Sh:414

Acta Math. Hung.
61 (1–2) (1993), 115–120.

A CONSISTENT EDGE PARTITION THEOREM FOR INFINITE GRAPHS

P. KOMJÁTH (Budapest) and S. SHELAH (Jerusalem)

0. Introduction

The fundamental problem of partition theory of infinite graphs is if for every graph Y and cardinal μ there exists a graph X such that if the vertices (or edges) of X are colored with μ colors then there is a copy of Y with all the vertices (edges) getting the same color. This is denoted as $X \to (Y)^1_\mu$ and $X \to (Y)^2_\mu$; if these statements fail, then, of course, the arrow is crossed. Let $K(\alpha)$ denote the complete graph on α vertices, and let $K(\alpha) \leqq X$ denote that the graph contains $K(\alpha)$ as subgraph. If κ is an infinite cardinal, then obviously $K(\kappa^+) \to K(\kappa^+)^1_\kappa$, and by the Erdős–Rado theorem [6], $K((2^\kappa)^+) \to (K(\kappa^+))^2_\kappa$, and this result gives the existence of X for any Y, μ.

To make the problem harder, one might require the copy to be *induced.* This relation is denoted as $\rightarrowtail$. Though the vertex problem is still fairly easy, the edge case even for finite X, μ was only solved around 1973 by Deuber, Nesetril–Rödl, and Erdős–Hajnal–Pósa [1, 11, 5]. The latter authors even showed that for μ finite, Y countable, there is an appropriate X. Hajnal and Komjáth [9] proved that it is consistent that there exists a Y of size $\aleph_1$ such that no X (of any size) has $X \rightarrowtail (Y)^2_2$. Shelah [14] proved that it is consistent that for any Y, μ there is an X with $X \rightarrowtail (Y)^2_\mu$. Hajnal recently proved [8] that if Y is finite, μ is infinite, an appropriate X exists, in ZFC.

Another way of making the problem harder is to pose restrictions on X. We may require that if $K(\alpha) \nleqq Y$, then $K(\alpha) \nleqq X$, either. This excludes the possibility of getting an easy solution by using the above-mentioned Erdős–Rado theorem. For finite X, μ, Folkman showed the existence of such an X with $X \to (Y)^1_\mu$ and also, for finite α, μ the existence of a finite X with $K(\alpha+1) \nleqq X \to (\alpha)^2_\mu$. [7]. Nesetril and Rödl solved the edge case, for finite Y, μ [12]. The infinite case for vertices, but if α is finite, was solved by Komjáth and Rödl [10]. The case of general α is given by Hajnal and Komjáth [9]. As for the edge coloring, Hajnal and Komjáth proved in [9] that it is consistent that there is a Y of size $\aleph_1$, with $K(3) \nleqq Y$ and if $X \to (Y)^2_\omega$ then $K(\omega) \leqq X$. It was an old problem of Erdős and Hajnal if a graph Y

with $K(4) \nleqq Y \to K(3)^2_\omega$ exists. S. Shelah in [14] proved that such a Y may consistently exist. Another old Erdős–Hajnal question was if a Y with $K(\omega_1) \nleqq Y \to (K(\omega))^2_\omega$ may exist. Here we solve (at least consistently) this problem by showing the consistency of the statement that if Y is a graph, μ a cardinal, then there exists a graph X with $X \rightarrowtail (Y)^2_\mu$ and if $K(\alpha) \nleqq Y$ then $K(\alpha) \nleqq X$, either.

We first show that if $2^\mu = \mu^+$, $\kappa > \mu$ is measurable, Y is a graph on μ, then there is a $\leqq \mu^+$-closed poset of size κ, adding a graph X on κ as above. From this, we can get the general result, if we assume that $\{\kappa_\alpha : \alpha \text{ ordinal}\}$ is a class of measurable cardinals, and take the iteration $\{P_\alpha, Q_\alpha : \alpha \text{ ordinal}\}$ of posets, where Q_α is the poset of Theorem 1 with $\mu = \kappa_\alpha^+$, $\kappa = \kappa_{\alpha+1}$, and Y is some graph on μ. We take inverse limits at singular ordinals, direct limits otherwise. This will guarantee enough closure properties for getting a model of ZFC, and for that the graphs preserve their partition property at later iterations.

1. The consistency proof

THEOREM 1. *If $2^\mu = \mu^+$, Y is a graph on μ, $\kappa > \mu$ is a measurable cardinal, then there exists a $\leqq \mu^+$-closed partial order P, $|P| = \kappa$, adding a graph X such that $X \rightarrowtail (Y)^2_\mu$, and whenever $K(\alpha) \leqq X$, then $K(\alpha) \leqq Y$.*

PROOF. The vertex set of X will be $[\kappa]^2$. We define a partial ordering $<$ on it by putting $\{\beta_0, \alpha_0\}_< < \{\beta_1, \alpha_1\}_<$ iff $\beta_0 < \beta_1$ and $\alpha_0 < \alpha_1$. A condition is of the form $p = (s, g, \varphi)$ where $s \subseteqq [\kappa]^2$, $|s| \leqq \mu^+$, $g \subseteqq [s]^2$. If $\{\{\beta_0, \alpha_0\}_<, \{\beta_1, \alpha_1\}_<\} \in g$, then either $\beta_0 < \beta_1 < \alpha_0 < \alpha_1$ or $\beta_1 < \beta_0 < \alpha_1 < \alpha_0$. φ is a function with $\mathrm{Dom}(\varphi) = \{A \leqq s : |A| > 2, [A]^2 \subseteqq g\}$. For $A \in \mathrm{Dom}(\varphi)$, $\varphi(A) \leqq \mu$ spans a complete graph in Y, $|\varphi(A)| = |A|$. We also require that if B properly end-extends A, then $\varphi(B)$ should properly end-extend $\varphi(A)$.

Condition $p' = (s', g', \varphi')$ extends $p = (s, g, \varphi)$ if $s' \supseteqq s$, $g = [s]^2 \cap g'$, $\varphi' \supseteqq \varphi$, and if $A \subseteqq s$, $|A| > 2$ spans a complete graph in g, $x \in s' - s$, and $A \cup \{x\}$ is complete in g', then $A < x$, i.e. $y < x$ holds for every $y \in A$.

Let P be the set of conditions defined so far.

LEMMA 1. *$(P, \leqq)$ is transitive.*

PROOF. Straightforward.

LEMMA 2. *If $p = (s, g, \varphi) \in P$, $A \subseteqq \kappa$, then $p \mid A \in P$. If $A \cap s$ is an initial segment in s, then $p \leqq p \mid A$.*

PROOF. Immediate from the definitions.

LEMMA 3. *$(P, \leqq)$ is $\leqq \mu^+$-closed.*

PROOF. Assume that $p_\xi = (s_\xi, g_\xi, \varphi_\xi)$ is a decreasing, continuous sequence of conditions ($\xi < \xi_0 \leqq \mu^+$). Take $p = (s, g, \varphi)$, where $s = \bigcup\{s_\xi : \xi < \xi_0\}$, $g = \bigcup\{g_\xi : \xi < \xi_0\}$, and whenever $A \subseteqq s$ spans a complete subgraph in

g, $|A| > 2$, then $\varphi(A) = \bigcup\{\varphi_\xi(A \cap s_\xi): \xi < \xi_0, |A \cap s_\xi| > 2\}$. For $\xi < \zeta < \xi_0$, $\varphi_\zeta(A \cap s_\zeta)$ end-extends $\varphi_\xi(A \cap s_\xi)$, so $\varphi(A)$ induces a complete subgraph in Y, and $|\varphi(A)| = |A|$. If B end-extends A, then select $\xi < \xi_0$ with $s_\xi \cap \cap (B - A) \neq \emptyset$. By the definition of order on P, $A \subsetneqq s_\xi$, so $\varphi(A) = \varphi_\xi(A)$ and $\varphi(B)$ end-extends $\varphi_\xi(B \cap s_\xi)$ which in turn end-extends $\varphi(A)$. To check $p \leqq p_\xi$ $(\xi < \xi_0)$, the only nontrivial thing is the clause on $A \cup \{x\}$. If $A \in \in \mathrm{Dom}(\varphi_\xi)$, $x \in s - s_\xi$, we can assume that $x \in s_{\xi+1} - s_\xi$, so $A < x$, and we are done.

LEMMA 4. *If $p_i = (s_i, g_i, \varphi_i)$ are conditions for $i < 2$, they agree on $s_0 \cap s_1$, then $q = (s_0 \cup s_1, g_0 \cup g_1, \varphi_0 \cup \varphi_1)$ is a condition. If $s_0 \cap s_1 < < (s_0 - s_1) \cup (s_1 - s_0)$, then $q \leqq p_0, p_1$.*

PROOF. Straightforward.

If $G \subsetneqq P$ is a generic subset, we let $X = \bigcup\{g: (s, g, \varphi) \in G\}$.

LEMMA 5. *If $K(\alpha) \leqq X$ for some α, then $K(\alpha) \leqq Y$.*

PROOF. $K(\mu + 1) \not\leqq X$, as if $A \subsetneqq [\kappa]^2$ spans a complete graph of type $\mu + 1$, pick $p = (s, g, \varphi) \in G$ fixing A. This is possible by Lemma 3. But then, $\varphi(A)$ would give a $K(\mu+1)$ in Y, a contradiction. If $K(\alpha) \leqq X$, $\alpha \leqq \mu$, argue similarly.

In order to finish the proof of Theorem 1, assume without loss of generality that $1 \Vdash F: X \to \mu$. By Fact 2.4 in [14] there is a set A of measure one, $\{N_s: s \in [A]^{<\omega}\}$ such that

(1) $N_s \prec (H(2^\kappa); \in, F, \Vdash, \ldots)$;
(2) $[N_s]^{\mu^+} \subsetneqq N_s$;
(3) $|N_s| = 2^{\mu^+}$;
(4) $N_{s_0} \cap N_{s_1} = N_{s_0 \cap s_1}$;
(5) there is an isomorphism $H(N_{s_0}, N_{s_1})$ between N_{s_0} and N_{s_1} for $|s_0| = |s_1|$, mapping s_0 onto s_1;
(6) $N_s \cap A = s$;
(7) if s_0 is end-extended to s_1, then N_{s_0} is end-extended by N_{s_1}.

Let $A' \subsetneqq A$ be a set of indiscernibles for $\{N_s: s \in [A]^{<\omega}\}$. Enumerate the first $\mu 2$ elements of A' in increasing order as $\{\beta(i) : i < \mu\} \cup \{\alpha(i) : i < \mu\}$. Put $t(i) = \{\beta(i), \alpha(i)\}$, $M_i = N_{t(i)}$ for $i < \mu$.

DEFINITION. For $p, q \in P$, $p \sim q$ denotes that $p \mid N_\emptyset = q \mid N_\emptyset$.

LEMMA 6. *If $p(i) \in M_i$, $p(j) \in M_j$, $p(i) \sim p(j)$, then $p(i)$, $p(j)$ are compatible.*

PROOF. By (4), the non-edge amalgamation works.

We next show that one-edge amalgamation can also be constructed.

DEFINITION. If $i < j < \mu$, $p(i) = (s(i), g(i), \varphi(i)) \in M_i$, $p(j) = = (s(j), g(j), \varphi(j)) \in M_j$, $p(i) \sim p(j)$, then put $p(i) + p(j) = (s, g, \varphi)$ with $s = s(i) \cup s(j)$, $g = g(i) \cup g(j) \cup \{\{t(i), t(j)\}\}$, $\varphi = \varphi(i) \cup \varphi(j)$.

LEMMA 7. *$p(i)+p(j)$ is a condition, extending both $p(i)$ and $p(j)$.*

PROOF. As $\beta(i) < \beta(j) < \alpha(i) < \alpha(j)$, it is possible to join $t(i)$ and $t(j)$. As $\sup(N_\emptyset) < \beta(i) < \beta(j)$, $t(i)$ and $t(j)$ are not joined into $N_\emptyset$, so no new complete subgraph with more than two elements is formed.

DEFINITION. If $i < j < \mu$, $p(i) \in M_i$, $p(j) \in M_j$, $\xi < \mu$, we call the pair $(p(i),p(j))$ ξ-good, if $p(i) \sim p(j)$, and for every selection of $p'(i) \leqq p(i)$, $p'(j) \leqq p(j)$ with $p'(i) \in M_i$, $p'(j) \in M_j$, $p'(i) \sim p'(j)$, there is a $q \leqq p'(i)+p'(j)$ such that $q \Vdash F(\{t(i),t(j)\}) = \xi$.

LEMMA 8. *If $i < j < \mu$, $p(i) \in M_i$, $p(j) \in M_j$, $p(i) \sim p(j)$, then there exist $\xi < \mu$, $p'(i) \leqq p(i)$, $p'(j) \leqq p(j)$, $p'(i) \in M_i$, $p'(j) \in M_j$ such that $(p'(i),p'(j))$ is ξ-good.*

PROOF. Assume that the statement is false. Put $p(i,0)=p(i)$, $p(j,0)=p(j)$, and we are going to construct decreasing, continuous sequences $p(i,\xi)$, $p(j,\xi)$ for $\xi \leqq \mu$. If $p(i,\xi)$, $p(j,\xi)$ are defined, let $p(i,\xi+1) \sim p(j,\xi+1)$ be such that no $q \leqq p(i,\xi+1)+p(j,\xi+1)$ can force $F(\{t(i),t(j)\}) = \xi$. If $q \leqq p(i,\mu)+p(j,\mu)$ determines $F(\{t(i),t(j)\})$, then we get a contradiction.

By transfinite recursion on $\alpha < \mu^+$, we select, for every $f\colon \alpha \to 2$, a condition $p(i,f) \in M_i$, and an ordinal $\xi(f) < \mu$ such that

(8) $H(M_i,M_j)(p(i,f)) = p(j,f) \quad (i < j < \mu)$;
(9) $(p(i,f^\frown 0),p(j,f^\frown 1))$ is $\xi(f)$-good $(i < j)$;
(10) $p(i,f') \leqq p(i,f)$ when $f' \supseteqq f$;
(11) $p(i,f) \sim p(j,g)$ when $f,g\colon \alpha \to 2$, $i < j$.

For α limit, we can take unions. Given $\{p(i,f)\colon f\colon \alpha \to 2,\ i < \mu\}$ we select $p(i,f^\frown 0)$, $p(i,f^\frown 1)$ by a transfinite recursion of length $|2^\alpha| \leqq \mu^+$, using Lemma 8. To insure (11), we must keep extending $p(i,f) \mid N_\emptyset$, this can be done by Lemmas 3 and 4.

By the Baire category theorem, there exist $\xi < \mu$, and increasing $\tau_i < \mu^+$ $f_i\colon \alpha \to 2$ $(i < \mu)$ for some $\alpha < \mu^+$, such that

(12) $f_i(\tau_i) = 0$, $f_j(\tau_i) = 1$, $f_i \mid \tau_i \subsetneqq f_j \mid \tau_j$ $(i < j)$;
(13) $\xi(f_i \mid \tau_i) = \xi$.

Put $Y = \{\{\delta(i),\varepsilon(i)\} : i < \mu\}$.

We are going to construct $q(\gamma,i)$ for $\gamma \leqq \mu$, $i < \mu$. Put $q(0,i) = p(i,f_i)$, for γ limit, $q(\gamma,i) = \bigcup\{q(\gamma',i)\colon \gamma' < \gamma\}$. If the construction is given, up to the γth level, let $u(\gamma) \in N_{t(i)\cup t(j)}$ be such that

$$u(\gamma) \leqq q(\gamma,\delta(\gamma)) + q(\gamma,\varepsilon(\gamma))$$

and $u(\gamma) \Vdash F(\{t(\delta(\gamma)),t(\varepsilon(\gamma))\}) = \xi$. We then take $q(\gamma+1,i) = q(\gamma,f) \cup \cup\, u(\gamma) \mid M_i$.

LEMMA 9. *$u(\gamma)$ exists.*

PROOF. By Lemma 8 and by $q(\gamma,i) \sim q(\gamma,j)$. This latter property holds for γ limit by continuity, for $\gamma = 0$ by definition and (11), and for $\gamma+1$ by definition.

LEMMA 10. $q(\gamma+1,i) \leqq q(\gamma,i)$.

PROOF. By Lemma 4.

If $u(\gamma) = (s(\gamma), g(\gamma), \varphi(\gamma))$ for $\gamma < \mu$, then put $u = (s, g, \varphi)$ where $s = \bigcup \bigcup\{s(\gamma) : \gamma < \mu\}$, $g = \bigcup\{g(\gamma) : \gamma < \mu\}$, and φ is such that it extends all $\varphi(\gamma)$, and $\varphi(\{t(i) : i \in A\}) = A$, when $|A| > 2$, and A spans a complete subgraph in Y.

LEMMA 11. $u \in P$.

PROOF. It suffices to show that if $B \subseteqq s$, $|B| > 2$, spans a complete subgraph then it is either in the domain of some $\varphi(\gamma)$ or it is of the form $B = \{t(i) : i \in A\}$ for some $A \subseteqq \mu$.

If two M_i-s cover B, then one of them covers, too, or else $\{t(i), t(j)\} \subseteqq B$, but then $B \cap N_\emptyset = \emptyset$, so $B = \{t(i), t(j)\}$. If no two M_i-s cover B, then $B \subseteqq \{t(i) : i < \mu\}$, and we are done, again.

LEMMA 12. $u \leqq u(\gamma)$.

PROOF. There is no complete subgraph in u which is extended the wrong way. The only candidate for this is a set of type $\{t(i) : i \in A\}$ of which only two vertices are in $u(\gamma)$.

LEMMA 13. $u \Vdash \{t(i) : i < \mu\}$ *span a monocolored copy of* Y.

PROOF. Obvious.

Clearly, Lemma 13 concludes the proof of Theorem 1.

THEOREM 2. *If the existence of class many measurable cardinals is consistent, then it is consistent that for every* Y, μ *there exists an* X *with* $X \rightarrowtail \rightarrowtail (Y)^2_\mu$ *such that if* $K(\alpha) \leqq X$, *then* $K(\alpha) \leqq Y$.

PROOF. By iterating the poset in Theorem 1.

The assumption on the existence of measurables can be eliminated, see [14] Sections 3, 4.

References

[1] W. Deuber, Partitionstheoreme für Graphen, *Math. Helv.*, **50** (1975), 311–320.

[2] P. Erdős and A. Hajnal, On decompositions of graphs, *Acta Math. Acad. Sci. Hung.*, **18** (1967), 359–377.

[3] P. Erdős and A. Hajnal, Unsolved problems in set theory, part I, *Proc. Symp. Pure Math.*, **13** (1971), 17–48.

[4] P. Erdős and A. Hajnal, Unsolved and solved problems in set theory, *Proc. Symp. Pure Math.*, **25** (1974), 269–287.

[5] P. Erdős, A. Hajnal and L. Pósa, Strong embeddings of graphs into colored graphs, in *Infinite and Finite Sets* (Keszthely, 1973), Coll. Math. Soc. J. Bolyai, 10, pp. 585–595.

[6] P. Erdős and R. Rado, A partition calculus in set theory, *Bull. Amer. Math. Soc.*, **62** (1956), 427–489.

[7] J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, *SIAM J. Appl. Math.*, **18** (1970), 19–24.

[8] A. Hajnal, Embedding finite graphs into graphs colored with infinitely many colors, *Israel J. Math.*, **73** (1991), 309–319.

[9] A. Hajnal and P. Komjáth, Embedding graphs into colored graphs, *Trans. Amer. Math. Soc.*, **307** (1988), 395–409.

[10] P. Komjáth and V. Rödl, Coloring of universal graphs, *Graphs and Combinatorics*, **2** (1986), 55–61.

[11] J. Nesetril and V. Rödl, Partitions of vertices, *Comm. Math. Univ. Carolin.*, **17** (1976), 85–95.

[12] J. Nesetril and V. Rödl, The Ramsey property for graphs with forbidden subgraphs, *J. Comb. Th. (B)*, **20** (1976), 243–249.

[13] J. Nesetril and V. Rödl, Partitions of finite relational and set systems, *J. Comb. Th. (A)*, **22** (1977), 289–312.

[14] S. Shelah, *Consistency of Positive Partition Theorems for Graphs and Models*, Springer Lect. Notes, 1401, 167–193.

(Received August 2, 1990; revised January 28, 1991)

DEPARTMENT OF COMPUTER SCIENCE
R. EÖTVÖS UNIVERSITY
BUDAPEST, MÚZEUM KRT. 6–8
H-1088
HUNGARY

INSTITUTE OF MATHEMATICS
HEBREW UNIVERSITY
JERUSALEM, GIVAT RAM
ISRAEL