alpha \mid \alpha < j(\kappa) \rangle$ and elements $\langle x_\alpha \mid \kappa \leqq \alpha < j(\kappa) \rangle$ are all new distinct reals. Since $j(\kappa) > \kappa^+$ and the forcing with measure algebra $\{0, 1\}^\lambda$ satisfies c.c.c., there exists $C \subseteq \kappa^+$, $C \in V$, $|C| = \kappa^+$ so that all x_α's with $\alpha \in C$ are added by the forcing with measure algebra $\{0, 1\}^\delta$ for some $\delta < \kappa$.

The rest of the proof is similar to Theorem 1.2, only Theorem 2.1 should be used instead of Theorem 1.1 there. Some of the arguments are even simpler, since we deal here only with the random real forcing. Note that the condition

† We are grateful to A. Kamburrelis for pointing out a gap in a "short cut" used in an earlier version of the proof.

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(4) of 2.1 is satisfied by $B(J)$ with $\mu = \aleph_0$. Just use Lemma 2.4 and the fact that $2^{\aleph_0} \geqq \kappa \geqq \lambda$. □

3. Hechler and Sacks reals

Let's state some other conditions on forcing notions which prevent them from being isomorphic to the forcing with an ideal.

Let $\mathbb{B}$ be a complete Boolean algebra satisfying c.c.c. Suppose that a generic object for $\mathbb{B}$ can be reconstructed from a real. We consider the ideal $J_{\aleph_1}(\mathbb{B})$ of $\mathbb{B}$-null sets defined in the beginning of Section 2. Each sentence φ of $L_{\aleph_1}$ can be viewed as a real. In the present context let us confuse them. Thus for real x corresponding to a sentence φ we shall denote by E_x the set $E_\varphi = \{r \mid r \text{ is a real and } V \vDash \varphi(r)\}$. For $n < \omega$ define $\mathbb{B}^n$ by induction as follows: $\mathbb{B}^1 = \mathbb{B}$, $\mathbb{B}^{n+1} = \mathbb{B}^n * \mathbb{B}^{V^{\mathbb{B}^n}}$.

THEOREM 3.1. *Let $\kappa > \aleph_0$ be a regular cardinal. Suppose that $\mathbb{B}$ is a complete Boolean algebra satisfying the following conditions*:

(1) $\mathbb{B}$ *has absolute definition*;

(2) $\mathbb{B}$ *satisfies c.c.c.*;

(3) $\mathbb{B}$ *is generated by a name of a real*;

(4) *there exists a set $\langle \sigma_\alpha \mid \alpha < \alpha^* \rangle$, $\alpha^* < \kappa$ so that*:

(i) *for every $\alpha < \alpha^*$ there is n_α, $0 < n_\alpha < \omega$ such that σ_α is a $\mathbb{B}^{n_\alpha}$-name of a real so that*

$$\| E_{\sigma_\alpha(\underset{\sim}{r}_1,\dots,\underset{\sim}{r}_{n_\alpha})} \in J_{\aleph_1}(\mathbb{B}) \|^{\mathbb{B}^{n_\alpha}} = 1,$$

(ii) *for every $\mathbb{B}$-name τ of a real there exists $\alpha < \alpha^*$ such that*

$$\| \tau(\underset{\sim}{r}_{n_\alpha}) \in E_{\sigma_\alpha(\underset{\sim}{r}_1,\dots,\underset{\sim}{r}_{n_\alpha})} \|^{\mathbb{B}^{n_\alpha}} = 1.$$

Then there is no κ-complete ideal I over a set X so that the forcings with $\mathbb{B}$ and the I are isomorphic.

REMARK. For $\mu < \kappa$ we can weaken the condition (2) to μ^+-c.c. and use $\varphi \in L_{\mu^+}$; the proof would not change.

PROOF. Suppose otherwise. Let I, X, κ, $\mathbb{B}$ and $\langle \sigma_\alpha \mid \alpha < \alpha^* \rangle$ be witnessing this. Shrinking X, if necessary, let us assume that every subset of X of cardinality less than $|X|$ belongs to I. Force with $\mathbb{B}$. Let $r(\mathbb{B})$ be the real generating the generic object. Using it, find the generic elementary embedding $j: V \to M$. Then, by $\aleph_1$-saturateness of I, $r(\mathbb{B}) \in M$. Pick in V a sequence $\langle r_x \mid x \in X \rangle$ representing $r(\mathbb{B})$. Since $r(\mathbb{B})$ is generic over V, it will also be generic over $L[\langle\sigma_\alpha \mid \alpha<\alpha^*\rangle, \langle r_x \mid x\in X\rangle]$. Using this, pick a subsequence $\langle r_n \mid n<\omega\rangle$ of $\langle r_x \mid x\in X\rangle$ so that for every n, r_n is $\mathbb{B}$-generic over $L[\langle\sigma_\alpha \mid \alpha<\alpha^*\rangle, \langle r_m \mid m<n\rangle]$. Since in V, for every $x\in X$ there is $y\in X$ so that r_y is $\mathbb{B}$-generic over $L[\langle\sigma_\alpha \mid \alpha<\alpha^*\rangle, \langle r_n \mid n<\omega\rangle, r_x]$, in M there is $y\in j(X)$ such that r_y is $\mathbb{B}$-generic over $L[\langle\alpha_\alpha \mid \alpha<\alpha^*\rangle, \langle r_n \mid n<\omega\rangle, r(\mathbb{B})]$. Since $V[G]$ is generated by $r(\mathbb{B})$, there exists a $\mathbb{B}$-name τ of r_y. Using 4(ii), find $\alpha<\alpha^*$ so that

$$\| \tau(\underset{\sim}{r}_{n_\alpha})\in E_{\sigma_\alpha(\underset{\sim}{r}_1,\dots,\underset{\sim}{r}_{n_\alpha})} \|^{\mathbb{B}^{n_\alpha}} = 1.$$

Then $\tau(r(\mathbb{B}))\in E_{\sigma_\alpha(r_1,\dots,r_{n_\alpha-1},r(\mathbb{B}))}$. But, on the other hand, $\tau(r(\mathbb{B}))=r_y$ which is $\mathbb{B}$-generic over $L[\langle\sigma_\beta \mid \beta<\alpha^*\rangle, \langle r_n \mid n<\omega\rangle, r(\mathbb{B})]$, and hence cannot belong to a set in $J_{\aleph_1}(\mathbb{B})$ with a code in $L[\langle\sigma_\beta \mid \beta<\alpha^*\rangle, \langle r_n \mid n<\omega\rangle, r(\mathbb{B})]$. Contradiction. $\square$ of 3.1.

Condition (5) of Theorem 2.1 and condition (4) of Theorem 3.1 look quite complementary. Actually, the first claims the symmetry of forcing with $\mathbb{B} * \mathbb{B}$ and the second an asymmetry in a strong form. We do not know if there exists a forcing notion with a simple definition which falls in between.

Let us give now an example of a forcing notion satisfying the conditions of Theorem 3.1.

For functions $f, g\in {}^\omega\omega$ let us say that f dominates g and write $f^*\geqq g$ if, for all but finitely many n's, $f(n)\geqq g(n)$. The Hechler real forcing is defined as follows. The set of conditions P consists of ordered pairs $\langle s,f\rangle$, where $f\in {}^\omega\omega$ and $s\in {}^n\omega$, for some $n<\omega$. A condition $\langle s,f\rangle \leqq \langle t,g\rangle$ ($\langle t,g\rangle$ is stronger than $\langle s,f\rangle$) if $t \restriction \operatorname{dom} s = s$, for every $m\geqq \operatorname{length}(t)$, $g(m)\geqq f(m)$ and for every $m\in \operatorname{dom} t\setminus \operatorname{dom} s$, $t(m)\geqq f(m)$. This forcing was introduced by S. Hechler [H]. It adds a real which dominates all the old reals.

It is clear that the conditions (1), (2) and (3) of Theorem 3.1 are satisfied by the Hechler real forcing. The following shows that also condition (4) is satisfied.

Lemma 3.2. *Let P be the Hechler real forcing. Then there is $n<\omega$ and P^n-names of reals $\langle \sigma_m \mid m<\omega\rangle$ so that for every P-name of a real τ there is $m<\omega$ such that*

$$\varnothing \Vdash_{P_n} \tau(\underset{\sim}{r}_n) \leqq^* \sigma_m(\underset{\sim}{r}_1,\dots,\underset{\sim}{r}_n).$$

Proof. Force with P. Let $f^*\in {}^\omega\omega$ be a function dominating all the functions of V. Let us change every dense open set $\tau\in V$ to a predense set $\tau^*\in V[f^*]$ consisting of conditions of the form $\langle t, f^*\rangle$. We proceed as follows.

For every $\langle t, g\rangle \in \tau$ find $k < \omega$ so that $k \geqq$ length t and $f^*(m) > g(m)$ for every $m \geqq k$. Take all possible extensions of $\langle t, g\rangle$ to conditions of the form $\langle t', g\rangle$ with t' of length k. Change g by f^* in each of them. Let τ^* be a set in $V[f^*]$ consisting of such conditions.

CLAIM 1. τ^* is a predense set in the Hechler forcing in $V[f^*]$.

PROOF. Suppose otherwise. Let $\langle t, h\rangle \in P$ in $V[f^*]$ be incompatible with every element of τ^*. Let r be a Cohen real over $V[f^*]$.

Let $h^* \in {}^{\omega}\omega$ be defined as follows: $h^*(n) = h(n) + f^*(n) + 1 + r(n)$. Then, by [Tr], h^* is Hechler over V. Also $h^{**} = t \cup (h^* \setminus h^* \restriction |t|)$ is Hechler over V. Clearly $\langle t, h^{**}\rangle > \langle t, h\rangle$. Pick $\langle s, g\rangle \in \tau$ which belongs to the V-generic set defined by h^{**}. Then $h^{**} \mid \text{length}(s) = s$. By the definition of τ^* for some $k \geqq \text{length}(s)$ for some s' of length k such that $s' \mid k = s$ and $s' \mid (k \setminus \text{length}(s)) = h^{**} \mid (k \setminus \text{length}(s))$, $\langle s', f^*\rangle \in \tau^*$. But $\langle s', f^*\rangle$ and $\langle t, h\rangle$ are compatible, since $s'(m) > h(m)$ for every $m \in \text{dom}\, s' \setminus \text{dom}\, t$. Contradiction. □ of the claim.

Consider now in $V[f^*]$ the following set of Hechler's conditions:

$$F^* = \{\langle t, f^*\rangle \in P \mid t(\text{length}(t) - 1) < f^*(\text{length}(t) - 1) \text{ or } t = \varnothing\}.$$

It is easy to see that F^* is a maximal antichain. Also every τ^* above $\langle t, f^*\rangle \in F^*$ is actually a predense set in the Cohen real forcing.

Let $\langle t_k \mid k < \omega\rangle$ be an enumeration of all t's such that $\langle t, f^*\rangle \in F^*$. Denote by $T_{\tau,k}$, for a dense open set $\tau \in V$, the following set:

$$\{t \in {}^{\omega>}\omega \mid \langle t, f^*\rangle \geqq \langle t_k, f^*\rangle \text{ and for some } t' \leqq t \text{ such that } \langle t', f^*\rangle \in \tau^*, \langle t, f^*\rangle \text{ is compatible with } \langle t_k, f^*\rangle\}$$

Then every $T_{\tau,k}$ is a dense open set in the Cohen real forcing above t_k with finite sequences bigger than f^*. More precisely, we should first remove f^* and then replace ${}^{\omega>}\omega$ by ${}^{\omega>}2$. Force with P over $V[f^*]$. Let f_1^* be the Hechler real. By J. Truss [Tr], there is a Cohen real over $V[f^*]$, real $x \in V[f^*, f_1^*]$. Using x, produce $\aleph_0$ Cohen reals $\langle x_k \mid k < \omega\rangle$ over $V[f^*]$ so that x_k extends t_k and x_k is bigger than f^* above t_k. Splitting each of the x_k's define a sequence $\langle x_{k,m} \mid m < \omega\rangle$ of Cohen reals over $V[f^*]$ so that $x_{k,m}$ extends t_k for every $m < \omega$ and the set $\{x_{k,m} \mid m \geqq m_0\}$ is dense in $\mathscr{U}_{t_k}$ for every $m_0 < \omega$, where $\mathscr{U}_{t_k}$ is the basic clopen set generated by t_k, i.e., all the branches extending t_k and bigger than f^*.

Then for every $m, k < \omega$ and a dense open $\tau \in V$ there exists $h_{\tau,k}(m) < \omega$ so

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that $x_{k,m} \mid h_{\tau,k}(m) \in T_{\tau,k}$, since $x_{k,m}$ is Cohen over $V[f^*]$ containing all dense open $T_{\tau,k}$'s. Force one more Hechler real f_2^*. Then for every $k < \omega$, dense open $\tau \in V$, there is $g_\tau(k) < \omega$ so that $f_2^*(i) > h_{\tau,k}(i)$ for every $i \geqq g_\tau(k)$. This implies $x_{k,i} \mid f_2^*(i) \in T_{\tau,k}$ for every $k < \omega$ and $i \geqq g_\tau(k)$. Forcing an additional Hechler real f_3^* we can dominate all g_τ's. So, for every τ for all but finitely many k's, $f_3^*(k) \geqq g_\tau(k)$. Let $\langle g_n \mid n < \omega \rangle$ be the functions in ${}^\omega\omega$ which are obtained from f_3^* by all possible changes of initial segments of f_3^*. Then for every τ there exists $n < \omega$ so that $g_n > g_\tau$.

Let us consider now the following sets:

$$X_{k,n} = \{x_{ki} \mid f_2^*(i) \mid i \geqq g_n(k)\}$$

where $k, n > \omega$.

It follows that for every $\tau \in V$ dense open in the Hechler forcing there exists $n < \omega$ so that $X_{k,n}$ is a dense subset of $T_{\tau,k}$ for every $k < \omega$. Note that $\langle X_{k,n} \mid k, n < \omega \rangle \in V[f^*, f_1^*, f_2^*, f_3^*]$. Returning to V and picking a P^4-name σ_n for every $\bigcup_{k<\omega} X_{k,n}$ $(n < \omega)$, we obtain $\aleph_0$ P^4-names of maximal antichains in the Hechler forcing which are forced to refine every maximal antichian τ in the Hechler forcing in V.

Suppose $\langle \tau_n \mid n < \omega \rangle \in V$ is a sequence of open sets. For every $n < \omega$ there is $k_n < \omega$ so that $f_2^*(i) \geqq h_{\tau_n k}(i)$ for every $k \geqq k_n$ and every $i \geqq f_3^*(k)$. For $k < k_n$ let $i_n(k)$ be so that for every $i \geqq i_n(k)$, $f_2^*(i) \geqq h_{\tau_n k}(i)$. Set $i_n = \max\{i_n(k) \mid k < k_n\}$. Force a Hechler real f_4^*. Then starting with some $n_0 < \omega$, $f_4^*(n) > i_n, k_n$. So for every $n \geqq n_0$, for every k and every $i \geqq f_3^*(k + f_4^*(n))$, $f_2^*(i) \geqq h_{\tau_n,k}(k)$. This produces a dense set which refines all $\langle \tau_n \mid n \geqq n_0 \rangle$. By the previous argument it is possible to refine finitely many dense sets $\langle \tau_n \mid n < n_0 \rangle$. So we obtain countably many dense sets so that every sequence $\langle \tau_n \mid n < \omega \rangle \in V$ of dense sets in V is refined by one of them.

The arguments above apply now easily in order to dominate the values of every Hechler name $\tau \in V$ of a real by a fixed countable set of P^{10} names of Hechler reals. We leave the checking of details to the reader. □ Lemma 3.2.

The following is now immediate.

Theorem 3.3. *Suppose that I is a σ-complete ideal over a set X. Then the forcing with I cannot be isomorphic to the Hechler real forcing.*

Let us now turn to other forcings which for different reasons cannot be isomorphic to the forcing with an ideal.

The Sacks forcing is a forcing P with pairs $\langle t, T \rangle$ where T is a perfect tree in

${}^{\omega}2$ with a trunk t, i.e., $T \subseteq {}^{\omega>}2$ such that for every $\eta \in T$, η extends t and there are $\nu_1 \neq \nu_2 \in T$, $\nu_2, \nu_{1T} > \eta$ and ν_1, ν_2 are on the same level. For $\langle t_i, T_i \rangle \in P$ $(i \in 2)$, $\langle t_0, T_0 \rangle \leqq \langle t_1, T_1 \rangle$ iff $t_1 \in T_0$ and $T_1 \subseteq T_0$.

This forcing does not satisfy c.c.c. And this will be crucial for further considerations.

Lemma 3.4. *There exists a sequence* $\langle \langle \varnothing, T_\alpha \rangle \mid \alpha < 2^\omega \rangle$ *of Sacks conditions so that for every* $\alpha \neq \beta < 2^\omega$, T_α *and* T_β *do not have a common branch. Actually, every Sacks condition can be extended to* $2^{\aleph_0}$*-conditions with this property.*

Proof. Let $\langle f_\alpha \mid \alpha < 2^\omega \rangle$ be 2^ω different functions from ω into 2. For $\alpha < 2^\omega$ we define T_α by induction as follows: if $\eta \in T_\alpha$ is defined and it lies on the level $2n$, then let $\eta^\frown\langle 0 \rangle$ and $\eta^\frown\langle 1 \rangle$ be in T_α. If $\eta \in T_\alpha$ is on the level $2n+1$, then let only $\eta^\frown f_\alpha(\nu)$ be in T_α. It is easy to see that such defined T_α have the desired property. □ Lemma 3.4.

If the forcing with an ideal does not satisfy c.c.c., then, in general, the ideal may fail to be precipitous or it can be precipitous but the generic ultrapower may not contain the generic real generating the forcing. In our case, this does not happen and the reason is the properness of the Sacks forcing.

An ideal I is called ω_1-preserving if the forcing with I preserve ω_1. Motivated by J. Baumgartner and A. Taylor [B–T] such ideals were considered in [G–Sh]. It is unknown if ω_1-preserving implies precipitousness. But if an ideal is ω_1-preserving and precipitous, then a generic extension of the world by the forcing with the ideal has the same reals as a generic ultrapower.

Proposition 3.5. *Suppose that* I *is a* σ*-complete ideal over a set* X *so that the forcing with* I *is proper. Then* I *is precipitous and* ω_1 *preserving.*

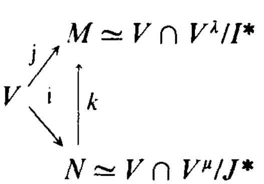
Proof. Let $\langle W_n \mid n < \omega \rangle$ be a sequence consisting of maximal antichains for I. Let $W_n = \langle A_{n\alpha} \mid \alpha < \alpha(n) \rangle$ for $n < \omega$. Pick a countable submodel N of $\langle V_\chi, < \rangle$ for χ big enough with well ordering $<$ so that I, $\langle W_n \mid n < \omega \rangle \in N$ and $N \cap \omega_1$ is an ordinal containing N. Using the properness of the forcing, pick a (N, I^+)-generic condition A. Then, for every $n < \omega$, $A \cap A_{n\alpha} \in I$ for every $\alpha \notin N$, hence the restriction of W_n's to A is countable. It implies precipitousness and ω_1-preserving of I. □

Now, returning to the Sacks forcing we can show the following:

THEOREM 3.6. *Suppose that I is a σ complete ideal over a set X. Then the forcing with I cannot be isomorphic to the Sacks real forcing.*

PROOF. Suppose otherwise. Let I be a σ-complete ideal over a set X witnessing this. Let $\lambda = |X|$. Shrinking X, if necessary, we can assume that every subset of X of cardinality less than λ beongs to I. By Proposition 3.5, I is precipitous and ω_1-preserving. Suppose that the empty condition forces that a regular cardinal κ is the critical point of generic embeddings. Since I is nowhere prime $2^{\aleph_0} \geqq \kappa$. But actually $2^{\aleph_0}$ should be bigger than κ since, otherwise, in a generic ultrapower $2^{\aleph_0} = (j((2^{\aleph_0})^V)) = j(\kappa) \geqq \kappa^+$. This means that one Sacks real adds at least κ^+ new reals. It is impossible, since the Sacks forcing is a proper forcing and in a proper forcing only countably many things are needed in order to determine a real. So $2^{\aleph_0} > \kappa$. If $\lambda = \kappa$, then we can proceed as follows. Force a Sacks real r. Let $j: V \to M$ be the generic elementary embedding defined by it. Then $r \in M$. Pick a sequence $\langle r_\alpha \mid \alpha < \kappa \rangle \in V$ representing r, i.e., $j(\langle r_\alpha \mid \alpha < \kappa \rangle)([\mathrm{id}]) = r$. Suppose that the empty condition forces this. Let $\langle \langle \emptyset, T_\beta \rangle \mid \beta < 2^{\aleph_0} \rangle$ be as in Lemma 3.4. Then for every $\beta < 2^{\aleph_0}$ we can find a Sacks real r which is a branch in T_β. Just pick a generic set for the Sacks forcing with $\langle \emptyset, T_\beta \rangle$ belonging to it. Then, in the corresponding generic ultrapower the same is true. Hence, back in V, there are many α's below κ so that r_α is a branch through T_β. But since $2^{\aleph_0} > \kappa$, there is $\alpha_0 < \kappa$ so that r_{α_0} is a branch through $2^{\aleph_0}$ T_β's, which is impossible.

Let us now remove the assumption $\lambda = \kappa$ that we made above. Pick in V an enumeration $\langle r_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ of the reals. Let r be a Sacks real. Then in the corresponding ultrapower r appears in $j(\langle r_\alpha \mid \alpha < 2^{\aleph_0} \rangle) = \langle r_\alpha \mid \alpha < (2^{\aleph_0})^M \rangle$. By properness $(2^{\aleph_0})^M < ((2^{\aleph_0})^+)^V$. Let $r = r_\delta$ for some $\delta < (2^{\aleph_0})^M$. Let μ be the least ordinal so that $j(\mu) > \delta$. Suppose for simplicity that the empty condition decides the values of μ, and δ. Define now an ideal J over μ in V. Let $Y \subseteq \mu$ be in J if $\emptyset \Vdash_{\text{Sacks}} \delta \notin j(Y)$. It is not hard to see that J is a κ-complete uniform (i.e., all subsets of cardinality less than μ are in J) ideal over μ. Such an ideal need not be precipitous. But generically extending I to a prime ideal, we extend also J to a prime ideal over μ with well founded ultrapower. After forcing Sacks real the following diagram is commutative:



where I^* is the prime ideal extending I, J^* is the prime ideal extending J defined as follows:

$$Y \in J^* \quad \text{iff}\ \delta \in j(Y) \quad \text{and} \quad k([f]_{J^*}) = j(f)(\delta).$$

Then, $k(r_{[\mathrm{id}]}) = r_\delta$ and since $k \restriction \kappa = \mathrm{id}$, $r_{[\mathrm{id}]} = r_\delta$. So $r_\delta \in N$. Hence the sequence $\langle r_\alpha \mid \alpha < \mu \rangle$ represents the generating Sacks real r_δ in N. Since J is uniform ideal, $i(\mu) \geqq (\mu^+)^V$.

So $2^{\aleph_0}$ of V should be bigger than μ. But now we can repeat the argument given for the case $\kappa = \lambda$. Just replace there κ by μ. □

The same arguments work for the Laver, Mathias, rational perfect forcing of Miller and actually for every Borel forcing notion satisfying the conclusion of Lemma 3.4.

4. Some related results

Grinblat [Gr] proved the following generalization of 1.8 to real valued measures:

THEOREM 4.1. *Suppose that $\langle \mu_n \mid n < \omega \rangle$ is a sequence of σ-additive real valued nowhere total measures over the set of reals. Then there exists a set of reals which is nonmeasurable in each of the measures.*

Let us show that in contrast to 0–1 measures (see 1.7, 1.8) it is impossible to generalize Grinblat's theorem to higher cardinals. Namely, *it is possible to have $\aleph_1$ $\aleph_2$-additive real valued nowhere total measures so that every set of reals is measurable in one of them*. This holds in the Solovay model [So1] with $2^{\aleph_0}$ real valued measurable. Let us briefly describe Solovay's construction. Let κ be a measurable cardinal $j : V \to M \simeq V^\kappa / U$ be an elementary embedding, where U is a normal ultrafilter over κ. Force with the measure algebra $\{0, 1\}^\kappa$. Let G be a generic object. Then j can be extended in a generic fashion to $j^* : V[G] \to M[G^*]$, where G^* is generic for $\{0, 1\}^{j(\kappa)}$. So, in order to extend j we need to force with the measure algebra $\mathbb{B} = \{0, 1\}^{j(\kappa) \setminus \kappa}$. Let $\nu : \mathbb{B} \to [0, 1]$ denotes its measure. Define a real valued total measure μ over $2^{\aleph_0}$ in $V[G]$ as follows: for $A \subseteq 2^{\aleph_0} = \kappa$

$$\mu(A) = \nu(\| \kappa \in j(\underset{\sim}{A})[G] \|_{\mathbb{B}}).$$

Let us define now a sequence $\langle \mu_i \mid i < \aleph_1 \rangle$ of κ-additive nowhere total real valued measures so that

(a) μ_i extends μ_j whenever $i \geqq j$,

(b) μ extends every μ_i and, for every A, $\mu(A)=0$ implies $\mu_i(A)=0$ $(i<\omega_1)$,

(c) every $A\subseteq 2^{\aleph_0}$ is μ_i-measurable for some $i<\omega_1$.

Pick S to be a subset of $j(\kappa)\setminus\kappa$ of order type ω_1. Let $\langle\alpha_i \mid i<\omega_1\rangle$ be its enumeration. Denote $S_i=\{\alpha_j \mid j<i\}$. Let $\mathbb{B}_i$ be the measure algebra $\{0,1\}^{(j(\kappa)\setminus S)\cup S_i}$ and ν_i its measure. Define measures $\langle\mu_i \mid i<\omega_1\rangle$ as follows: $A\subseteq 2^{\aleph_0}$ is μ_i-measurable iff $\|\kappa\in j(\underset{\sim}{A})[G]\|_{\mathbb{B}}\in\mathbb{B}_i$, or $\mu(A)=0$ and in this case set $\mu_i(A)=\nu(\|\kappa\in j(\underset{\sim}{A})[G]\|_{\mathbb{B}})$. The sequence $\langle\mu_i \mid i<\omega_1\rangle$ is as desired. Note that (c) holds since every $b\in\mathbb{B}$ depends only on $\aleph_0$-coordinates so, for some $i<\omega_1$, $b\in\mathbb{B}_i$.

Let us use the construction in order to show the following:

Proposition 4.2. *It is consistent to have a sequence $\langle\mu_n \mid n<\omega\rangle$ of σ-additive nowhere total real valued measures on the real line so that every set which is nonmeasurable in every μ_n $(\kappa<\omega)$ or is not of measure 0 for some μ_n has a positive inner measure in one of the μ_n's.*

Proof. Let V, $V[G]$, $\langle\mu_i \mid i<\omega_1\rangle$, μ be as in A.1. Collapse ω_1 to ω. Let $H:\omega\leftrightarrow(\omega_1)^{V[G]}$. We claim that $\langle\mu_i \mid i<(\omega_1)^{V[G]}\rangle$ extended in the obvious way is as desired. Suppose $A\in V[G,H]$ is a set of reals nonmeasurable in each μ_i. There exists $A'\subseteq A$, $A'\in V[G]$ which is not of measure 0 for μ_0, since μ_0 is κ-additive and the forcing for collapsing ω_1 is of cardinality $\omega_1<\kappa$. By the choice of $\langle\mu_i \mid i<(\omega_1)^{V[G]}\rangle$ there is i so that $\mu_i(A')>0$. But this implies that the μ_i-inner measure of A is >0. □

Working a little harder, it is possible to construct a sequence $\langle\mu_n \mid n<\omega\rangle$ so that for every set A which is not μ_n-measurable for all $n<\omega$, for every $m<\omega$ there is $n(m)<\omega$ so that the $\mu_{n(m)}$-inner measure of A is $>m$.

Let us sketch the proof of the following.

Theorem 4.3. *The forcing with a κ-complete ideal over κ cannot be isomorphic to (Cohen) $*$ (random).*

Sketch of the Proof. Suppose otherwise. Let I be such an ideal, $j:V\to M$ a generic elementary embedding generated by a pair of reals $\langle r_0,r_1\rangle$ generic for the forcing (Cohen) $*$ (random). Let $\langle r_{0\alpha},r_{1\alpha}\rangle \mid \alpha<\langle\kappa\rangle$ be a sequence of pairs of reals in V representing $\langle r_0,r_1\rangle$ in the generic ultrapower. Then in V the following holds:

(a) There exists a sequence of reals $\langle S_\alpha \mid \alpha<\kappa\rangle$ so that $2^{\aleph_0}\cap L[\langle S_\alpha \mid \alpha<\kappa\rangle]$ is nonmeager.

(b) For every $\lambda<\kappa$, every sequence of reals $\langle S'_\beta \mid \beta<\lambda\rangle$ $2^{\aleph_0}\cap L[\langle S'_\beta \mid \beta<\lambda\rangle]$ is meager.

Just $\langle r_{0\alpha} \mid \alpha<\kappa\rangle$ will satisfy (a), and for (b) note that adding a random real turns the old reals into a meager set (see [K3]). So in M, (a) + (b) are true with κ replaced by $j(\kappa)$. But let us construct in $V[r_0, r_1]$ a sequence of reals of the length κ satisfying (a). It will give the contradiction since $V[r_0, r_1]$ and M have the same reals.

Since after adding a Cohen real the set of old reals remains nonmeager. (a) holds in $V[r_0]$. Let us show that it is still true after adding the random real r_1. View each $r_{0\alpha}$ as name of a real in the random real forcing (just deal with basic clopen sets). Define the desired sequence $\langle S_\alpha \mid \alpha<\kappa\rangle$ by $S_\alpha = r_{0\alpha}(r_1)$. Now let $\underset{\sim}{T}\in V[r_0]$ be a name in the random real forcing of nowhere dense set. For every $\varepsilon>0$ we can find a Borel set A of measure $\geqq 1-\varepsilon$ which forces finitely many possibilites for splittings in $\underset{\sim}{T}$ for level high enough. Most of the $r_{0\alpha}$'s ($\alpha<\kappa$) are Cohen generic over $L[r_0, \underset{\sim}{T}]$. So it is not hard for them to run away with this finite set.

5. Concluding remarks and questions

By 1.3, forcing with a κ-complete ideal cannot be isomorphic to a nontrivial forcing of cardinality less than κ. By Kunen [K2], it is impossible to improve this even to forcings of cardinality κ satisfying κ-c.c. In 1.2 it is shown that some addition requirement of forcing notion can prevent it from being isomorphic to the forcing with an ideal. But the following question is natural:

PROBLEM 1. Let I be a κ-complete nowhere prime ideal over κ. Can the forcing with I be isomorphic to a forcing notion of cardinality κ satisfying α-c.c. for some $\alpha<\kappa$?

It was shown in 1.4 that the forcing with a κ-complete ideal cannot be isomorphic to the Cohen forcing for adding less than κ^+-Cohen reals. It is natural to try to replace κ^+ by 2^κ. If $2^\kappa>\kappa^{+\omega}$, then this may be wrong, as was noted after 1.4 in Section 1. But the situation is unclear if $2^\kappa<\kappa^{+\omega}$.

PROBLEM 2. Suppose $2^\kappa<\kappa^{+\omega}$. Can the forcing with a κ-complete ideal over κ be isomorphic to the Cohen forcing for adding less than 2^κ Cohen reals?

A similar question can be stated for the random reals. Thus 2.6 says that $d(\mu)>\mathrm{add}(\mu)$ for a σ-additive, total, nowhere prime probability measure μ.

PROBLEM 3. Suppose $2^{\mathrm{add}(\mu)}<(\mathrm{add}(\mu))^{+\omega}$. Does then the following hold: $d(\mu)\geqq 2^{\mathrm{add}(\mu)}$?

It would be interesting to characterise or find a general condition on forcing preventing it from being isomorphic to a σ-ideal. On these lines let us state the following more concrete question:

PROBLEM 4. Show that forcing with a σ-ideal cannot be isomorphic to a nontrivial Borel or having a simple absolute definition forcing notion.

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References

[B–T] J. Baumgartner and A. Taylor, *Saturation properties of ideals in generic extensions, II*, Trans. Am. Math. Soc. **271** (1982), 587–609.

[E] P. Erdös, *Some remarks on set theory*, Proc. Am. Math. Soc. **1** (1950), 127–141.

[F–Gu] R. Frankiewicz and A. Gutek, *Some remarks on embeddings of Boolean algebras and the topological spaces I*, Bull. Acad. Polon. Sci. (Math.) **29** (1982), 471–476.

[Fr] D. Fremlin, *List of problems*, circulated notes.

[Fr2] D. Fremlin, *Consequences of Martin's Axiom*, Cambridge University Press, 1984.

[G–Sh] M. Gitik and S. Shelah, *Cardinal preserving ideals*, J. Symb. Logic, submitted.

[Gr] L. Grinblat, *On σ-algebras of abstract sets*, Isr. J. Math., to appear.

[H] S. Hechler, *Classifying almost-disjoint families with applications to $\beta N \setminus N$*, Isr. J. Math. **10** (1971), 413–432.

[J] T. Jech, *Set Theory*, Academic Press, 1978.

[J–P] T. Jech and K. Prikry, *Cofinality of the partial ordering of functions from ω_1 into ω under eventual domination*, Math. Proc. Camb. Phil. Soc. **95** (1984), 25–32.

[Je–So] R. Jensen and R. Solovay, *Some applications of almost disjoint sets*, in *Mathematical Logic and Foundations of Set Theory* (Y. Bar-Hillel, ed.), North-Holland, Amsterdam, 1968, pp. 84–104.

[Ka–M] S. Kanamori and M. Magidor, *The evolution of large cardinal axioms in set theory*, in *Higher Set Theory*, Lecture Notes in Math. #669, Springer-Verlag, Berlin, 1978, pp. 99–275.

[Kr–P] A. Krawczyk and A. Pelc, *On families of σ-complete ideals*, Fund. Math. **CIX** (1980), 155–161.

[K1] K. Kunen, *Set Theory*, North-Holland Studies in Logic and Foundations of Math., Vol. 102 (1980).

[K2] K. Kunen, *Saturated ideals*, J. Symb. Logic **43** (1978), 65–76.

[K3] K. Kunen, *On measure and category*, in *Handbook of Set Theoretic Topology* (K. Kunen and J. Vaughan, eds.), North-Holland, Amsterdam, 1984.

[M] M. Magidor, *On the existence of non-regular ultrafilters and the cardinality of ultrapowers*, Trans. Am. Math. Soc. **249** (1979), 97–111.

[Sh1] S. Shelah, *Proper forcing*, Lecture Notes in Math. **940**, Springer-Verlag, Berlin, 1982.

[Sh2] S. Shelah, *Iterated forcing and normal ideals on ω_1*, Isr. J. Math. **60** (1987), 345–380.

[Sh3] S. Shelah, *On normal ideals and Boolean algebras*, in *Around Classification Theory of Models*, Lecture Notes in Math. **1182**, Springer-Verlag, Berlin, 1986, pp. 247–260.

[Sol] R. Solovay, *Real-valued measurable cardinals*, in *Axiomatic Set Theory* (D. Scott, ed.), Proc. Symp. Pure Math. **13** (I) (1971), 397–428.

[So2] R. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. Math. **92** (1970), 1–56.

[T] A. Taylor, *On saturated sets of ideals and Ulam's problem*, Fund. Math. CIX (1980), 37–53.

[Ta] A. Tarski, *Some problems and results relevant to the foundations of set theory*, in *Logic, Methodology and Philosophy of Science* (E. Nagel, P. Suppes and A. Tarski, eds.), Stanford, California, 1962, pp. 125–135.

[Tr] J. Truss, *Sets having calibre* $\aleph_1$, in *Logic Colloquium 76* (R. Gandy and M. Hyland, eds.), North-Holland, Amsterdam, 1977, pp. 595–612.

[W] H. Woodin, $\aleph_1$*-dense ideal on* $\aleph_1$, unpublished.